Computer Science at Kent

Travelling Salesman Heuristics: Exercises in Haskell

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Technical Report No. 10- 04
June 2004

Published by the Computing Laboratory,
University of Kent, Canterbury, Kent, CT2 7NF, UK

1 Introduction

This document contains a collection of programming exercises in the functional programming language Haskell. The exercises are all concerned with the infamous Travelling Salesman Problem (TSP for short), both its exact solution and heuristic approximations. Full solutions to the exercises have not been included (these are available on request). However, the document contains both hints for students and comments for teachers (*the latter in small italics*). The next section gives the teaching context in which this material was used – in particular, it lists assumptions made about the students' knowledge. Section 3 describes TSP informally. The following sections present some preliminary material: on recursion over lists (Section 4), on type "implementation" through type aliases (Section 5), and on computing the maximum of a list "under a function" (Section 6). The concept of a distance matrix is discussed in Section 7. After a short discussion of permutations, the exact, "brute force", TSP algorithm is given in Section 8. The following sections consider a number of heuristic approximations to TSP. The final section contains samples of the exam questions that were asked about model solutions to these exercises.

2 History

The module CO312 Case Studies was introduced into the first year of the BSc Computer Science programme at the University of Kent in 1996. Its purpose was to reinforce teaching in other first year modules through the use of larger case studies. I was responsible for the "functional programming" component of the module from the start, to supplement the first year module on Functional Programming and Logic. Initially this was taught in Miranda; in recent years, Haskell was used. Textbooks by Simon Thompson were used throughout – since 1999, the second edition of "Haskell: The Craft of Functional Programming" [1]. This edition differs from the previous one and the Miranda textbook by starting with a combinator library for "pictures" before the traditional build up through topics such as recursion and list algorithms. A notation from this book used here is that `~>` means "evaluates to".

By the time the students started on their CO312 Haskell case studies, they would typically have covered:

- basic types `Bool`, `Int` and `Char` with operations;
- tuples; lists, list comprehensions, list library functions;

- list programs through pattern matching and recursion.

They would have seen library functions on lists, but often not the higher order ones, and without much awareness of parametric polymorphism; they would *not* have seen algebraic types, abstract types or type classes.

The module was assessed through 50% coursework and 50% exam. A series of seven lectures would build up to the coursework assessment, by offering preparatory exercises, to be solved individually or in a class. Correct and incorrect solutions to these exercises would be discussed in the following lecture. Students would then have a few weeks to do the final assessment. The exam would consist of a model solution to this assessment (also handed out well in advance) with a number of questions related to modifications or extensions of the code, e.g. based on changes to the problem or the suggested solution.

3 What is TSP?

Given a list of towns to be visited and a method of determining the distances between towns, the travelling salesman problem is to find a route that visits all required towns, and of those routes, the shortest one. The particular variant we are looking at here assumes those distances are represented in a matrix, and that we need to start at one of the towns and end up there again after visiting all the others.

This is a known hard problem – it is a so-called "NP-complete" one, which implies that there is no known algorithm that is guaranteed always to give the best possible solution in a reasonable[1] amount of time. Furthermore, if anyone finds such an efficient algorithm for TSP, it can be modified to solve a large class of similarly difficult problems.

4 Recursion over lists

You will have seen the following three ways of writing programs over lists:

- list comprehensions, for example:

```
[dist y x | y <- ys, y/=x ]
```

---

[1] In technical terms, an algorithm whose running time is bounded by a polynomial in the size n of the input – here, the number of towns. The brute force TSP algorithm runs in time proportional to $n!$, which is not bounded by any polynomial.

Only functions returning a list can be computed by using (only) a list comprehension.

- Higher order functions such as `map, filter, foldl` usually lead to short programs, but how clear these appear depends on experience.
- With explicit recursion, anything is possible. Depending on the structure of your data, you might use *patterns* or *guards* and conditionals to distinguish the various cases (for example, when to make a recursive call and when not to).

The exercises in this document assume only limited experience with functional programming, for this reason we look at explicit recursion rather than higher order functions, giving a quick rundown of the common patterns to use in recursions over lists.

The standard pattern has a case for `[]` and one for `(x:xs)` in terms of a recursive call using `xs`:

```
f []     = 0
f (x:xs) = x + f xs
```

For functions which only make sense on non-empty lists, the standard pattern is:

```
g [x]    = x
g (x:xs) = max (g xs) x
```

However, sometimes both the empty *and* the singleton list need to be singled out.

```
h []     = 0
h [x]    = x + 1
h (x:xs) = 1 + h xs
```

The next few patterns involve looking at the first *two* elements of the list. The important decision there is whether, in the recursive call, *one* or *two* elements are removed from the list.

An example of losing the first two elements is as follows. Note that this also implies that you need special cases for *both* empty and singleton list (or alternatively: for lists of length 1 and 2).

```
odds []       = []
odds [x]      = [x]
odds (x:y:xs) = x: odds xs
```

The list of differences between adjacent elements needs to preserve the second element as the next head of the list in the recursive call:

```
diffs []       = []
diffs [x]      = []
diffs (x:y:xs) = (x-y):diffs (y:xs)
```

In the above examples, it is convenient that you can write `(x:y:xs)` for `(x:(y:xs))`.

5 Type aliases

Thompson [1, section 5.1] says: "we can give names to types in Haskell, so that types are easier to read". That is one aspect of their use – we might also look at it as giving an "implementation" of a more abstract type that we need in our program. For example, if we needed to discuss *time*, we might say

```
type Time = (Int,Int,Int)
```

intending to interpret the first number as hours, the second as minutes, and the third as seconds. Another example might be representing sets of numbers by lists, i.e.,

```
type SetInt = [ Int ]
```

This sort of thing can be done in a more general way with type parameters; Haskell also offers more advanced ways of "implementing abstract data types". This mechanism will suffice for now, though.

If we implement more "abstract" values this way, we need to remember the relation between those values and the Haskell values, e.g., for sets,

`[]`	represents	$\emptyset$
`[1]`	represents	$\{1\}$
`[3,1]`	represents	$\{1,3\}$

This relation may be partially documented in definitions. The names of the functions and constants defined form part of the *interface* to the type, e.g.,

```
empty :: SetInt
empty = []

union :: SetInt -> SetInt -> SetInt
union xs ys = xs ++ ys
```

```
noon :: Time
noon = (12,0,0)

hours, mins, secs :: Time -> Int
hours (h,m,s) = h
mins (h,m,s) = m
secs (h,m,s) = s
```

Also, we might define a straight type alias such as

```
type Town = Int
```

With this definition, Haskell will not find errors when you use a `Town` where an `Int` is expected or vice versa. It serves mainly as documentation (adding up `Towns` probably is not meaningful), and may make it easier to change your implementation of the newly defined type.

There are two main issues with this sort of implementation: *junk* and *confusion*. (These are both technical terms!) Neither of these needs to be avoided, but they need to be taken into account in programs.

5.1 Confusion

Confusion is when multiple Haskell values represent the same abstract value. For example, `[1,1,3]` and `[1,3]` *also* represent $\{1,3\}$. There is no confusion for `Time` unless one took `(0,0,0)` and `(24,0,0)` both to represent midnight.

