Geometry and Topology of Horofunction Compactifications

Kieran Power

A Thesis Submitted for the Degree of

Doctor of Philosophy in Mathematics

19 December, 2024

The University of Kent

School of Mathematics, Statistics, and Actuarial Science

Supervisor: Bas Lemmens

Abstract

This thesis concerns the global geometry and topology of the horofunction compactification of various metric spaces. In Chapters 4-6 we study the horofunction compactification of various homogeneous Finsler metric spaces, and establish a homeomorphism between the horofunction compactification and the dual unit ball in the tangent space at the base point. This homeomorphism establishes a one-to-one correspondence between the geometric parts of the horoboundary and the relative interiors of faces of the dual ball. In Chapter 7 we build on the work of Gutiérrez, and explore the topology and geometry of the horofunction compactification of infinite dimensional ℓ^p spaces for $1 \leq p < \infty$. We show a clear disconnect between the global geometry and topology of the horofunction compactification in the infinite dimensional case versus the finite dimensional case. We also establish a marked difference in the behaviour of the horoboundary of ℓ^1 versus ℓ^p for $1 < p < \infty$. Chapter 8 deals with the horofunction compactification of infinite dimensional spin factors considered as JB-algebras. We show that the exponential map extends to a geometry preserving homeomorphism on the boundary, mapping the horofunction compactification of the spin factor homeomorphically onto the horofunction compactification of the positive cone equipped with the Thompson metric. We conclude by showing that, considering an infinite dimensional Hilbert space as the tangent space at the identity of infinite dimensional real hyperbolic space, the exponential extends to a homeomorphism between the horofunction compactification of infinite dimensional Hilbert space and infinite dimensional real hyperbolic space.

Acknowledgments

My doctoral studies were kindly funded by the Engineering and Physical Sciences Research Council, and I gratefully acknowledge their support (EPSRC Grant EP/V520093/1).

Although it will be my name on the degree certificate, the completion of this thesis would not have been possible without the invaluable contributions of many others, and I would like to extend my heartfelt thanks to these people, namely, but not limited to:

- My supervisor, Bas Lemmens, for introducing me to the wonderful world of horofunctions. He gave of his time generously, and his mathematical insight freely, both of which contributed immensely to a truly rewarding PhD experience. I will always look back fondly on our many hours spent discussing mathematics on his blackboard.

- During my bachelor's days in Pretoria, I fell under the tutelage of wonderful mathematical mentors, who I am now fortunate enough to be able to call friends. I must therefore thank Jan Harm van der Walt, Miek Messerschmidt, and Eder Kikianty, who all helped foster my love for mathematics, and encouraged me to pursue it overseas.

- Following their advice I moved to Universiteit Utrecht in the Netherlands, where my master's thesis was supervised by Karma Dajani, who I am still incredibly grateful to four years later. She made my first taste of novel mathematical research a complete joy, and went above and beyond to support me during some tough personal challenges. I once again thank her for all her help, both mathematical and personal, which played no small part in allowing me to pursue this PhD.

- Of course, I must thank my friends, both old and new, without whom this journey called life would be much poorer. There are too many to name, but I would especially like to thank my South African friends, Taygen, Devon, Stacey, Walt, Jayden and Marco, for supporting me from thousands of kilometres away, and for making me feel like I'd never left whenever I visited. I also want to thank my office mates at the university: Morgan, Edoardo, Painos, Tommy, Ben, Giannis, and James. We engaged in many interesting mathematical discussions, but more importantly we shared much laughter and pontification on the vagaries of life. Our esoteric chats and dart games were crucial to my continued sanity.

- The writing stage of a PhD is a gruelling affair, and I am very thankful for the support of my partner, Danni, during this trying time. She handled my many, (many) late nights and erratic schedule without complaint, and her support eased the stress considerably.

- I will be forever grateful to my family in the UK- my aunt and uncle Adele and Patrick, and my cousins Sinead, Scott, Chris and Cara. They always made me feel welcome in their homes, and provided a little slice of home to help sustain me whenever life abroad seemed particularly daunting.

- Last, but definitely not least, I am eternally indebted to my parents, Tom and Sandy Power, without whose endless love and support in all things and all times, none of this would have been possible. It cannot be overstated how grateful I am to them for all that they have done, and continue to do for me.

...the feeling of mathematical beauty, of the harmony of numbers and of forms, of geometric elegance. It is a genuinely aesthetic feeling, which all mathematicians know.

- Henri Poincaré

Contents

Chapter 1

Introduction and Summary of Results

A basic notion in point set topology is the concept of the Alexandroff or one-point compactification of a non-compact topological space X. This compactification is obtained by adjoining a "point at infinity", normally denoted by ∞, and considering the set $X \cup \{\infty\}$. The set $X \cup \{\infty\}$ is turned into a topological space, by declaring a set $U \subseteq X \cup \{\infty\}$ open if $U \subseteq X$ is open in X, or if $U = X \backslash K \cup \{\infty\}$ for some closed and compact $K \subseteq X$. Under this topology, $X \cup \{\infty\}$ becomes a compact topological space in which X is an open dense subset. If X is Hausdorff and locally compact, then $X \cup \{\infty\}$ is also Hausdorff, and is up to homeomorphism the minimal compactification of X [54, Theorem 29.1]. As metric spaces are topological spaces, the Alexandroff compactification can be used to embed a non-compact metric space in a compact topological space. However, the Alexandroff compactification loses a lot of information about the metric structure of a space. Consider, for example, the metric space $\mathbb{R}^n$ with the standard Euclidean norm $\|\cdot\|_2$. The Alexandroff compactification of $\mathbb{R}^n$ is homeomorphic to the unit sphere of $\mathbb{R}^{n+1}$, denoted by S^n. To define the homeomorphism, first recall the definition of the inverse stereographic projection $\sigma^{-1}\colon \mathbb{R}^n \to S^n \backslash \{(1, 0, \ldots, 0)\}$. If $\{e_0, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^{n+1}$, then for $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$

$$\sigma^{-1}(y) = \frac{\|y\|_2^2 - 1}{\|y\|_2^2 + 1} e_0 + \sum_{i=1}^{n} \frac{2y_i}{\|y\|_2^2 + 1} e_i.$$

We can then define the homeomorphism $f\colon \mathbb{R}^n \cup \{\infty\} \to S^n$ by $f(x) = \sigma^{-1}(x)$ if $x \in \mathbb{R}^n$, and $f(\infty) = (1, \ldots, 0)$. The Alexandroff compactification thus provides a "nice" embedding of Euclidean $\mathbb{R}^n$ into a compact topological space. However, every norm $\|\cdot\|$ on $\mathbb{R}^n$ generates the usual Euclidean topology on $\mathbb{R}^n$, so the function f is a homeomorphism from the Alexandroff compactification of any n-dimensional normed space, X, onto S^n. If all we are given is the Alexandroff compactification of X, we can deduce nothing about the metric structure of X. This is unsurprising, because if γ is an unbounded geodesic in X, then $f \circ \gamma(t)$ converges to the pole $(1, 0, \ldots, 0) \in S^n$, which reflects the fact that $\gamma(t)$ converges to ∞ in $X \cup \{\infty\}$. The Alexandroff compactification is therefore incapable of distinguishing even geodesics moving in opposite directions, so has little chance of capturing any of the underlying metric structure. As the Alexandroff compactification is defined purely topologically, this is to be expected. The above discussion suggests that if one would want to define a compactification of a normed space that preserves some metric structure, the compactification at, the very least, needs to be able to distinguish geodesics moving in different directions.

If (M, d) is a proper complete CAT(0) metric space (as defined in [10, Page 159]), such a compactification exists, and is constructed by appending the boundary at infinity, also known as the visual boundary, ∂M to M. This boundary consists of equivalence classes of geodesics $\gamma\colon [0, \infty) \to M$, where γ_1 is equivalent to γ_2 if $\sup_t d(\gamma_1(t), \gamma_2(t)) < \infty$. The set $M \cup \partial M$ equipped with the cone topology is a compact Hausdorff space containing M as a densely embedded subspace [10, 8.5]. This compactification $M \cup \partial M$ is referred to as the geodesic compactification of M. If M is any complete n-dimensional Riemannian manifold of non-positive sectional curvature, it is homeomorphic to the closed unit ball of Euclidean $\mathbb{R}^n$, and the homeomorphism maps ∂M homeomorphically onto the sphere S^{n-1} [10, Example 8.11]. In light of this observation, some authors refer to ∂M as the sphere at infinity. By its construction, we see that $M \cup \partial M$ distinguishes between unbounded geodesics, unlike the Alexandroff compactification. Furthermore, an isometry f

on M can naturally be extended to a map on ∂M, because isometries map geodesics to geodesics and preserve equivalence classes, and this extension is a homeomorphism on ∂M [10, Corollary 8.9]. This extension of isometries to ∂M actually gives some information about the metric structure of M. If f is a parabolic isometry of M, by which we mean $d(f(x), x) > \inf_y d(f(y), y)$ for all $x \in M$, then its extension to ∂M must have a fixed point in ∂M [10, Proposition 8.25]. As a simple example, this provides another proof that rotation around any point in $(\mathbb{R}^n, \|\cdot\|)$ is not a parabolic isometry, as its extension to the boundary acts as rotation on the sphere. The geometric compactification of CAT(0) spaces naturally leads to the notion of the horofunction compactification of general metric spaces.

The modern notion of the horofunction compactification of a metric space (M, d) is generally attributed to Gromov's 1978 work [25], where he uses them to study hyperbolic manifolds and their associated groups and actions. To define the horofunction compactification of (M, d), we first fix an arbitrary $b \in M$, and define $\mathrm{Lip}_b(M)$ to be the collection of all real valued Lipschitz-1 functions on M evaluating to 0 at b. This space is compact when equipped with the topology of pointwise convergence. The map $\iota \colon M \to \mathrm{Lip}_b(M)$ defined by

$$\iota(x) = d(\cdot, x) - d(b, x)$$

injects M into $\mathrm{Lip}_b(M)$, and the closure of $\iota(M)$ in $\mathrm{Lip}_b(M)$ with respect to the topology of pointwise convergence is a compact Hausdorff space, which is called the horofunction compactification of M, denoted by $\overline{M}^h$. Functions in $\partial\overline{M}^h$ are called horofunctions. The choice of base point does not matter, as different base points lead to the same horofunctions just shifted by a constant. If M is a proper CAT(0) space, then ι extends uniquely to a homeomorphism between $\overline{M}^h$ and $M \cup \partial M$ [10, Theorem 8.13]. Horofunctions have since appeared in multiple contexts and found many different applications. The application of horofunctions started with their use in the study of non-positively curved manifolds in [25] but has since branched out to include diverse areas, from dynamical systems [7, 23, 35, 44, 51], to Teichmüller theory [17, 36, 53, 67], to complex analysis [1, 6, 8, 9, 11, 70].

Rieffel used the horofunction compactification in his study of non-commutative geometry [60], where he introduced a special class of horofunctions, known as Busemann points, which arise as limits of almost-geodesics. It is known that if (M, d) is a complete CAT(0) space, then every horofunction is a Busemann point (see [10, Corollary 8.19] for example), but in other settings far less is known. Rieffel asked the question in [60] whether every horofunction of a finite dimensional normed space is a Busemann point. In [38], Karlsson, Metz, and Noskov showed that the horofunctions of any polyhedral normed space are all Busemann points. In [66] Walsh answers this question completely, where he provides a topological condition on the collection of faces of the dual unit ball which is satisfied if and only if every horofunction is Busemann point, and provides examples of normed spaces where this condition does not hold. The study of Busemann points is particularly useful, as they can be used to study the group of isometries of metric spaces and the isometric embeddings between metric spaces [41, 49, 68, 69].

The horofunctions of various spaces have been identified with varying degrees of specificity. This includes the horofunctions of various normed spaces [26, 27, 28, 38, 64, 66, 69], the horofunctions of cones with the projective Hilbert or Thompson metrics [43, 49, 69], the horofunctions of certain Teichmüller spaces with the Teichmüller and Thurston metrics [24, 67], the horofunctions of real infinite dimensional hyperbolic space [13], the horofunctions of Schatten p-metrics on the symmetric cone of Hermitian matrices [21], and the horofunctions of symmetric spaces of non-compact type [12, 29, 64]. In many cases, even though an explicit representation of the horofunctions is known, not much can be said about the global geometry and topology of the horofunction compactification. For any metric space (M, d), the horoboundary $\partial\overline{M}^h$ can be partitioned into equivalence classes, where $f, g \in \partial\overline{M}^h$ are in the same equivalence class if $\sup_{x\in M} |f(x) - g(x)| < \infty$. Kapovich and Leeb called this partition of the boundary its *stratification*, and they asked in [34, Question 6.18], whether for any finite dimensional normed space X, there exists a homeomorphism between $\overline{X}^h$ and the closed unit ball in the dual space X^*, which maps horofunctions onto the sphere of the dual unit ball, and maps all horofunctions in a given equivalence class of the stratification bijectively onto the relative interior of a face of the dual unit ball. Schilling and Ji [32, 33] showed that this holds true for every finite dimensional space with a polyhedral norm. This question is still open in general. Inspired by the observation that the horofunction compactification of symmetric Riemannian spaces of non-positive sectional curvature is homeomorphic to the Euclidean ball, and each equivalence class in the stratification is a singleton, this question of Kapovich and Leeb can be generalised to (finite dimensional) homogeneous Finsler metric spaces. We say a Finsler metric space, (M, F), is homogeneous if, for every $x, y \in M$, there is a linear bijection between the tangent spaces T_x and T_y which maps the collection of relative interiors of the faces of the unit ball in $(T_x, F(x, \cdot))$ bijectively onto the collection of relative interiors of the faces of the unit ball in $(T_y, F(y, \cdot))$. For example, if the group of Finsler isometries of (M, F) acted transitively on (M, F), then (M, F) would be homogeneous.

Question 1.0.1. When does there exist a homeomorphism from the horofunction compactification of a homogeneous Finsler metric space onto the closed dual unit ball of the norm on the tangent space at the basepoint, which preserves the natural stratification of the equivalence classes?

Question 1.0.1 is the motivation behind much of this thesis. Busemann points play an important role in the investigation of the question. There exists an extended metric, δ, (possibly infinitely valued) on the collection of Busemann points, called the detour distance. The detour cost induces a partition of the Busemann points into equivalence classes, called parts of the boundary, where Busemann points f, g are in the same part if $\delta(f, g) < \infty$. Busemann points f and g are in the same part of the boundary if and only if $\sup_{x \in M} |f(x) - g(x)| < \infty$. Thus, if every horofunction of a normed space is a Busemann point, Question 1.0.1 can be reframed entirely in terms of parts of the boundary, instead of in terms of its stratification.

Question 1.0.1 obviously does not make much sense in the infinite dimensional setting, because the dual ball in infinite dimensions is never compact, whereas the horofunction compactification always is. However, the dual ball equipped with the weak* topology *is* always compact. As infinite dimensional normed spaces aren't proper, the horofunction compactification is often more complicated than their finite dimensional analogues. Gutiérrez [27] provided a nice description of the horofunctions of infinite dimensional ℓ^p spaces for $1 \leq p < \infty$. We built on his work to identify the Busemann points of these spaces, which motivated the question:

Question 1.0.2. For which infinite dimensional normed spaces X does there exist a homeomorphism from the Busemann points of $\partial\overline{X}^h$ onto the dual unit sphere S_{X^*} equipped with the weak* topology, which maps parts of the Busemann boundary bijectively onto the relative interiors of faces of the dual ball?

While Questions 1.0.1 and 1.0.2 were being studied during this thesis, Lemmens [42] noticed that the exponential map on a finite dimensional JB-algebra V can be extended to a homeomorphism between the horofunction compactification of V and the horofunction compactification of the interior of the positive cone of V equipped with the Thompson metric, and the homeomorphism maps parts to parts. He also showed that if the projectivised positive cone PV_+ is equipped with the Finsler metric H defined by $H(w, v) = |U_{w^{-1/2}}v|_e$, where $|v|_e = \text{diam}(\sigma(v))$ is the spectral radius of v, and U_x is the quadratic representation of x, then the exponential on the tangent space T_e extends as a homeomorphism between $\overline{T_e}^h$ and the horofunction compactification of PV_+ equipped with the Finsler distance, and parts are mapped to parts. Concurrently, Duchesne [16] released a paper stating that the horofunction compactification of infinite dimensional separable real hyperbolic space is homeomorphic to the horofunction compactification of a separable infinite dimensional Hilbert space. This lead to the final question investigated in this thesis:

Question 1.0.3. Can the work of Lemmens in [42] be extended to the setting of infinite dimensional spin factors, and in so doing provide an alternative proof for Duschesne's statement that the horofunction compactification of infinite dimensional real hyperbolic space is homeomorphic to the horofunction compactification of a separable infinite dimensional Hilbert space?

