Maintenance Importance Measures Incorporating Tripartite Component Dependence

Boyuan Zhang[a,c], Yi-Xuan Zheng[a], Shaomin Wu[c], and Yu Liu *[a,b]

[a]School of Mechanical and Electrical Engineering, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, PR China
[b]Center for System Reliability and Safety, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, PR China
[c]Kent Business School, University of Kent, Canterbury, Kent CT2 7PE, United Kingdom

Abstract

Importance measures are very useful for reliability enhancement of engineered systems at both design and maintenance phases, and they have therefore been extensively studied. The existing importance measures are largely introduced from the perspective of system structure, reliability, and maintenance related costs. It is known that three types of component dependence: stochastic, economic, and structural, exist in many engineering scenarios and have to be taken into account in optimising maintenance policies. Neglecting this tripartite dependence may result in erroneous evaluations of the benefit, time, and cost associated with a maintenance policy, thereby causing negative consequences. Nevertheless, to the best of our knowledge, all the three types of dependence measures are not considered jointly in the existing importance measures. To fill this research gap, a novel importance measure accounting for all the three types of dependence is proposed in this paper and further extend it to a joint importance measure of two components to quantify their interaction due to tripartite component dependence. Some interesting properties regarding the proposed importance measures, such as the relationship between both importance measures, the influence of the neglect of stochastic dependence, the monotonicity of the proposed measures, and the value range of the joint importance measure, are proved. Illustrative examples, including a numerical case and a landing gear system, illustrate the proposed importance measures and show that the proposed measures can effectively quantify the effectiveness of maintenance policies when the maintenance on a multi-component system is affected by the tripartite component dependence.

Keywords: Importance measures; stochastic dependence; structural dependence; economic dependence.

1 Introduction

Engineered systems are subject to inevitable degradation resulting from usage and environmental factors, causing system failure and thus incurring economic losses and safety risks

*Corresponding author: yuliu@uestc.edu.cn

for personnel (Lisnianski and Levitin, 2003; Wu et al., 2022). Such degradations can be deferred by performing maintenance (Hashemi and Asadi, 2021; Wu and Asadi, 2024; Peng et al., 2021). For a multi-component system, reliability importance measures can prioritise components for maintenance and have been therefore extensively studied to support decision-makers in obtaining an optimal maintenance policy (Chen and Feng, 2022; Du et al., 2025).

It is worth noting that the optimization of maintenance policies for a multi-component system can inevitably encounters three types of component dependence: stochastic (Zhang et al., 2023; Xu et al., 2024), structural (Zhou et al., 2015), and economic (Bansal et al., 2025; Kivanç et al., 2024). Stochastic dependence, also referred to as failure interaction or probabilistic dependence, is caused by common cause failures (Mi et al., 2018), loading sharing (Keizer et al., 2018; Fang and Li, 2016), and cascading failures (Guo et al., 2017), and it is identified through analyzing the correlated failures or degradation data from monitoring the degradation coupling between paired components(Wu et al., 2023). It implies that the state of one component can influence that of the other components. Neglecting the stochastic interrelations among components may result in a biased estimate of their reliability, thereby reducing the effectiveness of maintenance policies (Do et al., 2015). In the maintenance of a particular component, the constraints imposed by the system's structure often require the disassembly of its preceding components or sub-assemblies. This constraint relationship is termed structural dependence (Dao and Zuo, 2017), and it is found through disassembly precedence relationships documented in maintenance manuals (Zhou et al., 2015). The disassembly of a system takes the maintenance personnel's time and, therefore, incurs additional costs. Failing to take account of the structural dependence will cause an underestimation of the time required for maintenance, leading to the inability to complete the scheduled maintenance policy in a timely manner (Leppinen et al., 2025). Maintenance of multiple components may also relate to economic dependence due to the share of facilities, materials, and personnel (Yang et al., 2020). The economic dependence is presented through shared maintenance costs and facilities (Keizer et al., 2018). When economic dependence is not considered, the total cost of the maintenance schedule becomes higher, because downtime losses are repeatedly incurred or opportunities for joint maintenance are missed (Oakley et al., 2022). Consequently, ignoring the aforementioned tripartite dependence in maintenance optimization may result in erroneous evaluations of system reliability, maintenance time, and maintenance cost. It is thus imperative to put forth an importance measure that comprehensively incorporates the three types of component dependence together to evaluate the effectiveness of maintaining components.

This paper proposes a novel importance measure to assess the effectiveness of maintaining one component for a multi-component system with the consideration of three distinct dimensions of component dependence (3D). The importance measure will be referred to as 3D importance measure hereafter. The concept of the three dimensions of dependence was a result of a critical and comprehensive literature review (Nicolai and Dekker, 2008). We therefore believe that 3D importance measure is representative and suitable for both the academia and practitioners. It should be noted that the maintenance of one component at a time is common in many engineering applications where maintenance tasks are executed sequentially due to disassembly precedence and limited maintenance resources (Leppinen et al., 2025; Geurtsen et al., 2023). Meanwhile, a joint importance measure of two components is introduced to evaluate the interaction of both components. The joint importance measure will be referred to as joint 3D importance measure. The contributions of this paper are listed in the following:

1) To the best of our knowledge, this is the first paper that proposes a 3D importance measure accounting for three types of dependence in maintenance optimization: the component degradation interaction caused by stochastic dependence, the disassembly sequence induced by structural dependence, and the shared costs due to economic dependence.

2) A joint 3D importance measure of two components is extended via the proposed 3D importance measure to quantify the effect of the importance measure of maintaining one component due to the change of maintaining another component.

3) Some interesting and important properties of 3D importance measures, including the relationship between both 3D importance measures, the influence of the neglect of stochastic dependence, the monotonicity of the proposed measures, and the value range of joint 3D importance measure, are proved.

The remainder of this paper is rolled out as follows. Section 2 conducts a literature review. Section 3 devotes to describing the notations and assumptions. Section 4 demonstrates the procedure steps of the both 3D importance measures. Section 5 examines the effectiveness of the proposed method via a case study on a landing gear system. Section 6 concludes the paper and proposes future research.

2 Literature Review

Although system performance is determined by the collective behavior of all components, the concept of importance measures can quantify the influence of individual components on the overall system performance (Dui et al., 2023). Importance measures in reliability engineering are extensively studied to identify the weakest component of a system and support system enhancement activities. These measures can also provide valuable information to facilitate the system's reliable and efficient operation at different phases. Importance measures used at the system maintenance phase help engineers minimize maintenance costs and prolong the lifetime of the system. In the literature, there are enormous studies investigating the importance measures of components in the context of multi-component systems (see (Chen and Feng, 2022; Dui and Wu, 2024; Aven et al., 2025), for instance). Based on the types of system states, the component importance measure can be evaluated for binary-state systems (Miziuła and Navarro, 2019), multi-state systems (Cao et al., 2024), and continuous-state systems (Peng et al., 2012). It can also be time-dependent (Zhu and Boushaba, 2017) or time-independent (Borgonovo et al., 2016), depending on whether the importance measure is changed over time. Time-independent importance measures can be further divided into a comprehensive consideration of component significance across various time frames, such as mean time to failure (MTTF) based (Borgonovo et al., 2016), Barlow-Proschan (Barlow and Proschan, 1975), time-independent lifetime based (Kuo and Zhu, 2012), and the system's topological structure based importance measures (Meng, 1996). From the perspective of the number of components evaluated, existing literature focuses on the perspectives of a single component (Wu et al., 2022), pairs of components (Yi and Balakrishnan, 2024), and sets of components (Dui et al., 2017). Additionally, a multitude of studied works have conducted on importance measures from diverse viewpoints of system utility, encompassing reliability-based (Borgonovo et al., 2016), cost-based (Wu and Coolen, 2013; Dui et al., 2017), and resilience-based (Xu et al., 2020; Dui and Wu, 2024).

It is worth noting that some papers on the importance measures within three types of component dependence in a separate manner. Economic dependence can be categorized into positive setup (Chen and Feng, 2022) and negative downtime costs (Do and Bérenguer, 2020). For example, Chen and Feng (2022) introduced a group importance measure to jointly combine system reliability and maintenance cost, considering the saved setup cost among a group of components. Do and Bérenguer (2022) designed an importance measure to evaluate the improvement of the system's residual useful life via the replacement of one component, extending to consider total maintenance cost with a positive setup cost. Do and Bérenguer (2020) introduced a time-dependent conditional reliability-based importance measure for multi-component systems and extend it by analyzing economic dependence and incorporating maintenance cost. According to the principles of component failure interdependence, stochastic dependence has different modeling methods, such as load sharing, failure-induced damage, and common-mode deterioration. For example, Arriaza et al. (2023) extended a variance-based importance for coherent systems with dependent and heterogeneous components, modeling the dependence structure via copulas to account for stochastic dependence among component lifetimes. Cao et al. (2024) developed importance measures for multi-state systems by explicitly modeling hierarchical dependences among components, thereby capturing stochastic interactions induced by correlated degradation or failure mechanisms. Yi and Balakrishnan (2024) designed two importance measures to evaluate the importance of shared components for semi-coherent systems. It is unfortunate that, presently, no literature addresses structural dependence in the evaluation of component importance. Yet, some studies have taken into account the relative importance of components in the topological structure of the system. For example, Nguyen et al. (2017) designed a maintenance threshold that was established via the structural importance measure. Salmasnia et al. (2024) leveraged a structural importance measure to assess component condition and to guide the joint optimization of maintenance and production strategies.

The foregoing literature on the importance measures has solely considered economic or stochastic dependence without the consideration of structural dependence. Furthermore, it has not comprehensively taken into account multiple types of component dependence. The proposed 3D importance measure jointly incorporates stochastic, economic, and structural dependence. Indeed, ignoring tripartite dependence may result in erroneous evaluations of system reliability, maintenance time, and maintenance cost, thereby impeding the implementation of maintenance activities. This integrated formulation enables a more comprehensive evaluation of component significance under practical maintenance constraints, where cost sharing, disassembly order, and degradation interaction coexist. To the best of our knowledge, none of the existing works proposed importance measures with three types of component dependence.

3 Problem Descriptions and Model Assumptions

The reliability of an engineered system can be improved by maintaining its components. Importance measures can be utilized during the system maintenance phase to plan effective maintenance policies. In the procedure of maintaining components, the presence of stochastic, economic, and structural dependence between components need considering. Without loss of generality, we make the following assumptions.

1) Each component encompasses two states: working and failed.

2) The stochastic dependence exists among components. The system concludes both statistically independent and stochastically dependent components. Suppose that the reliability of each component is dependent with one other component at most. The failure intensity function of one component increases due to the failure of its dependent component.

3) A minimal maintenance activity is conducted to restore the state of a component, which only brings the component from a failed state back to an operational state without altering its age.

4) The associated costs include a maintenance cost of the component and a fixed setup cost. If components share the same maintenance facility, then the fixed setup cost is accounted for only once, reflecting the economic dependence among components.

5) Due to the structural dependence, the maintenance of a component requires the preliminary disassembly of preceding nodes including components and sub-assemblies.

These assumptions are adopted to balance modeling realism and analytical tractability in maintenance decision-making applications. In many engineering maintenance settings, decisions are primarily driven by observable failure events, limited maintenance resources, shared maintenance facilities, and accessibility constraints, which justify the use of binary-state representations, minimal repair policies, and dependence structures reflecting stochastic, economic, and structural interactions.

If some assumptions are violated in practice, the proposed framework can be adapted accordingly. For example, (1) multi-state degradation behavior can be incorporated by extending the state space, (2) more complex dependence patterns can be modeled through generalized transition intensities or graphical dependence structures; (3) alternative maintenance policies such as perfect or imperfect repair can be accommodated by modifying state-transition probabilities, and (4) dynamic or uncertain cost and disassembly relationships can be represented through time-dependent or stochastic parameters. These extensions do not alter the core importance formulation but modify the underlying transition quantities used in the analysis.

The symbol definitions corresponding to the assumptions are as follows.

1) The system under study is composed of m components. $C_\ell(t)$ denotes the state of component ℓ at time instant t. $C_\ell(t) = 1$ represents component ℓ is working and $C_\ell(t) = 0$ represents component ℓ is failed.

2) The system state at time t, denoted by $S(t)$, is determined by its components' states and its structure function, that is, $S(t) = \phi(\mathbf{C}(t)) = \phi(C_1(t), \ldots, C_m(t))$, where $\mathbf{C}(t)$ denotes the components' state combination at time t, and $\phi(\cdot)$ is the system's structure function. The state space of the system is denoted by $S(t) = \{0, 1\}$.

3) The initial failure intensity function of component ℓ is denoted by $\lambda_\ell(t)$. The failure intensity function of component ℓ increases to $\bar{\lambda}_\ell(t)$ due to the failure of its dependent component, where $\bar{\lambda}_\ell(t) > \lambda_\ell(t)$. Suppose that there are m_1 pairs of stochastic dependent components and m_2 stochastic independent components, and thus $2m_1 + m_2 = m$. $\mathbf{D} = \{d_{1,1}, d_{1,2}; d_{2,1}, d_{2,2}; \ldots; d_{m_1,1}, d_{m_1,2}\}$ denotes the serial number set of the dependent components, where $d_{i,1}$ and $d_{i,2}$ is the serial numbers of the ith pair of dependent components, respectively. $\mathbf{I} = \{d_1, \ldots, d_{m_2}\}$ encompasses the serial number set of the independent components, where d_i is the serial number of the ith independent component. $\mathbf{D} \cup \mathbf{I} = \{1, \ldots, m\}$ represents the complete set of components.

4) The maintenance cost of component ℓ is denoted by c_ℓ. The fixed setup cost of component ℓ is $c^e_{h(\ell)}$, where $h(\ell)$ is the serial number of the maintenance facility for component ℓ.

5) The disassembly sequence of component ℓ is denoted by $\mathbf{d}_\ell$.

4 The Proposed Importance Measure

This section will propose a novel 3D importance measure, which incorporates three types of dependence: stochastic, economic, and structural dependence. Meanwhile, a joint 3D importance measure of two components is extended by the proposed importance measure to evaluate their interaction. A nomenclature table is added in Appendix O to summarize all key symbols.

4.1 *Procedure of 3D importance measure*

Stochastic dependence impacts the reliability of a system; economic dependence affects the cost of maintaining components; structural dependence relates to the sequence of disassembly required for maintaining components. Therefore, the system reliability, maintenance cost, and the disassembly procedure number are utilized as utilities to evaluate the importance of components under stochastic, economic, and structural dependence, respectively. However, the measures of the system reliability, maintenance cost, and disassembly procedure number are different. Meanwhile, the system reliability is a positive utility, whereas the maintenance cost and disassembly number are negative utilities. Denote $h^r(\cdot)$, $h^e(\cdot)$, and $h^s(\cdot)$ as the unified system utilities associated with system reliability, maintenance cost, and disassembly procedure number, respectively, corresponding to the three types of component dependence.

4.1.1 *Reliability improvement with stochastic dependence*

Inspired by the Birnbaum importance measure (Birnbaum et al., 1968), the system reliability improvement associated with component ℓ is defined as the marginal benefit of restoring the component to the working state at time t_k, and it is given by,

$$H_\ell^r(t_k) = h^r(1_\ell, \mathbf{p}_k) - h^r(\mathbf{p}_k), \tag{1}$$

The classical Birnbaum importance is adopted as the starting point because it provides a clear and well-established marginal-effect interpretation of component importance. This interpretation serves as a transparent foundation for extending the importance concept to incorporate multiple forms of dependence. While many modern variants focus on a particular aspect of dependence, the proposed 3D importance measure integrates stochastic, economic, and structural dependence simultaneously within a unified framework. From this perspective, the contribution of this article lies not in modifying the Birnbaum formula itself, but in extending the importance concept to a broader maintenance-oriented dependence structure. $h^r(1_\ell, \mathbf{p}_k)$ represents the system utility for stochastic dependence after maintaining component ℓ at time t_k. $h^r(\mathbf{p}_k)$ indicates the system utility for stochastic dependence without any maintenance activities at time t_k. Hence, Eq. (1) denotes the difference between the system utility with maintaining component ℓ and that without any maintenance activities at time t_k. A greater value of $H_\ell^r(t_k)$ represents a greater utility improvement of component ℓ. To account for stochastic dependence among components, the reliability improvement is evaluated based on the probability of components' state combination. $\mathbf{p}_k = \{p_{1,k}, p_{2,k}, \ldots, p_{2^m,k}\}$ denotes the probability of components' state combination, where $p_{i,k} = \Pr\{\mathbf{C}(t_k) = \mathbf{X}_i\} = \Pr\{\{C_1(t_k), \ldots, C_m(t_k)\} = (X_{i,1}, \ldots, X_{i,m})\}$. $\mathbf{X}_i$ is the ith state combination of all components, and $X_{i,\ell}$ is the state of component ℓ when the system stays in $\mathbf{X}_i$. $(X_{i,1}, \ldots, X_{i,m}) \in \mathbf{\Omega}$, where $\mathbf{\Omega} = \{(0_1, 0_2, \ldots, 0_m), (1_1, 0_2, \ldots, 0_m), \ldots, (1_1, 1_2, \ldots, 1_m)\}$, 0_ℓ and 1_ℓ denote that component ℓ stays in failed and working states, respectively, and there are 2^m elements in $\mathbf{\Omega}$. $h^r(\mathbf{p}_k)$ is given by,

$$h^r(\mathbf{p}_k) = \sum_{i=1}^{2^m} p_{i,k} \phi(\mathbf{X}_i), \tag{2}$$

where $\phi(\mathbf{X}_i)$ is an indicator function and equates to 1 when the system staying in $\mathbf{X}_i$ is working, and zero otherwise.