For `SetInt`, we could reduce confusion by only considering lists without duplicates (some functions are defined in `List.hs` for this), or even sorted lists, but that complicates the code for the set operations. In general, for representing sets that way, we would need to be able to define equality and ordering on the elements.

The above remarks refer to type class constraints `Ord` *and* `Eq`*, of course.*

The consequence of having confusion in your data type is that `==` does not tell you much about equality of *abstract* values – it may return `False` for conceptually equal values.

For `SetInt`, equality might be defined by

```
equal :: SetInt -> SetInt -> Bool
equal xs ys = and [ elem x ys | x <- xs] &&
              and [elem y xs | y <- ys]
```

(each of the `and` expressions represents an inclusion of sets).

With type classes this will show up as an instance of `Eq`.

5.2 Junk

Junk is when some Haskell values do *not* represent any abstract value. If we do not insist on ordering and removal of duplicates, there is no junk for `SetInt`. For `Time`, there is plenty:

```
nnn :: Time
nnn = (0,9,99)
```

is dubious already: 9 minutes and 99 seconds? I own a microwave oven which treats this as confusion rather than as junk, and will happily heat for 10 minutes and 39 seconds if I enter '999'. However, it is even harder to interpret

```
zmtmohts :: Time
zmtmohts = (0,-12,-127)
```

(14 minutes and 7 seconds before midnight!?), but Haskell will not produce a type error for this.

The consequence of having junk in your data type is that you might need to check for validity of inputs. If you're writing a function on such a type, you need to make sure that you don't test it with values which are "junk". One way of preventing that involves validation functions, e.g.:

```
validTime :: Time -> Bool
validTime (h,m,s)
  = h>=0 && m>=0 && s>=0 && h<24 && m<60 && s<60
```

5.3 Two-dimensional arrays

A type which we will be using later is that of "matrices", or two-dimensional arrays, containing numbers. You can also think of them as rectangular areas in a spreadsheet. We will borrow mathematical notation for them, writing

$$\begin{pmatrix} 1 & 2 & 6 \\ 3 & 4 & 8 \end{pmatrix}$$

for a 2-by-3 matrix containing, in the first row, numbers 1, 2 and 6; in the second row, numbers 3, 4 and 8. (Alternatively, in the first column it has numbers 1 and 3, etc.) Also,

$$\begin{pmatrix}1\\3\\17\end{pmatrix}$$

is a matrix with three rows, containing a single element each (i.e., it has one column). Thus, it is different from

$$\begin{pmatrix}1&3&17\end{pmatrix}$$

which has a single row and three columns.

These are a bit like Java `int [] []`, and indeed we will implement them as `[[Int]]`, but we will insist on them being "rectangular", i.e., every row should have the same number of elements.

```
type Matrix = [ [ Int ] ]
```

We take `Matrix` as list of rows, i.e.

`[[1,2] , [3,4]]` represents $\begin{pmatrix}1&2\\3&4\end{pmatrix}$

`[[1,4,2]]` represents $\begin{pmatrix}1&4&2\end{pmatrix}$

`[[1],[4],[2]]` represents $\begin{pmatrix}1\\4\\2\end{pmatrix}$

There is **junk** for this representation of matrices: some concrete values *do not* represent abstract values, e.g.

`[[1] , [3,4]]` $\begin{pmatrix}1&\\3&4\end{pmatrix}$???

There is also (less obviously) confusion: for the matrix containing no values at all: how many rows and columns might it have?

`[[]]` = `[]` = `[[],[]]`
$1 \times 0 = 0 \times 0 = 2 \times 0$

5.4 Indexing

For getting a value out of a matrix, *indexing* is the natural approach. Indices on a list `xs` run from `0` to `(length xs)-1`.

```
[ [ 1, 2, 6 ], [3, 4, 8 ] ] !! 1 !! 2 == 8
```

Note: **Indexing is often the wrong method of dealing with lists.**

Do not get tempted to use indexing for iterative/recursive/loop programs over lists (*unlike* for Java arrays). Recursion over lists is usually much better, less error prone. Compare the following two programs for merging sorted lists:

```
merger :: [Int] -> [Int] -> [Int]
merger xs [] = xs
merger [] ys = ys
merger (x:xs) (y:ys)
 | x<y       = x: merger xs (y:ys)
 | otherwise = y: merger (x:xs) ys

merge2 xs ys
 = mergeit 0 0
   where
     mergeit ix iy
      | ix == length xs && iy == length ys = []
      | ix == length xs     = ys!!iy : mergeit ix (iy+1)
      | iy == length ys     = xs!!ix : mergeit (ix+1) iy
      | (xs!!ix)<(ys!!iy)   = xs!!ix : mergeit (ix+1) iy
      | otherwise           = ys!!iy : mergeit ix (iy+1)
```

5.5 Exercises

1. Define a function

   ```
   numRows :: Matrix -> Int
   ```

   giving the number of rows of a matrix. (Remember a matrix is a list of rows.)

2. Define a function

   ```
   isEmpty :: Matrix -> Bool
   ```

   The following definition is *not* good enough:

   ```
   isEmpty m = m == []
   ```

   Why? There is *confusion*.

3. Define a function

   ```
   numCols :: Matrix -> Int
   ```

   giving the number of *columns* of a matrix (still a list of rows, unfortunately). Ensure that

   ```
   numCols []
   ```

   does *not* give an error message.

   Different answers to this will lead to different results for "junk" inputs, could explain that this is OK.

4. Define a function

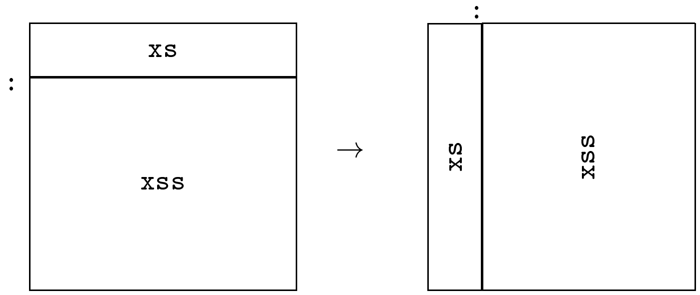
   ```
   rectangular :: Matrix -> Bool
   ```

   that checks whether all rows in the matrix have the same length.

   You could do this by defining an auxiliary function `allequal` that checks whether all numbers in a list are equal.

5. (Advanced) What is wrong with the following program for matrix transposal (swapping rows and columns, e.g., `[[1,2],[3,4]]` to `[[1,3],[2,4]]`)?

   ```
   tp :: Matrix  -> Matrix
   tp xss
    | isEmpty xss = []
    | otherwise =   [ a:as | (a,as) <- zip (head xss)
                                              (tp (tail xss))]
   ```



There is a function `transpose` *in* `List.hs`.

6 Maximum under a function

These exercises have also been used as coursework rather than voluntary exercises. There are many instances of this problem in later exercises, so it is good for the students to be aware of ways of solving it both correctly and efficiently.