Structure of the Thesis

The thesis is structured to (partly) address Questions 1.0.1-1.0.3. We provide a brief breakdown of the chapters below. Chapters 4,5, and 6 consist of joint work with Bas Lemmens, which has already been published [47], and concerns Question 1.0.1. Each of these chapters correspond to a main section of the paper, and we have left them mostly unchanged. Chapters 7 and 8 consist of solo work by the author dealing with Questions 1.0.2 and 1.0.3.

Chapter 2

This chapter is the preliminary chapter, and is a collection of content needed to understand later chapters that can be found in various textbooks. Most proofs are omitted, but references to the relevant textbooks are given for the interested reader. The exception to this are the sections on order-unit spaces and Jordan algebras, where the author provides certain proofs, mainly for his own edification. The main purpose of this chapter is to standardise notation and definitions that will be used throughout, and to serve as a convenient repository of standard results that will be used in later chapters.

Chapter 3

Chapter 3 serves as the full introduction to the theory of horofunctions that is needed to fully appreciate the thesis. All the relevant theory is introduced fully and discussed, and some examples are worked out in detail. Most of what is found in this chapter can be found in various published research papers. Some results are well known, but the author provided his own proofs to be more consistent with the terminology of the thesis. In particular the author is not aware of a reference using the same proof for Lemma 3.2.1. The results in Section 3.4 on the horofunction compactification of finite ℓ^1 sums of arbitrary metric spaces does not appear in the literature as far as the author knows. Analogous results for the case of proper geodesic metric spaces are contained in [45], which is joint work with Bas Lemmens and Cam Milliken .

Chapter 4

This chapter corresponds precisely to Section 3 in the joint paper with Lemmens [47]. We study the horofunction compactification of product domains $B = \prod_{i=1}^{r} B_{n_i}$ equipped with the Kobayashi distance, where each B_{n_i} is the open unit ball in $\mathbb{C}^{n_i}$ for some strictly convex C^3 norm on $\mathbb{C}^{n_i}$. The two main results are Theorem 4.1.1 and Theorem 4.1.4. In Theorem 4.1.1 we classify the horofunctions of B° and show that they are all Busemann points, and in Theorem 4.1.4 we show that $\overline{B}^h$ is homeomorphic to the closed unit ball in the dual $\bar{B}^*$, and this homeomorphism maps parts of the boundary bijectively to the relative interiors of faces of $\bar{B}^*$.

Chapter 5

Chapter 5 corresponds to Section 4 of [47]. In it we study the horofunction compactification of finite dimensional JB-algebras. We use the fact that the JB-algebra norm is equal to the spectral radius norm, and exploit the fact that finite dimensional JB-algebras correspond to Euclidean Jordan algebras. This allows us to prove the three main theorems of the chapter: Theorem 5.2.1, Theorem 5.2.2, and Theorem 5.2.4. Theorem 5.2.1 provides a complete characterisation of the horofunctions of any finite dimensional JB-algebra, and proves that every horofunction is a Busemann point. Theorem 5.2.2 provides an analytical expression for the detour distance between any two Busemann points, and characterises the parts of the boundary. Building upon these two theorems, Theorem 5.2.4 shows that the horofunction compactification of a finite dimensional JB-algebra is homeomorphic to the closed dual unit ball, and maps parts of the boundary to the relative interiors of faces of the dual.

Chapter 6

This chapter corresponds to Section 5 of [47], where we consider the horofunction compactifications of symmetric Hilbert geometries, which are the intersection of a hyperplane with a symmetric cone equipped with the (projective) Hilbert metric. The horofunction compactification is always considered with base point being the unit e of the JB-algebra. There is one noticeable difference, however, as we include the proof of Proposition 6.1.2, which shows that a symmetric Hilbert geometry can be realised as a Finsler distance. The horofunctions of symmetric Hilbert geometries were already classified in [44], so the main results of this chapter utilise that result, and are Proposition 6.2.2 and Theorem 6.3.1. Proposition 6.2.2 completely characterises the detour distance and parts of the boundary of the horofunction compactification of a symmetric Hilbert geometry, and Theorem 6.3.1 shows that the horofunction compactification of a symmetric Hilbert geometry is homeomorphic to the dual unit ball of the tangent space at e (with norm coming from the Finsler structure used in Proposition 6.1.2).

Chapter 7

We now shift our attention to non-proper metric spaces, and try to address Question 1.0.2. We build on the work of Gutiérrez in [27], and investigate the geometry and topology of various infinite dimensional Banach spaces. The chapter is divided into three sections, each one dealing with a specific class of space. Section 7.1 deals with infinite dimensional Hilbert spaces $\mathcal{H}$. We classify all the Busemann points of $\partial\overline{\mathcal{H}}^h$, which Gutiérrez does not consider in [27], and then prove the main result of this section, Theorem 7.1.5. This theorem answers Question 1.0.2 in the affirmative, and shows that the Busemann boundary of a Hilbert space is homeomorphic to the dual sphere equipped with the weak* topology. Corollary

7.1.9 follows from this proof, and shows that the class of functions in $\partial\overline{\mathcal{H}}^h$ arising from bounded weakly convergent nets, which Gutiérrez classifies in [27], are in fact horofunctions, and not just different representations of internal metric functionals. This is taken for granted in [27], but was not a priori clear to the author. Although simple to prove, Proposition 7.1.11 is interesting, because it shows a fundamental difference between finite and infinite dimensional normed spaces. Namely it shows that there can be no homeomorphism between $\overline{\mathcal{H}}^h$ and the closed dual unit ball equipped with the weak* topology that maps parts of the Busemann boundary bijectively onto the sphere. Section 7.2 generalises the results of Section 7.1 to the case of uniformly smooth and strictly convex infinite dimensional Banach spaces. We classify the Busemann points of uniformly smooth and strictly convex infinite dimensional Banach spaces, and then prove Theorem 7.2.5, which shows that the Busemann points of uniformly smooth and strictly convex infinite dimensional Banach spaces are homeomorphic to the unit sphere in the dual, and parts are mapped onto the relative interiors of faces of the dual ball. If we restrict our attention to infinite dimensional ℓ^p spaces $(1 < p < \infty)$, Proposition 7.2.9 then shows that there exists no homeomorphism between the horofunction compactification and the dual unit ball which maps Busemann points bijectively onto the sphere, unlike in the finite dimensional case. Section 7.3 is perhaps the most surprising of the section, as we deal with the horofunction compactification of infinite dimensional ℓ^1 spaces over any arbitrary index set I. We first note that every horofunction in $\partial\overline{\ell^1}^h$ is a Busemann point, and completely classify the parts of the boundary in Proposition 7.3.4. The highlight of this section is Theorem 7.3.5, where we prove that, unlike in the smooth case in the previous sections, $\overline{\ell^1}^h$ is homeomorphic to the closed dual unit ball, and Busemann points are mapped bijectively onto the dual sphere. Where the infinite dimensional case differs from the finite dimensional case, is that this homeomorphism maps infinitely many parts of the boundary into the relative interior of a single face of the dual ball.

Chapter 8

Chapter 8 is the longest chapter of the thesis, and completely answers Question 1.0.3. The chapter is divided into four sections. Throughout we consider an infinite dimensional spin factor V with JB-algebra norm, and positive cone V_+. In Section 8.1, we completely characterise the horofunction compactification of V in Theorem 8.1.1, as well as classify the Busemann points and parts of the boundary in Propositions 8.1.3 and 8.1.6.In Theorem 8.1.5 we also show how the Busemann points of $\partial\overline{V}^h$ can be represented in the same form as the horofunctions of finite dimensional JB-algebras as in Chapter 5. Section 8.2 consists entirely of classifying the horofunctions of the open cone V_+° equipped with the Thompson metric. The main results are Theorem 8.2.4, where we give an analytical form for all the horofunctions, and Theorem 8.2.5, where we classify all the Busemann points. In Section 8.3 we construct an extension of the exponential $\exp\colon V \to V_+^0$, and prove in Theorem 8.0.1 that this extension is a parts preserving homeomorphism from $\overline{V}^h$ onto the horofunction compactification of V_+° equipped with the Thompson metric, which extends Theorem 4.3 of [42] to infinite dimensions. In Section 8.4 we consider the hyperboloid model of real infinite dimensional hyperbolic space, which can be considered as an infinite dimensional Finsler manifold embedded in V with the projective Hilbert metric as its Finsler distance. The tangent space T_e at the unit e of V is precisely the Hilbert space underlying V, and we prove Theorem 8.0.3, which states that there exists a parts preserving homeomorphism between the horofunction compactification of T_e and the horofunction compactification of infinite dimensional hyperbolic space. This fully answers Question 1.0.3.

Chapter 9

This is the concluding chapter, where we provide some reflections on the results obtained and how they relate to Questions 1.0.1 and 1.0.2, and discuss avenues for further research on these questions, as well as other natural questions about the global geometry and topology of horofunction compactifications.

Chapter 2

Preliminaries

For the convenience of the reader, we introduce the basic definitions and theorems that will be ubiquitous throughout this thesis. All that we discuss here is well known, and there exist multiple textbooks covering each section in great detail. If the reader is already familiar with the areas discussed, they can comfortably skip it, and only refer back when theorems are cited, or they need clarification on a choice of definition or terminology.

2.1 Topological Preliminaries

In this section we shall briefly recall some important topological notions, but a recap of all the relevant topological notions is out of the scope of this thesis, so if a reader is unsure of any topological term used we refer them to [54], as all topological notions used throughout this thesis should be consistent with the definitions and terminology introduced there. A *topological space* (X, τ), consists of a set X and a *topology* on that set, $\tau \subseteq 2^X$ (the powerset of X), satisfying:

(i) $X, \emptyset \in \tau$.

(ii) If, for any arbitrary index set I, $A_i \in \tau$ for all $i \in I$, then $\bigcup_{i \in I} A_i \in \tau$.

(iii) If, for each $i \in \{1, \ldots n\}$ for $n \in \mathbb{N}$, $A_i \in \tau$, then $\bigcap_{i=1}^{n} A_i \in \tau$.

We call elements of τ the *open sets* of the topological space, and their complements are called the *closed sets* of the space. It follows from the above properties of τ that an arbitrary intersection of closed sets is closed, and a finite union of closed sets is closed. If the topology on a set X is understood, we often drop the notation (X, τ), and just refer to the topological space X. Recall that a space X is said to be *Hausdorff* if any two distinct points are elements of two disjoint closed sets. For any point $x \in X$, we call an open set U containing x a *neighbourhood* of x. The collection of neighbourhoods of x is generally denoted by $\mathcal{N}_x$. We say a sequence $(x_n) \subseteq X$ *converges* to a point $x \in X$, if for every neighbourhood U of x, there exists an $N \in \mathbb{N}$, such that $x_n \in U$ for every $n \geq N$. We say a point x is an *interior point* of $A \subseteq X$, if there exists a neighbourhood of x, say U, such that $U \subseteq A$. We define the *interior* of a set $A \subseteq X$, to be the union of all open sets contained in A, and we denote it by $\text{int}(A)$ or A°. Unsurprisingly, A° is equal to the union of all interior points of A. For any $A \subseteq X$, we say x is a *limit point* of A, if for every neighbourhood U of x, $U \cap A \backslash \{x\} \neq \emptyset$. Some authors prefer to call limit points *accumulation points* or *cluster points*. We define its *closure* $\bar{A}$, to be the intersection of all closed sets containing A. A set is closed if and only if it equals its closure, and if and only if it contains all its limit points. The *boundary* of $A \subseteq X$, denoted by ∂A is defined to be $\partial A = \bar{A} \backslash A^\circ$. We say $A \subseteq X$ is dense in X, if $\bar{A} = X$. If the space X contains a countable dense subset, we call it *separable*.

We call a function $f \colon X \to Y$ between topological spaces *continuous*, if $f^{-1}(U)$ is an open set in X for all open sets $U \subseteq Y$. We declare f *sequentially continuous* if, for any sequence $(x_n) \subseteq X$ converging to $x \in X$, the corresponding sequence $(f(x_n)) \subseteq Y$ converges to $f(x)$. Any continuous function between topological spaces is sequentially continuous, but the converse isn't always true. If $f \colon X \to Y$ is a continuous bijection, with a continuous inverse, it is called a *homeomorphism*, and in this case we say X and Y are *homeomorphic*. If (X, τ) is a topological space, we can turn any non-empty subset $A \subseteq X$ into its own topological space by equipping it with the *subspace topology*, by declaring $B \subseteq A$ open if and only if B is of the form $C \cap A$ for some $C \in \tau$. It is a useful fact that, if $\{\tau_i\}_{i \in I}$ is an arbitrary non-empty collection of topologies on a set X, then $\bigcap_{i\in I} \tau_i$ is also a topology on X. Furthermore, for any collection $\mathcal{C} \subseteq 2^X$, the collection of topologies on X containing $\mathcal{C}$ is non-empty (as it must consist of at least 2^X), which means we can define the topology *generated* by $\mathcal{C}$ as the intersection over all topologies containing $\mathcal{C}$. If τ_1 and τ_2 are both topologies on X, we say that τ_1 is *coarser* than τ_2 if $\tau_1 \subseteq \tau_2$, and in this case we say that τ_2 is *finer* than τ_1. Thus, the topology generated by a collection of subsets is the coarsest topology containing the collection. We call a collection $\mathcal{C} \subseteq 2^X$ a *basis* for a topological space X, if every open set in X is equal to a union of elements of $\mathcal{C}$. We say $\mathcal{C} \subseteq 2^X$ is a *subbasis* of X if every open set is a union of finite intersections of elements of $\mathcal{C}$. Thus the topology generated by a basis or subbasis of a topological space X is the same as the original topology on X, and any topology generated by a collection makes that collection a subbasis for that topology. A topological space is called *second countable* if it possesses a countable basis (or equivalently subbasis). For example, $\mathbb{R}$ as it is usually considered (where convergence is given with respect to the absolute value) is a topological space given by taking the collection of all open intervals as its basis. The density of the rationals in $\mathbb{R}$ means we can equivalently take the collection of all open intervals with rational end points as a basis, and so $\mathbb{R}$ is second countable.

Of particular importance to us in this thesis are functions between topological spaces, and the different topologies induced by collections of functions, as well as topologies on the function spaces themselves. If X is a set, and $\mathscr{F}$ is a collection of functions $f\colon X \to Y_f$, where each Y_f is a topological space, we can define the *$\mathscr{F}$-weak topology* on X. This is the coarsest topology on X for which every $f \in \mathscr{F}$ is continuous, and corresponds to the topology generated by the collection $\{f^{-1}(V)\colon V \text{ is open in } Y_f,\ f \in \mathscr{F}\}$. For any sets X and Y, recall that we denote by Y^X the set of all functions from X to Y. The space Y^X can be identified with the product space $\prod_{x\in X} Y$. We can define, for every $x \in X$ the projection $\pi_x\colon Y^X \to Y$ defined by $\pi_x(f) = f(x)$. If Y is a topological space, then Y^X can be equipped with the *product topology*, which is the coarsest topology making each projection π_x continuous. When thinking of Y^X as a space of functions, the product topology is generally known as the *topology of pointwise convergence*, because a sequence $(f_n) \subseteq Y^X$ converges to $f \in Y^X$ in the product topology if and only if $(f_n(x)) \subset Y$ converges to $f(x)$ in Y for every $x \in X$. For any point $x \in X$ and open set $U \subseteq Y$ we can define the set

$$S_{x,U} = \{f \in Y^X : f(x) \in U\}. \tag{2.1.1}$$

The collection of all $S_{x,U}$ forms a subbasis for the topology of pointwise convergence on Y^X [54, Section 46].

This thesis is about compactifications, so we shall devote a bit more space to the notion of compactness. We call a topological space (X, τ) *compact*, if every open cover has a finite subcover. By this we mean, for every collection $\{U_i\}_{i\in I} \subseteq \tau$ such that $X = \bigcup_{i\in I} U_i$ there must exist a finite sub-collection $\{U_{i_j}\}_{j=1}^n \subseteq \{U_i\}_{i\in I}$, such that $X = \bigcup_{j=1}^n U_{i_j}$. We say that a subset $A \subseteq X$ is a compact subset of X if A is a compact topological space when inheriting the subspace topology on X. Equivalently, A is a compact subset of X, if for every $\{U_i\}_{i\in I} \subset \tau$ such that $A \subseteq \bigcup_{i\in I} U_i$, there exists a finite sub-collection $\{U_{i_j}\}_{j=1}^n \subseteq \{U_i\}_{i\in I}$, such that $A \subseteq \bigcup_{j=1}^n U_{i_j}$. There is also the notion of *sequential compactness*, where a space X is said to be sequentially compact if every sequence in X has a subsequence converging to a $x \in X$. This definition applies just as readily to subsets of a space. In general, compactness and sequential compactness are not equivalent. As we shall see in the next section, any unbounded metric space is not compact, which means, for example, that $\mathbb{R}$ with the usual topology is not compact. It is, however, *locally compact*, which means that for any point x, there exists a neighbourhood U of x, and a compact set K, such that $U \subseteq K$.