$h^r(1_\ell, \mathbf{p}_k)$ is given by,

$$h^r(1_\ell, \mathbf{p}_k) = \sum_{i=1}^{2^m} p_{i,k} \phi(\mathbf{X}_{i\setminus\{\ell\}}), \tag{3}$$

where $\mathbf{X}_{i\setminus\{\ell\}} = \{X_{i,1}, \ldots, X_{i,\ell-1}, 1_\ell, X_{i,\ell+1}, \ldots, X_{i,m}\}$, and it indicates that component ℓ is maintained to the working state, and other components' states are not changed. Therefore, $H_\ell^r(t_k)$ is given by,

$$H_\ell^r(t_k) = \sum_{i=1}^{2^m} p_{i,k} (\phi(\mathbf{X}_{i\setminus\{\ell\}}) - \phi(\mathbf{X}_i)). \tag{4}$$

Eq. (4) indicates that the system utility improvement depends on the change of the system state by maintaining component ℓ.

Due to the stochastic dependence among components, the estimation of $p_{i,k}$ needs to consider components' degradation process and their interactions jointly. Denote $\lambda_\ell(t_k)$ as the initial failure intensity of component ℓ at time t_k. If components i and j are dependent, the failure of component j will increase the failure intensity of component i to $\bar{\lambda}_i(t_k)$. There are m_1 pairs of dependent components and m_2 independent components. Let $\mathbf{p}_{k-1} = \{p_{1,k-1}, p_{2,k-1}, \cdots, p_{2^m,k-1}\}$ be the distribution of components' state combination at time t_{k-1}. Therefore, the distribution of components' state combination at t_k can be deduced via that at t_{k-1} and the components' degradation models, that is,

$$\begin{aligned} p_{i,k} &= \Pr\{\mathbf{C}(t_k) = \mathbf{X}_i\} \\ &= \sum_{j=1}^{2^m} \Pr\{\mathbf{C}(t_k) = \mathbf{X}_i | \mathbf{C}(t_{k-1}) = \mathbf{X}_j\} p_{j,k-1}, \end{aligned} \tag{5}$$

where $\Pr\{\mathbf{C}(t_k)=\mathbf{X}_i|\mathbf{C}(t_{k-1})=\mathbf{X}_j\}$ is the probability that the components' states transits from $\mathbf{X}_j$ at t_{k-1} to $\mathbf{X}_i$ at t_k, and it is given by,

$$\begin{aligned}&\Pr\{\mathbf{C}(t_k)=\mathbf{X}_i|\mathbf{C}(t_{k-1})=\mathbf{X}_j\}\\&=\prod_{v=1}^{m_1}\underbrace{\Pr\{C_{d_{v,1}}(t_k)=X_{i,d_{v,1}},C_{d_{v,2}}(t_k)=X_{i,d_{v,2}}|C_{d_{v,1}}(t_{k-1})=X_{j,d_{v,1}},C_{d_{v,2}}(t_{k-1})=X_{j,d_{v,2}}\}}_{\text{Part 1}}\times\\&\prod_{u=1}^{m_2}\underbrace{\Pr\{C_{d_u}(t_k)=X_{i,d_u}|C_{d_u}(t_{k-1})=X_{j,d_u}\}}_{\text{Part 2}},\end{aligned}\tag{6}$$

where "Part 1" in Eq. (6) possesses four cases based on the states of dependent components at time t_{k-1}.

Case 1: Components $d_{v,1}$ and $d_{v,2}$ are failed. $\Pr\{C_{d_{v,1}}(t_k)=0,C_{d_{v,2}}(t_k)=0|C_{d_{v,1}}(t_{k-1})=0$, and $C_{d_{v,2}}(t_{k-1})=0\}=1$.

Case 2: Component $d_{v,1}$ is failed, and component $d_{v,2}$ is functioning. The failure intensity of component $d_{v,2}$ will increase to $\bar{\lambda}_{d_{v,2}}(t)$ due to the failure of component $d_{v,1}$. Then, "Part 1" in Eq. (6) is given by,

$$\begin{aligned}&\Pr\{C_{d_{v,1}}(t_k)=0,C_{d_{v,2}}(t_k)=1|C_{d_{v,1}}(t_{k-1})=0,C_{d_{v,2}}(t_{k-1})=1\}=e^{-\int_{t_{k-1}}^{t_k}\bar{\lambda}_{d_{v,2}}(y)dy}\\&\Pr\{C_{d_{v,1}}(t_k)=0,C_{d_{v,2}}(t_k)=0|C_{d_{v,1}}(t_{k-1})=0,C_{d_{v,2}}(t_{k-1})=1\}=1-e^{-\int_{t_{k-1}}^{t_k}\bar{\lambda}_{d_{v,2}}(y)dy}\end{aligned}\tag{7}$$

It should be noted that the proposed framework models the component degradation process via the time-dependent failure intensity function. As such, it is applicable to a wide range of lifetime distributions, including Weibull, log-normal, and gamma, by substituting the corresponding failure intensity functions.

Case 3: Component $d_{v,1}$ is functioning, and component $d_{v,2}$ is failed. Then, "Part 1" in Eq. (6) can be computed in the same way as Eq. (7).

Case 4: Components $d_{v,1}$ and $d_{v,2}$ are functioning. If both components are functioning at t_k, the failure intensity of components are unchanged. In this scenario, "Part 1" in Eq. (6) is given by,

$$\begin{aligned}&\Pr\{C_{d_{v,1}}(t_k)=1,C_{d_{v,2}}(t_k)=1|C_{d_{v,1}}(t_{k-1})=1,C_{d_{v,2}}(t_{k-1})=1\}\\&=e^{-\int_{t_{k-1}}^{t_k}\lambda_{d_{v,1}}(y)+\lambda_{d_{v,2}}(y)dy}\end{aligned}\tag{8}$$

Suppose that one component is functioning, and the other is failed at time t_k. In this scenario, the failure intensity of the functioning component will increase when the other component fails. The probability density function (PDF) of the lifetime of component i being failed is $\lambda_{d_{v,i}}(t_k)e^{-\int_{t_{k-1}}^{t_k}\lambda_{d_{v,i}}(y)dy}$. In this scenario, "Part 1" in Eq. (6) in Case 4 is given by,

$$\begin{aligned}
&\Pr\{C_{d_{v,1}}(t_k)=1, C_{d_{v,2}}(t_k)=0|C_{d_{v,1}}(t_{k-1})=1, C_{d_{v,2}}(t_{k-1})=1\}\\
&\quad=\int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x}\lambda_{d_{v,1}}(y)dy}e^{-\int_{x}^{t_k}\bar{\lambda}_{d_{v,1}}(y)dy}\lambda_{d_{v,2}}(x)e^{-\int_{t_{k-1}}^{x}\lambda_{d_{v,2}}(y)dy}dx\\
&\quad=e^{-\int_{t_{k-1}}^{t_k}\bar{\lambda}_{d_{v,1}}(y)dy}\int_{t_{k-1}}^{t_k}\lambda_{d_{v,2}}(x)e^{-\int_{t_{k-1}}^{x}\left(\lambda_{d_{v,1}}(y)+\lambda_{d_{v,2}}(y)-\bar{\lambda}_{d_{v,1}}(y)\right)dy}dx,
\end{aligned}\tag{9}$$

or

$$\begin{aligned}
&\Pr\{C_{d_{v,1}}(t_k)=0, C_{d_{v,2}}(t_k)=1|C_{d_{v,1}}(t_{k-1})=1, C_{d_{v,2}}(t_{k-1})=1\}\\
&\quad=e^{-\int_{t_{k-1}}^{t_k}\bar{\lambda}_{d_{v,2}}(y)dy}\int_{t_{k-1}}^{t_k}\lambda_{d_{v,1}}(x)e^{-\int_{t_{k-1}}^{x}\left(\lambda_{d_{v,1}}(y)+\lambda_{d_{v,2}}(y)-\bar{\lambda}_{d_{v,2}}(y)\right)dy}dx.
\end{aligned}\tag{10}$$

The probability of both components being failed at t_k can be computed by the equation that $\sum_{i=0}^{1}\sum_{j=0}^{1}\Pr\{\ C_{d_{v,1}}(t_k)=i, C_{d_{v,2}}(t_k)=j|C_{d_{v,1}}(t_{k-1})=1, C_{d_{v,2}}(t_{k-1})=1\}=1$.

"Part 2" in Eq. (6) encompasses two cases based on the state of component d_u at time t_{k-1}.

Case 1: Component d_u is failed. $\Pr\{C_{d_u}(t_k)=0|C_{d_u}(t_{k-1})=0\}=1$.

Case 2: Component d_u is functioning. "Part 2" in Eq. (6) is given by,

$$\begin{aligned}
&\Pr\{C_{d_u}(t_k)=1|C_{d_u}(t_{k-1})=1\}=e^{-\int_{t_{k-1}}^{t_k}\lambda_{d_{v,2}}(y)dy},\\
&\Pr\{C_{d_u}(t_k)=0|C_{d_u}(t_{k-1})=1\}=1-e^{-\int_{t_{k-1}}^{t_k}\lambda_{d_{v,2}}(y)dy}.
\end{aligned}\tag{11}$$

Therefore, Eq. (6) can be evaluated based on Eqs. (7) to (11). Then, according to Eq. (5), the distribution of components' state combination at time t_k can be computed via that at time t_{k-1} and Eq. (6). Consequently, the system utility improvement of component ℓ can be computed via Eq. (4).

It should be noted that evaluating system reliability on the full joint dependence state space may suffer from computational challenges. In our setting, the state probabilities and transition probabilities of each dependent pair, and that of independent components can be computed separately. The complexity of joint dependence state space is equal to $\mathcal{O}(4^m)$, and that of pairwise decomposition equates to $\mathcal{O}(16m_1+4m_2)$ which is manageable for large-scale systems.

Furthermore, the proposed framework requires the transition probabilities (or intensities) among component-state combinations. A wide range of dependence models, such as shared frailty models, conditional intensity (hazard) models, copula-based dependence for failure times, and graphical dependency structures, can be mapped into the required joint transition quantities and then handled by Eq. (6). In other words, Eq. (6) provides a unifying computational interface for dependence, while Eq. (8) corresponds to a simple special case within this interface. Therefore, the proposed framework is not limited to intensity dependence modeling, and it accommodates more systematic dependence representations if they can be translated into joint state-transition quantities.

4.1.2 *Reduced cost with economic dependence*

Since minimizing cost is a pivotal objective in maintenance, cost reduction is utilized as a utility to quantify the effect of maintaining components. Drawing on the concept of marginal benefit, the unified cost reduction of component ℓ is given by,

$$H_\ell^e = h^e(1_\ell, \mathbf{c}) - h^e(\mathbf{c}), \tag{12}$$

where $h^e(1_\ell, \mathbf{c})$ and $h^e(\mathbf{c})$ depend on their corresponding total maintenance costs, and $\mathbf{c} = \{c_1, c^e_{h(1)}, \ldots, c_m, c^e_{h(m)}\}$. c_ℓ denotes the maintenance cost of component ℓ, and $c^e_{h(\ell)}$ is the fixed setup cost of component ℓ whose corresponding maintenance facility number is $h(\ell)$. Let $c^e(\ell)$ and $c^e(0)$ be the total maintenance cost associated with maintaining component ℓ and the total maintenance cost without maintenance, respectively. Observably, $c^e(0) = 0$. Since the cost reduction differ from reliability improvement in both scale and preference direction, $h^e(\cdot)$ is unified by,

$$\begin{aligned} h^e(1_\ell, \mathbf{c}) &= \frac{c_0 - c^e(\ell)}{c_0}, \\ h^e(\mathbf{c}) &= \frac{c_0 - c^e(0)}{c_0} = 1. \end{aligned} \tag{13}$$

To explicitly relate the H_ℓ^e and H_{ij}^e, c_0 is defined as the maximum cost of maintaining two components and can be evaluated by Eq. (26). Then, $c_0 - c^e(\ell)$ can be regarded as a positive cost utility of component ℓ. Moreover, c_0 is further utilized as a denominator to unify the cost utility of one component. When a maintenance activity is conducted on one component, the following costs include a maintenance cost of the component and a fixed setup cost. Therefore, the total maintenance cost of component ℓ is $c_\ell + c^e_{h(\ell)}$, that is,

$c^e(\ell) = c_\ell + c^e_{h(\ell)}$. Therefore, Eq. (12) can be evaluated by,

$$H^e_\ell = \frac{c_0 - c^e(\ell)}{c_0} - \frac{c_0 - c^e(0)}{c_0} = \frac{c^e(0) - c^e(\ell)}{c_0} = \frac{-(c_\ell + c^e_{h(\ell)})}{c_0}. \tag{14}$$

Equation (14) indicates that a greater value of H^e_ℓ represents a less maintenance cost of component ℓ.

4.1.3 *Reduced disassembly number with structural dependence*

To maintain a component, its upstream assembly nodes must be disassembled, with fewer disassembly number being more desirable. The reduced disassembly number is used as a utility to assess the effect of maintaining components. Drawing on the concept of marginal benefit, the unified disassembly number reduction of component ℓ is given by,

$$H^s_\ell = h^s(1_\ell, \mathbf{d}) - h^s(\mathbf{d}), \tag{15}$$

where $h^s(1_\ell, \mathbf{d})$ and $h^s(\mathbf{d})$ depend on their disassembly sequences. Since the disassembly number reduction differ from reliability improvement in both scale and preference direction, $h^s(\cdot)$ is unified by,

$$\begin{aligned} h^s(1_\ell, \mathbf{d}) &= \frac{n_0 - n(\mathbf{d}_\ell)}{n_0}, \\ h^s(\mathbf{d}) &= \frac{n_0 - n(\emptyset)}{n_0}. \end{aligned} \tag{16}$$

where n_0 is the maximum disassembly number of maintaining two components and can be evaluated via Eq. (30). The effect of n_0 in Eq. (16) is the same as that of c_0 in Eq. (13). $n(\mathbf{d}_\ell)$ and $n(\emptyset)$ are the disassembly number with maintaining component ℓ and the disassembly number without maintenance, respectively. $\mathbf{d}_\ell$ is the disassembly sequence of maintaining component ℓ. $\emptyset$ represents an empty set. Observably, $n(\emptyset) = 0$. The calculation of the disassembly number is defined as follows. Before one component is maintained, some preceding disassembly nodes including other components and sub-assemblies need to be disassembled. $\mathbf{d}$ denotes the disassembly sequence of a system and is demonstrated as follows. The disassembly sequence of the system can be drawn as a directed graph (Dao and Zuo, 2017) depicted in Fig. 1.

As shown in Fig. 1, there are three types of nodes, encompassing (1) node 0 which indicates the system; (2) node S_j which represents sub-assembly j; (3) node i which denotes component i. Particularly, there are three relationships between both nodes, including (1)

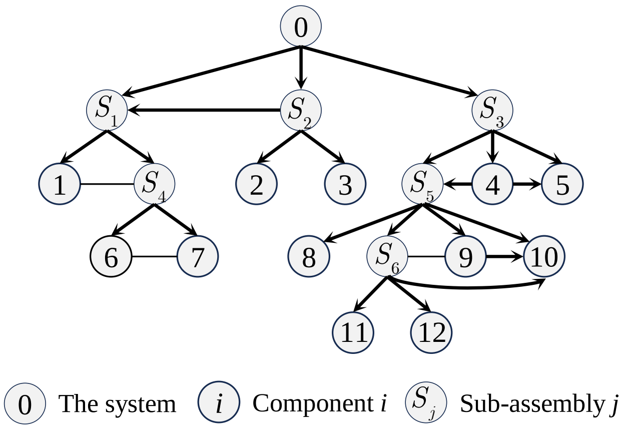

Figure 1: The disassembly sequence of the system

a disassembly sequence order from one node to another node, such as nodes 9 and 10; (2) both nodes are restricted, such as nodes 6 and 7; (3) both nodes are irrelevant, such as nodes 11 and 12. Therefore, the relationship between nodes i and j can be defined by $r(i,j)$ as follows:

$$r(i,j) = \begin{cases} 0, & \text{nodes } i \text{ and } j \text{ are irrelevant} \\ 1, & \text{a sequence order from node } i \text{ to node } j \\ -1, & \text{a sequence order from node } j \text{ to node } i \\ 2, & \text{nodes } i \text{ and } j \text{ are restricted} \end{cases} \tag{17}$$

$r(i,j) = 1$ or $r(j,i) = -1$ implies that node i needs to be disassembled before node j is disassembled. $r(i,j) = 2$ means that both nodes i and j need to be disassembled simultaneously. The disassembly sequence of node i is denoted by $\mathbf{d}_i$. For example, $\mathbf{d}_6 = (0, S_2, S_1, 1, S_4, 7, 6)$. The disassembly number of node i is equal to the number of the elements in the $\mathbf{d}_i$, and it is denoted by $n(\mathbf{d}_i)$. For example, $n(\mathbf{d}_6) = 7$. H_ℓ^s is given by,

$$\begin{aligned} H_\ell^s &= \frac{n_0 - n(\mathbf{d}_\ell)}{n_0} - \frac{n_0 - n(\emptyset)}{n_0} \\ &= \frac{n(\emptyset) - n(\mathbf{d}_\ell)}{n_0} \\ &= \frac{-n(\mathbf{d}_\ell)}{n_0}. \end{aligned} \tag{18}$$

Eq. (18) indicates that a greater value of H_ℓ^s represents a less disassembly number of maintaining component ℓ.