The function `max` computes the maximum of two numbers, and `maximum` gives the maximum of a list. They might have been defined as follows:

```
max :: Int -> Int -> Int
max a b
 | a <= b = b
 | otherwise = a
maximum :: [Int] -> Int
maximum [x] = x
maximum (x:xs) = max x (maximum xs)
```

Note that the latter does not work for empty input, and is characterised by the fact that if

```
maximum [x1,x2,...,xn] ~> xi
```

then

```
elem xi [x1,x2,...,xn] ~> True
xi >= x1
xi >= x2
...
xi >= xn
```

If we have a function `f`, and a list `xs`, we could find the member of `xs` for which `f` is maximal by taking the `maximum` of all `x` in `xs` – but if we do this in the obvious way, we will have lost the information “which” `x` this maximum belonged to. Here, we work out a strategy to avoid this problem by keeping pairs of `x` and `f x` – a “tupling” strategy. It is all presented here as an exercise concerning a meaningless function `f`, but the strategy can be reused often in the various TSP programs.

Given is a function `f` defined by

```
f :: Int -> Int
f x = 8*(x^2)-2*(x^3)+4*x-17
```

A more advanced version has the function `f` *as a parameter, passed to all relevant functions.*

The final aim is to write a relatively efficient[2] function

```
maxf :: [Int] -> Int
```

such that if `maxf [x1,...,xn] = xi` then

```
elem xi [x1,...,xn] == True
f xi >= f x1
f xi >= f x2
...
f xi >= f xn
```

This is a generalisation of the definition of `maximum`, where we do not compare the elements of the list but their images under a given function `f`. It will be constructed bottom-up in the next few exercises.

6. Define a function

   ```
   tupleWithf :: [Int] -> [(Int,Int)]
   ```

   such that

   ```
   tupleWithf [x1,x2,...,xn]
   = [ (f x1, x1), (f x2, x2), ..., (f xn, xn) ]
   ```

   The particular order `(f x,x)` *is chosen to give students the option of exploiting the instance of* `Ord` *for tuples. A pitfall here is mixtures of patterns and recursion/comprehension which ignore the first element of the input.*

7. Define a function

   ```
   maxFirst :: (Int,Int) -> (Int,Int) -> (Int,Int)
   ```

   which returns, of its two input tuples, the one whose first component is the highest. For example,
   `maxFirst (1,2) (3,4)` returns `(3,4)`
   `maxFirst (5,3) (4,6)` returns `(5,3)`
   For `maxFirst (0,1) (0,2)` it does not matter whether your program returns `(0,1)` or `(0,2)`.

---

[2] In the sense that it does not evaluate `f x` twice unless `x` occurs more than once in the input list.

8. Define a function

   ```
   maximumFirst :: [(Int,Int)] -> (Int,Int)
   ```

   which returns, from its list of input tuples, the tuple whose first component is highest. You may assume that the input list is non-empty.

   Pitfall: defining it for empty lists anyway, and then picking the wrong unit of `max` *on* `(Int,Int)`*.*

9. Using the functions defined or otherwise, define the function `maxf` (described earlier) which returns the element of its non-empty input list for which `f` is maximal.

   Most solutions which do not use tupling will end up computing `f` *on average twice for all elements.*

   Questions 10 and 11 are two variations of this.

10. Using the functions defined above or otherwise, define the function `minf` which returns the element from its non-empty input list for which `f` is minimal.

    Two kinds of reuse possible: by analogy, or using reflection in the x-axis.

11. Using the functions defined above or otherwise, define the function

    ```
    maxIndexf :: [Int] -> Int
    ```

    such that if

    ```
    maxIndexf xs ~> i
    ```

    for non-empty `xs`, then

    ```
    0 <= i < length xs
    f (xs!!i) >= f (xs!!0)
    f (xs!!i) >= f (xs!!1)
    ...
    f (xs!!i) >= f (last xs)
    ```

i.e., it returns the *index* of the element of its input list for which `f` is maximal.

> *A modified tupling strategy is more efficient but possibly less clear than looking up the index of* `maxf xs` *in* `xs` *(*`List.hs` *provides a number of ways, some involving* `Maybe`*).*

7 Distance Matrices

For TSP, we need to represent distance information somehow. A natural way of doing this is in a *distance matrix*. A sample distance matrix is the following:

	0	1	2	3
Faversham = 0	0	7	12	10
Whitstable = 1	7	0	5	7
Herne Bay = 2	12	5	0	10
Canterbury = 3	10	7	10	0

This could be represented in Haskell by

```
[ [0,7,12,10], [7,0,5,7], [12,5,0,10], [10,7,10,0] ]
```

where we might store the information about which town names correspond to which indices separately.

Haskell arrays would be an alternative if they had been taught to the students already. Maybe even with constant retrieval cost?

Distance matrices have various properties. First, they are *square*: they have as many rows as columns, as we record all distances between a fixed set of towns. Second, they are likely to be *symmetric*: the distance from A to B is the same as the distance from B to A. As a consequence, most road atlases only present half of the distance matrix information in a triangular table[3]. Third, the distance between a town and itself is always 0. These distances can be found on what is called the *main diagonal*, which consists of all positions `m!!i!!i`. Finally, the distance of getting from A to B via C, i.e. the distance from A to C plus that from C to B, is never less than the

---

[3] The AA road atlas for the UK has such a triangular table – as a consequence, it does not represent the fact that crossing the Severn near Bristol is a different distance going into Wales or going into England.

recorded distance from A to B “directly”, i.e., “a detour is never shorter”. This is represented by the “triangular property”:

$$\forall i, j, k : d!!i!!k \leq d!!i!!j + d!!j!!k$$

$$\begin{tikzcd}
 & k \\
i \arrow[ur, "d!!i!!k"] \arrow[r, "d!!i!!j"'] & j \arrow[u, "d!!j!!k"']
\end{tikzcd}$$

12. Define a function

    ```
    square :: Matrix  -> Bool
    ```

    which checks that a matrix is rectangular, and has equal numbers of rows and columns.

    As in Exercise 3, there is a risk of a run-time error for empty list input.

13. Define a function

    ```
    symmetric :: Matrix -> Bool
    ```

    which tests if the matrix is symmetric (you may use `transpose` from `List.hs`).

14. Define a function

    ```
    all0 :: [Int] -> Bool
    ```

    which checks whether all numbers in a list are 0.

    Some students would disagree with logicians on the answer for the empty list, as do many solutions using `nub` *or recursion. This exercise is also likely to provide illustrations for gratuitous use of guards or conditionals (rather than Boolean operators) and phrases such as* `== True`*.*

15. Define a function

    ```
    maindiag :: Matrix -> [Int]
    ```

which returns the main diagonal of a square matrix. For example, the main diagonal of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is `[1,5,9]`.

Here, and the next exercise, is when indexing is useful; should probably give bonus marks for correct recursive solutions. Non-square matrices are "junk" here so the run-time errors they may cause are not an issue.

16. Define a function

```
triangle :: Matrix -> Bool
```

which checks a matrix for the triangular property.

8 Brute Force Travelling Salesman

The exact solution to the TSP can be obtained by generating all possible permutations of the list of towns to be visited, and then selecting the one which has the lowest cost. The cost is relative to a given distance matrix.

We call this a "brute force" solution because Hugs crashes, if we do this naively, with a control stack overflow, at a list of about 8 towns (40,320 possible tours). A slightly less naive version (see Section 8.4) still takes too long to compute any output for about 12 towns.