As we shall see in later sections, the compactness of a topological space allows powerful tools, which is why there is the notion of compactification:

Definition 2.1.1. A compact topological space Y is said to be a *compactification* of a topological space X, if there exists a continuous injection $\iota \colon X \to Y$, such that $\iota(X)$ is a homeomorphism onto $\iota(X)$ with the subspace topology, and $\iota(X)$ is dense in Y.

In general, a function $f \colon X \to Y$ which is a homeomorphism onto its image is known as an *embedding* of X into Y.

2.2 Metric Space Preliminaries

The horofunction compactification is a method of compactifying metric spaces, so before we introduce it we recall some of the relevant notions. We once again don't have space to introduce all the relevant notions, so we refer the interested reader to [40] and [54] if any of the metric space theory used in the thesis is unclear. A metric space is a pair (M, d), where M is a set and $d \colon M \times M \to [0, \infty)$ satisfies:

(i) $d(x, x) = 0$ for all $x \in M$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in M$ (symmetry).

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$ (the triangle inequality).

(iv) $d(x, y) = 0$ implies $x = y$ for all $x, y \in M$.

The full definition of a metric is a rather strong requirement, and there are many cases where it is useful to consider spaces where some of the requirements are weakened. We call (M, d) a *pseudo-metric* space if d satisfies (i)-(iii) above, but we don't require that separate points have a strictly positive distance between them. We call (M, d) a *hemi-metric* if d satisfies (i), (iii), (iv) above, but is not symmetric. In this thesis we are primarily interested in metric spaces, but the horofunction compactification has been studied in the context of both pseudo-metric spaces and hemi-metric spaces (for examples see [33, 22, 44, 69, 67]). Unless specified otherwise, in all that follows, we shall be considering (M, d) to be a metric space, although a lot of the terminology is the same for hemi/pseudo-metric spaces.

For any $x \in (M, d)$, and $r > 0$, we define the *open ball* of radius r centred at x by $B^\circ(x; r) = \{y \in M : d(x, y) < r\}$, and similarly the *closed ball* of radius r centred at x by $B(x; r) = \{y \in M : d(x, y) \leq r\}$. We define the *sphere* of radius r centered at x by $S(x; r) = B(x; r) \backslash B^\circ(x; r)$. The set of all open balls generates a topology on M, and unless otherwise specified, this is the topology we will be referring to when referring to topological properties about (M, d). Recall that a topological space (X, τ) is said to be *metrisable* if there exists a metric d on X generating τ. We say two metrics on a set are *equivalent* if they both generate the same topology. Metrisable spaces are particularly useful due to the following theorem, which is a collection of well known topological facts, which can be found in [54, Section 20-21].

Theorem 2.2.1. *Let X be a metrisable space, then the following hold:*

(i) A function $f : X \to Y$, for any topological space Y, is continuous if and only if it is sequentially continuous.

(ii) A point $x \in X$ is a limit point of a set A if and only if there exists a sequence $(x_n) \subseteq A$ such that $x_n \to x$.

(iii) X is compact if and only if it is sequentially compact. Similarly, $A \subseteq X$ is compact if and only if it is sequentially compact.

Recall that a metric space is called *complete* if every Cauchy sequence converges to an element in the metric space.

Example 2.2.2 (Homeomorphisms aren't enough)**.** Equip the sets $\mathbb{R}$ and $(-1, 1)$ with the usual metric $d(x, y) = |x-y|$. These are thus two distinct metric spaces, with corresponding metric topologies. $\mathbb{R}$ and $(-1, 1)$ under these metric topologies are homeomorphic. Indeed, it is a common exercise to show that $x \mapsto \tanh(x)$ is a homeomorphism from $\mathbb{R}$ onto $(-1, 1)$. However, as a metric space, $\mathbb{R}$ is unbounded and complete, whereas $(-1, 1)$ is bounded and not complete.

The above example shows that homeomorphisms aren't, predictably, enough to preserve the structure of a metric space. A map $f\colon M \to N$ between two metric spaces (M, d) and (N, ρ) is called an *isometry* if $d(x, y) = \rho(f(x), f(y))$ for all $x, y \in M$. We say M and N are *isometric* if there exists a bijective isometry between them.

We call (M, d) a *proper metric space* if all closed balls are compact. A prototypical example is any finite-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$, which we know is proper thanks to the famous Heine-Borel theorem. Proper metric spaces are complete and separable [56, page 28]. A *path* in a metric space is a continuous function $\gamma\colon I \to M$, where $I \subseteq \mathbb{R}$ is an interval. A path γ is called *geodesic* if $d(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in I$. We call M a *geodesic space* if for every $x, y \in M$ there exists a geodesic path $\gamma\colon [0, d(x, y)] \to M$ with $\gamma(0) = x$ and $\gamma(d(x, y)) = y$.

If we are given a finite collection of metric spaces, (M_i, d_i), for $i \in \{1, \ldots, m\}$, we can form the ℓ^p product metric space of these metric spaces, for $p \in [1, \infty]$, by defining $M = \prod_{i=1}^m M_i$, and for any $(x, y) = ((x_1, \ldots, x_m), (y_1, \ldots, y_m))$,

$$d^p(x, y) = \left(\sum_{i=1}^m d_i(x_i, y_i)^p \right)^{\frac{1}{p}}, \quad \text{for } p \in [1, \infty), \text{ and } d^\infty(x, y) = \max_i d_i(x_i, y_i).$$

It is useful that the ℓ^p product space generally inherits nice properties from its constituent factors. The following lemma [56, Proposition 2.6.6] is particularly useful:

Lemma 2.2.3. *If (M_i, d_i) are proper geodesic metric spaces for $i \in \{1, \ldots, m\}$, then the ℓ^p product space, (M, d^p), of these metric spaces is also a proper geodesic metric space for all $p \in [1, \infty]$.*

2.3 Nets

In practice, investigating whether a set is closed or compact, or a function is continuous, is very difficult simply using the topological definition. If a space is metrisable, Theorem 2.2.1 means that we can investigate the topological properties of it, its subsets, and functions on it through analysing the behaviours of sequences, which is a lot more tractable. It also guarantees that if the space is compact, any sequence must have a convergent subsequence, which is often a crucial argument in the study of horofunctions. Some of the spaces dealt with in this thesis are not metrisable, which is why we introduce the notion of nets, which can be thought of as a generalisation of sequences. Our exposition closely aligns with that of Megginson [50, Chapter 2]. Recall that a *directed set* is a set with a relation, $(A, \leq)$, where $\leq$ is reflexive and transitive, and for every $\alpha, \beta \in A$, there exists a $\gamma \in A$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. A *net* in a topological space X consists of a directed set $(A, \leq)$ and a function $f\colon A \to X$. We generally denote $f(\alpha)$ by x_α for all $\alpha \in A$, and refer to the net $(x_\alpha)_{\alpha\in A}$, or just (x_α) if the index set is understood. We say that a net $(x_\alpha)_{\alpha\in A}$ converges to a point $x \in X$ if, for every neighbourhood U of x there exists an $\alpha \in A$ such that for all $\alpha \leq \beta$, $x_\beta \in U$. The following example is important enough that it can be thought of motivating the definition of a net.

Example 2.3.1. Let (X, τ) be any topological space. We can order τ with the relation $\leq$, by saying that $A \leq B$ if $A \supseteq B$. It follows immediately from the definition of set inclusion that $\leq$ is transitive and reflexive. By the definition of a topology, if $A, B \in \tau$, the $A \cap B \in \tau$, and $A \supseteq A \cap B$ and $B \supseteq A \cap B$, meaning that $(\tau, \leq)$ is a directed set. Furthermore, for any $x \in X$, the same reasoning shows that the collection of all neighbourhoods of x is also a directed set under $\supseteq$. Now assume that $x \in X$ is a limit point for some $A \subseteq X$. Thus, for every neighbourhood U of x, $U \cap A \backslash \{x\} \neq \emptyset$. If $\mathcal{N}_x$ is the collection of all neighbourhoods of x, we can use the axiom of choice to define $f\colon \mathcal{N}_x \to X$ by $f(U) = x_U \in U \cap A \backslash \{x\}$. From what we have just shown, $(x_U)_{U\in\mathcal{N}_x}$ is a net in X, and by construction $(x_U) \subseteq A$, and $x_U \to x$. We can thus conclude that a point is a limit point of a subset of a topological space if and only if there exists a net contained in the set converging to the point. This is not, in general, the case for sequences; there exists topological spaces where a point can be a limit point of a set, but there exists no sequence in that set converging to the point.

Just as in the case for sequences, it is desirable to have a notion of a subnet. Unlike in the case of sequences, it is not immediately obvious what notion one should choose for the definition of a subnet. There are in fact various definitions of subnets, which are similar, but not equivalent. A full discussion of the different types of subnets can be found in [63, Chapter 7]. In this thesis we exclusively what are known as Willard subnets, which we now define. Recall that a subset of a directed set, say $B \subseteq A$, is *cofinal* if for every $\alpha \in A$ there exists a $\beta \in B$ such that $\alpha \leq \beta$. We say that x is a *cluster point* of the net $(x_\alpha)_{\alpha \in A}$ if, for every $U \in \mathcal{N}_x$, the set $\{\alpha \in A : x_\alpha \in U\}$ is cofinal in A. We say that a map $f \colon A \to A'$ between two directed sets is a *cofinal map*, if, for every cofinal $B \subseteq A$, $f(B)$ is cofinal in A'. A map $f \colon A \to A'$ between two directed sets is known as *monotone* if $\alpha \leq \beta$ implies that $f(\alpha) \leq f(\beta)$. A net $J \colon B \to X$ is said to be a *subnet* of $I \colon A \to X$, if there exists a monotone function $\varphi \colon B \to A$ such that $J = I \circ \varphi$, and $\varphi(B)$ is cofinal in A. The following example shows that care needs to be taken when defining properties of nets in metric spaces.

Example 2.3.2. The set $\mathbb{N}$ with the usual ordering is a directed set, from which we see that any sequence in a topological space is a net. Conversely, there are many nets which aren't sequences, some of which can have surprising behaviour. Consider for example the set $\mathbb{N} \cup \{\infty\}$, with the ordering $\leq_\infty$ defined by $n \leq n$ and $n \leq \infty$ for all $n \in \mathbb{N}$. This is, by construction, a directed set, so we can consider the net $(x_\alpha)_{\alpha \in \mathbb{N} \cup \{\infty\}}$ in $\mathbb{R}$ with the usual topology, where $x_n = n$, and $x_\infty = 0$. It is clear that x_∞ is contained in any neighbourhood of 0, but also, $\{\alpha \in \mathbb{N} \cup \{\infty\} : \infty \leq_\infty \alpha\} = \{\infty\}$, so (x_α) actually converges to 0, but for every $K \in \mathbb{R}$ there exists an α with $x_\alpha > K$. Thus we see that, unlike in the case of sequences, convergent nets in a metric space are not necessarily bounded.

We say that a net $(x_\alpha) \subseteq M$ in a metric space (M, d) *converges to infinity* if there exists a $b \in M$, such that for every $n \in \mathbb{N}$ there exists a β so that for all $\alpha \geq \beta$, $d(x_\alpha, b) \geq n$.

To avoid the issue highlighted in Example 2.3.2, we define a net $(x_\alpha) \subseteq M$ in a metric space (M, d) to be an *unbounded net* if there exists a subnet (x_β) converging to infinity. If the net is not unbounded, we call it an *eventually bounded net.* By this definition, the net in Example 2.3.2 is eventually bounded. We should note that this definition is slightly different than what some authors use, but for setup is more convenient. What is important for our setup, is that our definition of eventually bounded nets means that any eventually bounded net $(x_\alpha) \subseteq M$ has a subnet (x_β) that is entirely contained in some ball $B(d; r)$, and an unbounded net has a subnet converging to infinity. The main utility of nets are encapsulated by the following theorem, which is a combination of Proposition 2.1.21, Corollary 2.1.36, and Proposition 2.1.37 in [50].

Theorem 2.3.3. *For any topological space X:*

(i) X is compact if and only if every net in X has a convergent subnet.

(ii) $A \subseteq X$ is closed if and only if for every net $(x_\alpha) \subseteq X$ with $x_\alpha \to x \in X$, we have $x \in A$.

(iii) A function $f\colon X \to Y$ is continuous at a point $x \in X$ if and only if every net $f(x_\alpha) \to f(x)$ for every net (x_α) converging to x.

The proof of Theorem 2.3.3, needs the following lemma, which is interesting for its own sake [50, Proposition 2.1.35]:

Lemma 2.3.4. *If $x \in X$ is a cluster point of the net $(x_\alpha)_{\alpha \in A}$, then it has a subnet converging to x. Conversely, if $(x_\alpha)_{\alpha \in A}$ has a subnet converging to x, then x is a cluster point of $(x_\alpha)_{\alpha \in A}$.*

This lemma is interesting because the analogous statement does not hold for sequences. In general, just because a sequence has an accumulation point, it does not need to have a subsequence converging to that point.

Remark 2.3.5. A subsequence of a sequence $(x_n) \subseteq X$ can be thought of as a sequence (y_n) and a monotone, injective function $\varphi\colon N \to \mathbb{N}$ such that $y_n = x_{\varphi(n)}$ for all $n \in \mathbb{N}$.

Thus, every subsequence of a sequence is also a subnet. Conversely, not every subnet of a sequence is also a subsequence, as a subnet of a sequence is a net (y_α), together with a directed set A and monotone cofinal map $\phi\colon A \to N$ such that $y_\alpha = x_{\phi(\alpha)}$, which means that (y_α) could very easily not even be a sequence, for example if $A = \mathbb{R}$. This distinction is important to keep in mind in light of part *(i)* of Theorem 2.3.3; just because a sequence must have a convergent subnet, does not mean it must have a convergent subsequence.

In this thesis we will be dealing with product spaces, in which we will need the following lemma [50, Proposition 2.1.16]:

Lemma 2.3.6. *If $X = \prod_{i\in I} X_i$ is a product of topological spaces, and $x \in X$, then a net $(x^\alpha) = (x_i^\alpha)_{i\in I}$ converges $x \in X$, if and only if for every $i \in I$ the net (x_i^α) converges to x_i.*

2.4 Normed Spaces and Topological Vector Spaces

The study of normed spaces and topological vector spaces falls under the umbrella of functional analysis. There are many excellent texts covering the foundations of the subject, but our main points of reference for this thesis are [62] and [50]. Recall that a normed space is a pairing $(X, \|\cdot\|)$, where X is a vector space over a field $\mathbb{F}$ (technically it could be any field, but in our case, and in most cases in the literature, it is either $\mathbb{C}$ or $\mathbb{R}$), and $\|\cdot\|\colon X \to \mathbb{R}$ satisfies the following:

(i) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{F}$ (homogeneity).

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (the triangle inequality).

(iii) $\|x\| = 0$ implies $x = 0$ for all $x \in X$.

Conditions (i) and (ii) imply that $\|x\| \geq 0$ for all $x \in X$, and $\|0\| = 0$. If we drop requirement (iii), the resulting $\|\cdot\|$ is known as a *seminorm*. Every normed space $(X, \|\cdot\|)$ has a natural metric structure, where the metric is given by $X \times X \ni (x, y) \mapsto \|x - y\|$. The norm topology on a normed space is simply the metric topology generated by this metric.

For convenience, if the norm on $(X, \|\cdot\|)$ is understood, we will just refer to the normed space X. We define $B_X^\circ = B^\circ(0;1)$, and $B_X = B(0;1)$, and call them the open and closed unit balls of X, respectively. We call $S_X = B_X \backslash B_X^\circ$ the unit sphere of X.