Consequently, $H_\ell^r(t_k)$, H_ℓ^e, and H_ℓ^s can be computed. Then, a novel 3D importance measure is defined as Definition 1.

Definition 1. *The 3D importance measure of component ℓ is defined by,*

$$I_\ell(t_k) = \alpha \underbrace{[h^r(1_\ell, \mathbf{p}_k) - h^r(\mathbf{p}_k)]}_{\text{Part1}} + \beta \underbrace{[h^e(1_\ell, \mathbf{c}) - h^e(\mathbf{c})]}_{\text{Part2}} + \gamma \underbrace{[h^s(1_\ell, \mathbf{d}) - h^s(\mathbf{d})]}_{\text{Part3}}, \tag{19}$$

where $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\alpha + \beta + \gamma = 1$.

α, β, and γ can be determined by multi-criteria decision-making methods, such as analytic hierarchy process (Salomon and Gomes, 2024), entropy weight method (Feng et al., 2024), and expert elicitation method (Rietbergen et al., 2016). The specific choice of weighting method does not affect the validity of the 3D importance formulation. It should be noted that Parts 2 and 3 of Eq. (19) are negative, and they can be interpreted as a penalty induced by cost and disassembly number. Under this sign structure, aggregation rules such as multiplicative or harmonic-mean forms are not suitable, as it may lead to undefined results when negative values are present. In contrast, a weighted additive aggregation naturally accommodates both benefit-type and penalty-type components and allows the overall importance to be interpreted as a net benefit. Therefore, a weighted average is utilized to aggregate the three components, and the resulting overall importance may be negative.

Definition 2. *Joint 3D importance measure, I_{ij}, of components i and j is defined by,*

$$\begin{aligned} I_{ij}(t_k) = {} & \alpha \underbrace{[h^r(1_i, 1_j, \mathbf{p}_k) - h^r(1_i, \mathbf{p}_k) - h^r(1_j, \mathbf{p}_k) + h^r(\mathbf{p}_k)]}_{\text{Part1}} \\ & + \beta \underbrace{[h^e(1_i, 1_j, \mathbf{c}) - h^e(1_i, \mathbf{c}) - h^e(1_j, \mathbf{c}) + h^e(\mathbf{c})]}_{\text{Part2}} \\ & + \gamma \underbrace{[h^s(1_i, 1_j, \mathbf{d}) - h^s(1_i, \mathbf{d}) - h^s(1_j, \mathbf{d}) + h^s(\mathbf{d})]}_{\text{Part3}}, \end{aligned} \tag{20}$$

where $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, and $\alpha + \beta + \gamma = 1$.

Joint 3D importance measure is used to quantify the influence of maintaining one component on 3D importance measure of the other component. The details on the relationship between 3D importance measure and joint 3D importance measure is demonstrated in Proposition 1. The three parts in Eq. (20) are explained below.

Part 1 in Eq. (20) is given by,

$$H^r_{ij}(t_k) = h^r(1_i, 1_j, \mathbf{p}_k) - h^r(1_i, \mathbf{p}_k) - h^r(1_j, \mathbf{p}_k) + h^r(\mathbf{p}_k), \tag{21}$$

where $h^r(1_i, \mathbf{p}_k)$, $h^r(1_j, \mathbf{p}_k)$, and $h^r(\mathbf{p}_k)$ can be evaluated by Eqs. (2) and (3). $h^r(1_i, 1_j, \mathbf{p}_k)$ is the system reliability after maintaining components i and j at time t_k, and it is given by,

$$h^r(1_i, 1_j, \mathbf{p}_k) = \sum_{u=1}^{2^m} \phi(\mathbf{X}_{u\backslash\{i,j\}}) p_{u,k}, \tag{22}$$

where $\mathbf{X}_{u\backslash\{i,j\}} = \{\mathbf{X}_{u,1}, \dots, \mathbf{X}_{u,i-1}, 1_i, \mathbf{X}_{u,i+1}, \dots, \mathbf{X}_{u,j-1}, 1_j, \mathbf{X}_{u,j+1}, \dots, \mathbf{X}_{u,2^m}\}$. Therefore, $H_{ij}^r(t_k)$ is given by,

$$\begin{aligned} H_{ij}^r(t_k) = & \sum_{u=1}^{2^m} \left(\phi\left(\mathbf{X}_{u\backslash\{i,j\}}\right) + \phi(\mathbf{X}_u)\right) p_{u,k} \\ & - \sum_{u=1}^{2^m} \left(\phi\left(\mathbf{X}_{u\backslash\{i\}}\right) + \phi\left(\mathbf{X}_{u\backslash\{j\}}\right)\right) p_{u,k}. \end{aligned} \tag{23}$$

where $p_{u,k}$ is the probability that the system stays at uth components state combination at time t_{k-1}, and it can be calculated by Eq. (5).

Part 2 in Eq. (20) is given by,

$$H_{ij}^e = h^e(1_i, 1_j, \mathbf{c}) - h^e(1_i, \mathbf{c}) - h^e(1_j, \mathbf{c}) + h^e(\mathbf{c}), \tag{24}$$

where $h^e(1_i, \mathbf{c})$, $h^e(1_j, \mathbf{c})$, and $h^e(\mathbf{c})$ can be evaluated by Eq. (13). $h^e(1_i, 1_j, \mathbf{c})$ depends on the cost of maintaining components i and j, and it is given by,

$$h^e(1_i, 1_j, \mathbf{c}) = \frac{c_0 - c^e(i,j)}{c_0}, \tag{25}$$

where $c^e(i,j)$ is the cost of maintaining components i and j, and it is given by,

$$c^e(i,j) = \begin{cases} c_i + c_j + c_{h(i)}^e + c_{h(j)}^e & h(i) \neq h(j) \\ c_i + c_j + c_{h(i)}^e & h(i) = h(j) \end{cases}. \tag{26}$$

It indicates that the maintenance cost will be saved if both components share the same facility. c_0 is the maximum cost of maintaining two components, that is, $c_0 = \max_{i,j} c^e(i,j)$. Then, H_{ij}^e is given by,

$$
\begin{aligned}
H_{ij}^{e} &= h^{e}(1_i, 1_j, \mathbf{c}) - h^{e}(1_i, \mathbf{c}) - h^{e}(1_j, \mathbf{c}) + h^{e}(\mathbf{c}) \\
&= \frac{c^{e}(i) + c^{e}(j) - c^{e}(i,j) - c^{e}(0)}{c_0} \\
&= \frac{c_i + c_{h(i)}^{e} + c_j + c_{h(j)}^{e} - c^{e}(i,j) - 0}{c_0} \\
&= \begin{cases} 0, & h(i) \neq h(j) \\ \dfrac{c_{h(i)}^{e}}{c_0}, & h(i) = h(j) \end{cases}.
\end{aligned} \tag{27}
$$

Part 3 in Eq. (20) is given by,

$$
H_{ij}^{s} = h^{s}(1_i, 1_j, \mathbf{d}) - h^{s}(1_i, \mathbf{d}) - h^{s}(1_j, \mathbf{d}) + h^{s}(\mathbf{d}), \tag{28}
$$

where $h^{s}(1_i, \mathbf{d})$, $h^{s}(1_j, \mathbf{d})$, and $h^{s}(\mathbf{d})$ can be evaluated by Eq. (18). $h^{s}(1_i, 1_j, \mathbf{d})$ depends on the disassembly number of components i and j, and it is given by,

$$
h^{s}(1_i, 1_j, \mathbf{d}) = \frac{n_0 - n(\mathbf{d}_i \cup \mathbf{d}_j)}{n_0}, \tag{29}
$$

where $n(\mathbf{d}_i \cup \mathbf{d}_j)$ is the disassembly number of maintaining components i and j, and it is given by,

$$
n(\mathbf{d}_i \cup \mathbf{d}_j) = n(\mathbf{d}_i) + n(\mathbf{d}_j) - n(\mathbf{d}_i \cap \mathbf{d}_j), \tag{30}
$$

where $\mathbf{d}_i \cup \mathbf{d}_j$ and $\mathbf{d}_i \cap \mathbf{d}_j$ are the union and intersection sets of $\mathbf{d}_i$ and $\mathbf{d}_j$. n_0 is the maximum disassembly number of disassembling two components, that is, $n_0 = \max_{i,j} n(\mathbf{d}_i \cup \mathbf{d}_j)$. $h^{s}(\cdot)$ equates to one when the disassembly number is zero, and it equates to zero when the disassembly number reaches its maximum value. Thus, H_{ij}^{s} can be further simplified as:

$$
\begin{aligned}
H_{ij}^{s} &= h^{s}(1_i, 1_j, \mathbf{d}) - h^{s}(1_j, \mathbf{d}) - h^{s}(1_i, \mathbf{d}) - h^{s}(\mathbf{d}) \\
&= \frac{n(\mathbf{d}_i) + n(\mathbf{d}_j) - n(\mathbf{d}_i \cup \mathbf{d}_j) - n(\emptyset)}{n_0} \\
&= \frac{n(\mathbf{d}_i \cap \mathbf{d}_j)}{n_0}.
\end{aligned} \tag{31}
$$

4.2 *Properties of the proposed 3D importance measures*

This subsection presents some properties of the proposed 3D importance measures.

Proposition 1. *The relationship between 3D importance measure and the joint form is that* $I_{ij}(t_k) = I_{i|j}(t_k) - I_i(t_k)$ *or* $I_{ij}(t_k) = I_{j|i}(t_k) - I_j(t_k)$. $I_{i|j}(t_k)$ $(I_{j|i}(t_k))$ *is that 3D importance measure of component* i (j) *when component* j (i) *has been maintained at time*

t_k.

The proof of Proposition 1 is given in Appendix A. Proposition 1 shows that joint 3D importance measure is equal to the different between 3D importance measure of one component when the other component has been maintained and that of one component without other maintenance activities. The 3D joint importance represents the interaction effect between two components in maintenance. A positive value indicates that maintaining the two components jointly provides additional benefits beyond separate maintenance actions, while a negative value implies that joint maintenance is disadvantageous. It possesses three cases: (1) $I_{ij}(t_k) > 0$; (2) $I_{ij}(t_k) < 0$; (3) $I_{ij}(t_k) = 0$. Case 1 implies that 3D importance measure of one component will increase with the maintenance of the other component. Case 2 implies that both components have a negative relationship, whereby Case 3 implies that 3D importance measure of one component is unchanged with the maintenance of the other component. A greater value of $I_{ij}(t_k)$ represents a greater 3D importance measure improvement of one component when the other component has been maintained.

Proposition 2. *Suppose that a series system consists of m components, and components i and j are stochastic dependent. $H_i^r(t_k)$ and $H_j^r(t_k)$ are no more than that without the consideration of stochastic dependence, that is, $H_i^r(t_k) \leq \bar{H}_i^r(t_k)$, and $H_j^r(t_k) \leq \bar{H}_j^r(t_k)$, where $\bar{H}_i^r(t_k)$ and $\bar{H}_j^r(t_k)$ are 3D importance measures of components i and j at time t_k without the consideration of stochastic dependence.*

The proof of Proposition 2 is given in Appendix B.

Lemma 1. *Suppose that a series system consists of m components, and components i and j are stochastic dependent. "Part 1" of 3D importance measure of component j (i) is non-decreasing with respect to $\bar{\lambda}_i(t_k)$ $(\bar{\lambda}_j(t_k))$ and independent of $\bar{\lambda}_j(t_k)$ $(\bar{\lambda}_i(t_k))$.*

The proof of Lemma 1 is given in Appendix C.

Proposition 3. *Suppose that a parallel system consists of m components, and components i and j are stochastic dependent. Then, $H_i^r(t_k) \geq \bar{H}_i^r(t_k)$, and $H_j^r(t_k) \geq \bar{H}_j^r(t_k)$, where $\bar{H}_i^r(t_k)$ and $\bar{H}_j^r(t_k)$ are 3D importance measures of components i and j without the consideration of stochastic dependence.*

The proof of Proposition 3 is given in Appendix D.

Lemma 2. *Suppose that a parallel system consists of m components, and components i and j are stochastic dependent. "Part 1" of 3D importance measure of component j is the same as that of component i, that is, $H_j^r(t_k) = H_i^r(t_k)$.*

The proof of Lemma 2 is given in Appendix E.

Lemma 3. *Suppose that a parallel system consists of m components, and components i and j are stochastic dependent. "Part 1" of 3D importance measure of component j (i) is non-increasing with respect to $\bar{\lambda}_i(t_k)$ and $\bar{\lambda}_j(t_k)$.*

The proof of Lemma 3 is given in Appendix F.

Proposition 4. *$H^r_{ij}(t_k) \geq 0$ when the system is a series system.*

The proof of Proposition 4 is given in Appendix G.

Lemma 4. *$H^r_{ij}(t_k) \leq 0$ when the system is a parallel system.*

The proof of Lemma 4 is given in Appendix H.

Proposition 5. *Suppose a series system consists of m components, and components i and j are stochastic dependent. Then, $H^r_{ij}(t_k) \leq \bar{H}^r_{ij}(t_k)$, where $\bar{H}^r_{ij}(t_k)$ is joint 3D importance measures of components i and j without the consideration of stochastic dependence. Meanwhile, $H^r_{ij}(t_k)$ is non-increasing with respect to $\bar{\lambda}_i(t_k)$ and $\bar{\lambda}_j(t_k)$.*

The proof of Proposition 5 is given in Appendix I.

Lemma 5. *Suppose a parallel system consists of m components, and components i and j are stochastic dependent. Then, $H^r_{ij}(t_k) \leq \bar{H}^r_{ij}(t_k)$, where $\bar{H}^r_{ij}(t_k)$ is joint 3D importance measures of components i and j without the consideration of stochastic dependence. Meanwhile, $H^r_{ij}(t_k)$ is non-decreasing with respect to $\bar{\lambda}_i(t_k)$ and $\bar{\lambda}_j(t_k)$.*

The proof of Lemma 5 is given in Appendix J.

Proposition 6. *H^e_{ij} is no less than zero, that is, $H^e_{ij} \geq 0$.*

The proof of Proposition 6 is given in Appendix K. This proposition implies that the cost of maintaining one component will decrease with the maintenance of the other component when both components share the same facility, and it is unchanged when both components don't share the same facility. A greater value of H^e_{ij} represents a greater maintenance cost are saved when both components are maintained simultaneously.

Proposition 7. *H^s_{ij} is no less than zero, that is, $H^s_{ij} \geq 0$.*

The proof of Proposition 7 is given in Appendix L. This proposition suggests that the more common nodes there are to dismantle between two components, the greater the benefit of dismantling one component becomes when the other has already been dismantled. A greater value of H_{ij}^s represents a greater disassembly number is saved when both components are maintained simultaneously.

At this point, joint 3D importance measure can be computed. It involves the improvement of the system reliability, the saving of the maintenance cost, and the reduction of disassembly number. The value of the proposed joint 3D importance measure can guide decision-maker to trade off the benefit of maintaining both components.

Lemma 6. *For a series system, joint 3D importance measure is no less than zero, that is,* $I_{ij}(t_k) \geq 0$.

The proof of Lemma 6 is given in Appendix M.

Proposition 8. *When stochastic dependence is absent, the relationship between* $H_\ell^r(t_k)$ *and Birnbaum importance measure is shown as follow,*

$$H_\ell^r(t_k) = \Pr\{C_\ell(t_k) = 0_\ell\} I_\ell^b(t_k), \tag{32}$$

where $I_\ell^b(t_k)$ *is Birnbaum importance measure of component* ℓ*.*

The proof of Proposition 8 is given in Appendix P.

From a practical standpoint, the derived properties of the proposed importance measures provide actionable insights for maintenance planning. The monotonicity properties guide the ranking of components under different system structures, while the joint measure quantifies the synergy of concurrent maintenance actions. These properties help practitioners determine which components should be prioritized, combined, or postponed during maintenance scheduling.

4.3 Algorithm for calculating 3D importance measure

The pseudocode of the algorithm for calculating 3D importance measure is given in Appendix N. It should be noted that the proposed 3D importance measure is utilized to quantify the effectiveness of maintaining one component with three types of component dependence. The proposed 3D importance measure can be extend to quantify the effectiveness of maintaining multiple components via the following steps: (1) change the elements in $\mathbf{X}_i$ to 1 if the corresponding components are maintained, and then substitute the changed $\mathbf{X}_i$ into Section 4.1.1 to calculate the reliability improvement; (2) compute the total cost of maintaining multiple components and the maximum maintenance cost, and use them in Section 4.1.2 to calculate the unified cost reduction; (3) evaluate the union sets of the disassembly sequence of the components and the maximum disassembly number, and apply them to Section 4.1.3 to calculate the unified disassembly number reduction; (4) obtain 3D importance measure of multiple components via Eq. (19).