8.1 Permutations

The *permutations* of a list are all lists with all the same elements occurring equally often (and no others). In other words, a permutation is any list you can get by 0 or more times swapping elements in the list (you could imagine that this does not make for an efficient algorithm though!).

The permutations of `[1,3]` are {`[1,3]`, `[3,1]`}. Those of `[1,2,3]` are: {`[1,2,3]`, `[2,1,3]`, `[2,3,1]`, `[1,3,2]`, `[3,1,2]`, `[3,2,1]`}. More interestingly, the permutations of `[1,1,3]` are {`[1,1,3]`, `[1,3,1]`, `[3,1,1]`}. However, as this is a set we may actually include some of these elements more than once in the result, by not taking into consideration that `1` occurs twice. (Note: {} is not really Haskell, in real code we would have to use `[ ]` instead.)

The approach we will take to defining a function to generate permutations

```
perms :: [Int] -> [[Int]]
```

is a "psychic bottom-up" solution: we'll first define an auxiliary function for no reason at all, and then it will turn out to be an essential piece of the required function. (Though, as it happens, it will come in handy elsewhere, too.)

The auxiliary function is

```
ins :: Int -> [Int] -> [[Int]]
```

such that `ins y xs` is the set (list) of all possible ways of inserting `y` into `xs`, e.g.

```
ins 3 [2,4,5]
~>
[ [3,2,4,5], [2,3,4,5], [2,4,3,5], [2,4,5,3] ]
```

(or these same lists in some different order – duplicates, if any, need not be removed, so `ins a [b,c]` will end up having three elements even when `a`, `b` and `c` are all the same.)

We solve this by answering a list of questions. For different Haskell problems, you might ask yourself similar questions in order to get started on solutions.

- *What is the* type *of* `ins 2 []`*?*
  The type of any result of `ins` is `[[Int]]` so that must be the type here as well.
  This may have looked too obvious, but asking yourself this stops you from forgetting some square brackets later, hopefully.

- *How many elements has* `ins 2 []`*? Why?*
  In how many ways can you insert an element into an empty list? Only one: `[2]` ....

- *So what* is `ins x []`*?*
  So this is changing `2` to `x`. Probably the result is `[x]`, but that's not of type `[[Int]]`. So, the result must be `[[x]]` – the list of different ways of inserting `x` into an empty list is a singleton list containing the only way of doing it:

  ```
  ins x [] = [[x]]
  ```

- *Does* `ins 3 [4,5]` *relate to* `ins 3 [2,4,5]`*? How?*
  Taking some "sensible" ordering of the various ways of inserting, we might have

  ```
  ins 3 [4,5] ~>           [ [3,4,5],  [4,3,5],  [4,5,3]]
  ins 3 [2,4,5] ~>[[3,2,4,5],[2,3,4,5],[2,4,3,5],[2,4,5,3]]
  ```

  (Nicely lined up!) So, going from `ins 3 [4,5]` to `ins 3 [2,4,5]`, we put the new value 2 in front of each result, and add one more list: the one we get by inserting 3 before 2.

- *So is there a general recursive relation?*
  To insert `x` into `y:ys`, we either put `x` right before `y:ys`, or we insert `x` into `ys`, and put `y` in front of the result:

  ```
  ins x (y:ys) = (x:y:ys): [ y:xs | xs <- ins x ys ]
  ```

  or, using `++` rather than `:` and `map` instead of the list comprehension

  ```
  ins x (y:ys) = [ [x,y]++ys ] ++ map (y:) (ins x ys)
  ```

  or, using an auxiliary function to do the "put `y` in front of each . . . " bit

  ```
  ins x (y:ys) = (x:y:ys) : consall y (ins x ys)
  consall y [] = []
  consall y (xs:xss) = (y:xs) : consall y xss
  ```

Having defined `ins`, we can now define

```
perms : [Int] -> [[Int]]
```

that gives all permutations of a list. We will not worry about removing duplicates or the ordering of results, as before.

- *What is the* type *of* `perms []`*?*
  Again, it must be `[[Int]]` like any other `perms` result.

- *How many elements does* `perms []` *have? Why?*
  There's only 1 way of (possibly) reordering an empty list . . .

- *So what is* `perms []`*?*
  Not `[]` as that has the right type (as an empty list of lists), but too few elements (it has none).

```
perms [] = [[]]
```

- *What is* `perms [2,3]` *?*
  That will be a list containing the elements `[2,3]` and `[3,2]`, in some order (and possibly multiple times).
- *Which elements of* `perms [1,2,3]` *relate to which elements of* `perms [2,3]`*?*
  `perms [1,2,3]` has six different elements: `[1,2,3]`, `[1,3,2]`, `[2,1,3]`, `[2,3,1]`, `[3,1,2]`, `[3,2,1]`, which may occur in any order. Three of these have `3` before `2`, and three have `2` before `3`. The latter are `[ [1,2,3], [2,1,3], [2,3,1]]`.
- *What is* `ins 1 [2,3]`*?*
  It is that same list `[ [1,2,3], [2,1,3], [2,3,1]]`.
- *In general, can you find a way to relate* `perms (x:xs)` *to* `perms xs`*, using* `ins`*? You need to get the types right –* `ins` *returns a list of lists.*

  ```
  perms (x:xs) = concat [ins x ys | ys <- perms xs]
  ```

  Without the `concat`, the right hand side is a `[[[Int]]]` rather than a `[[Int]]` as required. (Note that `[concat(ins x ys)| ys <- perms xs]` also corrects this type error, but gives a different result.)

  In words: once we know how to permute `xs`, we can permute a list with an additional element `x` by putting it in any possible place (using `ins`) in every possible permutation of the shorter list `xs`.

8.2 Cheap tours

A town is an integer (an index into a distance matrix), a tour is a list of towns, interpreted in a circular way: `[1,2,3]` represents going from `1` to `2` to `3` to `1`. As a consequence, there is no fundamental difference between the tours `[1,2,3,4]` and `[3,4,1,2]` ("confusion").

```
type Town = Int
type Tour = [Town]
```

If the type `[Town]` is used below, it denotes a list of towns that should *not* be interpreted as a circular tour – but possibly even as a *set* of towns.

17. Define a function

```
cost :: Matrix -> Tour -> Int
```

such that

```
cost dist [a,b,c,d]
~> dist!!a!!b + dist!!b!!c + dist!!c!!d + dist!!d!!a
```

If you want to do this recursively, you might want to define an auxiliary function to do most, but not all, of the work. This is because `cost dist [a,b,c,d]` does not rely on `cost dist [b,c,d]`: the latter uses `dist!!d!!b` which is not included in the former!

There is no need to define `cost` for the empty tour.

18. Given a function `cost` as above, define a function

    ```
    cheapest :: Matrix -> [Tour] -> Tour
    ```

    which, given a list of tours (so a list of lists of numbers), select the (a) cheapest of them using the function `cost` and the given matrix.

    (Use the ideas from Section 6!)

    Using `sortBy` *is not a great idea as it does not implement tupling and as a consequence leads to recomputed* `cost`*. Students may not realise the positive consequences that lazy evaluation has on the efficiency of using sorting in place of minimum, anyway.*

8.3 Using the brute force program

With these ingredients, this is the "brute force" TSP program:

```
tsp :: Matrix -> [Int] -> Tour
tsp m xs = cheapest m (perms xs))
```

This is not the most efficient way of computing the exact solution, but it is a simple one. The more complicated "branch-and-bound" algorithm will do much better on average, though not necessarily in the worst case.