We call a normed space a *Banach space* if it is complete under this metric. We call two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on the same vector space X *equivalent*, if there exists $\alpha, \beta \in [0,\infty)$, such that, for all $x \in X$

$$\alpha\|x\|_2 \leq \|x\|_1 \leq \beta\|x\|_2.$$

It is well known that equivalent norms generate equivalent metrics, so the norm topology on a space is invariant under equivalent norms. The main structure preserving maps between normed spaces are the so called *bounded linear operators*. If $T\colon X \to Y$ is a linear map between normed spaces, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively, we call it bounded if

$$\sup_{x\in B_X} \|Tx\|_Y < \infty.$$

We call this quantity the *operator norm* of T, and if we equip the space of all bounded linear operators between X and Y with this norm, it becomes a normed space, which we denote by $L(X;Y)$. It is an standard exercise to show that an operator between normed spaces is continuous with respect to the norm topology if and only if it is bounded. An important space in functional analysis is $L(X;\mathbb{F})$, the set of bounded linear operators from X to its field. This is known as the *dual space* of X, and we denote it by X^*. The elements of X^* are known as the *continuous linear functionals*. Regardless of the completeness of X, X^* is always a Banach space. We can also talk about the *bi-dual* of X, which is simply the dual of the dual, and denoted by X^{**}. It is always possible to isometrically embed X into X^{**} by defining $J\colon X \to X^{**}$ by

$$J(x) = \phi_x \in X^{**}, \text{ where } \phi_x(f) = f(x). \tag{2.4.1}$$

If J is an isomorphism, we say that X is reflexive, and will write $X = X^{**}$, where this equality is to be understood through the action of J. If X and Y are Banach spaces, we can associate to every $T \in L(X;Y)$ a unique $T^* \in L(Y^*;X^*)$ defined by $f(Tx) = T^*f(x)$ for every $x \in X$ and $f \in Y^*$. We call T^* the *adjoint* of T, and $\|T^*\| = \|T\|$. In the case of finite dimensional real vector spaces, the adjoint of a linear map is simply given by its matrix transpose.

There are some strong topological differences between normed spaces over finite and infinite-dimensional vector spaces, which we collect in the following theorem [62]:

Theorem 2.4.1. *(i) A vector space X is finite-dimensional if and only if every norm on it is equivalent to every other norm.*

(ii) A vector space X is finite-dimensional if and only if every linear functional on it is continuous.

(iii) A normed space X is finite-dimensional if and only if its closed unit ball is compact in the norm topology.

The topology induced by a norm is fairly strong, as seen by the above theorem. In some cases it is useful to weaken the topology on a normed space. We call a vector space X equipped with a topology a *topological vector space* (TVS) if it is a Hausdorff topological space satisfying:

(i) The addition map $+\colon X \times X \to X$ given by $(x, y) \mapsto x + y$ is continuous with respect to the product topology on $X \times X$.

(ii) The scalar multiplication map $\cdot\colon \mathbb{F} \times X \to X$ given by $(\lambda, x) \mapsto \lambda x$ is continuous with respect to the product topology on $\mathbb{F} \times X$.

If X is a finite dimensional TVS, there always exists a norm generating its topology, which is why the study of topological vector spaces as objects in themselves only really occurs in the infinite dimensional setting. If X and Y are topological vector spaces, it no longer makes sense to talk about bounded linear operators between them, but it still makes sense to talk about continuous linear operators between them. As in the normed space, we denote by $L(X;Y)$ the space of all continuous linear operators by $L(X;Y)$. There are various ways to put topologies on $L(X;Y)$ to make it into a TVS, but this is out of the scope of this thesis, and we refer the interested reader to [62]. It is clear by the definition of a TVS, that if (x_n) and (y_n) are sequences in some TVS X, and $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$. The following shows that something similar is true for nets:

Lemma 2.4.2. *If $(x_\alpha)_{\alpha \in A}$ and $(y_\beta)_{\beta \in B}$ are nets in a topological vector space V, converging to $x \in V$ and $y \in V$ respectively, then we can construct subnets with the same indexing set $(x_\gamma)_{\gamma \in \Gamma}$ and $(y_\gamma)_{\gamma \in \Gamma}$ such that $x_\gamma + y_\gamma$ converges to $x + y$.*

Proof. Let $\Gamma = A \times B$ be equipped with the ordering $(\alpha, \beta) \le (\alpha', \beta')$ if and only if $\alpha \le \alpha'$ in A and $\beta \le \beta'$ in B. It is straightforward to check that this makes Γ a directed set. Now define $\varphi_1 \colon \Gamma \to A$ by $\varphi_1(\alpha, \beta) = \alpha$, and $\varphi_2 \colon \Gamma \to B$ by $\varphi_2(\alpha, \beta) = \beta$. It is a routine calculation to check that these cofinal maps make the nets $(x_{(\alpha,\beta)})$ and $(y_{(\alpha,\beta)})$ subnets of (x_α) and (y_β) respectively. Thus $\lim_{(\alpha,\beta)} x_{(\alpha,\beta)} = x$ and $\lim_{(\alpha,\beta)} y_{(\alpha,\beta)} = y$. This also shows for any neighbourhood of the form $U \times W$ of the point $(x, y) \in V \times V$ there exists an $(\alpha', \beta') \in \Gamma$ so for all $(\alpha, \beta) \ge (\alpha', \beta')$ we have $(x_{(\alpha,\beta)}, y_{(\alpha,\beta)}) \in U \times V$. Thus $\lim_{(\alpha,\beta)}(x_{(\alpha,\beta)}, y_{(\alpha,\beta)}) = (x, y)$, and because addition is continuous it follows that $\lim_{(\alpha,\beta)} x_{(\alpha,\beta)} + y_{(\alpha,\beta)} = x + y$. □

Perhaps the most important examples of topological vector spaces not arising from a norm, and the only ones we will need in this thesis, are the *weak* and *weak** topologies on a normed space X. The weak topology on X is simply an abbreviation for the X^*-weak topology, as defined in section 2.1, and the weak* topology on X^* is likewise an abbreviation for the $J(X)$-weak topology, with J as in (2.4.1). Convergence in these spaces is pointwise, by which we mean that a net $(x_\alpha) \subseteq X$ converges weakly to $x \in X$ if and only if $f(x_\alpha) \to f(x)$ for every $f \in X^*$, and a net $(f_\alpha) \subseteq X^*$ converges to f in the weak* topology on X^* if and only if $f_\alpha(x) \to f(x)$ for all $x \in X$. These topologies will be particular useful to us, in light of the following famous theorem, a proof of which can be found in [62]:

Theorem 2.4.3 (Banach-Alaoglu Theorem). *If X is a normed space, then B_{X^*} is closed in the weak* topology on X^*.*

2.5 Convex Geometry

There is a close relationship between the horofunction compactification of normed spaces, and the *convex geometry* of the closed dual ball. For the convenience of the reader we shall briefly recall the relevant facts about convex geometry in this section. We refer the reader to [61] for a full account. Let V be an $\mathbb{R}$ vector space. We say that a set $C \subseteq V$ is *convex*, if the line segment between any two points in C is contained in C. Precisely, C is convex if, for all $x, y \in C$, and all $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in C$. We use $[x, y]$ to denote such a line segment. Similarly, we shall use (x, y) to denote the open line segment, which is the collection of points $\lambda x + (1-\lambda)y$, for $\lambda \in (0,1)$. We call a set $A \subseteq V$ an *affine subspace* of V if the line passing between any two distinct points in A is also contained in A, by which we mean, if $x, y \in A$, then we have $\{\lambda x + (1-\lambda)y \colon \lambda \in \mathbb{R}\} \subseteq A$. It is clear that an affine subspace must be convex. It is often useful to make use of the fact that an affine subspace is just a translated vector subspace of V. That is: A is an affine subspace of V if and only if there exists a subspace of V, say W, and an $x \in V$, such that $A = x + W$. We collect some useful facts about convex sets and affine subspaces in the following proposition:

Proposition 2.5.1. *Let V and W be $\mathbb{R}$ vector spaces, let $T \colon V \to W$ be a linear operator $\{C_i\}_{i\in I}$ and $\{A_i\}_{i\in I}$ be arbitrary collections of convex subsets and affine subspaces of V respectively, then:*

(i) The sets $\bigcap_{i\in I} C_i$ and $\bigcap_{i\in I} A_i$ are convex and affine, respectively.

(ii) If $C \subseteq V$ and $C' \subset W$ are convex, then both $T(C)$ and $T^{-1}(C')$ are convex. The same holds if we replace convex with affine.

(iii) If V is a topological vector space, then both the interior and closure of a convex set is again a convex set, and similarly for an affine subspace.

We call an affine subspace, A, an *affine hyperplane*, if $A = x + \ker(f)$ for some non-zero linear functional f acting on V. If V is a topological vector space, all the closed affine hyperplanes are precisely those hyperplanes A of the form $A = x + \ker(f)$ for $0 \neq f \in V^*$. To any hyperplane $A = x + \ker(f)$ we can define the associated *closed half-space* in V, which is the set $\{v \in V : f(v) \leq f(x)\}$, and the associated open half-space, which is the set $\{v \in V : f(v) > f(x)\}$. Thus each hyperplane splits V into two disjoint convex sets. If C is a non-empty convex set, we say that H is a *supporting hyperplane* for C if $H \cap \bar{C} \neq \emptyset$, and C is contained in the closed half-space generated by H. If $C \subseteq V$ is a convex set, and $x_0 \in C$, we say that $f \in V^*$ is a *supporting functional* of C at the point x_0 if $f \neq 0$, and $f(x) \leq f(x_0)$ for all $x \in C$. Supporting functionals are closely related to supporting hyperplanes. Indeed, if f is a supporting functional for C at x_0, and $f(x_0) > \inf_{x \in C} f(x)$, then $H = f^{-1}(f(x_0))$ is a supporting hyperplane for C. If V is a complex vector space, we define the supporting functional of a convex set $C \subseteq V$ similarly, except we say f supports C at $x_0 \in C$ if $\operatorname{Re} f(x_0) \geq \operatorname{Re} f(x)$ for all $x \in C$.

Given any non-empty set $S \subseteq V$, there are useful ways to generate a convex or affine set containing S. The *convex hull* of S, denoted by $\operatorname{conv}(S)$ is defined as the smallest convex set containing S, and similarly the *affine hull* of S is defined to be the smallest affine set containing S, which we denote by $\operatorname{aff}(S)$. It follows immediately from part *(ii)* of Proposition 2.5.1 that $\operatorname{conv}(S)$ is precisely the intersection of all convex sets containing S, and likewise that $\operatorname{aff}(S)$ is the intersection of all affine subspaces containing S.

Important information about convex sets is encoded in their so-called facial structure. If C is a convex subset of an $\mathbb{R}$ vector space V, we call a convex set $F \subseteq C$ a face of C, if for every $[x,y] \subseteq C$ such that $(x,y) \cap F \neq \emptyset$, $[x,y] \cap C \subseteq F$. Each convex set has itself and the empty set as trivial faces. If $F \subseteq C$ is a face that is not trivial it is called a *proper face*. If the singleton $\{x\}$ is a face of C, it is called an *extreme point* of C. If there exists a linear function f on V and an $a \in \mathbb{R}$ such that $F = f^{-1}(a) \cap C$, and $f(x) \leq f(a)$ for all $x \in C$, then F is a face of C, and it is known as an *exposed face*. If V is finite dimensional we say that a face F is a *facet* if $\dim(F) = \dim(C) - 1$, where we use dim to denote the dimension of the affine span of a set. We also say that C is *polyhedral* if it is bounded and is precisely the intersection of finitely many half-planes

Important examples of convex sets are *cones*. Let V be a real vector space. We call a set $C \subseteq V$ a cone if it is convex and $\lambda C \subseteq C$ for all $\lambda \geq 0$. We call C *proper* if $C \cap -C = \{0\}$. Some authors use the term cone and proper cone interchangeably, and refer to a cone that isn't proper as a *wedge*. If V is a topological vector space, the *dual cone* of a closed cone C is defined by

$$C^* = \{\varphi \in V^* : \varphi(x) \geq 0 \text{ for all } x \in C, \}$$

whereas the dual cone of an open cone Ω is defined as

$$\Omega^* = \{\varphi \in V^* : \varphi(x) > 0 \text{ for all } x \in \Omega\}.$$

The *automorphism group* of such an Ω is defined by

$$\mathrm{Aut}(\Omega) = \{T \in L(V;V) : T \text{ is an isomorphism and } T(\Omega) = \Omega\}.$$

If V is a finite dimensional real Hilbert space and $\Omega \subseteq V$ is an open cone, we can make the identification

$$\Omega^* = \{y \in V : \langle x, y \rangle > 0 \text{ for all } x \in \Omega\}.$$

We say that such an Ω is *self-dual* if $\Omega = \Omega*$. An open cone Ω is called a *symmetric cone* if it is self-dual, and $\mathrm{Aut}(\Omega)$ acts transitively on Ω (by which we mean for all $x, y \in \Omega$ there exists a $T \in \mathrm{Aut}(\Omega)$ such that $Tx = y$).

As cones are convex, the notion of faces defined above applies to them. A face F is called an exposed face if there exists a $\varphi \in C^*$ such that $F = C \cap \varphi^{-1}(0)$, which is equivalent to the definition for exposed faces of general convex sets. Similarly, we say that C is *polyhedral* if there exists $\varphi_1, \ldots, \varphi_n \in V$ such that

$$C = \bigcap_{i=1}^{n} \varphi_i^{-1}([0, \infty)).$$

The intersection of cones is also a cone, which allows us to define the *conical hull* of any non-empty $S \subseteq V$ as the intersection of all cones containing S. There are useful analytic descriptions of the hulls we have introduced, which we collect below:

Proposition 2.5.2. *Let $S \subseteq V$ be an arbitrary subset of a real vector space. Then*

(i)

$$\mathrm{conv}(S) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in S,\ \lambda_i \geq 0,\ n \in \mathbb{N},\ \textit{and}\ \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

(ii)

$$\mathrm{aff}(S) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in S,\ \lambda_i \in \mathbb{R},\ n \in \mathbb{N},\ \textit{and}\ \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

(iii)

$$\mathrm{cone}(S) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in S,\ \lambda_i \geq 0,\ \textit{and}\ n \in \mathbb{N} \right\}.$$

Sums of the form found in (i) and (ii) above are referred to as, respectively, a convex and affine combination of the elements x_i. There are various different ways of defining the interior of a subset of a vector space with respect to its convex structure, which we collect in the following definition:

Definition 2.5.1. Let V be an $\mathbb{R}$ vector space, and X an $\mathbb{R}$ topological vector space. We then define:

(i) We say $x \in V$ is in the *algebraic relative interior* of $A \subseteq V$, if for every straight line L that passes through x and is contained in $\mathrm{aff}(A)$, there exists an open segment $(a, b) \subseteq L$, such that $x \in (a, b)$ and $(a, b) \subseteq A$.

(ii) We say that $x \in X$ is in the *relative interior* of $A \subseteq X$, if x is in the topological interior of A, considered as a subset of $\mathrm{aff}(A)$ equipped with the subspace topology. We denote the collection of all such points as $\mathrm{ri}(A)$.