5 Illustrative examples

5.1 A numerical case

To systematically evaluate the behavior of the proposed importance framework under different dependency configurations and system scales, we conduct a controlled numerical study prior to the illustrative case presented in Section 5.2. The objective is to investigate how stochastic degradation coupling, economic resource sharing, and structural disassembly hierarchy influence component ranking, and to compare the proposed three-dimensional importance measure with classical reliability-based metrics under parameter perturbation and system expansion.

We first consider a three-component system with a mission time of five years. The detailed parameter settings are summarized in Table 1. Components 1 and 2 are stochastically dependent, while Component 3 is independent. Specifically, if Component 2 fails, the failure intensity of Component 1 increases to $0.06t^2$. Components 1 and 3 are economically related because they share the same facility (F1). The maintenance process follows a strict disassembly order $\{1 \rightarrow 2 \rightarrow 3\}$, meaning that maintenance of Component 3 requires prior dismantling of Components 1 and 2 due to structural dependence. The weighting coefficients of reliability, economic, and structural importance, denoted by α, β, and γ, are set to 0.6, 0.25, and 0.15, respectively. Both series and parallel system configurations are examined. In each experiment, six importance evaluation schemes are compared. Method 1 corresponds to the proposed framework incorporating all three types of dependencies. Methods 2-4 are ablation variants in which only two dependency types are retained. In Method 2 (without stochastic dependence), component failures do not affect the degradation rates of other components, and failure intensities remain independent. In Method 3 (without economic dependence), facility sharing is removed, and each maintenance action independently incurs its setup cost, thereby eliminating cost coupling. In Method 4 (without structural dependence), the disassembly hierarchy is ignored, and the structural weight is proportionally redistributed among the remaining dimensions, effectively removing assembly constraints from the evaluation. Methods 5 and 6 correspond to the classical Birnbaum and Fussell-Vesely importance measures, which do not incorporate dependency effects.

Table 1: Component parameters of the three-component system

Component	Intensity	Stochastic dependence	Fixed cost (unit: US dollors)	Facility and its setup cost (unit: US dollors)
1	$0.01t^2$	2 fails: $0.06t^2$	3500	F1: 3000
2	$0.05t$	1 fails: $0.2t$	3000	F2: 2000
3	0.04	No	2500	F1: 3000

Before analyzing importance values, we first compare baseline component reliabilities under independence and stochastic dependence. Without stochastic coupling, the reliabilities of Components 1, 2, and 3 are 0.6592, 0.5353, and 0.8187, respectively. When stochastic dependence is incorporated, the reliabilities of Components 1 and 2 decrease to 0.4488 and 0.4441, while Component 3 remains at 0.8187. This confirms that mutual degradation amplification between Components 1 and 2 significantly reduces their effective reliability levels, and consequently affect system-level performance. For the three-component series system, the reliability-based marginal scores under stochastic dependence are 0.0747, 0.0786, and 0.0640 for Components 1, 2, and 3, respectively. When stochastic dependence is ignored, the corresponding scores become 0.1493, 0.2508, and 0.0640. This indicates that, for Components 1 and 2, the reliability-based scores under stochastic coupling are lower than those obtained under independence. These results are consistent with Proposition 2. For the three-component parallel system, the reliability-based marginal scores under stochastic dependence are 0.0834 for all components, whereas under independence they are 0.0287 for all components. In other words, for both Component 1 and Component 2, the reliability-based scores under stochastic coupling are higher than those obtained under independence. These outcomes align with Proposition 3.

Figure 2 presents the component ranking results under six importance evaluation schemes for the series configuration. The leftmost panel corresponds to the baseline case without parameter perturbation. Considering that, in practical engineering applications, degradation parameters are typically estimated from data and are subject to uncertainty due to limited sample sizes and measurement noise, robustness analysis is necessary. To assess the stability of the ranking outcomes, multiplicative perturbations of 5% and 10% are applied to the failure intensity parameters. The four panels on the right-hand side of Figure 2 display the corresponding ranking results under these perturbation scenarios. The Birnbaum importance (Method 5) ranks components in decreasing order of reliability in the series configuration, while the Fussell-Vesely measure (Method 6) yields identical rankings. These rankings coincide with those obtained from reliability magnitudes under independence. A more insightful observation concerns the relative ordering of Components 1 and 2. Taking the baseline case without parameter perturbation as an example, the corresponding values are 0.1493 and 0.2508 for Components 1 and 2, respectively. When stochastic coupling is incorporated, the two values become nearly equal (0.0747 versus 0.0786). As a result, compared with Method 2 (without stochastic), the proposed method (Method 1) changes the ranking of Component 1 from second to first. Furthermore, when structural constraints are ignored (Method 4), Component 3 may attain a higher ranking position. Once the disassembly hierarchy is incorporated (Method 1), the necessity of dismantling preceding components reduces the relative priority of Component 3, resulting in a consistent ranking shift. This demonstrates that structural hierarchy can alter priorities when reliability contributions are comparable. Across all perturbation levels, ranking results remain largely stable for both the proposed and classical measures, indicating that moderate uncertainty in parameter estimation does not materially affect priority ordering. Moreover, the rankings obtained from the proposed method differ from those of the classical measures, indicating that reliability magnitude alone does not fully determine maintenance importance.

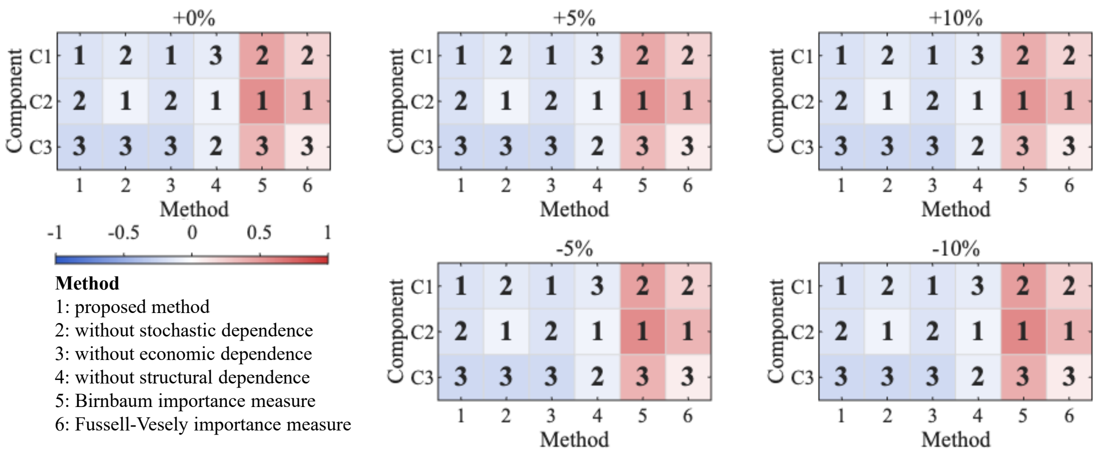

Figure 2: Importance ranking results of the three-component series system with parameter perturbations

Figure 3 shows the ranking results for the three-component parallel system under the same perturbation levels. In contrast to the series configuration, the Birnbaum importance ranking in the parallel system is reversed, consistent with theoretical expectations. The Fussell-Vesely measure assigns identical importance values across components in the parallel configuration because its value equals one for all components. However, when structural dependence is incorporated within the proposed framework (Method 1), ranking differences emerge even in the parallel system. Accounting for assembly constraints shifts Component 3 to a higher priority position compared to the case where structural dependence is ignored. Similar to the series case, ranking outcomes remain robust under parameter perturbations, suggesting that dependency structure, particularly the maintenance hierarchy, rather than small parameter fluctuations governs ranking behavior.

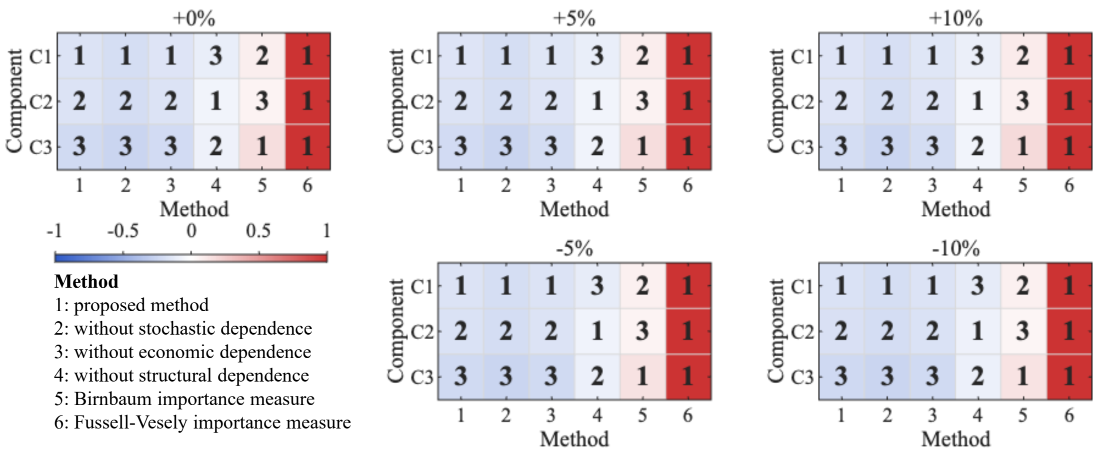

Figure 3: Importance ranking results of the three-component parallel system with parameter perturbations

We further extend the analysis to larger systems to investigate scalability. Figure 4 presents the ranking results for three-, five-, and ten-component systems under both series and parallel configurations without parameter perturbation. In the five-component case, additional components are introduced without adding new dependency types, and the assembly sequence becomes $\{1 \to 2 \to 3 \to 4 \to 5\}$. In the ten-component configuration, additional economic sharing relationships are incorporated among newly added components, with the assembly sequence extended to $\{1 \to 2 \to 3 \to \cdots \to 9 \to 10\}$. The detailed component information of five-component system is summarized in Table 2. For the ten-component system, the parameter settings follow the same structure. Specifically, in Table 2, Component 4 is extended to represent Components 4-6, and Component 5 is extended to represent Components 7-10, while all other parameter values remain unchanged. As observed in Fig. 4, the classical Birnbaum and Fussell-Vesely measures produce rankings that remain invariant as system size increases, since they depend solely on reliability magnitudes and system topology under independence assumptions. In contrast, the proposed method exhibits size-dependent ranking changes. For example, in both series and parallel configurations, the importance rankings under the proposed method differ between the three- and five-component systems. Such changes are not observed under the classical measures, demonstrating that the proposed framework captures coupling effects that become increasingly significant as system complexity grows. The proposed importance measure differentiates components in ways that classical reliability-based measures cannot, while remaining robust under moderate parameter perturbations and responsive to system expansion.

Table 2: Component parameters of the five-component system

Component	Intensity	Stochastic dependence	Fixed cost (unit: US dollors)	Facility and its setup cost (unit: US dollors)
1	$0.01t^2$	2 fails: $0.06t^2$	3500	F1: 3000
2	$0.05t$	1 fails: $0.2t$	3000	F2: 2000
3	0.04	No	2500	F1: 3000
4	$0.015t^{0.7}$	No	3000	F1: 1500
5	0.03	No	4000	F1: 2200

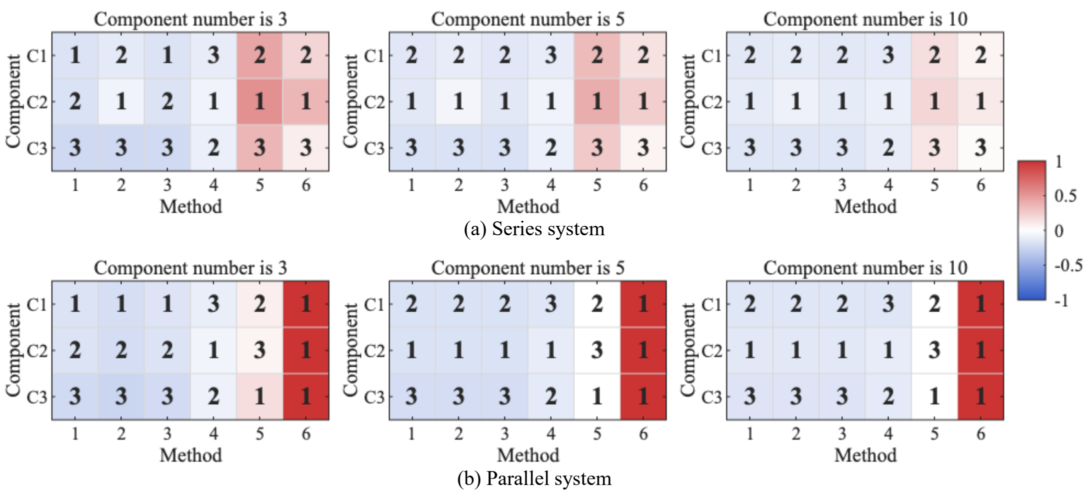

Figure 4: Importance ranking results for three-, five-, and ten-component systems under series and parallel configurations

5.2 A landing gear system

A landing gear system is utilized to illustrate the effectiveness of the proposed method, as shown in Fig. 5. The landing gear system analyzed in this study is constructed based on the typical architecture and maintenance characteristics of aircraft landing gear assemblies, as commonly observed in aviation reliability practice (Boniol et al., 2017). It consists of eleven main components, including a gear actuator, a drag strut, two link rods, two wheels, two torque links, a wheel axle, and a shock strut. The shock strut can be further divided into an outer cylinder and a piston.The parameters of the landing-gear system are assumed because the real data are confidential and are therefore unavailable to us. Nevertheless, The chosen of parameters reflect realistic engineering magnitudes. The configuration and dependency relationships were determined through engineering judgment and domain expertise, reflecting the realistic physical coupling and maintenance interactions among mechanical and hydraulic subsystems. The component parameters such as maintenance cost and failure intensity were set according to commonly accepted ranges in mechanical reliability studies (Roth et al., 2019; Deng et al., 2025), adjusted to preserve their relative magnitudes and ensure interpretability. Stochastic dependence was represented by degradation coupling between paired components (e.g., link rods and wheels), economic dependence was modeled through shared maintenance facilities (strut and universal service stations), and structural dependence was captured via the hierarchical disassembly sequence.

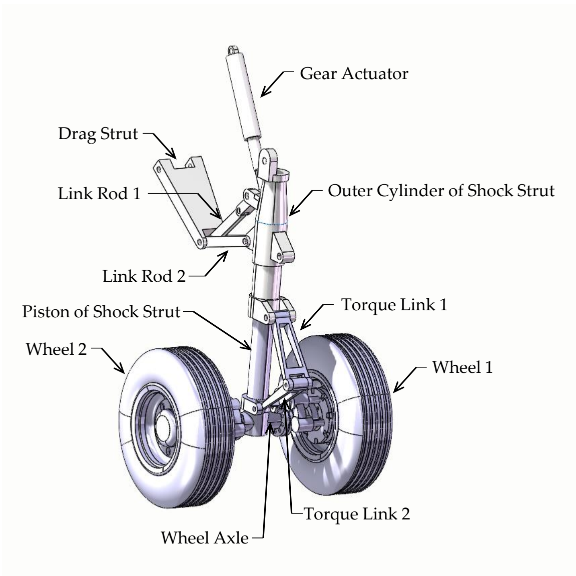

Figure 5: The disassembly sequence of the landing gear system

The disassembly sequence of the landing gear system is depicted in Fig. 6. For example, the disassembly of the shock strut necessitates the prior dismantling of the gear actuator, the link rods, the torque links, and the wheel axle. Moreover, disassembling the wheel axle involves dismantling two wheels first and subsequently removing the components on the wheel axle. The maintenance cost of each component is tabulated in Table 3. Meanwhile, the maintenance of the shock strut needs the strut service facility, whereas the maintenance of the gear actuator and wheels requires the universal service facility. The costs of the strut and universal service facilities are 2000 and 3000 US dollars, respectively. The initial failure intensity function of each component is tabulated in Table 4. If a link rod (wheel) fails, the failure intensity function of the other link rod (wheel) will increase to $0.2x^{0.5}$ $(0.3x)$. The reliability block diagram of the landing gear system is shown in Fig. 7. Meanwhile, according to the analytic hierarchy process method based on expert judgement, the weights α, β, and γ are set to be 0.85, 0.1, and 0.05, respectively. Regarding the weights, operational reliability takes precedence over maintenance cost and disassembly number. The impact of these weights on the proposed 3D importance measures will be further investigated in the subsequent content. The state of each component at 0.5 years is in its working state. The elapsed time is 0.5 years.