A Haskell module containing a definition of a sample 50x50 distance matrix called `dm` is at `www.cs.kent.ac.uk/people/staff/eab2/tsp/DM.hs`. The next few exercises assume you have downloaded this, and created a module `Brute.hs` containing the following lines:

```
module Brute
where
import DM
```

plus the definition of `tsp` as given above and all necessary auxiliary functions, given above or developed in previous exercises.

19. What is `tsp dm [2,4,6,8,10]`?
20. Looking at `names:: [String]` in `Dm.hs`, what does the result to the previous question represent in terms of travel in the UK?
21. What is the highest value of `n` such that

    ```
    tsp dm [1..n]
    ```

    is computed by Hugs without error messages?

8.4 Small Improvements

This section lists two small improvements on the brute force TSP. It is not essential to use them or understand them, but they may allow the exact TSP solution to be computed for a slightly larger number of towns.

For the purpose of the TSP on a symmetric distance matrix, all permutations of 3 (or fewer) towns are equivalent: there is only one triangle connecting the three towns, and it does not matter in which order it is traversed. So instead of the function `perms`, we could use

```
perms' (x:xs)
 | (length xs) < 3 = [(x:xs)]
 | otherwise       = concat [ ins x p | p <- perms' xs]
perms' [] = [[]]
```

This reduces the number of possibilities to be considered by a factor of 6.

Also, `tsp` suffers from control stack overflows in Hugs: too many comparisons between costs are postponed by the lazy evaluation strategy. The following variant of `cheapest` alleviates this problem. It is not necessary to understand this code – just that `cheapest'` can be used in place of `cheapest` (and `foldl1'` in place of `foldl1`) in this context.

```
cheapest' m xs = snd (foldl1' (<) [(cost m x, x) | x <- xs])
foldl1' f (x:xs) = foldl' f x xs
foldl' f a [] = a
foldl' f a (h:t) = (foldl' f $! f a h) t
```

22. Repeat Exercise 21 with `cheapest'` for `cheapest` and `perms'` for `perms`.

9 Nearest Neighbour Heuristic

A *heuristic* algorithm for an optimisation problem is a method that gives a good (but not necessarily the best) solution (relatively) quickly. One obvious heuristic for TSP is to start somewhere, and then always to pick the nearest unvisited town.

23. Define a function

    ```
    nearest :: Matrix -> Town -> [Town] -> Town
    ```

    such that `nearest dm x ys` gives the town from `ys` which is nearest to the town `x` according to distance matrix `dm`.

24. Look up the function `delete` in `List.hs`. What does it do?

25. Use these to define a function

    ```
    nn :: Matrix -> [Town] -> Tour
    ```

    such that `nn dm xs` returns a tour containing all towns from `xs`, starting from the first town in `xs` and then picking the town nearest to the last town visited at every step.

    It is probably useful to define an auxiliary function which has the same functionality but takes in the tour constructed so far as an extra argument, i.e.

    ```
    nnLoop :: Matrix -> [Town] -> Tour -> Tour
    ```

    The accumulating argument (or at the very least: its last town) is necessary to determine the next town to visit. A direct recursive version is likely to construct the tour "inside-out" using the wrong selection criterion.

    A somewhat analogous problem is the following. Assume we want to sum all numbers in a list, adding from left to right. The following function does not *do that:*

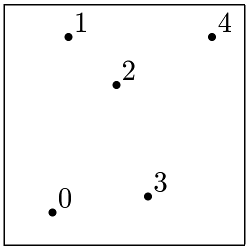
    ```
    sumall1 [x]    = x
    sumall1 (x:xs) = x + sumall1 xs
    ```

because it doesn't actually sum from left to right, but from right to left – the first addition performed in `suma1l1 [3,8,9]` *is* `8+9`*. A version using a "result so far" can sum easily from left to right:*

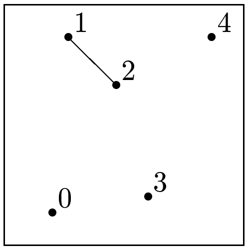
```
sumall2 xs  = sumaux 0 xs
sumaux sumsofar []     = sumsofar
sumaux sumsofar (x:xs) = sumaux (sumsofar+x) xs
```

10 Combining tour segments

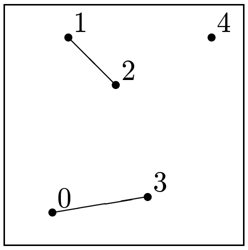
A second heuristic works by considering "segments" – partial tours each containing some of the towns. The whole tour is obtained by starting with each town in a single segment, and then gradually combining the segments until a single segment is formed containing *all* the towns. The obvious decision for combining segments is to pick those which are "close" to each other. For example:



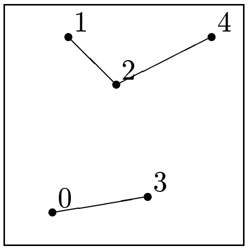
`[ [0], [1], [2], [4], [3] ]`



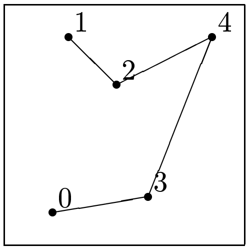
`[[0],[1,2],[4],[3]]`



`[ [0,3], [1,2], [4] ]`



`[ [0,3], [1,2,4] ]`



`[ [0,3,4,2,1] ]`

A `Segment` is a list of `Town`s, representing a partial tour (in order) – *not* interpreted as a "circular" sequence. We will only consider *non-empty* `Segment`s.

```
type Segment = [Town]
```

A `Segmentation` is a list of `Segment`s such that it contains all relevant `Town`s, each in exactly one of the segments. (In intermediate results some towns may be missing.)

```
type Segmentation = [Segment]
```

The segmentation

```
[ [1,2], [0], [3,4] ]
```

could be said to represent a collection of possible tours of the towns `[0..4]`, namely those where where `1` is visited immediately before *or after* `2`, and similarly for `3` and `4`.

Some of the next few exercises are intended in the first place to get used to the `Segmentation` *type, and have very simple answers.*

26. Define a function

    ```
    size :: Segmentation -> Int
    ```

    which reports the number of towns that occur in a segmentation. (E.g. the segmentations in the example all have size 5.)

27. Define a function

    ```
    endpoints :: Segmentation -> [Town]
    ```

    which returns the list of all endpoints (i.e., *first or* last elements) of the given segmentation, without duplicates. For example,

```
endpoints [ [1,2,4,5], [3,6,7], [0], [8,9]]
~> [ 1, 5, 3, 7, 0, 8, 9 ]
```

(Note that `0` occurs only once.)

Allowing duplicates and then removing them is inefficient; segmentations do not contain duplicate towns.

28. Define a function

```
element :: Town -> Segmentation -> Bool
```

which reports whether the given town occurs anywhere in the segmentation.

Blind use of a library function is likely to lead to a type error.