(iii) We define the *quasi-relative interior* of $A \subseteq X$ by, if $\overline{\mathrm{cone}}(B)$ denotes the closure in X of the conic hull of $B \subseteq X$,

$$\mathrm{qri}(A) = \{x \in A : \overline{\mathrm{cone}}(A - x) \text{ is a subspace of } X\}.$$

If X is a finite dimensional normed space, and $C \subseteq X$ is convex, then all three of the notions of relative interior coincide. We are particularly interested in these notions of relative interior in light of the following theorem, a proof of which can be found in [61, Theorem 18.2],

Theorem 2.5.3. *If C is a compact convex subset of a finite dimensional normed space X, then $\mathrm{ri}(F_1) \cap \mathrm{ri}(F_2) = \emptyset$ for all distinct faces $F_1, F_2 \subseteq C$, and*

$$C = \bigcup_{\substack{F \text{ face} \\ \text{of } C}} \mathrm{ri}(F).$$

2.6 Smoothness and Convexity of Normed Spaces

The topic of convexity and smoothness of normed spaces is a rich area, and in this section we only introduce the bare necessities that we will need later on. The interested reader is referred to Chapter 5 of [50] and Chapters 8 and 9 of [19] for further reading. We say that a normed space X is *strictly convex* if, for any $x, y \in S_X$ satisfying $\|x + y\| = 2$, then $x = y$. Strictly convex spaces are also known as *rotund* spaces. There is a stronger notion of convexity, and we say that a normed space X is *uniformly convex* if for each $0 < \varepsilon \leq 2$ there exists a $\delta > 0$ so for all $x, y \in S_X$ with $\|x - y\| \geq \varepsilon$ we have $\|\frac{x+y}{2}\| \leq 1 - \delta$. Every uniformly convex space is strictly convex [50, Proposition 5.2.6] and strictly convex finite dimensional normed spaces are uniformly convex [50, Proposition 5.2.14]. In the 1930's, Milman [52] and Pettis [58] independently proved the following theorem, which bears their name:

Theorem 2.6.1. *(The Milman-Pettis Theorem) Every uniformly convex Banach space is reflexive.*

Proof. See [50, Theorem 5.2.15] for a more modern proof. □

The notion of supporting hyperplanes and supporting functionals introduced in Section 2.5 play an important role in the study of convexity and smoothness of normed spaces. Of particular importance are the notion of norming functionals. As a consequence of the Hahn-Banach extension theorem, if X is a normed space, for any $x \in X$, there exists a $f \in X^*$ such that $\|f\|_* = 1$, and $f(x) = \|x\|$. We call f a *norming functional* for x. We denote by x^* a norming functional for x, which in general does not need to be unique. For any $x \in S_X$, its norming functionals are precisely the supporting functionals of B_X at x of norm one. Later we shall need the following fact about strictly convex spaces and norming functionals:

Lemma 2.6.2. *If X is a strictly convex space, and x^* and y^* are norming functionals for $x, y \in X \backslash \{0\}$ with $\|x\| = \|y\|$, and $x^* = y^*$, then $x = y$.*

Proof. Without loss of generality we might as well assume that $x, y \in S_X$. Assume that $x \neq y$, so by definition of a strictly convex space, $\|x + y\| < 2$. As $\|x^*\| = 1$ it follows that $\|x + y\| \geq |x^*(x + y)| = |x^*x + y^*y| = \|x\| + \|y\|$, from which we deduce that $\|x + y\| = \|x\| + \|y\| = 2$, contradicting strict convexity. □

We will also need the following fact about uniformly convex normed spaces:

Lemma 2.6.3. *If X is a uniformly convex normed space, and $(x_\alpha) \subseteq S_X$ and $(y_\alpha) \subseteq S_X$ are nets indexed by the same directed set, and $\|x_\alpha + y_\alpha\| \to 2$, then $\|x_\alpha - y_\alpha\| \to 0$.*

Proof. Assume by way of contradiction that $\|x_\alpha - y_\alpha\| \not\to 0$. There must thus exist a $\varepsilon > 0$ and subnets (x_β) and (y_β) such that $\|x_\beta - y_\beta\| \geq \varepsilon$ for all β. As X is uniformly convex, there exists a $\delta > 0$ such that $\|x_\beta + y_\beta\| \leq 2 - \delta$ for all β, a contradiction. □

We now turn our attention to the notion of smoothness. There are different definitions of a smooth normed space in the literature, but as the word smooth is often associated with differentiability, we shall use the definition involving Gateaux derivatives. For a function $f\colon X \to \mathbb{R}$ we say that f is *Gateaux differentiable* at $x \in X$ if

$$\lim_{t\to 0} \frac{f(x+ty) - f(x)}{t}$$

exists for all $y \in X$. This quantity is then called the *directional derivative* of f at x in the direction of y. If f is Gateaux differentiable at x, we use $\nabla f_x\colon X \to \mathbb{R}$ to denote the *Gateaux derivative* of f at x, which is defined, for all $y \in X$, by:

$$\nabla f_x(y) = \lim_{t\to 0} \frac{f(x+ty) - f(x)}{t}.$$

We use the notation $\nabla\|x\|$ to denote the Gateaux derivative of $\|\cdot\|$ at the point $x \in X$. Note that for any $0 \neq x \in X$, $y \in X$, and $t \in \mathbb{R}\backslash\{0\}$,

$$\left| \frac{\|x+ty\| - \|x\|}{t} \right| \leq \|y\|,$$

and if we choose $y = x$ the inequality becomes an equality, meaning that $\nabla\|x\| = x^*$. We say that X is a *smooth* normed space if its norm $\|\cdot\|$ is Gateaux differentiable on $X\backslash\{0\}$. This condition is equivalent to requiring that, for each $x \in X\backslash\{0\}$, there exists a unique norming functional $x^* \in S_{X^*}$ [19, Lemma 8.4], characterised by $x^*(x) = \|x\|$. Thus, if X is a smooth normed space, then for any non-zero $x \in X$ and $c > 0$, $(cx)^* = x^*$. Indeed:

$(cx)^*(cx) = c\|x\|$, meaning that $(cx)^*x = \|x\|$, which by uniqueness enforces the conclusion. This characterisation of smoothness allows us to prove the following:

Proposition 2.6.4. *A normed space is strictly convex if its dual space is smooth, and it is smooth if its dual space is strictly convex.*

Proof. Suppose that X is not smooth. By the above characterisation of smoothness, there must then exist some $x \in S_X$ with two norming functionals, say $x_1^*, x_2^* \in S_{X^*}$ such that $x_1^*x = x_2^*x = 1$. As B_{X^*} is convex, $\|x_1^* + x_2^*\|_* \leq 2$, and by the definition of the dual norm $\|x_1^* + x_2^*\|_* \geq (x_1^* + x_2^*)(x) = 2$, so $\|x_1^* + x_2^*\|_* = 2$, but $x_1^* \neq x_2^*$, so X^* is not strictly convex. Next, suppose that X is not strictly convex. There thus exists $x, y \in S_X$ with $\|x + y\| = 2$, but $x \neq y$. Thus $z = \frac{1}{2}(x + y) \in S_X$. As $z^* \in S_{X^*}$ and $x, y \in S_X$ we must have $z^*x \leq 1$ and $z^*y \leq 1$, so the only way that $z^*z = 1$ is if z^* is also a norming functional for x and y. If J denotes the natural embedding of X into X^{**} as defined in Section 2.4, it thus follows that $J(x)z^* = 1$ and $J(y)z^* = 1$, but as $x \neq y$, $J(x) \neq J(y)$, so z^* has two distinct norming functionals in X^{**}. Therefore X^* is not smooth. □

Similarly as in the case of smoothness, there are different definitions of uniform smoothness of normed spaces in the literature, and we choose the one based on a differentiability criterion. We say that a function $f\colon X \to \mathbb{R}$ is *Fréchet differentiable* at a point $x \in X$ if there exists a linear functional $Df_x \in X^*$ such that

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - Df_x(h)|}{\|h\|} = 0.$$

We call the functional Df_x the Fréchet derivative of f at x. We denote by $D\|x\|$ the Fréchet derivative of $\|\cdot\|$ at x. If $f\colon X \to \mathbb{R}$ is Fréchet differentiable at $x \in X$ then it is also Gateaux differentiable, and $Df_x = \nabla f_x$. We say that $f\colon X \to \mathbb{R}$ is *uniformly Fréchet differentiable* on an open set $U \subseteq X$ if f is Fréchet differentiable at every point $x \in U$, and the map $x \mapsto Df_x$ is a uniformly continuous map. We say that a normed space X is *uniformly smooth* if its norm is uniformly Fréchet differentiable on $X \backslash \{0\}$. In 1940, Šmulian [65] proved an analogue to close relationship between convexity and smoothness:

Theorem 2.6.5. *A normed space is uniformly convex if and only if its dual space is uniformly smooth, and it is uniformly smooth if and only if its dual space is uniformly convex.*

Proof. See [50, Theorem 5.5.12] for a modern proof. □

As a corollary to the above theorem and the Milman-Pettis Theorem (Theorem 2.6.1) Šmulian [65] proved an analogue to the Milman-Pettis Theorem:

Theorem 2.6.6. *Every uniformly smooth Banach space is reflexive.*

To end this section, we prove a result about the facial structure of the dual unit balls of strictly convex and uniformly smooth normed spaces.

Proposition 2.6.7. *If X is a uniformly smooth and strictly convex Banach space, then the only proper faces of B_{X^*} are extreme points $\{f\} \subset S_{X^*}$, and every $f \in S_{X^*}$ is an extreme point.*

Proof. Theorem 2.6.6 means that X is reflexive, and Theorems 2.6.4 and 2.6.5 mean that X^* is smooth and uniformly convex. Thus for any $f \in S_{X^*}$, there exists a unique $x \in S_X$ such that $f(x) = 1$. Lemma 2.6.2 means that for any $y \neq x$ with $y \in B_X$, $f(y) < 1$. Thus $\{f\}$ is exposed by $x \in X^{**}$, so it is a face. Conversely, suppose that $F \subseteq S_{X^*}$ is a proper face of B_{X^*}, and $f, g \in F$. Let x be the unique norming functional for f as above. As F is convex, we must then have $\frac{1}{2}(f+g) \in S_{X^*}$, but

$$\|f + g\|_* \leq f(x) + g(x) = 1 + g(x),$$

so for $\frac{1}{2}(f+g)$ to be in S_{X^*} we must have $g(x) = 1$, which by Lemma 2.6.2 means $f = g$. □

James Clarkson proved, in his 1936 paper [14] introducing the notion, that $L^p(X, \mathcal{F}, \mu)$ is uniformly convex for any measure space $(X, \mathcal{F}, \mu)$ and any $1 < p < \infty$. This, in combination with Theorem 2.6.5, and the fact that the dual of $L^p(X, \mathcal{F}, \mu)$ is $L^q(X, \mathcal{F}, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, shows that $L^p(X, \mathcal{F}, \mu)$ is also uniformly smooth. As a corollary, all ℓ^p sequence spaces, for $1 < p < \infty$, are also uniformly convex and uniformly smooth. Another classic example of uniformly smooth and uniformly convex normed spaces are Hilbert spaces:

Example 2.6.8. Let $\mathcal{H}$ be an arbitrary Hilbert space. Fix arbitrary $0 < \varepsilon \leq 2$ and $x, y \in S_{\mathcal{H}}$. The parallelogram law means that

$$\|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x - y\|^2,$$

so if $\|x - y\| \geq \varepsilon$,

$$\frac{1}{2}\|x + y\| \leq \sqrt{1 - \frac{\varepsilon^2}{4}} < 1 - \frac{\varepsilon^2}{4}.$$

Thus by definition $\mathcal{H}$ is uniformly convex. The Riesz-Representation Theorem means that $\mathcal{H}^* = \mathcal{H}$, so the dual of $\mathcal{H}$ is also uniformly convex, which by Theorem 2.6.5 means that $\mathcal{H}$ is uniformly smooth.

2.7 Order-unit Spaces

Let V be a real vector space. We say that $(V, \leq)$ is an *ordered vector space* if $\leq$ is a partial ordering on V satisfying the following two properties:

(i) If $x \leq y$, then $x + z \leq y + z$ for all $x, y, z \in V$.

(ii) If $x \leq y$, then $\lambda x \leq \lambda y$ for all $x, y \in V$ and $\lambda \geq 0$.

A classical example of an ordered vector space is $(\mathbb{R}^n, \leq)$, where we define $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_2)$ if $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$ (where $\leq$ on $\mathbb{R}$ is the usual ordering). We say that $(V, \leq)$ is *Archimedean* if, for any $x, y \in V$ such that $x \leq \delta y$ for all $\delta > 0$, then $x \leq 0$. A proper cone $C \subseteq V$ induces a partial order $\leq$ on V, where $x \leq y$ if and only if $y - x \in C$. It follows almost immediately from the definition of a cone that this partial ordering in fact makes $(V, \leq)$ an ordered vector space (where we need the fact that C is proper to ensure $\leq$ is antisymmetric). In fact, the converse is also true:

Lemma 2.7.1. *Let V be an $\mathbb{R}$ vector space. The set of partial orderings on V that turn V into an ordered vector space is in one-to-one correspondence with the collection of proper cones, C, on V.*

Proof. As we have just seen, given a proper cone C, we can associate a partial order $\leq_C$ on V by setting $x \leq_C y$ if and only if $y - x \in C$. Conversely, given a partial order $\leq$ on V, which makes $(V, \leq)$ an ordered vector space, let us define the set $C_{\leq} = \{x \in V : 0 \leq V\}$. We claim that $C_{\leq}$ is a cone. Indeed, axiom (ii) of ordered vector spaces immediately means that $\lambda x \in C$ for all $x \in C_{\leq}$ and $\lambda \geq 0$. Now let $x, y \in C_{\leq}$, and $\lambda \in (0, 1)$. We have just shown that $\lambda x \in C_{\leq}$ and $(1 - \lambda)y \in C_{\leq}$, and axiom (i) of ordered vector spaces thus means that $\lambda x \leq \lambda x + (1 - \lambda)y$, and the transitivity of $\leq$ thus means that $\lambda x \leq \lambda x + (1 - \lambda)y \in C_{\leq}$, making $C_{\leq}$ convex. Thus indeed $C_{\leq}$ is a cone. To see that it is proper, suppose that $x \in C_{\leq} \cap -C_{\leq}$, meaning that $0 \leq x$ and $0 \leq -x$, which means $x \leq 0$, so by the antisymmetry of $\leq$, we must have $x = 0$. Furthermore $x \leq_{C_{\leq}} y$ if and only if $y - x \in C_{\leq}$, if and only if $0 \leq y - x$, if and only if $x \leq y$, which establishes a unique correspondence between cones and partial orderings turning V into an ordered vector space. □

The cone $C_{\leq}$ in the above proof is often referred to as the positive cone of $(V, \leq)$, and is denoted by V_+. In light of the above lemma we often refer to an ordered vector space $(V, \leq)$ as the vector space V with positive cone V_+. We say V_+ is an *Archimedean cone* if the order associated with it is Archimedean. Let us now fix some $(V, \leq)$ with associated positive cone V_+. We call an element $e \in V_+$ an *order-unit* if, for every $x \in V$, there exists a $\lambda \geq 0$ such that $-\lambda e \leq x \leq \lambda e$. If V_+ is Archimedean, and such an e exists, we can define the *order-unit norm* $\|\cdot\|_e$ on V by

$$\|x\|_e = \inf\{\lambda \geq 0 : -\lambda e \leq x \leq \lambda e\}.$$

We should note that even if V_+ is not Archimedean, $\|\cdot\|_e$ is still well-defined and is a semi-norm, but it may fail to be a full norm. If V_+ is Archimedean, and an order-unit e

exists, we call the triple (V, V_+, e) an *order-unit space*, and e the *distinguished order-unit*. If (V, V_+, e) is an order-unit space, we denote by V_+° the interior of V_+ with respect to the order-unit norm. We have the following useful fact:

Lemma 2.7.2. *In an order-unit space (V, V_+, e), the positive cone V_+ is closed in the norm topology induced by $\|\cdot\|_e$, and $e \in V_+^\circ$.*

Proof. Let $(x_n) \subseteq V_+$ converge to some $x \in V$ in $\|\cdot\|_e$. Thus, for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ so that for all $n \geq N$, $-\varepsilon e \leq x - x_n \leq \varepsilon e$, but as each $x_n \in V_+$, this means that $-\varepsilon e \leq x$ for all $\varepsilon > 0$, which means $0 \leq x$ as V_+ is Archimedean, meaning that $x \in V_+$. If $x \in B(e, \frac{1}{2})$, then $\frac{1}{2}e \leq x$, so indeed $e \in V_+^\circ$. □

We say that $y \in V_+$ *dominates* $x \in V$ if there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha y \leq x \leq \beta y$. We say $x \sim y$ if x dominates y, and y dominates x. This partitions V_+ into equivalence classes under $\sim$, which we call the *parts* of V_+. We can define the so called *gauge function*, $M \colon V \times V_+ \to \mathbb{R} \cup \{\infty\}$ and *reverse gauge function*, $m \colon V \times V_+ \to \mathbb{R} \cup \{-\infty\}$ by

$$M(x/y) = \inf\{\lambda \in \mathbb{R} : x \leq \lambda y\}, \text{ and } \quad m(x/y) = \sup\{\lambda \in \mathbb{R} : \lambda y \leq x\}, \tag{2.7.1}$$

where we use the standard convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

Corollary 2.7.3. *Lemma 2.7.2 immediately implies that if $(x_\alpha), (y_\alpha)$ are two nets in V converging to x and y respectively in norm, and $x_\alpha \leq y_\alpha$ for all α, then $x \leq y$.*

It would be useful if the gauge functions were real valued functions, which is why we prove the following.

Lemma 2.7.4. *If (V, V_+, e) is an order-unit space, then every $y \in V_+^\circ$ dominates every $x \in V$. Furthermore, if $x, y \in V_+\backslash\{0\}$, then $x \sim y$ if and only if there exists $0 < \alpha \leq \beta$ such that $\alpha x \leq y \leq \beta x$*

Proof. Fix some $y \in V_+^\circ$ and $x \in V$. Note that for all $n \in \mathbb{N}$, $\|y - (y - \frac{1}{n}x)\|_e = \frac{1}{n}\|x\|_e \to 0$ and $\|y - (y + \frac{1}{n}x)\|_e = \frac{1}{n}\|x\|_e \to 0$. As y is an interior point of V_+, this must mean that for all n large enough, $y \pm \frac{1}{n}x \in V_+$, meaning $x \leq ny$, and $-ny \leq x$. Now let us fix $x, y \in V_+\backslash\{0\}$ with $x \sim y$. By assumption we know that $0 \leq y \leq \alpha x$ for some $\alpha \geq 0$, because $x \in V_+$, so if $\alpha < 0$, $-x \in V_+$, a contradiction. In fact, $\alpha > 0$, because else anti-symmetry means $y = 0$. Similarly, we know that $x \leq \beta y$ for $\beta > 0$. Thus $x \leq \beta\alpha x$, meaning $\beta^{-1} \leq \alpha$, as V_+ is a proper cone, meaning $\beta^{-1}x \leq y \leq \alpha x$. Conversely, suppose $x, y \in V_+\backslash\{0\}$, and that there exists $0 < \alpha \leq \beta$ such that $\alpha x \leq y \leq \beta x$. So x dominates y, but as $\alpha > 0$ clearly $\beta^{-1}y \leq x \leq \alpha y$, so indeed $x \sim y$. □

This immediately implies that both the gauge functions are real-valued functions when restricted to $V_+^\circ \times V$. In some sense the reverse gauge function is superfluous, because, for any $x, y \in V_+^\circ$, if $\lambda y \leq x$ for some $\lambda > 0$, then $y \leq \lambda^{-1}x$, so $m(x/y) = M(y/x)^{-1}$.