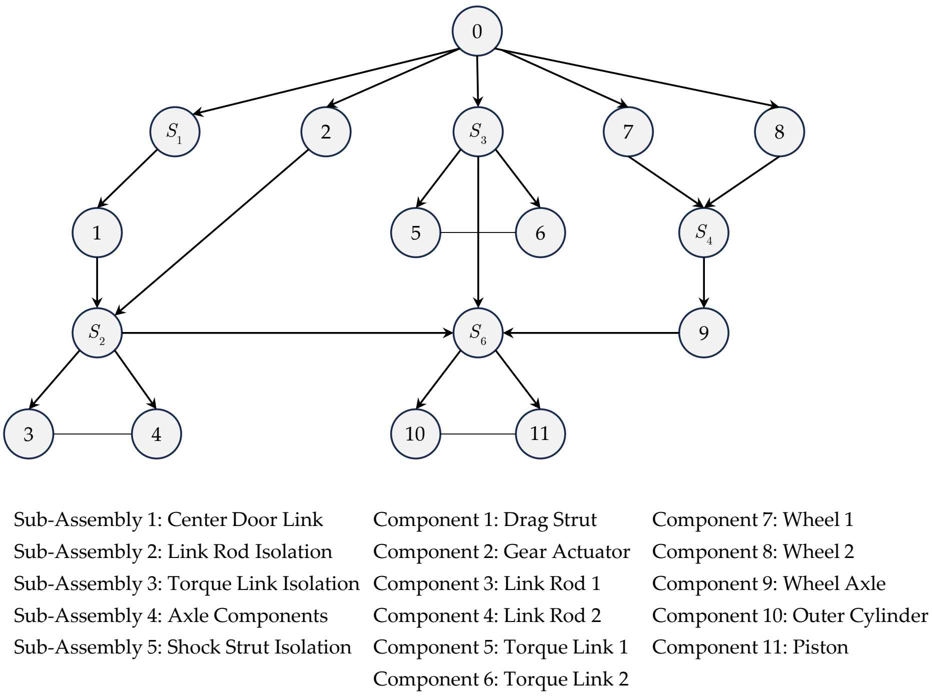

Figure 6: The disassembly sequence of the landing gear system

Table 3: The maintenance cost of each component (unit: US dollars)

Component ID	1	2	3, 4	5, 6	7, 8	9	10	11
Maintenance Cost	5000	20000	1000	1500	2000	3000	30000	15000

Table 4: The initial failure intensity function of each component (unit: per year)

Component ID	1	2	3, 4	5, 6	7, 8	9	10	11
Initial Failure Intensity	$0.05x^2$	$0.1x$	$0.1x^{0.5}$	$0.04x^2$	$0.15x$	$0.04x^2$	$0.02x^{0.7}$	0.04

Table 5: 3D importance measure of each component at $t_k = 1$ year

Component ID	1	2	3, 4	5, 6	7, 8	9	10	11
$H_\ell^r(t_k)$	0.0129	0.0335	0.0031	0.0103	0.0052	0.0155	0.0072	0.0177
Rank	4	1	10	5	8	3	7	2
H_ℓ^e	-0.1129	-0.371	-0.0161	-0.0242	-0.0806	-0.0968	-0.5161	-0.2742
Rank	8	10	1	3	5	7	11	9
H_ℓ^s	-0.2	-0.1333	-0.4667	-0.2667	-0.1333	-0.3333	-0.8667	-0.8667
Rank	4	1	8	5	1	7	10	10
$I_\ell(t_k)$	-0.01033	-0.0153	-0.0233	-0.007	-0.01032	-0.0132	-0.0888	-0.0557
Rank	5	8	9	1	3	6	11	10

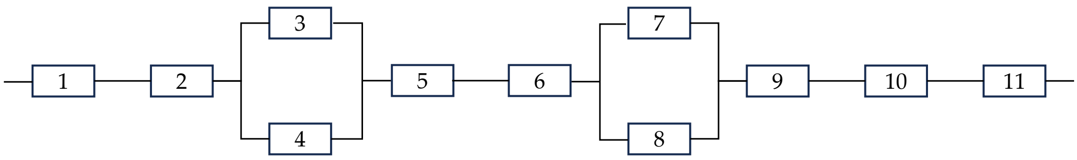

Figure 7: The reliability block diagram of the system

3D importance measure of each component is tabulated in Table 5. As shown in Table 5, maintaining the gear actuator yields the highest improvement in system reliability. Additionally, the link rods have the lowest maintenance cost, while the wheels involve the least disassembly number. Overall, the torque links are the most critical. Although the torque links do not rank highest in terms of reliability improvement alone, they achieve the highest overall importance by balancing reliability, maintenance cost, and disassembly number. In the case study, prioritizing the torque links leads to a lower expected maintenance cost and reduced disassembly burden while still achieving a comparable level of system reliability improvement. In contrast, strategies focusing only on reliability improvement tend to select components with higher maintenance costs or greater structural complexity, resulting in less efficient maintenance actions. The proposed measure produces rankings consistent with engineering intuition, that is, components with large marginal reliability impact and receive high priority, while components with high maintenance cost and disassembly number receive low priority. Joint 3D importance measure of each component is tabulated in Table 6. As observed from Table 6, joint 3D importance measures of maintaining any two components are positive, indicating that repairing any pair of components will produce beneficial outcomes. Particularly, maintaining the outer cylinder and piston of the shock strut simultaneously leads to the highest value of joint 3D importance measures.

Table 6: Joint 3D importance measure of both components $t_k = 1$ year

ID	1	2	3	4	5	6	7	8	9	10	11
1	-	0.0038	0.0100	0.0100	0.0035	0.0035	0.0034	0.0034	0.0035	0.0133	0.0134
2	0.0038	-	0.0068	0.0068	0.0037	0.0037	0.0083	0.0083	0.0087	0.0069	0.0072
3	0.0100	0.0068	-	0.0207	0.0034	0.0034	0.0033	0.0033	0.0034	0.0167	0.0167
4	0.0100	0.0068	0.0207	-	0.0034	0.0034	0.0033	0.0033	0.0034	0.0167	0.0167
5	0.0035	0.0037	0.0034	0.0034	-	0.0134	0.0034	0.0034	0.0035	0.0067	0.0068
6	0.0035	0.0037	0.0034	0.0034	0.0134	-	0.0034	0.0034	0.0035	0.0067	0.0068
7	0.0034	0.0083	0.0033	0.0033	0.0034	0.0034	-	0.0038	0.0116	0.0067	0.0068
8	0.0034	0.0083	0.0033	0.0033	0.0034	0.0034	0.0038	-	0.0116	0.0067	0.0068
9	0.0035	0.0087	0.0034	0.0034	0.0035	0.0035	0.0116	0.0116	-	0.0168	0.0169
10	0.0133	0.0069	0.0167	0.0167	0.0067	0.0067	0.0067	0.0067	0.0168	-	**0.0467**
11	0.0134	0.0072	0.0167	0.0167	0.0068	0.0068	0.0068	0.0068	0.0169	**0.0467**	-

As mentioned in Section 4.1.1, the initial time and the elapsed time will affect the system reliability improvement $H_\ell^r(t_k)$. To explore the influence of initial and elapsed time on the value of $H_\ell^r(t_k)$, we set the initial and elapsed time values in the range [0,2] years with the intervals of 0.2 years, as shown in Fig. 8. As observed from Fig. 8, the value of $H_\ell^r(t_k)$ first increases and then decreases to zero with the increase of the initial and elapsed time. Specifically, the value of $H_\ell^r(t_k)$ is equal to zero when the elapsed time is zero years. Meanwhile, to observe the ranking changes of the components, we depict the ranking of $H_1^r(t_k)$ and $H_{11}^r(t_k)$ specifically, as illustrated in Fig. 9. The ranking of each component changes with various settings of the initial and elapsed time. With the increase of initial and elapsed time, the relative importance of Component 1 continuously increases, whereas that of Component 11 gradually decreases.

By changing the maintenance cost of Component 1, the value of H_ℓ^e of each component is tabulated in Table 7. Meanwhile, the value of H_ℓ^e of each component with various costs of the strut service facility is tabulated in Table 8. As observed in Table 7, the value of H_1^e increases to one with the increase of the maintenance cost of Component 1, whereas the values of H_ℓ^e of other components decrease to zero. Similar results can be found in other components. As shown in Table 8, the values of H_ℓ^e of Components 1, 10, and 11 approach one when the cost of the strut service facility equates to 10^7 US dollars. This is because Components 1, 10, and 11 share the strut service facility.

To demonstrate how the disassembly number affects the value of H_ℓ^s, 10^3 disassembly steps are added for each component before its disassembly. The results are tabulated in Table 9. The values of H_1^s, H_3^s, H_4^s, and H_{10}^s approach one when 10^3 disassembly steps are added for Component 1 as the disassembly of Components 3, 4, 10, and 11 requires disassembling Component 1 first. Similar results can be found when 10^3 disassembly steps

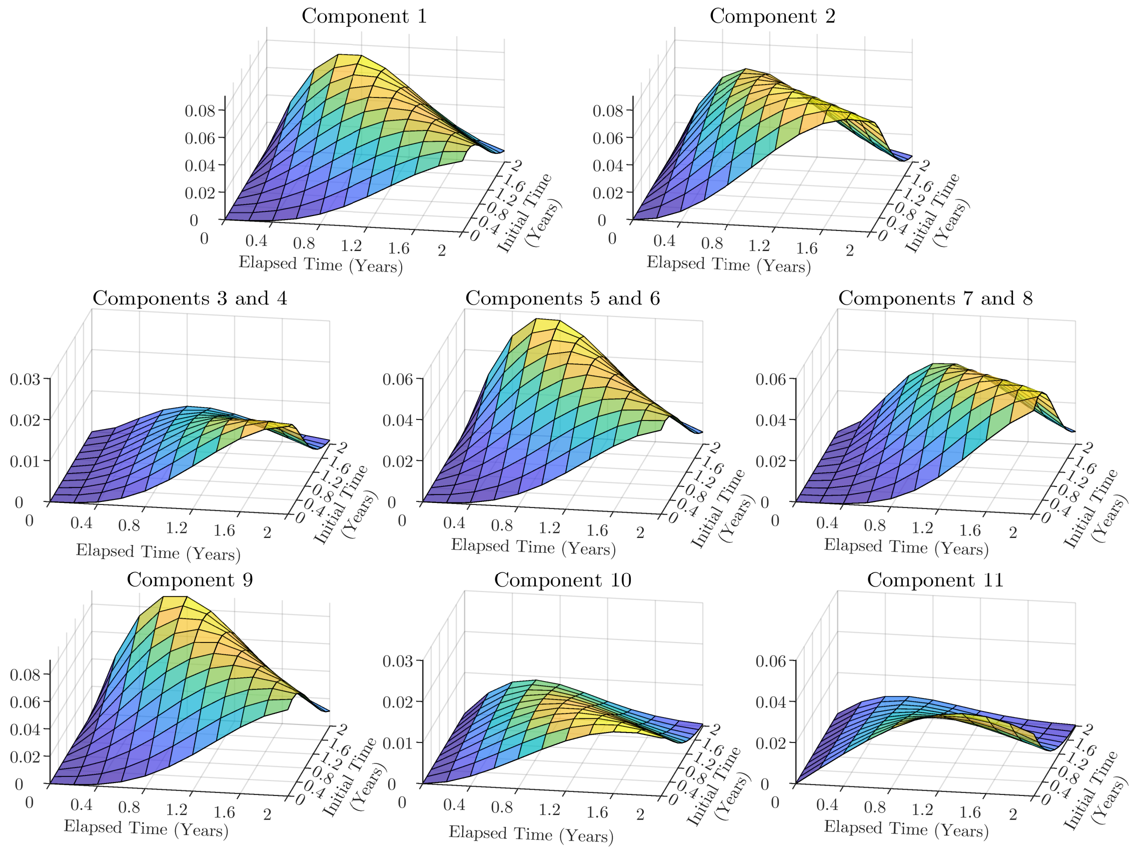

Figure 8: The value of H_ℓ^r of each component with various settings of the initial and elapsed time (time unit: years)

are added for other components.

To investigate the impact of weights on the proposed 3D importance measure, the ranking of $I_\ell(t_k)$ with various settings of weights is shown in Fig. 10. Due to space constraints, only a subset of components are presented in the figure. As observed from Fig. 10, the following conclusions can be drawn: (1) Component 2 becomes more important as weights α and γ increase, while its ranking of $I_2(t_k)$ decreases with the increase of weight β. It implies that the maintenance of Component 2 contributes significantly to enhancing system reliability, and its disassembly procedure is uncomplicated. In contrast, a reverse trend is observed for Components 3 and 4; (2) Components 7 and 8 remain important in most weight settings, while Component 11 yields an opposing conclusion.

6 Conclusion, Discussion, and Future Work

This article proposed a 3D importance measure to assess the effectiveness of maintaining one component with tripartite component dependence. The proposed measure included the

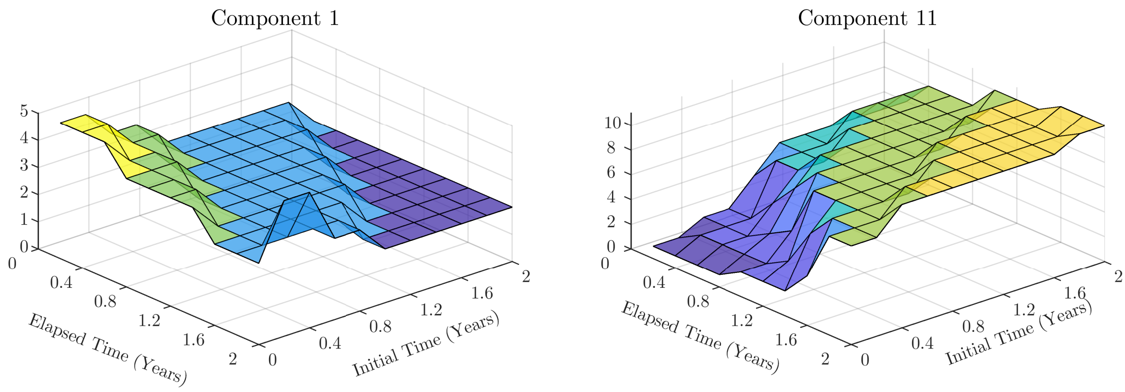

Figure 9: The ranking of H_1^r and H_{11}^r with various settings of the initial and elapsed time (time unit: years)

Table 7: The value of H_ℓ^e with various maintenance costs of Component 1 (cost unit: US dollars)

Cost	Component ID							
	1	2	3, 4	5, 6	7, 8	9	10	11
10^3	-0.0545	-0.4182	-0.0182	-0.0273	-0.0909	-0.1091	-0.5818	-0.3091
10^4	-0.2182	-0.4182	-0.0182	-0.0273	-0.0909	-0.1091	-0.5818	-0.3091
10^5	-0.7727	-0.1742	-0.0076	-0.0114	-0.0379	-0.0455	-0.2424	-0.1288
10^6	-0.9709	-0.0223	-0.0010	-0.0015	-0.0048	-0.0058	-0.0310	-0.0165
10^7	-0.9970	-0.0023	-0.0001	-0.0001	-0.0005	-0.0006	-0.0032	-0.0017

system reliability improvement, the maintenance cost reduction, and the disassembly number reduction. The system reliability improvement incorporated the component degradation interaction caused by stochastic dependence. The maintenance cost reduction involved the shared costs due to economic dependence. The disassembly number reduction included the disassembly sequence induced by structural dependence. Meanwhile, 3D importance measure was extended to a joint form to evaluate the maintenance effectiveness change of maintaining one component after the other component has been maintained. Some interesting properties of the proposed 3D importance measures, including the relationship between both 3D importance measures, the influence of the neglect of stochastic dependence, the monotonicity of the proposed measures, and the value range of joint 3D importance measure, were proved. Moreover, the pseudocode of the algorithm for calculating the proposed 3D importance measures was provided. A numerical case and a landing gear system were analyzed to showcase the efficacy of the proposed method. The conclusions showed that (1) three parts of the proposed measure were sensitive to the operation time of the system, the cost of maintaining each component, the disassembly steps of each component, respectively; (2) the proposed measures can effectively quantify the effectiveness of maintenance

Table 8: The value of H^e_ℓ with various costs of the universal facility (cost unit: US dollars)

Cost	Component ID							
	1	2	3, 4	5, 6	7, 8	9	10	11
10^3	-0.1111	-0.4259	-0.0185	-0.0278	-0.0926	-0.1111	-0.5741	-0.2963
10^4	-0.2381	-0.3651	-0.0159	-0.0238	-0.0794	-0.0952	-0.6349	-0.3968
10^5	-0.6863	-0.1503	-0.0065	-0.0098	-0.0327	-0.0392	-0.8497	-0.7516
10^6	-0.9544	-0.0218	-0.0009	-0.0014	-0.0047	-0.0057	-0.9782	-0.9639
10^7	-0.9952	-0.0023	-0.0001	-0.0001	-0.0005	-0.0006	-0.9977	-0.9962

Table 9: The value of H^s_ℓ with changing disassembly steps of each component

Changed ID	Component ID							
	1	2	3,4	5,6	7	8	9	10,11
1	-0.9882	-0.0020	-0.9921	-0.0039	-0.0020	-0.0020	-0.0049	-0.9980
2	-0.0030	-0.9872	-0.9921	-0.0039	-0.0020	-0.0020	-0.0049	-0.9980
3	-0.0030	-0.0020	-0.9921	-0.0039	-0.0020	-0.0020	-0.0049	-0.0128
4	-0.0030	-0.0020	-0.9921	-0.0039	-0.0020	-0.0020	-0.0049	-0.0128
5	-0.0030	-0.0020	-0.0069	-0.9892	-0.0020	-0.0020	-0.0049	-0.0128
6	-0.0030	-0.0020	-0.0069	-0.9892	-0.0020	-0.0020	-0.0049	-0.0128
7	-0.0030	-0.0020	-0.0069	-0.0039	-0.9872	-0.0020	-0.9901	-0.9980
8	-0.0030	-0.0020	-0.0069	-0.0039	-0.0020	-0.9872	-0.9901	-0.9980
9	-0.0030	-0.0020	-0.0069	-0.0039	-0.0020	-0.0020	-0.9901	-0.9980
10	-0.0030	-0.0020	-0.0069	-0.0039	-0.0020	-0.0020	-0.0049	-0.9980
11	-0.0030	-0.0020	-0.0069	-0.0039	-0.0020	-0.0020	-0.0049	-0.9980

policies when the maintenance on a multi-component system is affected by the tripartite component dependence.