29. Define a function

```
splitOnTown :: Town -> Segmentation -> (Segment,Segmentation)
```

which splits the segmentation into the segment which has the town as an endpoint, and all the other segments. E.g.,

```
splitOnTown 2 [ [1,2], [3], [0] ] ~> ([1,2], [[3],[0]])
```

You may assume that the town is an endpoint of one of the segments in the given segmentation.

(This is one example where an intermediate `Segmentation` does not contain all the relevant towns.)

30. Define a function

```
otherEnds :: Segmentation -> Town -> [Town]
```

such that `otherEnds ss t` returns all endpoints of `ss`, except for the endpoints of the segment that `t` is an endpoint of. For example,

```
otherEnds [ [1,2,4,5], [3,6,7], [0], [8,9]] 7
~> [ 1, 5, 0, 8, 9 ]
```

31. Define a function

```
initial :: [Town] -> Segmentation
```

such that `initial ts` returns a `Segmentation` containing all of `ts` in `length ts` *separate* segments.

32. Define a function

    ```
    complete :: Segmentation -> Bool
    ```

    which reports whether the segmentation is complete, in the sense that it puts all its elements in a *single* segment.

33. Define a function

    ```
    splitOnTowns::(Town,Town) -> Segmentation
                 -> (Segment,Segment,Segmentation)
    ```

    such that

    ```
    splitOnTowns (x,y) ss
    ```

    returns a triple: the segment of `ss` which has `x` as an endpoint, the segment of `ss` which has `y` as an endpoint, and all the other segments of `ss`. E.g.,

    ```
    splitOnTowns (2,3) [ [1,2], [3], [0] ]
    ~> ([1,2], [3], [[0]])
    ```

    You may assume that the towns are endpoints of two different segments in the given segmentation.

34. Define a function

    ```
    merge :: (Town,Segment) -> (Town,Segment) -> Segment
    ```

    which merges two segments. In a call

    ```
    merge (p1,s1) (p2,s2)
    ```

you should assume that `p1` is an endpoint of `s1`, and `p2` of `s2`. The result of this call should be a segment obtained by, if necessary, reversing one of `s1` and `s2`, and concatenating them in such a way that `p1` is next to `p2` in the resulting segment.

The acceptable outcomes for `merge (2,[1,2]) (3, [3])` are `[1,2,3]`, also `[3,2,1]`, but not `[3,1,2]`; for `merge (4,[1,4]) (6, [8,6])` they are `[1,4,6,8]` or `[8,6,4,1]`, but not `[1,4,8,6]`.

35. Define a function

```
join :: (Town,Town) -> Segmentation -> Segmentation
```

such that for a call

```
join (x,y) ss
```

(`x` and `y` may be assumed to be endpoints of different segments of `ss`) the result is a segmentation which is identical to `ss` except that the segments containing `x` and `y` have been `merge`d.

So a possible result of `join (3,4) [ [0,3], [1,2,4] ]` is `[ [0,3,4,2,1] ]`, and `join (0,3) [ [0] , [1,2] , [3] , [4] ]` could give `[ [0,3], [1,2], [4] ]` (see pictorial example).

A slightly more efficient overall solution can be obtained by not *using the functions suggested, as they lead to repeated retrieval of the selected segments.*

The functions defined so far could be used to implement any TSP heuristic that builds up the tour link by link. The determining decision is *which* two towns to use for the next `join`.

Our strategy is: “furthest town first”.

36. Define a function

```
howfar ::  Town -> Matrix -> Segmentation -> Int
```

which returns the sum of the distances of a given town to the endpoints of all other segments in the segmentation.

37. Define a function

```
furthest :: Segmentation -> Matrix -> Town
```

which returns the endpoint which has the *largest* `howfar` value of all endpoints.

38. Define a function

    ```
    closest:: Segmentation -> Matrix -> Town -> Town
    ```

    such that `closest ss dm t` is an endpoint in `ss`, not in the same segment as `t`; and of all such endpoints, it is the one with the *smallest* distance (according to `dm`) to `t`.

39. Define a function

    ```
    nextJoin :: Segmentation -> Matrix -> (Town,Town)
    ```

    which returns the next two towns to be joined. These should be the `furthest` town in the segmentation, and the town `closest` to it.

40. Define a function

    ```
    tsps :: Matrix -> [Town] -> Tour
    ```

    which returns a tour constructed using the segment heuristic, by starting from an `initial` segmentation, repeatedly performing the `join` determined by `nextJoin`, and extracting the tour from the segmentation once it's `complete`.

41. Compare the quality (cost) of tours generated using `tsps` to the exact solutions (`tsp`) and any other heuristic you have programmed for a representative number of (reasonably sized) inputs. Try to explain the results. (No more than 150 words.)

42. As the `nextJoin` function is the only "intelligence" of the `tsps` algorithm, turn it into a parameter of the algorithm so we can replace it by another. I.e., define a function

    ```
    tsph :: Matrix -> [Town] -> JoinFn -> Tour
    ```

    ```
    type JoinFn = Segmentation -> Matrix -> (Town,Town)
    ```

such that

`tsph m ts nextJoin`

gives the same result as `tsps m ts`.

43. Define a function

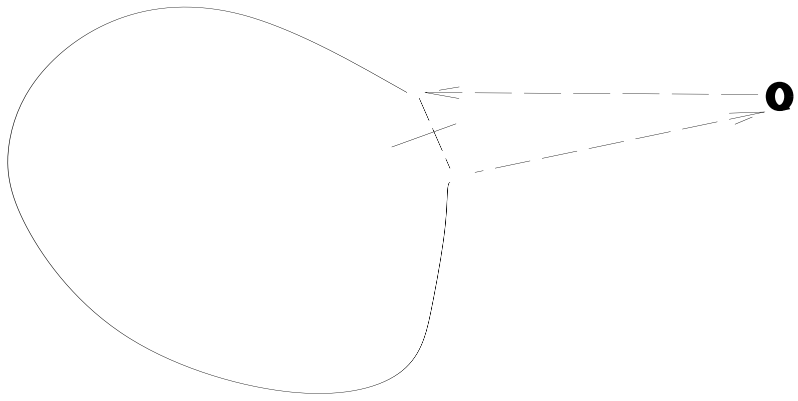
    `myNextJoin :: JoinFn`

    which can be used as a parameter to `tsph` to implement a different join selection strategy for the segment heuristic.

    Explain the idea of the heuristic, present some relevant results, and contrast these with the exact solution and other heuristics. (No more than 200 words.)

11 TSP by extending tours

Adding one town to an existing tour:



Given a tour, we insert a town into it in the place where it leads to the smallest increase in cost. Doing this for all remaining towns in sequence gives yet another TSP heuristic. Depending on the order in which the towns are inserted, this can be one of the better heuristics. The order in which the towns are added is determined in three different ways:

- by the order the towns are given initially;

- using remotest towns first;
- using remotest towns first (in a different way).

44. You will be able to use the functions `ins` and `cheapest` for this part.

    Define a function

    ```
    insTown :: Matrix -> Tour -> Town -> Tour
    ```

    such that `insTown dm ts t` returns a tour obtained by inserting `t` somewhere in `ts` (without reordering `ts` itself), choosing of all the possibilities for doing so one which has minimal total distance according to the distance matrix `dm`.

    For example, `insTown dm [1,2,3] 5` will return one of `[5,1,2,3]`, `[1,5,2,3]`, `[1,2,5,3]` or `[1,2,3,5]` – which one depends on which of these four is cheapest according to `dm`.