If (V, V_+, e) is an order unit space, we shall use V^* to denote its topological dual, equipped with the standard dual norm $\|\cdot\|_*$. It is a simple calculation to verify that the closed dual cone V_+^* is a proper cone, and we call its elements *positive functionals*. The Hahn-Banach theorem guarantees that V_+^* is non-empty, so V^* inherits an order from (V, V_+).

Lemma 2.7.5. *For any $x, y \in V$, $x \leq y$ if and only if $\varphi(x) \leq \varphi(y)$ for all $\varphi \in V_+^*$, and a functional $\varphi \in V^*$ is positive if and only if $\varphi(e) = \|\varphi\|_*$.*

Proof. It is clear that if $x \leq y$ then $\varphi(x) \leq \varphi(y)$ for all $\varphi \in V_+^*$. Conversely, suppose by way of contradiction that $\varphi(x) \leq \varphi(y)$ for all $\varphi \in V_+^*$ and some $x, y \in V$, but $x \not\leq y$, meaning $y - x \notin V_+$. The Hahn-Banach separation theorem, along with the fact that V_+ is closed and convex and $0 \in V_+$ means that there exists a $f \in V^*$ such that $f(z) \geq 0$ for all $z \in V_+$, and $f(z) < 0$ for all $z \notin V_+$. Thus $f \in V_+^*$, but $f(y - x) < 0$, a contradiction. To prove the second statement, suppose first that $\varphi \in V_+^*$. For any $x \in \mathbf{B}_V$, $-e \leq x \leq e$, which means, because φ is positive, $|\varphi(x)| \leq \varphi(e)$, showing that indeed $\varphi(e) = \|\varphi\|_*$. Conversely, suppose that $\varphi(e) = \|\varphi\|_*$. Fix some arbitrary $x \in V_+$. Then $0 \leq x \leq \|x\|_e e$, from which we deduce that $0 \leq \|x\|_e e - x \leq \|x\|_e e$. Thus $\varphi(\|x\|_e e - x) \leq \|\varphi\|_* \|x\|_e$, from which we deduce that $0 \leq x$. □

Of particular importance in the study of order-unit spaces are the so called states. We say $\varphi \in V_+^*$ is a *state* if $\varphi(e) = 1$, and we denote by $S(V)$ the set of all states.

Lemma 2.7.6. *For any order-unit space (V, V_+, e), and $x \in V$,*

$$\|x\|_e = \sup\{\varphi(x) : \varphi \in S(V) \cup -S(V)\},$$

and $S(V)$ is a convex weak compact subset of V^*.*

Proof. For a fixed $x \in V$, let us define $\xi = \sup\{\varphi(x) : \varphi \in S(V) \cup -S(V)\}$. We know that $|\varphi(x)| \leq \|\varphi\|_* \|x\|_e$ for every $\varphi \in V_*$. Lemma 2.7.5 means that $\|\varphi\|_* = 1$ for all states φ, so taking the supremum on both sides of the inequality shows that $\xi \leq \|x\|_e$. If $\xi e - x, \xi e + x \in V_+$, then by the definition of the order-unit norm, $\|x\|_e \leq \xi$, so we'd be done. Suppose by way of contradiction that $\xi e - x \notin V_+$. Once again, as in the proof of Lemma 2.7.5, the Hahn-Banach separation theorem means that there exists a non-zero $f \in V_+^*$ such that $f(z) < 0$ for all $z \notin V_+$. Furthermore, Lemma 2.7.5 means that $\varphi = f/\|f\|_* \in S(V)$. Thus, $\varphi(\xi e - x) < 0$, meaning that $\xi < \varphi(x)$, contradicting the definition of ξ, so indeed $\xi e - x \in V_+$. A similar argument shows that $\xi e + x \in V_+$, meaning that indeed $\xi = \|x\|_e$.

Now, for any $\varphi, \phi \in S(V)$, and $\lambda \in (0, 1)$, $[\lambda\varphi + (1 - \lambda)\phi](e) = 1$, so indeed $S(V)$ is convex. Furthermore, for any net $(\varphi_\alpha) \subseteq S(V)$, $\varphi_\alpha(e) = 1$, so if φ_α converges in the weak* topology to some $\varphi \in V$, we must have $\varphi(e) = 1$. Similarly, if $x \in V_+$, $\varphi_\alpha(x) \geq 0$ for all α, so $\varphi \in V_+^*$. Thus $S(V)$ is weak* closed, and is also norm bounded by Lemma 2.7.5. The Banach-Alaoglu Theorem thus means that $S(V)$ is weak* compact. □

In light of the above lemma, the Krein-Milman means that $S(V)$ is the weak* closure of the convex hull of its extreme points, and we call these extreme points the *pure states*. If (V, V_+, e) and (W, W_+, w) are two order-unit spaces, their product is a natural order-unit space:

Lemma 2.7.7. *Let (V, V_+, e) and (W, W_+, w) be two order-unit spaces. The triple $(V \times W, V_+ \times W_+, (e, w))$ is also an order-unit space, and moreover*

$$M((x_1, y_1)/(x_2, y_2)) = \max\{M(x_1/x_2), M(y_1/y_2)\}$$

for all $(x_1, y_1), (x_2, y_2) \in V_+^\circ \times W_+^\circ$.

Proof. It is clear from the definition of products of vector spaces that $V_+ \times W_+$ is a proper cone in $V \times W$, so indeed $(V \times W, V_+ \times W_+)$ is an ordered vector space, where $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ in V and $y_1 \leq y_2$ in W. Furthermore, if $-\lambda e \leq x \leq \lambda e$ for some $x \in V$ and $\lambda \geq 0$, and $-\gamma w \leq y \leq \gamma w$ for some $y \in W$ and $\gamma \geq 0$, then $-\max\{\lambda, \gamma\}(e, w) \leq (x, y) \leq \max\{\lambda, \gamma\}(e, w)$, proving that $(V \times W, V_+ \times W_+, (e, w))$ is an order-unit space, and also that $\|(x, y)\|_{(e,w)} = \max\{\|x\|_e, \|y\|_w\}$. The exact same logic shows that

$$M((x_1, y_1)/(x_2, y_2)) = \max\{M(x_1/x_2), M(y_1/y_2)\}$$

for all $(x_1, y_1), (x_2, y_2) \in V_+^\circ \times W_+^\circ$. □

We are now in a position to prove some useful facts about gauge functions.

Lemma 2.7.8. *Let (V, V_+, e) be an order-unit space, and consider $M \colon V \times V_+^\circ \to \mathbb{R}$ the restricted gauge function as in* (2.7.1).

(i)

$$M(x/y) = \sup_{\varphi \in S(V)} \frac{\varphi(x)}{\varphi(y)},$$

and the supremum is attained.

(ii) *$M \colon V \times V_+^\circ \to \mathbb{R}$ is continuous with respect to the product topology of the topology generated by $\|\cdot\|_e$ on each factor.*

(iii) *If $T \colon V \to V$ is a linear injection satisfying $T(V_+) \subseteq V_+$, then $M(x/y) = M(Tx/Ty)$.*

(iv) *For any $x, y \in V \times V_+^\circ$, $\alpha, \beta > 0$, $M(\alpha x/\beta y) = \alpha\beta^{-1} M(x/y)$*

Proof. *(i)* As $e, y \in V_+^\circ$, we know by Lemma 2.7.4 that there exists a $\lambda > 0$ such that $\lambda e \leq y$, meaning that $\varphi(y) \geq \lambda > 0$ for all $\varphi \in S(V)$. By the same lemma, we know there exists some $\lambda \in \mathbb{R}$ such $x \leq \lambda y$, so for any such λ we have $\lambda \geq \frac{\varphi(x)}{\varphi(y)}$ for any $\varphi \in S(V)$, meaning $M(x/y) \geq \sup_{\varphi \in S(V)} \frac{\varphi(x)}{\varphi(y)}$. Suppose by way of contradiction that the inequality above is strict, then there exists a $\varepsilon > 0$ such that $\varphi(x) \leq \varphi([M(x/y)+\varepsilon]y)$ for all $\varphi \in S(V)$, which by Lemma 2.7.5 means $x \leq (M(x/y)+\varepsilon)y$, contradicting $M(x/y)$ being the infimum. As $S(V)$ is weak* compact and $\varphi \mapsto \frac{\varphi(x)}{\varphi(y)}$ is continuous with bounded range the supremum is achieved.

(ii) Let $((x_\alpha, y_\alpha)) \subseteq V \times V_+^\circ$ be a net converging to some $(x, y) \in V \times V_+^\circ$. For each α, we have just proved that there exists a $\varphi_\alpha \in S(V)$ such that $M(x_\alpha/y_\alpha) = \frac{\varphi_\alpha(x_\alpha)}{\varphi_\alpha(y_\alpha)}$. Lemma 2.7.6 means that there exists a subnet $((x_\beta, y_\beta))$ and a $\varphi \in S(V)$ such that φ_β converges to φ in the weak* topology. Now, for any β,

$$
\begin{aligned}
|\varphi_\beta(x_\beta) - \varphi(x)| &\leq |\varphi_\beta(x_\beta - x)| + |(\varphi - \varphi_\beta)(x)| \\
&\leq \|\varphi_\beta\|_* \|x_\beta - x\|_e + |(\varphi - \varphi_\beta)(x)| \\
&= \|x_\beta - x\|_e + |(\varphi - \varphi_\beta)(x)| \\
&\to 0.
\end{aligned}
$$

Similarly we can show that $\varphi_\beta(y_\beta) \to \varphi(y)$. Thus $M(x_\beta/y_\beta) \to \frac{\varphi(x)}{\varphi(y)}$. Corollary 2.7.3 thus means that $x \leq \frac{\varphi(x)}{\varphi(y)} y$, which by the definition of the gauge function and part (i) we have just proved means that indeed $M(x/y) = \frac{\varphi(x)}{\varphi(y)}$. This argument shows that every subnet of $(M(x_\alpha/y_\alpha))$ has a further subnet converging to $M(x/y)$, so indeed M is continuous.

(iii)-(iv) If $\beta y - x \in V_+$, then $T(\beta y - x) \in V^+$, which by linearity implies that $M(Tx/Ty) = M(x/y)$. Furthermore if $x \leq \lambda y$ for $\lambda \in \mathbb{R}$, then $\alpha x \leq \beta^{-1}\alpha\lambda\beta y$, from which we deduce that $M(\alpha x/\beta y) = \alpha\beta^{-1} M(x/y)$. □

Lemma 2.7.9. *Let (V, V_+, u) be an order-unit space. If $v \in \partial V_+$ and $w_n \in \operatorname{int} V_+$ with*

$w_{n+1} \le w_n$ for all $n \ge 1$ and $w_n \to w \in \partial V_+$, then

$$\lim_{n\to\infty} M(v/w_n) = \begin{cases} M(v/w) & \text{if } v \neq 0 \text{ and } w \text{ dominates } v \\ \infty & \text{if } v \neq 0 \text{ and } w \text{ doesn't dominate } v \\ 0 & \text{if } v = 0. \end{cases}$$

Proof. Set $\lambda_n = M(v/w_n)$ for $n \ge 1$. Then for $n \ge m \ge 1$ we have that $0 \le \lambda_n w_n - v \le \lambda_n w_m - v$. This implies that $\lambda_m \le \lambda_n$ for all $m \le n$, hence (λ_n) is monotonically increasing.

Now suppose that $\lambda = M(v/w) < \infty$, i.e., w dominates v. Then $0 \le \lambda w - v \le \lambda w_n - v$, hence $\lambda_n \le \lambda$ for all n. This implies that $\lambda_n \to \lambda^* \le \lambda < \infty$. As $0 \le \lambda_n w_n - v$ for all n and V_+ is closed, we know that $\lim_{n\to\infty} \lambda_n w_n - v = \lambda^* w - v \in V_+$. So $\lambda^* \ge \lambda$, hence $\lambda^* = \lambda$. We conclude that if w dominates v, then $\lim_{n\to\infty} M(v/w_n) = M(v/w)$.

On the other hand, if w does not dominate v, then

$$\lambda w - v \notin V_+ \quad \text{for all } \lambda \ge 0. \tag{2.7.2}$$

Assume, by way of contradiction, that (λ_n) is bounded. Then $\lambda_n \to \lambda^* < \infty$, since (λ_n) is increasing, and $\lambda_n w_n - v \to \lambda^* w - v \in V_+$, as V_+ is closed. This contradicts (2.7.2), and hence $\lambda_n = M(v/w_n) \to \infty$, if w does not dominate v. Finally, $M(0/w_n) = 0$ for all $n \in \mathbb{N}$. □

2.8 The Hilbert and Thompson Metric on Cones

It is possible to equip any open and bounded convex set $C \subseteq \mathbb{R}^n$ with *Hilbert's metric*, d_h. Sometimes called Hilbert's projective metric, it was introduced by Hilbert to generalise Cayley's notion of distance in the Klein model of hyperbolic geometry. We set $d_h(x,x) = 0$ for all $x \in C$, and metric between two distinct points x and $y \in C$ is defined by first identifying x' as the unique point on the boundary of C intersected by the ray emanating from y and passing through x, and y' as the unique point on the boundary of C intersected by the ray emanating from x and passing through y, as illustrated in Figure 2.1, and defining

$$d_h(x,y) = \frac{1}{2}\log\left(\frac{\|x-y'\|\|y-x'\|}{\|x-x'\|\|y-y'\|}\right),$$

where $\|\cdot\|$ is the usual Euclidean norm.

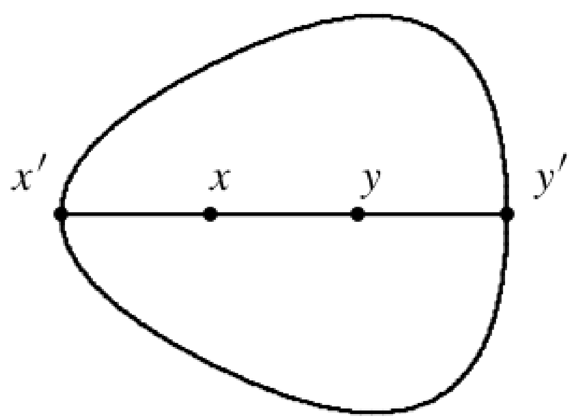

Figure 2.1: Illustration of the line segments used to define the Hilbert metric.

The factor of $\frac{1}{2}$ is often ignored when studying Hilbert's metric, as its only purpose is to set the sectional curvature of the space to -1 if C is the unit ball, to perfectly match Klein's model of hyperbolic geometry.