It should be noted that the current formulation assumes pairwise stochastic dependency between components. The proposed framework could be extensible to accommodate dependency among multiple components. The transition probability in Eq. (6) could be generalized to include multi-component conditional probabilities without altering the overall computational framework. However, such extensions would significantly increase computational complexity and reduce interpretability, as the number of component combinations grows combinatorially. Future research will investigate sparse evaluation and surrogate models to balance modeling fidelity and practical applicability.

Several potential issues merit in-depth research in our future works. First, this article presumed the degradation behavior of each component is binary-state. Indeed, components can encompass several intermediate states beyond mere working and failed. The extension of the proposed measure to multi-state components will be explored in the future. Moreover, this article assumed that the failure intensity of one component will increase when the stochastic dependent component fails. Other stochastic degradation dependency among

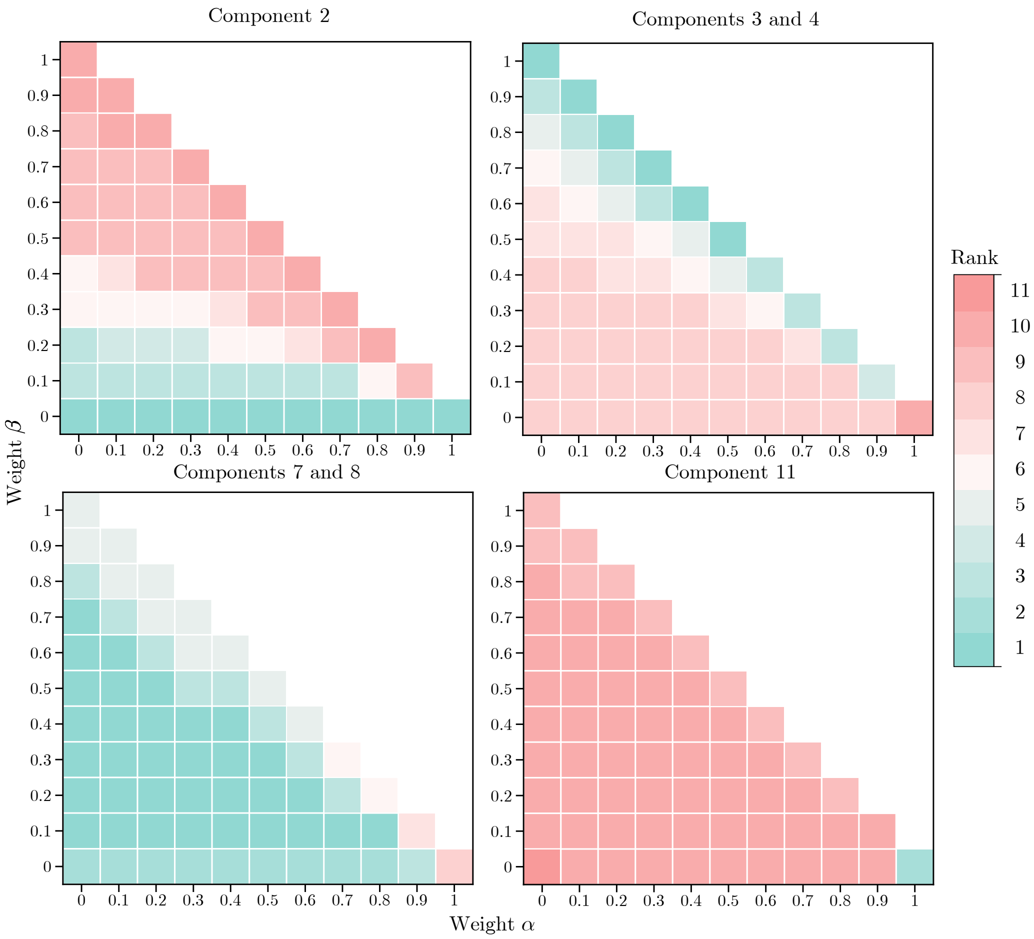

Figure 10: The ranking of $I_\ell(t_k)$ with various settings of weights

components, including common cause and cascading failures, can be considered in the future. Meanwhile, the maintenance epochs are predefined, rather than being optimized or condition-based. In our future research, we will research into the determination of maintenance epochs based on an optimization or condition-based decision framework, enabling adaptive scheduling that balances reliability improvement, maintenance cost, and system structure. Furthermore, the current study employs a conditional intensity formulation for clarity and analytical tractability, but the proposed framework can be extended to alternative dependence structures such as shared frailty models, copula-based representations, or graphical intensity networks. Future research will further investigate these dependence forms and assess their effects on maintenance decision-making. While this paper focuses on developing the maintenance importance framework with tripartite component dependence, future work will systematically investigate weight determination using various MCDM techniques comparing their stability and interpretability in real maintenance decisions. Moreover, the proposed importance measures are evaluated at discrete maintenance epochs and quantify instantaneous marginal effects of maintenance actions. Extending the framework to incorporate life-cycle economic performance and reliability evolution over an entire maintenance cycle is of practical interest, and it would be studied in the future.

References

Arriaza, A., Navarro, J., Sordo, M. A., and Suárez-Llorens, A. (2023). A variance-based importance index for systems with dependent components. *Fuzzy Sets and Systems*, 467:108482.

Aven, T., Insua, D. R., Soyer, R., Zhu, X., and Zio, E. (2025). Fifty years of reliability in operations research. *European Journal of Operational Research*, 324:361–381.

Bansal, V., Chen, Y., and Zhou, S. (2025). Component-wise markov decision process for solving condition-based maintenance of large multi-component systems with economic dependence. *IISE transactions*, 57(2):158–171.

Barlow, R. E. and Proschan, F. (1975). Importance of system components and fault tree events. *Stochastic Processes and their Applications*, 3(2):153–173.

Birnbaum, Z. W. et al. (1968). *On the Importance of Different Components in A Multicomponent System*. Academic Press.

Boniol, F., Wiels, V., Aït-Ameur, Y., and Schewe, K.-D. (2017). The landing gear case study: challenges and experiments. *International Journal on Software Tools for Technology Transfer*, 19(2):133–140.

Borgonovo, E., Aliee, H., Glaß, M., and Teich, J. (2016). A new time-independent reliability importance measure. *European Journal of Operational Research*, 254(2):427–442.

Cao, Y., Lu, C., and Dong, W. (2024). Importance measures for multi-state systems with multiple components under hierarchical dependences. *Reliability Engineering & System Safety*, 248:110142.

Chen, Y. and Feng, H. (2022). Maintenance strategy of multicomponent system based on structure updating and group importance measure. *Communications in Statistics-Theory and Methods*, 51(9):2919–2935.

Dao, C. D. and Zuo, M. J. (2017). Selective maintenance of multi-state systems with structural dependence. *Reliability Engineering & System Safety*, 159:184–195.

Deng, K., Ompusunggu, A. P., Xu, Y., Skote, M., and Zhao, Y. (2025). A review of material-related mechanical failures and load monitoring-based structural health monitoring (shm) technologies in aircraft landing gear. *Aerospace*, 12(3):266.

Do, P. and Bérenguer, C. (2020). Conditional reliability-based importance measures. *Reliability engineering & system safety*, 193:106633.

Do, P. and Bérenguer, C. (2022). Residual life-based importance measures for predictive maintenance decision-making. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, 236(1):98–113.

Do, P., Scarf, P., and Iung, B. (2015). Condition-based maintenance for a two-component system with dependencies. *IFAC-PapersOnLine*, 48(21):946–951.

Du, X., Liu, B., Shi, X., Zhou, C., and Chen, C. (2025). Selective maintenance decision of phased-mission system considering a novel importance measure. In *Equipment Intelligent Operation and Maintenance*, pages 50–60. CRC Press.

Dui, H., Liu, M., Song, J., and Wu, S. (2023). Importance measure-based resilience management: Review, methodology and perspectives on maintenance. *Reliability Engineering & System Safety*, 237:109383.

Dui, H., Si, S., and Yam, R. C. (2017). A cost-based integrated importance measure of system components for preventive maintenance. *Reliability Engineering & System Safety*, 168:98–104.

Dui, H. and Wu, S. (2024). *Importance Measures Informed Reliability Design*. Spriner.

Fang, R. and Li, X. (2016). On allocating one active redundancy to coherent systems with dependent and heterogeneous components' lifetimes. *Naval Research Logistics*, 63(4):335–345.

Feng, Z., Shen, X., Li, P., Zhao, J., Zhang, H., Xu, Y., and Yuan, J. (2024). Performance optimization and scheme evaluation of liquid cooling battery thermal management systems based on the entropy weight method. *Journal of Energy Storage*, 80:110329.

Geurtsen, M., Adan, J., and Akçay, A. (2023). Integrated maintenance and production scheduling for unrelated parallel machines with setup times. *Flexible Services and Manufacturing Journal*, pages 1–34.

Guo, H., Zheng, C., Iu, H. H.-C., and Fernando, T. (2017). A critical review of cascading failure analysis and modeling of power system. *Renewable and Sustainable Energy Reviews*, 80:9–22.

Hashemi, M. and Asadi, M. (2021). Optimal preventive maintenance of coherent systems: A generalized pólya process approach. *IISE Transactions*, 53(11):1266–1280.

Keizer, M. C. O., Teunter, R. H., Veldman, J., and Babai, M. Z. (2018). Condition-based maintenance for systems with economic dependence and load sharing. *International Journal of Production Economics*, 195:319–327.

Kivanç, İ., Fecarotti, C., Raassens, N., and van Houtum, G.-J. (2024). A scalable multi-objective maintenance optimization model for systems with multiple heterogeneous components and a finite lifespan. *European Journal of Operational Research*, 315(2):567–579.

Kuo, W. and Zhu, X. (2012). Some recent advances on importance measures in reliability. *IEEE Transactions on Reliability*, 61(2):344–360.

Leppinen, J., Punkka, A., Ekholm, T., and Salo, A. (2025). An optimization model for determining cost-efficient maintenance policies for multi-component systems with economic and structural dependencies. *Omega*, 130:103162.

Lisnianski, A. and Levitin, G. (2003). *Multi-State System Reliability: Assessment, Optimization and Applications*. World scientific.

Meng, F. C. (1996). Comparing the importance of system components by some structural characteristics. *IEEE Transactions on Reliability*, 45(1):59–65.

Mi, J., Li, Y.-F., Peng, W., and Huang, H.-Z. (2018). Reliability analysis of complex multi-state system with common cause failure based on evidential networks. *Reliability Engineering & System Safety*, 174:71–81.

Miziuła, P. and Navarro, J. (2019). Birnbaum importance measure for reliability systems with dependent components. *IEEE Transactions on Reliability*, 68(2):439–450.

Nguyen, K.-A., Do, P., and Grall, A. (2017). Joint predictive maintenance and inventory strategy for multi-component systems using birnbaum's structural importance. *Reliability Engineering & System Safety*, 168:249–261.

Nicolai, R. P. and Dekker, R. (2008). Optimal maintenance of multi-component systems: a review. *Complex system maintenance handbook*, pages 263–286.

Oakley, J. L., Wilson, K. J., and Philipson, P. (2022). A condition-based maintenance policy for continuously monitored multi-component systems with economic and stochastic dependence. *Reliability Engineering & System Safety*, 222:108321.

Peng, H., Coit, D. W., and Feng, Q. (2012). Component reliability criticality or importance measures for systems with degrading components. *IEEE Transactions on Reliability*, 61(1):4–12.

Peng, S., Jiang, W., and Zhao, W. (2021). A preventive maintenance policy with usage-dependent failure rate thresholds under two-dimensional warranties. *IISE Transactions*, 53(11):1231–1243.

Rietbergen, C., Groenwold, R. H., Hoijtink, H. J., Moons, K. G., and Klugkist, I. (2016). Expert elicitation of study weights for bayesian analysis and meta-analysis. *Journal of Mixed Methods Research*, 10(2):168–181.

Roth, M., Yanishevsky, M., and Beaudet, P. (2019). Failure analysis of aircraft landing gear components.

Salmasnia, A., Jafarian-Namin, S., and Khorasani, F. (2024). Integration of statistical process monitoring, maintenance and production planning for a multi-component system using the structural importance measure. *International Journal of Quality Engineering and Technology*, 10(2):119–150.

Salomon, V. A. P. and Gomes, L. F. A. M. (2024). Consistency improvement in the analytic hierarchy process. *Mathematics*, 12(6):828.

Wu, C., Pan, R., and Zhao, X. (2022). Reliability assessment of multi-state performance sharing systems with transmission loss and random shocks. *IISE Transactions*, 54(11):1060–1071.

Wu, S. and Asadi, M. (2024). A preventive maintenance policy and a method to approximate the failure process for multi-component systems. *European Journal of Operational Research*, 318(3):825–835.

Wu, S. and Coolen, F. P. (2013). A cost-based importance measure for system components: An extension of the birnbaum importance. *European Journal of Operational Research*, 225(1):189–195.

Wu, X., Huang, T., and Liu, J. (2023). Common stochastic effects induced multivariate degradation process with temporal dependency in degradation characteristic and unit dimensions. *Reliability Engineering & System Safety*, 239:109505.

Xu, J., Liu, B., Zhao, X., and Wang, X.-L. (2024). Online reinforcement learning for condition-based group maintenance using factored markov decision processes. *European Journal of Operational Research*, 315(1):176–190.

Xu, Z., Ramirez-Marquez, J. E., Liu, Y., and Xiahou, T. (2020). A new resilience-based component importance measure for multi-state networks. *Reliability Engineering & System Safety*, 193:106591.

Yang, L., Li, G., Zhang, Z., Ma, X., and Zhao, Y. (2020). Operations & maintenance optimization of wind turbines integrating wind and aging information. *IEEE Transactions on Sustainable Energy*, 12(1):211–221.

Yi, H. and Balakrishnan, N. (2024). Importance measures for two semi-coherent systems with shared components. *Reliability Engineering & System Safety*, 250:110306.

Zhang, N., Tian, S., Cai, K., and Zhang, J. (2023). Condition-based maintenance assessment for a deteriorating system considering stochastic failure dependence. *IISE Transactions*, 55(7):687–697.

Zhou, X., Huang, K., Xi, L., and Lee, J. (2015). Preventive maintenance modeling for multi-component systems with considering stochastic failures and disassembly sequence. *Reliability Engineering & System Safety*, 142:231–237.

Zhu, X. and Boushaba, M. (2017). A linear weighted (n, f, k) system for non-homogeneous markov-dependent components. *IISE Transactions*, 49(7):722–736.