    A more efficient solution does not reuse as suggested, but instead makes use of the fact that inserting a town involves removing one link, and replacing it by two others; the cost of these is the only thing that really needs to be computed. This is proportional to the size of the `Tour` *rather than quadratic. Taking that even further, one might optimise the next question by memoising these link costs for the tour constructed so far.*

45. Define a function

    ```
    addTowns :: Matrix -> Tour -> [Town] -> Tour
    ```

    such that `addTowns m ts us` adds all towns from `us` to the tour `ts`, at each step adding the head of `us` using `insTown` until `us` is empty (in which case it returns `ts`). As usual, you may assume that `us` has no duplicates.

    For this heuristic, the auxiliary "main loop" function with the tour so far as an accumulated argument is asked for explicitly although it is not strictly necessary; later variants need such auxiliary functions but it is left implicit there. See also the comment after Exercise 23.

46. Define a function (it has a short name to make your testing easier):

    ```
    aT :: Matrix -> [Town] -> Tour
    ```

such that `aT m ts` returns a tour constructed by repeated use of `insTown`; this can be achieved by calling `addTowns` with suitable arguments.

For efficiency reasons, you may use the fact that all tours of length 3 (or less) are equivalent (see Section 8.4).

The next few exercises construct the tour by inserting the remotest town first into the tour.

47. If we are not going to continuously insert the head of the list, it will be useful to have a function which removes an element from a list.

    Define a function

    ```
    removeTown :: Town -> [Town] -> [Town]
    ```

    such that `removeTown t ts` gives a list of towns containing all towns in `ts` except for `t`. You may assume that `t` occurs exactly once in `ts`.

    Recursive solutions often forget to include the segment just considered in the final result.

48. The following heuristics are based on the idea of *remoteness*: the remoteness of a town `t` with respect to a list of towns `ts` and a distance matrix `m` is the sum of the distances (according to `m`) between `t` and each town in `ts`.

    Define a function

    ```
    remoteness :: Matrix -> [Town] -> Town -> Int
    ```

    such that `remoteness m ts t` gives the remoteness of `t` with respect to `m` and `ts`. For example,

    ```
    remoteness dm [3,4,5] 7 ~> dm!!3!!7 + dm!!4!!7 + dm!!5!!7
    ```

49. We will decide which town to add depending on its remoteness – in particular, for version 1 we will pick the `remotest` town in each step.

    Define a function

    ```
    remotest :: Matrix -> [Town] -> [Town] -> Town
    ```

such that `remotest m ts us` returns the town `u` in `us` such that `remoteness m ts u` is the highest of all.

50. The solution to TSP is obtained by repeatedly adding the `remotest` town relative to the matrix and *the tour constructed so far.*

    Define a function

    ```
    aRT1 :: Matrix -> [Town] -> Tour
    ```

    such that `aRT1 m ts` returns a tour containing all towns of `ts`, constructed as follows:

    - initially, the tour contains a *single* town, which must be the remotest with respect to `m` and `ts`;
    - at every next step, the town from the remainder of the input is chosen which is remotest with respect to `m` and *the tour constructed so far* (rather than the input!), and added using `insTown`.

    Alternatively, we could consider the remoteness of towns *with respect to the original input.* Rather than by keeping the original input around in the program, this can be solved by sorting the input list, and then using the `aT` program.

51. Define a function

    ```
    sortByRemoteness :: Matrix -> [Town] -> [Town]
    ```

    such that `sortByRemoteness m ts` contains all the towns of `ts`, sorted by *decreasing* `remoteness` with respect to `m` and `ts`. For example, if `remoteness m [1,2,3] 1 ~> 17`, `remoteness m [1,2,3] 2 ~> 15`, and `remoteness m [1,2,3] 3 ~> 19`, then `sortByRemoteness m [1,2,3] ~> [3,1,2]`.

52. Define a function

    ```
    aRT2 :: Matrix -> [Town] -> Tour
    ```

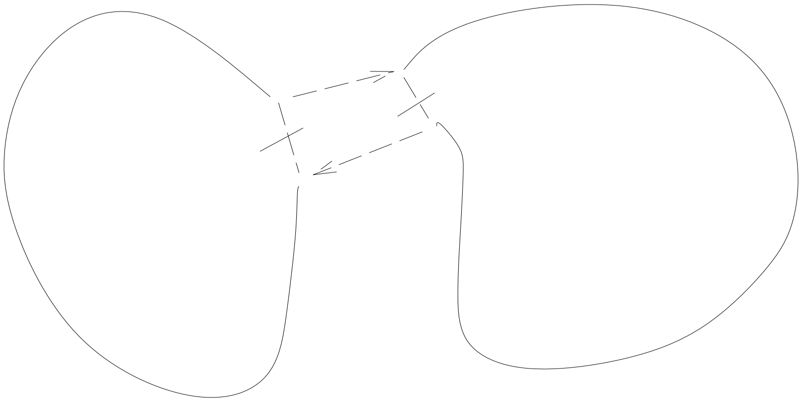
    which works by first sorting the input towns by decreasing remoteness, and then applying `aT` to the result.

53. Compare the quality (in terms of `cost` and running time) of tours generated using `aT`, `aRT1`, and `aRT2` with each other and the exact solutions (`tsp`) for a representative number of (reasonably sized) inputs. Explain and analyse the results. (No more than 200 words.)

12 TSP by combining tours

Rather than adding individual towns to existing tours, we may also choose to combine small tours into bigger ones – starting with tours which start and end in the same place, and ending up with one that contains all towns to be visited.

Combining two existing tours:



12.1 Some preliminary functions

The following general purpose function may be useful.

54. Define a function

    ```
    splits :: Tour -> [ (Tour,Tour) ]
    ```

    such that `splits ts` returns a list containing all pairs `(ts1,ts2)` such that `ts1++ts2==ts`. For example, `splits [1,2,3]` should contain the pairs `([],[1,2,3])`, `([1],[2,3])`, `([1,2],[3])`, `([1,2,3],[])` in some order.

    As a `Tour` represents a circular tour, there is some *confusion*: `[1,2,3]` represents the same tour as `[2,3,1]` and `[3,1,2]`.

55. Define a function

    ```
    rots :: Tour -> [Tour]
    ```

    which lists all the tours you get from the input by "rotating" the list, i.e., starting in a different town but following the same tour.

56. Also, because the matrix is symmetric, there is no difference to the cost if we reverse the tour.

    Define a function

    ```
    revs :: [Tour] -> [Tour]
    ```

    which reverses each of the tours in the input list. (Function `reverse` is predefined.)

57. For a list of three towns, rotating and reversing together give all permutations – for longer lists, the function `perms` would return other permutations as well.

    Define a function

    ```
    rotrevs :: Tour -> [Tour]
    ```

    which returns the list of all tours that can be obtained by rotations and/or reversals of the input list. (For a list without duplicates of length n, there should be $2n$ different ones.)

12.2 Merging tours

In this heuristic, we start with lots of small tours.

58. Define a function

    ```
    singletons :: [Town] -> [Tour]
    ```

    such that `singletons ts` returns a `[Tour]` with each town from `ts` in a *separate* tour.