In an ordered vector space (V,V_+), we can define *Birkhoff's version of the Hilbert metric*, on V_+, by defining $d_H\colon V_+\times V_+\to[0,\infty]$ by

$$d_H(x,y) = \begin{cases} \log(M(x/y)M(y/x)) & \text{if } x\sim y,\ y\neq 0 \\ 0 & \text{if } x=y=0 \\ \infty & \text{if } x \text{ and } y \text{ are not in the same part of } V_+ \end{cases}.$$

Lemma 2.7.8 means that $d_H(\alpha x/\beta y) = d_H(x/y)$ for all $\alpha,\beta>0$, meaning that d_H does not distinguish between points on rays, so it cannot be a full metric. If we restrict d_H to a slice of the cone, however, then it is a full metric as the next lemma shows:

Lemma 2.8.1. *Let (V, V_+, e) be an order unit space, and $\varphi \in S(V)$, then if we define $\Omega_V = \varphi^{-1}(1) \cap V_+^\circ$, the space (Ω_V, d_H) is a metric space.*

Proof. It is clear from the definition that $d_H(x, x) = 0$ and $d_H(x, y) = d_H(y, x)$ for all $x, y \in \Omega_V$. As $\Omega_V \subseteq V_+^\circ$, Lemma 2.7.4 shows that $d_H(x, y) \in [0, \infty)$ for all $x, y \in \Omega_V$. Now, suppose that $d_H(x, y) = 0$. Thus $M(x/y) = M(y/x)^{-1}$, so using the definition of the gauge function, this leads to $x \leq M(x/y)y \leq x$, which means that, using Lemma 2.7.5 and the definition of Ω_V, $M(x/y) = M(y/x) = 1$. Thus $x \leq y$ and $y \leq x$, which means $x = y$ by antisymmetry. Finally, fix any $x, y, z \in \Omega_V$. Transitivity and the definition of the gauge function as an infimum shows us that $M(x/z)M(z/y) \geq M(x/y)$ and $M(y/z)M(z/y) \geq M(y/x)$. Thus $d_H(x/y) \leq d_H(x/z) + d_H(z/y)$, and so we can conclude that (Ω_V, d_H) is a metric space. □

The study of Birkhoff's version of the Hilbert metric is motivated by the fact that, if V is finite dimensional, then $(\Omega_V, 2d_h) = (\Omega_V, d_H)$. A proof of this fact is given in [46, Theorem 2.1.2]. It will be useful for us to understand the Hilbert metric on a product of order unit-unit spaces, but this actually follows as a simple corollary of 2.7.7 combined with the fact that log is monotone increasing.

Corollary 2.8.2. *Let $(V \times W, V_+ \times W_+, (e, w))$ be a product of two order-unit spaces (V, V_+, e) and (W, W_+, w). If $x_1 \sim x_2$ in V, and $y_1 \sim y_2 \in W$, then*

$$d_H((x_1, y_1), (x_2, y_2)) = \max\{\log M(x_1/x_2), \log M(y_1/y_2)\} + \max\{\log M(x_2/x_1), \log M(y_2/y_1)\}.$$

Closely related to Birkhoff's Hilbert metric is the *Thompson metric* on a cone. If (V, V_+)

is an ordered vector space, the Thompson metric, $d_T \colon V_+ \times V_+ \to [0, \infty]$, is defined by

$$d_T(x,y) = \begin{cases} \max\{\log M(x/y), \log M(y/x)\} & \text{if } x \sim y \\ 0 & \text{if } x = y = 0 \\ \infty & \text{if } x \text{ and } y \text{ are not in the same part.} \end{cases}$$

If V_+ is Archimedean, d_T is a complete metric on each part of V_+ [46, Appendix A.2].

2.9 Jordan and JB Algebras

Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. An $\mathbb{F}$ Jordan algebra is a standard commutative algebra (i.e a vector space V with a commutative bilinear product $\cdot \bullet \cdot \colon V \times V \to V$), with the further property, that for all $x, y \in V$:

$$x \bullet (x^2 \bullet y) = x^2 \bullet (x \bullet y). \tag{2.9.1}$$

In general a Jordan algebra does not need to be associative. If $L_x \colon V \to V$ is the linear operator $L_x y = x \bullet y$, and $[T, S] = TS - ST$ for any two V endomorphisms we can rewrite condition (2.9.1) as

$$[L_x, L_{x^2}] = 0 \tag{2.9.2}$$

for all $x \in V$. We should note that any associative and commutative algebra is automatically a Jordan algebra. We say a Jordan algebra is *unital* if there exists some $e \in V$ such that $e \bullet x = x$, and e is called the *unit.* Throughout this thesis we shall only be dealing with unital Jordan algebras, so throughout the reader should read Jordan algebra as unital Jordan algebra. We say that y is the *inverse* of $x \in V$ if $x \bullet y = e$. If an inverse exists it is unique, and we denote it by x^{-1}. We define the spectrum of $x \in V$, denoted by $\sigma(x)$, in the usual way

$$\sigma(x) = \{\lambda \in \mathbb{F} : \lambda e - x \text{ is not invertible}\}.$$

Example 2.9.1. If A is an associative (but noncommutative) algebra over $\mathbb{F}$ we can equip it with the so called *Jordan product* $\bullet$, where

$$x \bullet y = \frac{1}{2}(xy + yx),$$

which turns A into a Jordan algebra.

Any Jordan algebra V which arises by equipping an associative algebra with the Jordan product is known as a *special Jordan algebra*. A Jordan algebra which cannot be constructed from an associative algebra structure on the underlying vector space via the Jordan product is called an *exceptional Jordan algebra*.

Example 2.9.2. If V is a linear subspace of an associative algebra A that is stable under squares, by which we mean that $x \in V \implies x^2 \in V$ then V equipped with the Jordan product is a Jordan algebra. A good example of this is the space of symmetric $n \times n$ matrices over $\mathbb{R}$, denoted by $\mathrm{Sym}(n, \mathbb{R})$.

We call an algebra V *power associative* if $x^{p+q} = x^p x^q$ for all $x \in V$ and $p, q \in \mathbb{N}$. All Jordan algebras are power associative [20, Proposition II.1.2].

2.9.1 The Minimal Polynomial

Let V be a finite dimensional power associative algebra. Let $\mathbb{F}[X]$ denote the algebra over $\mathbb{F}$ of all polynomials in one variable with coefficients in $\mathbb{F}$. Then naturally for any $x \in V$

$$\mathbb{F}[x] = \{p(x) : p \in \mathbb{F}[X].\}$$

Furthermore $\mathbb{F}[x]$ is isomorphic to $\mathbb{F}[X]/\mathcal{J}(x)$, where

$$\mathcal{J}(x) = \{p \in \mathbb{F}[X] : p(x) = 0.\}$$

This follows from the first isomorphism theorem for algebras, by choosing the natural homomorphism $p \mapsto p(x)$. Now as $\mathbb{F}[X]$ is a principal ideal domain, $\mathcal{J}(x)$ is generated by a single monic polynomial, which we call the *minimal polynomial* of x. With this we can define $m(x)$ to be the degree of the minimal polynomial of x. We have the following lemma:

Lemma 2.9.3. *For any $x \in V$,*

$$m(x) = \min\{k > 0 : \{e, x, \dots, x^k\} \text{ are linearly dependent}\}$$

Proof. Let p be the minimal polynomial of x. Then, for any $q \in \mathcal{J}(x)$, we know that $q = pr$ for some $r \in \mathbb{F}[X]$, so, if q is non-zero, then $\deg(q) \geq \deg(p)$. Furthermore if d is the degree of p then $\{e, x, \dots, x^d\}$ are linearly dependent as $p(x) = 0$ and p is non-zero. These two facts put together prove the claim. □

We can now define the rank of a power associative algebra:

$$r = \operatorname{rank} V = \max\{m(x) : x \in V\}$$

We say that $x \in V$ is *regular* if $m(x) = r$. We have the following theorem about regular elements of V [20, Proposition II.2.1]:

Theorem 2.9.4. *The set of regular elements is open and dense in V. Furthermore there exists unique $\mathbb{F}$ valued polynomials $a_1, \dots, a_r$ on V, with a_j homogeneous of degree j, so that the minimal polynomial f of every $v \in V$ is given by:*

$$f(X; v) = X^r - a_1(v)X^{r-1} + a_2(v)X^{r-2} + \cdots + (-1)^r a_r(v).$$

Let us now recall a basic linear algebra fact: If two matrices A and B are similar, then their determinants and traces are equal. Thus the determinant and trace is a well defined property of endomorphisms on V. We use $\operatorname{Det} T$ and $\operatorname{Tr} T$ to represent these values respectively. We can generalise this and define the *trace* and *determinant* of elements in V, by $\mathrm{tr}(x) = a_1(x)$ and $\det(x) = a_r(x)$. Note that the polynomials a_i are fixed and defined on all of V, so these notions are well defined for all elements, and not just the regular ones. This nomenclature is justified by the following observation. Let $L_0(x)$ denote the restriction of L_x to $\mathbb{F}[x]$. For a regular element x, the set $\{e, x, \dots, x^{r-1}\}$ forms a basis for $\mathbb{F}[x]$, and with respect to this basis $L_0(x)$ has matrix representation $A(x)$, where

$$A(x) = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^{r-1}a_r(x) \\ 1 & 0 & \dots & 0 & (-1)^{r-2}a_{r-1}(x) \\ 0 & 1 & \dots & 0 & (-1)^{r-3}a_{r-2}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_1(x). \end{pmatrix} \tag{2.9.3}$$

To see this consider any $v = \sum_{k=0}^{r-1} b_k x^k$, where $x^0 = e$. Clearly $L_0(x)v = \sum_{k=1}^{r} b_{k-1}x^k$, and we calculate

$$A(x)v = \begin{pmatrix} (-1)^{r-1}b_{r-1}a_r(x) \\ b_0 + (-1)^{r-2}b_{r-1}a_{r-1}(x) \\ b_1 + (-1)^{r-3}b_{r-1}a_{r-2}(x) \\ \vdots \\ b_{r-2} + b_{r-1}a_1(x) \end{pmatrix} = \sum_{k=1}^{r-1} b_{k-1}x^k + b_{r-1}\sum_{j=1}^{r}(-1)^{j-1}a_j(x)x^{j-r}.$$

Now note that because $f(x,x) = 0$ the above calculation in conjunction with Theorem 2.9.4 implies that

$$A(x)v = \sum_{k=1}^{r-1} b_{k-1}x^k + b_{r-1}(f(x,x) + x^r) = \sum_{k=1}^{r} b_{k-1}x^k = L_0(x)v.$$

With this matrix representation it is then clear to see that

$$\mathrm{Tr}(L_0(x)) = a_1(x) = \mathrm{tr}(x), \text{ and } \mathrm{Det}(L_0(x)) = a_r(x) = \det(x).$$

We can also define the *characteristic polynomial* of x over variable λ as $F(\lambda; x) = \mathrm{Det}(\lambda I - L_0(x))$. We can immediately prove the following:

Lemma 2.9.5. *For any $x \in V$ the minimal polynomial of x is a factor of the characteristic polynomial of x.*

Proof. The Cayley-Hamilton Theorem immediately gives that $F(L_0(x); x) = 0$ for all $x \in V$. Furthermore, for any polynomial $p \in \mathbb{F}[X]$ we know that $p(x) = p(L_0(x))e$ for all $x \in X$. Thus, for any $x \in V$, $F(x; x) = F(L_0(x); x)e = 0$, so $F(\lambda; x) \in \mathcal{J}(x)$, which gives the result. □

Lemma 2.9.6. *$F(\lambda; x) = f(\lambda; x)$ for any regular $x \in V$.*

Proof. We first show that if x is regular then so too is $\lambda e - x$. This is in fact quite clear, because if q is the minimal polynomial for $\lambda e - x$ with $\deg q < r$ then $q(\lambda e - x)$ can be expanded into $p \in \mathbb{F}[x]$ such that $p(x) = 0$, but $\deg p = \deg q < r$, which contradicts x being regular. Clearly $\lambda e - x \in \mathbb{F}[x]$. Thus

$$\det(\lambda e - x) = \mathrm{Det}(L_0(\lambda e - x)) = \mathrm{Det}(\lambda I - L_0(x)) = \mathrm{Det}(\lambda I - A(x)),$$

with $A(x)$ as in (2.9.3). By expanding along the last column it is clear that $\mathrm{Det}(\lambda I - A(x)) = f(\lambda; x)$.

□

Proposition 2.9.7. *For any $x \in V$ the characteristic polynomial of x is given by:*

$$F(\lambda; x) = \sum_{i=0}^{r} (-1)^i a_i(x) \lambda^{r-i},$$

where $a_0(x) = 1$.

Proof. If x is regular then this statement is precisely Lemma 2.9.6 in conjunction with Theorem 2.9.4. Now let $x \in V$ be non-regular. By Theorem 2.9.4 there exists a sequence

$(x_n) \subseteq V$ converging to x in norm. Thus for each $n \in \mathbb{N}$ and fixed $\lambda \in \mathbb{F}$

$$F(\lambda; x_n) = \sum_{i=0}^{r} (-1)^i a_i(x_n) \lambda^{r-i}.$$

The continuity of each a_i on V and Det on the space of matrices and the continuity of L_0 on V allows us to take the limit as $n \to \infty$ on both sides of the above equality to get the result. □

Theorem 2.9.4 can be used to show that an element x in a finite dimensional Jordan algebra V over $\mathbb{F}$ is invertible if and only if $\det(x) \neq 0$ [20, Proposition II.2.4]. Thus, for any $x \in V$,

$$\sigma(x) = \{\lambda \in \mathbb{F} : \det(\lambda e - x) = 0\}. \tag{2.9.4}$$

2.9.2 Euclidean Jordan Algebras

A Euclidean Jordan Algebra is a finite dimensional Jordan algebra over $\mathbb{R}$ equipped with an inner product $\langle \cdot, \cdot \rangle$ satisfying $\langle L_x u, v \rangle = \langle u, L_x v \rangle$ for all $u, v, x \in V$. An algebra is known as *formally real* if $x^2 + y^2 = 0$ if and only if $x = 0 = y$. All Euclidean Jordan algebras are formally real by the fact that L_x is self adjoint, which means that $\langle x^2 + y^2, e \rangle = \langle x, x \rangle + \langle y, y \rangle$. Conversely, any formally real finite dimensional Jordan algebra is a Euclidean Jordan algebra [20, Proposition VIII.4.2].

Recall that an element $p \in V$ satisfying $p^2 = p$ is called an *idempotent*. We call two elements $c, d \in V$ *orthogonal* if $c \bullet d = 0$. We call a collection of idempotents $\{p_1, \ldots, c_p\}$ a *complete system of orthogonal idempotents* if $p_i \bullet p_j = 0$ for all $i \neq j$ and $\sum_{i=1}^k p_i = e$, where e is the algebra identity. The next two theorems are fundamental in the study of Euclidean Jordan Algebras, and will be used extensively throughout the thesis.

Theorem 2.9.8. *(First Spectral Theorem) For any $x \in V$ there exists a unique system of complete idempotents $\{p_1, \ldots, p_k\}$ and corresponding unique collection of real numbers*

$\{\lambda_1, \ldots, \lambda_k\}$ *such that*

$$x = \sum_{i=1}^{k} \lambda_i p_i.$$

Furthermore each $p_j \in \mathbb{R}[x]$*, and we call* $\{\lambda_1, \ldots, \lambda_k\}$ *the eigenvalues of* x*.*

Proof. See [20, III.1.1]. □

We say that an idempotent is *primitive* if it is non-zero and cannot be written as a sum of two orthogonal non-zero idempotents. We say that a collection $\{p_1, \ldots, p_m\}$ is a *Jordan frame* if it is a complete system of orthogonal idempotents with each p_j primitive.

Theorem 2.9.9. *(Second Spectral Theorem) Let* V *have rank* r*. For* $x \in V$ *there then exists a Jordan Frame* $\{p_1, \ldots, p_r\}$ *and real numbers* $\{\lambda_1, \ldots, \lambda_r\}$ *uniquely determined by* x *so that*

$$x = \sum_{i=1}^{r} \lambda_j p_j.$$

Furthermore for any $x \in V$ *the minimal polynomial is given by*

$$f(X; x) = \prod_{i=1}^{k} (X - \lambda_i)$$

where the λ_i *are as in Theorem 2.9.8, and* $k \leq r$*, where equality is achieved precisely for regular elements. Finally:*

$$\det x = \prod_{i=1}^{r} \lambda_i, \quad \operatorname{tr} x = \sum_{j=1}^{r} \lambda_j.$$

Proof. See [20, III.1.2]. □

We should make a quick remark that calling the numbers λ_i in the above two theorems *eigenvalues* is well founded, because for every $x \in V$, if $\{\lambda_1, \ldots, \lambda_k\}$ are the unique eigenvalues of x guaranteed by Theorem 2.9.8, then

$$\sigma(x) = \{\lambda_1, \ldots, \lambda_k\}.$$

Indeed, by Theorem 2.9.8, $x = \sum_{i=1}^{k} \lambda_i p_i$, where $\{p_1, \dots, p_k\}$ is a complete set of orthogonal idempotents. Thus, for any $\lambda \in \mathbb{F}$, we can write

$$\lambda e - x = \sum_{i=1}^{k} (\lambda - \lambda_i) p_i.$$

The uniqueness of the spectral decomposition in Theorem 2.9.8 with respect to a complete system of orthogonal idempotents means that 0 is an eigenvalue of $\lambda e - x$ if and only if $\lambda \in \{\lambda_1, \dots, \lambda_k\}$, so by Theorem 2.9.9, $\det(\lambda e - x) = 0$, if and only if $\lambda \in \{\lambda_1, \dots, \lambda_k\}$ meaning that $\lambda \in \sigma(x)$ if and only if $\lambda \in \{\lambda_1, \dots, \lambda_k\}$ by (2.9.4). We now discuss an important example:

Example 2.9.10. Equip $V = \mathrm{Sym}(n, \mathbb{R})$ with the Jordan product $A \bullet B = \frac{1}{2}(AB + BA)$. It is then clearly a Jordan algebra, with $\operatorname{rank} V = n$. To prove this latter claim, we first note that $A \bullet A = AA$ for any $A \in V$, so we see that the minimal polynomial for any A over V is the same as the minimal polynomial for A over $\mathrm{Sym}(n, \mathbb{R})$ considered as an associative algebra with the standard matrix multiplication. From linear algebra we know that the minimal polynomial of any $n \times n$ matrix has degree less than or equal to n. Furthermore any diagonal matrix with each element on the diagonal having a different entry will have n distinct eigenvalues, and so its minimal polynomial will be of degree n.