Appendix A

Proof. $I_i(t_k)$ and $I_{i|j}(t_k)$ are given by,

$$I_i(t_k) = \alpha\left(h^r(1_i, \mathbf{p}_k) - h^r(\mathbf{p}_k)\right) + \beta\left(h^e(1_i, \mathbf{p}_k) - h^e(\mathbf{p}_k)\right) + \gamma\left(h^s(1_i, \mathbf{p}_k) - h^s(\mathbf{p}_k)\right), \quad \text{(A.1)}$$

$$\begin{aligned} I_{i|j}(t_k) =& \alpha\left(h^r(1_i, 1_j, \mathbf{p}_k) - h^r(1_j, \mathbf{p}_k)\right) + \beta\left(h^e(1_i, 1_j, \mathbf{p}_k) - h^e(1_j, \mathbf{p}_k)\right) \\ &+ \gamma\left(h^s(1_i, 1_j, \mathbf{p}_k) - h^s(1_j, \mathbf{p}_k)\right). \end{aligned} \quad \text{(A.2)}$$

Then, we can draw the following conclusion, that is,

$$\begin{aligned} I_{ij}(t_k) =& \alpha\left(h^r(1_i, 1_j, \mathbf{p}_k) - h^r(1_j, \mathbf{p}_k) - h^r(1_i, \mathbf{p}_k) + h^r(\mathbf{p}_k)\right) \\ &+ \beta\left(h^e(1_i, 1_j, \mathbf{p}_k) - h^e(1_j, \mathbf{p}_k) - h^e(1_i, \mathbf{p}_k) + h^e(\mathbf{p}_k)\right) \\ &+ \gamma\left(h^s(1_i, 1_j, \mathbf{p}_k) - h^s(1_j, \mathbf{p}_k) - h^s(1_i, \mathbf{p}_k) + h^s(\mathbf{p}_k)\right) \\ =& I_{i|j}(t_k) - I_i(t_k). \end{aligned} \quad \text{(A.3)}$$

Similarly, $I_{ij}(t_k) = I_{j|i}(t_k) - I_j(t_k)$. This establishes Proposition 1. $\square$

Appendix B

Proof. To simplify the derivation process, the following parameters are set:

$$\begin{array}{l} \prod_{\ell \neq i,j,} \Pr\{C_\ell(t_k) = 1\} = p_\alpha, \Pr\{C_j(t_k) = 1\} = p_j, \Pr\{C_i(t_k) = 1\} = p_i \\ \Pr\{C_j(t_k) = 1, C_i(t_k) = 1\} = p_{11}, \Pr\{C_j(t_k) = 0, C_i(t_k) = 1\} = p_{01} \\ \Pr\{C_j(t_k) = 1, C_i(t_k) = 0\} = p_{10}, \Pr\{C_j(t_k) = 0, C_i(t_k) = 0\} = p_{00} \end{array} \quad \text{(B.1)}$$

Case 1: Both dependent components are failed at time t_{k-1}, that is, $p_j = p_i = p_{11} = p_{01} = p_{10} = 0$.

$$\begin{aligned}
\bar{H}_j^r(t_k) &= h^r(1_j, \mathbf{p}_k) - h^r(\mathbf{p}_k) = p_i p_\alpha - p_i p_j p_\alpha = 0 \\
\bar{H}_i^r(t_k) &= h^r(1_i, \mathbf{p}_k) - h^r(\mathbf{p}_k) = p_j p_\alpha - p_i p_j p_\alpha = 0 \\
H_j^r(t_k) &= h^r(1_j, \mathbf{p}_k) - h^r(\mathbf{p}_k) = (p_{11} + p_{01})p_\alpha - p_{11}p_\alpha = 0 = \bar{H}_j^r(t_k) \\
H_i^r(t_k) &= h^r(1_i, \mathbf{p}_k) - h^r(\mathbf{p}_k) = (p_{11} + p_{10})p_\alpha - p_{11}p_\alpha = 0 = \bar{H}_i^r(t_k)
\end{aligned} \tag{B.2}$$

Case 2: Component j is functioning, and component i is failed at time t_{k-1}. $p_i = p_{11} = p_{01} = 0$. p_j can be evaluated by Eq. (11), and it is equal to $e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy}$. p_{10} can be computed by Eq. (7), and it is equal to $e^{-\int_{t_{k-1}}^{t_k} \bar{\lambda}_j(y)dy}$.

$$\begin{aligned}
\bar{H}_j^r(t_k) &= h^r(1_j, \mathbf{p}_k) - h^r(\mathbf{p}_k) = p_i p_\alpha - p_i p_j p_\alpha = 0 \\
\bar{H}_i^r(t_k) &= h^r(1_i, \mathbf{p}_k) - h^r(\mathbf{p}_k) = p_j p_\alpha - p_i p_j p_\alpha = e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy} p_\alpha \\
H_j^r(t_k) &= h^r(1_j, \mathbf{p}_k) - h^r(\mathbf{p}_k) = (p_{11} + p_{01})p_\alpha - p_{11}p_\alpha = 0 = \bar{H}_j^r(t_k) \\
H_i^r(t_k) &= h^r(1_i, \mathbf{p}_k) - h^r(\mathbf{p}_k) = (p_{11} + p_{10})p_\alpha - p_{11}p_\alpha \leq \bar{H}_i^r(t_k)
\end{aligned} \tag{B.3}$$

Case 3: Component j is failed, and component i is functioning at time t_{k-1}. $p_j = p_{11} = p_{10} = 0$. p_i can be computed via Eq. (11), and it equates to $e^{-\int_{t_{k-1}}^{t_k} \lambda_i(y)dy}$. p_{01} can be evaluated via Eq. (7), and it is equal to $e^{-\int_{t_{k-1}}^{t_k} \bar{\lambda}_i(y)dy}$. The results are similar to Case 2, that is,

$$H_j^r(t_k) \leq \bar{H}_j^r(t_k), H_i^r(t_k) = \bar{H}_i^r(t_k) \tag{B.4}$$

Case 4: Both dependent components are functioning at time t_{k-1}. $p_j = e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy}$. $p_i = e^{-\int_{t_{k-1}}^{t_k} \lambda_i(y)dy}$. p_{11} can be computed via Eq. (8), and it is equal to $e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)+\lambda_i(y)dy}$. p_{01} can be computed via Eq. (10), and it is equal to $\int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_i(y)dy} \lambda_j(x)$ $e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx$. p_{10} is equal to $\int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_j(y)dy} \lambda_i(x) e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} dx$.

$$\begin{aligned}
\bar{H}_j^r(t_k) &= h^r(1_j, \mathbf{p}_k) - h^r(\mathbf{p}_k) = p_i p_\alpha - p_i p_j p_\alpha = (1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy}) e^{-\int_{t_{k-1}}^{t_k} \lambda_k(y)dy} p_\alpha, \\
H_j^r(t_k) &= h^r(1_j, \mathbf{p}_k) - h^r(\mathbf{p}_k) = p_{01}p_\alpha = p_\alpha \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_i(y)dy} \lambda_j(x) e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx.
\end{aligned} \tag{B.5}$$

Then, we can draw the conclusion that $H_j^r(t_k) \leq \bar{H}_j^r(t_k)$ by the following equations.

$$\begin{aligned}
H_j^r(t_k) &= p_\alpha \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_i(y)dy} \lambda_j(x) e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx \\
&\leq p_\alpha \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} e^{-\int_x^{t_k} \lambda_i(y)dy} \lambda_j(x) e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx \\
&= e^{-\int_{t_{k-1}}^{t_k} \lambda_i(y)dy} \left(1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy}\right) p_\alpha = \bar{H}_j^r(t_k)
\end{aligned} \tag{B.6}$$

The similar conclusion can be drawn, that is, $H_i^r(t_k) \leq \bar{H}_i^r(t_k)$. This establishes Proposition 2. □

Appendix C

Proof. In Case 1 and Case 2 , $H_j^r(t_k) = 0$. Obviously, $H_j^r(t_k)$ is non-decreasing with respect to $\bar{\lambda}_i(t_k)$ and independent of $\bar{\lambda}_j(t_k)$.

In Case 3, $H_j^r(t_k) = e^{-\int_{t_{k-1}}^{t_k} \bar{\lambda}_i(y)dy} p_\alpha$. $H_j^r(t_k)$ is non-decreasing with respect to $\bar{\lambda}_i(t_k)$ and independent of $\bar{\lambda}_j(t_k)$.

In Case 4, $H_j^r(t_k) = p_\alpha \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_i(y)dy} \lambda_j(x) e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx$. $H_j^r(t_k)$ is non-decreasing with respect to $\bar{\lambda}_i(t_k)$ and independent of $\bar{\lambda}_j(t_k)$.

In conclusion, H_j^r is non-decreasing with respect to $\bar{\lambda}_i(t_k)$ and independent of $\bar{\lambda}_j(t_k)$. Analogously, H_k^r is also non-decreasing with respect to $\bar{\lambda}_j(t_k)$ and independent of $\bar{\lambda}_i(t_k)$. This establishes Lemma 1. □

Appendix D

Proof. To simplify the derivation process, the following parameters are set by

$$\prod_{\ell \neq i,j} (1 - \Pr\{C_\ell(t_k) = 1\}) = p_\beta. \tag{D.1}$$

The other parameters are set the same as Proposition 2.

Case 1: Both dependent components are failed at time t_{k-1}, that is, $p_j = p_i = p_{11} = p_{01} = p_{10} = 0$.

$$\begin{aligned}
\bar{H}_j^r(t_k) &= 1 - (1 - (1 - p_j)(1 - p_i)p_\beta) = p_\beta \\
\bar{H}_i^r(t_k) &= 1 - (1 - (1 - p_j)(1 - p_i)p_\beta) = p_\beta \\
H_j^r(t_k) &= 1 - (1 - p_{00}p_\beta) = p_\beta = \bar{H}_j^r(t_k) \\
H_i^r(t_k) &= 1 - (1 - p_{00}p_\beta) = p_\beta = \bar{H}_i^r(t_k)
\end{aligned} \tag{D.2}$$

Case 2: Component j is functioning, and component i is failed time t_{k-1}, that is, $p_i = p_{11} = p_{01} = 0$, $p_j = e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy}$, and $p_{10} = e^{-\int_{t_{k-1}}^{t_k} \bar{\lambda}_j(y)dy}$.

$$\begin{aligned}
\bar{H}_j^r(t_k) &= (1 - p_j)(1 - p_i)p_\beta = (1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy})p_\beta \\
\bar{H}_i^r(t_k) &= (1 - p_j)(1 - p_i)p_\beta = (1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy})p_\beta \\
H_j^r(t_k) &= p_{00}p_\beta = (1 - e^{-\int_{t_{k-1}}^{t_k} \bar{\lambda}_j(y)dy})p_\beta \geq (1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy})p_\beta = \bar{H}_j^r(t_k) \\
H_i^r(t_k) &= p_{00}p_\beta = (1 - e^{-\int_{t_{k-1}}^{t_k} \bar{\lambda}_j(y)dy})p_\beta \geq (1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy})p_\beta = \bar{H}_i^r(t_k)
\end{aligned} \tag{D.3}$$

Case 3: Component j is failed, and component i is functioning at time t_{k-1}, that is, $p_j = p_{11} = p_{10} = 0$, $p_i = e^{-\int_{t_{k-1}}^{t_k} \lambda_i(y)dy}$, and $p_{01} = e^{-\int_{t_{k-1}}^{t_k} \bar{\lambda}_i(y)dy}$. The results are similar to case 2, that is,

$$H_j^r(t_k) \geq \bar{H}_j^r(t_k), H_i^r(t_k) \geq \bar{H}_i^r(t_k) \tag{D.4}$$

Case 4: Both dependent components are functioning at time t_{k-1}, that is, $p_j = e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy}$, $p_i = e^{-\int_{t_{k-1}}^{t_k} \lambda_i(y)dy}$, $p_{11} = e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)+\lambda_i(y)dy}$, $p_{01} = \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_i(y)dy} \lambda_j(x)$ $e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx$, and $p_{10} = \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_j(y)dy} \lambda_i(x) e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} dx$.

$$H_j^r(t_k) = p_{00}p_\beta = (1 - p_{11} - p_{10} - p_{01})p_\beta$$
$$= \left(\begin{array}{l} 1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)+\lambda_i(y)dy} - \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_j(y)dy} \lambda_i(x) e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} dx \\ - \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_i(y)dy} \lambda_j(x) e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx \end{array} \right) p_\beta \tag{D.5}$$

Subsequently, $H_j^r(t_k) \geq \bar{H}_j^r(t_k)$ can be drawn based on the following equations.

$$\begin{aligned} H_j^r(t_k) &= \left(\begin{array}{l} 1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)+\lambda_i(y)dy} - \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_j(y)dy} \lambda_i(x) e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} dx \\ - \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_i(y)dy} \lambda_j(x) e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx \end{array} \right) p_\beta \\ &\geq \left(\begin{array}{l} 1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)+\lambda_i(y)dy} - \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy} \lambda_i(x) e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} dx \\ - \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{t_k} \lambda_i(y)dy} \lambda_j(x) e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx \end{array} \right) p_\beta \\ &= \left(1 + e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)+\lambda_i(y)dy} - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy} - e^{-\int_{t_{k-1}}^{t_k}} \lambda_i(y)dy \right) p_\beta \\ &= \left(1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)dy} \right) \left(1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_i(y)dy} \right) p_\beta = \bar{H}_j^r(t_k) \end{aligned} \tag{D.6}$$

In the same way, $H_i^r(t_k) \geq \bar{H}_i^r(t_k)$. This establishes Proposition 3.

□

Appendix E

Proof. The result can be observed in the proof of Proposition 3. The "Part 1" of the 3D importance measure of each component equates to the difference between the system's reliability with and without the maintenance of the component. The reliability of a parallel system with the maintenance of any component is equal to one. Its reliability without the maintenance of any component is determined by the reliability of each component without maintenance. Therefore, "Part 1" of the 3D importance measure of one component is the same as that of the other. This establishes Lemma 2. □

Appendix F

Proof. In Case 1, $H_j^r(t_k) = p_\beta$. Obviously, $H_j^r(t_k)$ is independent of $\bar{\lambda}_j(t_k)$ and $\bar{\lambda}_i(t_k)$.

In Case 2, $H_j^r(t_k) = (1 - e^{-\int_{t_{k-1}}^{t_k} \bar{\lambda}_j(y)dy})$. $H_j^r(t_k)$ is non-increasing with respect to $\bar{\lambda}_j(t_k)$ and independent of $\bar{\lambda}_i(t_k)$.

In Case 3, $H_j^r(t_k) = (1 - e^{-\int_{t_{k-1}}^{t_k} \bar{\lambda}_i(y)dy})$. $H_j^r(t_k)$ is non-increasing with respect to $\bar{\lambda}_i(t_k)$ and independent of $\bar{\lambda}_j(t_k)$.

In Case 4, for $\forall \bar{\lambda}'_i(t_k) \geq \bar{\lambda}_i(t_k) \geq \lambda_i(t_k)$, and $\forall \bar{\lambda}'_j(t_k) \geq \bar{\lambda}_j(t_k) \geq \lambda_j(t_k)$, one has

$$
\begin{aligned}
&H_j^r\left(\bar{\lambda}_j'(t_k), \bar{\lambda}_i'(t_k)\right) \\
&= \left(\begin{array}{l} 1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)+\lambda_i(y)dy} - \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_j'(y)dy} \lambda_i(x) e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} dx \\ - \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_i'(y)dy} \lambda_j(x) e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx \end{array}\right) p_\beta \\
&\geq \left(\begin{array}{l} 1 - e^{-\int_{t_{k-1}}^{t_k} \lambda_j(y)+\lambda_i(y)dy} - \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_j(y)dy} \lambda_i(x) e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} dx \\ - \int_{t_{k-1}}^{t_k} e^{-\int_{t_{k-1}}^{x} \lambda_i(y)dy} e^{-\int_x^{t_k} \bar{\lambda}_i(y)dy} \lambda_j(x) e^{-\int_{t_{k-1}}^{x} \lambda_j(y)dy} dx \end{array}\right) p_\beta \\
&= H_j^r\left(\bar{\lambda}_j(t_k), \bar{\lambda}_i(t_k)\right).
\end{aligned}
\tag{F.1}
$$

Consequently, $H_j^r(t_k)$ is non-increasing with respect to $\lambda_i(t_k)$ and $\lambda_j(t_k)$. According to Lemma 2, the "Part 1" of the 3D importance measure of one component is the same as that of each other. Therefore, the conclusion can be drawn in other components. This established Lemma 3. □

Appendix G

Proof. For a series system, it is reliable when it stays in $(1_1, \ldots, 1_m)$ only. Therefore, the system reliability without any maintenance activities at time t_k is given by,

$$h^r(\mathbf{p}_k) = p_{2^m,k}. \tag{G.1}$$

The system is reliable with the maintenance of component i when it stays in $(1_1, \ldots, 1_m)$ or $(1_1, \ldots, 0_i, \ldots, 1_m)$. Let $(1_1, \ldots, 0_i, \ldots, 1_m)$ denote $\mathbf{X}_{2^m-2^{i-1}+1}$, where $2^m - 2^{i-1} + 1$ is the serial number of $(1_1, \ldots, 0_i, \ldots, 1_m)$. Therefore, the system reliability with the maintenance of component i at time t_k is given by,

$$h^r(1_i, \mathbf{p}_k) = p_{2^m,k} + p_{2^m-2^{i-1}+1,k}. \tag{G.2}$$

The system reliability with the maintenance of component j at time t_k, $h^r(1_j, \mathbf{p}_k)$, can be also computed by Eq. (G.2).