    We will repeatedly combine these until we have included all of the towns.

59. Define a function

    ```
    complete :: [Tour] -> Bool
    ```

    which reports whether the list of tours is complete, in the sense that it puts all its towns in a *single* tour.

    For merging tours, there are *two* decisions to be taken: *which* two tours (from a list) to merge, and *how* to merge them.

60. For selecting two tours, we will take the tours in the list which are nearest to each other. We define the distance between two tours as the smallest distance between some town of one tour and some town of the other.

    Define a function

    ```
    distance :: Matrix -> Tour -> Tour -> Int
    ```

    that returns for `distance dm ts1 ts2` the distance according to `dm` between a town from `ts1` and a town from `ts2` that is minimal for all such pairs of towns.

61. Define a function

    ```
    nearest2 :: Matrix -> [Tour] -> (Tour,Tour)
    ```

    such that `nearest2 dm tss` gives a pair of tours `ts1` and `ts2` from `tss` for which their `distance` with respect to `dm` is minimal.

62. For the decision on how to merge two tours, imagine the problem of combining two paper loops. You would need to cut both in half, (so would need to decide at which point to cut each of them), and then stick them together at the cut, potentially reversing one of them.

    Define a function

    ```
    mergeTours :: Matrix -> Tour -> Tour -> Tour
    ```

    which merges two tours in all possible ways, returning the cheapest of those.

A suggestion for `mergeTours dm ts1 ts2` is to list all possibilities by splitting `ts1` in an arbitrary point, then inserting into `ts1` at that point all rotations and reverses of `ts2`. Selecting the cheapest of those should be a familiar problem.

For example, for `merge dm [1,2] [3,4]`, the variations on `[3,4]` are just `[3,4]` and `[4,3]`, each of which needs to be inserted at every possible point in `[1,2]`, leading to `[ [3,4,1,2], [4,3,1,2], [1,3,4,2], [1,4,3,2], [1,2,3,4], [1,2,4,3] ]`, of which a cheapest must be selected.

63. Define a function

    ```
    mergetsp:: Matrix -> [Town] -> Tour
    ```

    which returns a tour constructed starting from an `singletons` list of tours, repeatedly merging the nearest two tours, until it is `complete`.

64. Compare the quality (cost) of tours generated using `mergetsp` to the exact solutions (`tsp`) and other implemented heuristics for a representative number of (reasonably sized) inputs. Try to explain the results. (No more than 150 words.)

65. Discuss and define a reasonable alternative `distance` function, and use it in a variation on `mergetsp`.

66. Extend the comparison and analysis of Exercise 64 to also include your modification on `mergetsp`.

13 Exam Questions

Exam papers for CO312 consisted of a "case study" (a model solution to the year's major assessment) which had been handed out well in advance, with questions such as the following.

1. Select a part of the code that, in your view, has been done in a clumsy or inefficient way, and present an alternative solution for it. Explain in which respect your solution improves the solution given above.

2. The required properties of a distance matrix could be characterised as follows:

```
isDM :: Matrix -> Bool
isDM m = square m && symmetric m &&
         all0 (maindiag m) && triangle m
```

For each of these four properties, explain:

- what does it mean for a matrix to fail this property, and
- how using a "distance matrix" which failed this property would impact on the various TSP algorithms. (Would the results still be correct or optimal? If not, could they be fixed easily? Would it still be OK to use `perms'` instead of `perms`? etc.)

3. Distances between towns are represented by values of type `Int`. List all the changes that would need to be made to the code if we represented distances by values of the type `Float` instead.

4. Give a solution for the function `nearest` (Exercise 23) which is significantly different from the one presented in the case study. Discuss the relative merits of the two versions, addressing issues of clarity and efficiency.

5. The nearest neighbour heuristic (Exercise 25) could also be described as a program on segmentations. Rather than always picking the cheapest link connecting *any* of the segments, it always connects the end of the *first* segment to the start of a segment which is nearest to it in the distance matrix.

   Give an alternative definition of `nextJoin` (Exercise 39) which does this. If convenient, you may assume that all of the segments in the `Segmentation` argument to `nextJoin` are singletons, except for the first one. Point out some of the reasons why this variant of "nearest neighbour" is less efficient in terms of running time than `nn`.

6. Consider a variant on the TSP where we only need to visit the towns given, but we no longer need to return to our original starting point. The answer to our problem would still be represented by a permutation of the input, but now interpreted as a true sequence rather than a cycle.

   How would this problem solved by a modification of the code given

   - for the "brute force" TSP;
   - for the heuristic(s)?

Would you expect your heuristic(s) now to result in solutions which are closer to the optimal solution than they were for the original TSP? Why?

7. Define a function

   ```
   internal :: Segmentation -> Town -> Bool
   ```

   which returns `True` if the town occurs in the segmentation, but is not an endpoint of the segments in the segmentation, and which returns `False` in all other cases.

8. A simpler heuristic than "furthest first" implemented in the `tsps` heuristic (Exercise 40) is "nearest first" which simply chooses for every next step the two endpoints of different segments which are nearest to each other. This can be done by *only* changing the `nextJoin` function (Exercise 39).

   Define an alternative `nextJoin` function (same type as above) which encodes this strategy: it should return the pair of towns which, of all combinations of two endpoints of *different* segments, have the smallest distance between them according to the distance matrix.

9. Give a variant of the function `distance` (Exercise 60) which determines the distance between two tours not as the *minimum*, but as the *average* distance between pairs of towns from each of the tours. You may round down the average to an integer, i.e., use `div` rather than `/` for division.

10. If, in the function `nearest2` (Exercise 61), you had accidentally swapped the two arguments in its definition, i.e., you had written

    ```
    nearest2 :: Matrix -> [Tour] -> (Tour,Tour)
    nearest2 tss dm
    ```

    and left the rest unchanged, Hugs would not have reported a type error. Why not? How would you have discovered about the two arguments being swapped anyway?

11. Modify the `mergetsp` function (Exercise 63) such that, rather than selecting the nearest tours at every step, it simply selects the first two elements of the list of tours to merge.

Keeping in mind that the newly merged tour is inserted at the *front* of the list of tours, explain how the resulting algorithm relates to the `aT` algorithm (Exercise 46).

Online materials

This document is available through the departmental publications web page at `http://www.cs.kent.ac.uk/pubs/`. A sample distance matrix is available at `http://www.cs.kent.ac.uk/people/staff/eab2/tsp/DM.hs`. A program to visualise tours computed using this distance matrix is at `http://www.cs.kent.ac.uk/people/staff/eab2/tsp/mapper/`.

Acknowledgements

I would like to thank the CO312 students from 2000/01 to 2003/04 for their feedback and ideas. In particular, Adam Sampson wrote the "TSP mapper" (see URL above) and also inspired the "segments" heuristic. Neil Renaud implemented in 2001 the "remotest town first" strategy in Section 11 as his alternative heuristic – based on real field research, talking to a friend who had to solve TSP for his delivery job on a daily basis.

My colleague Claus Reinke suggested `foldl1'` as the solution to the control stack overflow problem in brute force TSP; Olaf Chitil made some useful comments on this technical report.

References

[1] Simon Thompson, "Haskell: The Craft of Functional Programming", 2nd edition, Addison-Wesley, 1999.