The system is reliable with the maintenance of components i and j when it stays in $(1_1, \ldots, 1_m)$, $(1_1, \ldots, 0_i, \ldots, 1_m)$, $(1_1, \ldots, 0_j, \ldots, 1_m)$, or $(1_1, \ldots, 0_i, \ldots, 0_j, \ldots, 1_m)$. Let $(1_1, \ldots, 0_i, \ldots, 0_j, \ldots, 1_m)$ denote $\mathbf{X}_{2^m-2^{i-1}-2^{j-1}+1}$. The system reliability with the maintenance of components i and j is given by,

$$h^r(1_i, 1_j, \mathbf{p}_k) = p_{2^m,k} + p_{2^m-2^{i-1}+1,k} + p_{2^m-2^{j-1}+1,k} + p_{2^m-2^{i-1},k-2^{j-1}+1,k}. \tag{G.3}$$

Then, $H_{ij}^r(t_k)$ is given by,

$$
\begin{aligned}
H_{ij}^r(t_k) &= h^r(1_i, 1_j, \mathbf{p}_k) - h^r(1_j, \mathbf{p}_k) - h^r(1_i, \mathbf{p}_k) + h^r(\mathbf{p}_k) \\
&= p_{2^m,k} + p_{2^m-2^{i-1}+1,k} + p_{2^m-2^{j-1}+1,k} + p_{2^m-2^{i-1}-2^{j-1}+1,k} \\
&\quad - p_{2^m,k} - p_{2^m-2^{i-1}+1,k} - p_{2^m,k} - p_{2^m-2^{j-1}+1,k} + p_{2^m,k} \\
&= p_{2^m-2^{i-1}-2^{j-1}+1,k} \geq 0.
\end{aligned}
\tag{G.4}
$$

This establishes Proposition 4. □

Appendix H

Proof. For a parallel system, it is failed when it stays in $(0_1, \ldots, 0_m)$ only. Therefore, the system reliability without any maintenance activities is given by,

$$h^r(\mathbf{p}_k) = 1 - p_{1,k}. \tag{H.1}$$

Particularly, the system reliability with any maintenance activities is equal to one, that is, $h^r(1_j, \mathbf{p}_k) = 1,\ h^r(1_j, \mathbf{p}_k) = 1,$ and $h^r(1_j, 1_j, \mathbf{p}_k) = 1$. Then, $H^r_{ij}(t_k)$ is given by,

$$\begin{aligned} H^r_{ij}(t_k) &= h^r(1_j, 1_j, \mathbf{p}_k) - h^r(1_i, \mathbf{p}_k) - h^r(1_j, \mathbf{p}_k) + h^r(\mathbf{p}_k) \\ &= 1 - 1 - 1 + 1 - p_{1,k} \\ &= -p_{1,k} \leq 0. \end{aligned} \tag{H.2}$$

It implies that the system reliability improvement of one component for a parallel system is decreased when the other components have already been maintained. This establishes Lemma 4. □

Appendix I

Proof. The setting of parameters can be found in Propositions 2 and 3. $\bar{H}^r_{ij}(t_k)$ and $H^r_{ij}(t_k)$ are given by,

$$\begin{aligned} \bar{H}^r_{ij}(t_k) &= p_\alpha - p_i p_\alpha - p_j p_\alpha + p_j p_i p_\alpha \\ &= p_\alpha(1 - p_j)(1 - p_i), \end{aligned} \tag{I.1}$$

$$\begin{aligned} H^r_{ij}(t_k) &= p_\alpha - (p_{11} + p_{01})p_\alpha - (p_{11} + p_{10})p_\alpha + p_{11}p_\alpha \\ &= p_{00}p_\alpha. \end{aligned} \tag{I.2}$$

The forms of $\bar{H}^r_{ij}(t_k)$ and $H^r_{ij}(t_k)$ are similar with Proposition 3. Therefore, the conclusions are the same as Proposition 3. This establishes Proposition 5. □

Appendix J

Proof. The parameters are set the same as Lemmas 1 and 2. $\bar{H}^r_{ij}(t_k)$ and $H^r_{ij}(t_k)$ are given by,

$$\begin{aligned} \bar{H}^r_{ij}(t_k) &= 1 - 1 - 1 + (1 - (1 - p_j)(1 - p_i)p_\beta) \\ &= -p_\beta(1 - p_j)(1 - p_i), \end{aligned} \tag{J.1}$$

$$\begin{aligned} H^r_{ij}(t_k) &= 1 - 1 - 1 + (1 - p_{00}p_\alpha) \\ &= -p_{00}p_\beta. \end{aligned} \tag{J.2}$$

The forms of $-\bar{H}^r_{ij}(t_k)$ and $-H^r_{ij}(t_k)$ are similar with Proposition 3. Therefore, the conclusions are the opposite of Proposition 3. This establishes Lemma 5. □

Appendix K

Proof. H^e_{ij} is given by,

$$\begin{aligned} H^e_{ij} &= h^e(1_i, 1_j, \mathbf{c}) - h^e(1_i, \mathbf{c}) - h^e(1_j, \mathbf{c}) + h^e(\mathbf{c}) \\ &= \frac{c^e(i) + c^e(j) - c^e(i,j) - c^e(0)}{c_0} \\ &= \frac{c_i + c^e_{h(i)} + c_j + c^e_{h(j)} - c^e(i,j) - 0}{c_0} \\ &= \begin{cases} 0, & h(i) \neq h(j) \\ \dfrac{c^e_{h(i)}}{c_0}, & h(i) = h(j) \end{cases}. \end{aligned} \tag{K.1}$$

This establishes Proposition 6. □

Appendix L

Proof. H^s_{ij} can be simplified as:

$$\begin{aligned} H^s_{ij} &= h^s(1_i, 1_j, \mathbf{d}) - h^s(1_j, \mathbf{d}) - h^s(1_i, \mathbf{d}) - h^s(\mathbf{d}) \\ &= \frac{n(\mathbf{d}_i) + n(\mathbf{d}_j) - n(\mathbf{d}_i \cup \mathbf{d}_j) - 0}{n_0} \\ &= \frac{n(\mathbf{d}_i \cap \mathbf{d}_j)}{n_0} \geq 0. \end{aligned} \tag{L.1}$$

This establishes Proposition 7. □

Appendix M

Proof. Based on Propositions 4, 6, and 7, the proposed joint 3D importance measure is given by,

$$I_{ij}(t_k) = \alpha H^r_{ij}(t_k) + \beta H^e_{ij} + \gamma H^s_{ij} \geq 0. \tag{M.1}$$

It implies that the benefit of maintaining one component will increase for a series system when other components have already been maintained. This establishes Lemma 6. □

Appendix N

Algorithm 1: Calculating the 3D importance measure

Require:

- The failure intensity function of each component $\lambda_\ell(t)(l \in \{1, \dots, m\})$.
- The increased failure intensity function of each dependent component.
- The elapsed time t_{k-1}, and the distribution of components' state combinations $\mathbf{p}_{k-1}$.
- The structure function of the system $\phi(\cdot)$.
- The maintenance cost c_ℓ and fixed setup cost $c^e_{h(\ell)}$ of each component.
- The disassembly relationship among nodes.
- Weights α, β, and γ.

Output :

- The 3D importance measure of component ℓ ($I_\ell(t_k)$).
- The joint 3D importance measure of components i and j ($I_{ij}(t_k)$).

$H^r_\ell(t_k) \leftarrow 0$ and $H^r_{ij}(t_k) \leftarrow 0$;
for $u \leftarrow 1$ **to** 2^m **do**
  $p_{u,k} \leftarrow 0$;
  for $v \leftarrow 1$ **to** 2^m **do**
    Calculate $\Pr\{\mathbf{C}(t_k) = \mathbf{X}_u | \mathbf{C}(t_{k-1}) = \mathbf{X}_v\}$ based on Eq. (6) ;
    $p_{u,k} \leftarrow p_{u,k} + \Pr\{\mathbf{C}(t_k) = \mathbf{X}_u | \mathbf{C}(t_{k-1}) = \mathbf{X}_v\} p_{v,k-1}$;
  $H^r_\ell(t_k) \leftarrow H^r_\ell(t_k) + \left(\left(\chi\left(\phi\left(\mathbf{X}_{u\setminus\{k\}}\right) = 1\right) - \chi\left(\phi(\mathbf{X}_u) = 1\right)\right) p_{u,k}\right.$;
  $H^r_{ij}(t_k) \leftarrow H^r_{ij}(t_k) + (\chi(\phi(\mathbf{X}_{u\setminus\{i,j\}}) = 1) + \chi(\phi(\mathbf{X}_u) = 1) - \chi(\phi(\mathbf{X}_{u\setminus\{i,j\}}) = 1) - \chi(\phi(\mathbf{X}_u) = 1)) p_{u,k}$;
for $i_1 \leftarrow 1$ **to** m **do**
  for $j_1 \leftarrow i_1$ **to** m **do**
    if $h(i_1) = h(j_1)$ **then**
      $c^e(i_1, j_1) \leftarrow c_{i_1} + c_{j_1} + c^e_{h(i_1)}$
    else
      $c^e(i_1, j_1) \leftarrow c_{i_1} + c_{j_1} + c^e_{h(i_1)} + c^e_{h(j_1)}$;
$c_0 \leftarrow \max_{i_1, j_1} c^e(i_1, j_1)$;
$H^e_\ell \leftarrow \frac{-(c_\ell + c^e_{h(\ell)})}{c_0}$;
$H^e_{ij} \leftarrow \frac{c_i + c^e_{h(i)} + c_j + c^e_{h(j)} - c^e(i,j)}{c_0}$;
for $i_2 \leftarrow 1$ **to** m **do**
  for $j_2 \leftarrow i_2$ **to** m **do**
    $n(\mathbf{d}_{i_2} \cup \mathbf{d}_{j_2}) \leftarrow n(\mathbf{d}_{i_2}) + n(\mathbf{d}_{j_2}) - n(\mathbf{d}_{i_2} \cap \mathbf{d}_{j_2})$;
$n_0 \leftarrow \max_{i_2, j_2} n(\mathbf{d}_{i_2}, \mathbf{d}_{j_2})$;
$H^s_\ell \leftarrow \frac{-n(\mathbf{d}_\ell)}{n_0}$;
$H^s_{ij} \leftarrow \frac{n(\mathbf{d}_i \cap \mathbf{d}_j)}{n_0}$;
$I_\ell(t_k) \leftarrow \alpha H^r_\ell(t_k) + \beta H^e_\ell + \gamma H^s_\ell$;
$I_{ij}(t_k) \leftarrow \alpha H^r_{ij}(t_k) + \beta H^e_{ij} + \gamma H^s_{ij}$;

Appendix O

Nomenclature

$C_\ell(t)$	State of component ℓ at time t
$S(t)$	System state at time t
$\mathbf{C}(t)$	Components' state combination at time t
$\lambda_\ell(t)$	Initial failure intensity function of component ℓ at time t
$\bar{\lambda}_\ell(t)$	Increased failure intensity function of component ℓ at time t
m_1	Pair number of stochastic dependent components
m_2	Number of stochastic independent components
m	Number of components
$\phi(\cdot)$	System's structure function
$d_{i,1}$	First serial number of the ith pair of dependent components
$d_{i,2}$	Second serial number of the ith pair of dependent components
d_j	Serial number of the jth of independent components
$\mathbf{D}$	Serial number set of dependent components
$\mathbf{I}$	Serial number set of independent components
c_ℓ	Maintenance cost of component ℓ
$h(\ell)$	Serial number of the maintenance facility for component ℓ
$c^e_{h(\ell)}$	Fixed setup cost of component ℓ
$\mathbf{d}_\ell$	Disassembly sequence of component ℓ
$h^r(\cdot)$	Unified system utilities associated with system reliability
$h^e(\cdot)$	Unified system utilities associated with maintenance cost
$h^s(\cdot)$	Unified system utilities associated with disassembly procedure number
$H^r_\ell(t_k)$	System reliability improvement of component ℓ at time t_k
$h^r(1_\ell, \mathbf{p}_k)$	System utility for stochastic dependence after maintaining component ℓ at time t_k
$h^r(\mathbf{p}_k)$	System utility for stochastic dependence without any maintenance activities at time t_k
$\mathbf{p}_k$	Probability of components' state combination at time t_k
$p_{i,k}$	Probability of ith components' state combination at time t_k
$\mathbf{X}_i$	ith state combination of all components
$\mathbf{\Omega}$	State combination set of all components
$\mathbf{X}_{i\setminus\{\ell\}}$	ith state combination of all components when component ℓ is maintained.
H^e_ℓ	Unified cost reduction of component ℓ
$h^e(1_\ell, \mathbf{c})$	Unified cost of component ℓ
$h^e(\mathbf{c})$	Unified cost without maintenance
c_0	Maximum cost of maintaining two components
$c^e(\ell)$	Total maintenance cost associated with maintaining component ℓ

$c^e(0)$	Total maintenance cost without maintenance
H^s_ℓ	Unified disassembly number of component ℓ
$h^s(1_\ell, \mathbf{d})$	Unified disassembly number of component ℓ
$h^s(\mathbf{d})$	Unified disassembly number without maintenance
n_0	Maximum disassembly number of maintaining two components
$n(\mathbf{d}_\ell)$	Total disassembly number associated with maintaining component ℓ
$n(\emptyset)$	Total disassembly number without maintenance
$\mathbf{d}$	Disassembly sequence of a system
$\mathbf{d}_\ell$	Disassembly sequence of maintaining component ℓ
$r(i, j)$	Relationship between nodes i and j.
$I_\ell(t_k)$	3D importance measure of component ℓ
$I_{ij}(t_k)$	Joint 3D importance measure of component ℓ

Appendix P

Proof. Suppose that $\mathbf{\Omega}_{1_\ell} = \{(0_1, 0_2, \ldots, 1_\ell, \ldots, 0_m), (1_1, 0_2, \ldots, 1_\ell, \ldots, 0_m), \ldots, (1_1, 1_2, \ldots, 1_\ell, \ldots, 1_m)\}$, and it indicates the set of components' state combination when component ℓ stays in working state. The number of components' state combination in $\mathbf{\Omega}_{1_\ell}$ is equal to 2^{m-1}. The ith components' state combination in $\mathbf{\Omega}_{1_\ell}$ is denoted by $\mathbf{X}^1_i$, and the corresponding probability at time t_k is denoted by $p^1_{i,k}$.

In the same way, suppose that $\mathbf{\Omega}_{0_\ell} = \{(0_1, 0_2, \ldots, 0_\ell, \ldots, 0_m), (1_1, 0_2, \ldots, 0_\ell, \ldots, 0_m), \ldots, (1_1, 1_2, \ldots, 0_\ell, \ldots, 1_m)\}$, and it indicates the set of components' state combination when component ℓ stays in failure state. The number of components' state combination in $\mathbf{\Omega}_{0_\ell}$ is equal to 2^{m-1}. The ith components' state combination in $\mathbf{\Omega}_{0_\ell}$ is denoted by $\mathbf{X}^0_i$, and the corresponding probability at time t_k is denoted by $p^0_{i,k}$. Except for the ℓth element, the elements of $\mathbf{X}^1_i$ and $\mathbf{X}^0_i$ are the same.

$h^r(1_\ell, \mathbf{p}_k)$ is given by:

$$\begin{aligned} h^r(1_\ell, \mathbf{p}_k) &= \sum_{i=1}^{2^m} p_{i,k} \phi(\mathbf{X}_{i\setminus\{\ell\}}) \\ &= \sum_{i=1}^{2^{m-1}} p^1_{i,k} \phi(\mathbf{X}^1_i) + \sum_{i=1}^{2^{m-1}} p^0_{i,k} \phi(\mathbf{X}^1_i) \\ &= \sum_{i=1}^{2^{m-1}} \left(p^1_{i,k} + p^0_{i,k}\right) \phi(\mathbf{X}^1_i) \end{aligned} \tag{P.1}$$

$h^r(\mathbf{p}_k)$ is given by:

$$\begin{aligned} h^r(\mathbf{p}_k) &= \sum_{i=1}^{2^m} p_{i,k} \phi(\mathbf{X}_i) \\ &= \sum_{i=1}^{2^{m-1}} p^1_{i,k} \phi(\mathbf{X}^1_i) + \sum_{i=1}^{2^{m-1}} p^0_{i,k} \phi(\mathbf{X}^0_i) \end{aligned} \tag{P.2}$$

$h^r(0_\ell, \mathbf{p}_k)$ is the system utility for stochastic dependence when component ℓ failed at time

t_k, and it is given by:

$$\begin{aligned} h^r(1_\ell, \mathbf{p}_k) &= \sum_{i=1}^{2^{m-1}} p_{i,k}^1 \phi(\mathbf{X}_i^0) + \sum_{i=1}^{2^{m-1}} p_{i,k}^0 \phi(\mathbf{X}_i^0) \\ &= \sum_{i=1}^{2^{m-1}} \left(p_{i,k}^1 + p_{i,k}^0\right) \phi(\mathbf{X}_i^0) \end{aligned} \tag{P.3}$$

Therefore, $H_\ell^r(t_k)$ is given by,

$$\begin{aligned} H_\ell^r(t_k) &= h^r(1_\ell, \mathbf{p}_k) - h^r(\mathbf{p}_k) \\ &= \sum_{i=1}^{2^{m-1}} \left(p_{i,k}^1 + p_{i,k}^0\right) \phi(\mathbf{X}_i^1) - \sum_{i=1}^{2^{m-1}} p_{i,k}^1 \phi(\mathbf{X}_i^1) - \sum_{i=1}^{2^{m-1}} p_{i,k}^0 \phi(\mathbf{X}_i^0) \\ &= \sum_{i=1}^{2^{m-1}} p_{i,k}^0 \left(\phi(\mathbf{X}_i^1) - \phi(\mathbf{X}_i^0)\right) \end{aligned} \tag{P.4}$$

$I_\ell^b(t_k)$ is given by,

$$\begin{aligned} I_\ell^b(t_k) &= h^r(1_\ell, \mathbf{p}_k) - h^r(0_\ell, \mathbf{p}_k) \\ &= \sum_{i=1}^{2^{m-1}} \left(p_{i,k}^1 + p_{i,k}^0\right) \phi(\mathbf{X}_i^1) - \sum_{i=1}^{2^{m-1}} \left(p_{i,k}^1 + p_{i,k}^0\right) \phi(\mathbf{X}_i^0) \\ &= \sum_{i=1}^{2^{m-1}} \left(p_{i,k}^1 + p_{i,k}^0\right) \left(\phi(\mathbf{X}_i^1) - \phi(\mathbf{X}_i^0)\right) \end{aligned} \tag{P.5}$$

Due to the absence of stochastic dependence, $p_{i,k}^1$ and $p_{i,k}^0$ are given by,

$$p_{i,k}^1 = \Pr\{C_l(t_k) = 1_\ell\} \prod_{j=1}^{j\neq\ell} \Pr\{C_j(t_k) = \mathbf{X}_i^1(j)\}, \tag{P.6}$$

$$p_{i,k}^0 = \Pr\{C_l(t_k) = 0_\ell\} \prod_{j=1}^{j\neq\ell} \Pr\{C_j(t_k) = \mathbf{X}_i^1(j)\} \tag{P.7}$$

Therefore, $H_\ell^r(t_k) = \Pr\{C_\ell(t_k) = 0_\ell\} I_\ell^b(t_k)$.

This establishes Proposition P. $\square$