We can also show that the "Jordan" eigenvalues of a matrix A are exactly its regular eigenvalues. This follows from the uniqueness of "Jordan" eigenvalues given by Theorem 2.9.9, as A can always be written as the sum $\sum_{i=1}^{n} \lambda_i P_i$ where the λ_i are the regular eigenvalues (including multiplicities) and the P_i are projections onto the eigenspace of the i-th linearly independent eigenvector of A. The P_is form a Jordan frame. This immediately implies that $\det(A) = \mathrm{Det}(A)$ and $\mathrm{tr}(A) = \mathrm{Tr}(A)$ for all $A \in V$.

Equipped with the form $A, B \mapsto \mathrm{tr}(A \bullet B)$, V is in fact a Euclidean Jordan algebra. This follows from utilizing the above observation, combined with the fact that $\mathrm{Tr}(A \bullet B) =$

$\mathrm{Tr}(AB)$ for symmetric matrices A and B, so we can calculate for any $X, Y, Z \in V$ and $\lambda \in \mathbb{R}$ that

$$\begin{aligned}
\mathrm{tr}(\lambda(X+Y) \bullet Z) &= \mathrm{Tr}\left(\frac{\lambda}{2}(X+Y)Z + \frac{\lambda}{2}Z(X+Y)\right) \\
&= \mathrm{Tr}\left(\frac{\lambda}{2}(XZ+YZ) + \frac{\lambda}{2}(ZX+ZY)\right) \\
&= \mathrm{Tr}\left(\frac{\lambda}{2}(XZ+ZX)\right) + \mathrm{Tr}\left(\frac{\lambda}{2}(YZ+ZY)\right) \\
&= \lambda\,\mathrm{tr}(X \bullet Z) + \lambda\,\mathrm{tr}(Y \bullet Z),
\end{aligned}$$

proving that indeed $\mathrm{tr}(\cdot \bullet \cdot)$ is a real inner-product on V. Furthermore

$$\begin{aligned}
\mathrm{tr}(L_X Y \bullet Z) &= \mathrm{tr}((X \bullet Y) \bullet Z) \\
&= \mathrm{tr}\left(\frac{1}{4}(XY+YX)Z + \frac{1}{4}Z(XY+YX)\right) \\
&= \mathrm{tr}\left(\frac{1}{4}(XYZ+YXZ+ZXY+ZYX)\right) \\
&= \frac{1}{4}\left(\mathrm{Tr}(XYZ)+\mathrm{Tr}(YXZ)+\mathrm{Tr}(ZXY)+\mathrm{Tr}(ZYX)\right) \\
&= \frac{1}{4}\left(\mathrm{Tr}(YXZ)+\mathrm{Tr}(YZX)+\mathrm{Tr}(ZXY)+\mathrm{Tr}(XZY)\right) \\
&= \mathrm{tr}\left(\frac{1}{4}(XZ+ZX)Y + \frac{1}{4}Y(XZ+ZX)\right) \\
&= \mathrm{tr}(Y \bullet L_X Z),
\end{aligned}$$

proving that indeed V equipped with $\mathrm{tr}(\cdot \bullet \cdot)$ is a Euclidean Jordan algebra.

The trace is not just important in the case of matrices. Indeed, a Jordan Algebra V is Euclidean if and only if the map $(x, y) \mapsto \mathrm{tr}(x \bullet y)$ is an inner product on V [20, Proposition III.1.5]. If V is a Euclidean Jordan algebra, the set $V_+ = \{x^2 : x \in V\}$ is a closed cone, known as the *cone of squares*, and V_+° is a symmetric cone [20, Theorem III.2.1]. The converse is also true, and every symmetric cone can be realised as the interior of the cone of squares of some Euclidean Jordan Algebra. The Koecher-Vinberg theorem states that this correspondence is one-to-one [20, Chapter 3].

2.9.3 The Quadratic Representation and Peirce Decomposition

Let V be a Euclidean Jordan algebra. If $p \in V$ is an idempotent, L_p is a root of the polynomial $2X^3 - 3X^2 + X$ [20, Proposition III 1.3], so the only possible eigenvalues of L_p are 0, $\frac{1}{2}$, and 1. By the definition of a Euclidean Jordan Algebra, L_p is self-adjoint, so the spectral theorem for self-adjoint matrices means that we have the orthogonal (with respect to the inner product) decomposition

$$V = V(p,1) \oplus V(p, \frac{1}{2}) \oplus V(p,0),$$

where $V(p,i)$ is the eigenspace of L_p for the eigenvalue i. This decomposition is called the *Peirce decomposition* of V with respect to p. It is useful to understand the projections associated to these eigenspaces. To that end, we define the *quadratic representation* of V to be the map $U\colon V \to V$, defined by

$$U_x = 2L_x^2 - L_{x^2} \tag{2.9.5}$$

We should note that the quadratic representation is defined as above for all Jordan algebras, not just Euclidean ones. If $p \in V$ is an idempotent, and $P_i\colon V \to V(p,i)$ for $i \in \{0, 1/2, 1\}$ denote the orthogonal projections onto the Peirce subspaces, then [20, Page 64]

$$P_1 = U_p,\ P_0 = U_{e-p},\ \text{and}\ P_{1/2} = I - U_p - U_{e-p}.$$

Furthermore, $V(p,1)$ and $V(p,0)$ are Jordan subalgebras of V, and they are orthogonal with respect to the Jordan product [20, Proposition IV.1.1]. If $\{p_1, \ldots, p_r\}$ is a Jordan frame for V, a direct calculation shows that each of the L_{p_i} commute with each other, allowing a simultaneous diagonalisation of the collection $\{L_{p_1}, \ldots, L_{p_r}\}$. If, for all $i, \in \{1, \ldots, r\}$ we define $V_{ii} = V(p_i, 1)$ and $V_{ij} = V(p_i, 1/2) \cap V(p_j, 1/2)$, this observation allows the following to be proved [20, Theorem IV.2.1]

Theorem 2.9.11. *If $\{p_1, \ldots, p_r\}$ is a Jordan frame for V, then V admits the orthogonal decomposition*

$$V = \bigoplus_{i \leq j} V_{ij},$$

with orthogonal projections $P_{ij}\colon V \to V_{ij}$ given by

$$P_i i = U_{p_i}, \text{ and } P_{ij} = 4L_{p_i}L_{p_j}.$$

Furthermore $V_{ij} \bullet V_{\ell k} = \{0\}$ if $\{i, j\} \cap \{\ell, k\} = \emptyset$.

The decomposition in Theorem 2.9.11 is called the *Peirce decomposition* of V with respect to the Jordan frame $\{p_1, \ldots, p_r\}$.

2.9.4 JB-Algebras

Let V be a Jordan algebra over $\mathbb{R}$. If $\|\cdot\|$ is a norm on V, we call the normed algebra $(V, \|\cdot\|)$ a *JB-algebra* if $(V, \|\cdot\|)$ is complete, and $\|\cdot\|$ satisfies the following for all $x, y \in V$

(i) $\|x \bullet y\| \leq \|x\|\|y\|$

(ii) $\|x^2\| = \|x\|^2$

(iii) $\|x^2\| \leq \|x^2 + y^2\|$.

Condition *(iii)* implies that every JB-algebra is formally real, so the finite dimensional JB-algebras are precisely the Euclidean Jordan algebras by remarks made in the beginning of subsection 2.9.2. Thus everything that follows applies to Euclidean Jordan algebras. Condition *(i)* implies that the operators L_x and U_x are bounded for every $x \in V$. In fact $\|U_x\| = \|x^2\| = \|x\|^2$ [4, Theorem 1.25]. Interestingly, every JB-algebra is actually an order unit space with positive cone $V_+ = \{x^2 : x \in V\}$ and order unit e coinciding with the Jordan algebra unit and order unit norm $\|\cdot\|_e$ coinciding with the JB-algebra norm [4, Theorem 1.11]. As a JB-algebra is an order unit space, everything in section 2.7 can be brought to bear on their study. In particular it makes sense to talk about positive elements, positive operators, and states. We can define the spectral radius norm on a JB algebra, denoted by $\|\cdot\|_\sigma$ in the usual way:

$$\|x\|_\sigma = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Intriguingly, the JB-algebra norm is equal to the spectral radius norm, and furthermore $x \in V_+$ if and only if $\sigma(x) \subseteq [0, \infty)$ [4, Corollary 1.22]. The quadratic representation (2.9.5) operators encode a lot of information about JB-algebras. Indeed, $x \in V$ is invertible if and only if U_x is invertible, in which case $U_x = U_{x^{-1}}$ [4, Lemma 1.23], and U_x is a positive operator for every $x \in V$. If $p \in V$ is an idempotent, $\|U_p\| = 1$, and for any $\varphi \in S(V)$, $\varphi(p) = 1$ if and only if $U_p^*(\varphi) = \varphi$ if and only if $\|U_p^*\varphi\|_* = 1$ [4, Proposition 1.41]. Furthermore, for any idempotent $p \in V$, $U_p(V)$ is a JB-subalgebra of V with unit p, where we define a *JB-subalgebra* to be a norm closed subspace that is also closed under the Jordan product.

2.9.5 Spin Factors

Classical examples of Jordan algebras are the so-called spin factors, which we study in depth in Chapter 8. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, with $\dim \mathcal{H} \geq 3$. Define $V = \mathbb{R} \oplus \mathcal{H}$. We define the product $(\cdot,\cdot) \bullet (\cdot,\cdot) \colon V \times V \to V$ by

$$(\lambda_1, x_1) \bullet (\lambda_2, x_2) = (\lambda_1\lambda_2 + \langle x_1, x_2\rangle, \lambda_1 x_2 + \lambda_2 x_1).$$

This operation turns V into a Jordan algebra, with identity element $e = (1, 0)$. This Jordan algebra is known as a *spin factor*. If we equip V with the standard product inner product it becomes a Hilbert space. To be clear this inner product is defined as:

$$\langle(\lambda_1, x_1), (\lambda_2, x_2)\rangle_V = \lambda_1\lambda_2 + \langle x_1, x_2\rangle.$$

This induces the norm $\|\cdot\|_s$ on V defined by $\|(\lambda, x)\|_s = \sqrt{|\lambda|^2 + \|x\|_{\mathcal{H}}^2}$. Thus if $\mathcal{H}$ is finite dimensional V is a Euclidean Jordan Algebra. However, even if $\mathcal{H}$ is infinite dimensional V still has some of the desirable properties of a Euclidean Jordan Algebra:

Theorem 2.9.12. *For any $v \in V$ there exists a Jordan frame of two idempotents, $\{p_1, p_2\}$ and two eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$, such that $v = \lambda_1 p_1 + \lambda_2 p_2$, and these eigenvalues uniquely depend on v.*

Proof. We first calculate that for any $v = (\lambda, x)$,

$$\begin{aligned} v^2 &= (\lambda^2 + \|x\|_{\mathcal{H}}^2, 2\lambda x) \\ &= 2\lambda v + (\|x\|_{\mathcal{H}}^2 - \lambda^2)e, \end{aligned}$$

meaning that $\operatorname{rank} V = 2$ (We note that this calculation does not rely on the dimension of $\mathcal{H}$ in anyway). In particular this means that $\mathbb{R}[v]$ is a finite dimensional Euclidean space for every $v \in V$. This then allows us to replicate the proofs of Theorem III.1.1 and Theorem III.2.1 of [20] to deduce the result. □

The above proof immediately implies that the minimal polynomial for any $v = (\lambda, x) \in V$ divides

$$f(X; v) = X^2 - 2\lambda X - (\|x\|_{\mathcal{H}}^2 - \lambda^2)e,$$

which has roots $\lambda \pm \|x\|_{\mathcal{H}}$. This implies that

$$\sigma(v) = \{\lambda + \|x\|_H, \lambda - \|x\|_{\mathcal{H}}\} \text{ for any } v = (\lambda, x) \in V. \tag{2.9.6}$$

This means, that if $\|\cdot\| = \max_{\gamma \in \sigma(\cdot)}\{|\gamma|\}$ is the spectral radius norm (JB algebra norm), then $\|v\| = |\lambda| + \|x\|_{\mathcal{H}}$ where $v = (\lambda, x)$. This immediately implies that $\|\cdot\|_s$ and $\|\cdot\|$ are equivalent norms on V. This also implies that the JB algebra V is just the direct sum of two normed spaces equipped with the ℓ^1 norm. We can calculate that, if $p^2 = p$ for some

$p = (\lambda, x) \in V$, then

$$2\lambda x = x, \text{ and } \lambda^2 + \langle x, x\rangle = \lambda,$$

which means that the only primitive idempotents are elements $p = (\frac{1}{2}, x)$, where $\|x\|_{\mathcal{H}} = \frac{1}{2}$. The following lemma will be used frequently in later chapters.

Lemma 2.9.13. *If V is a spin factor, and $v \in V$ has spectral decomposition $v = \lambda p + \mu(e - p)$ for $p = (\frac{1}{2}, x)$ for some $x \in \frac{1}{2}B_{\mathcal{H}}$, then*

$$v = \left(\frac{\lambda + \mu}{2}, (\lambda - \mu)x\right).$$

Conversely, if $v = (\gamma, x) \in V$ for some $\gamma \in \mathbb{R}$ and $x \in \mathcal{H}$, v has spectral decomposition $v = \lambda p + \mu(e - p)$, where

$$p = \left(\frac{1}{2}, \frac{1}{2\|x\|_{\mathcal{H}}}x\right), \ \lambda = \gamma + \|x\|_{\mathcal{H}}, \text{ and } \mu = \gamma - \|x\|_{\mathcal{H}}.$$

Proof. The first statement of the lemma is simple addition. Conversely, Theorem 2.9.12 means that any $v = (\gamma, x) \in V$ has spectral decomposition of the form $v = \lambda p + \mu(e - p)$, so the second statement can be verified by direct substitution of the claimed eigenvalues and primitive idempotents into the first equation of the lemma. □

The fact that V has rank 2 puts strong conditions on which elements in the cone V_+ are orthogonal to each other.

Lemma 2.9.14. *Let $u, v \in V_+$ have spectral representations $u = \lambda_1 p + \mu_1(e - p)$ and $v = \lambda_2 q + \mu_2(e - q)$, for $\lambda_i\mu_i \geq 0$ and $\lambda_i \geq \mu_1$ and p, q primitive idempotents. Then, $u \bullet v = 0$ if and only if either one of u or v is 0, or $p = e - q$ and $\mu_1 = \mu_2 = 0$.*

Proof. First suppose $u \bullet v = 0$. We know that $p = (\frac{1}{2}, x)$ with $x \in \frac{1}{2}S_{\mathcal{H}}$. It can then be calculated straight from the definition of the quadratic representation that, for any $w = (\gamma, z) \in V$,

$$U_p(w) = (\gamma + 2\langle z, x\rangle)p, \text{ and } U_{e-p}(w) = (\gamma - 2\langle z, x\rangle)(e - p). \tag{2.9.7}$$

Thus $U_p(V) = p\mathbb{R}$ and $U_{e-p} = (e-p)\mathbb{R}$. Theorem 2.9.11 thus means that v has Peirce decomposition

$$v = \alpha p + \beta(e-p) + z,$$

where $\alpha p = U_p v$, $\beta(e-p) = U_{e-p}v$, and $z = 4L_pL_{e-p}v$. As U_p and U_{e-p} are positive operators, $\alpha, \beta \geq 0$. As $u \bullet v = 0$, this decomposition means that

$$u \bullet z = -\lambda_1 \alpha p - \mu_1 \beta(e-p). \tag{2.9.8}$$

However, $z = 4L_pL_{e-p}v$, and if $v = (\gamma, w)$, we can directly calculate that

$$u \bullet z = (\lambda_1 + \mu_1)\left(0, \frac{1}{2}w - 2\langle w, x\rangle x\right),$$

so, because the eigenvalues found in (2.9.8) are unique, Lemma 2.9.13 thus means that $\lambda_1\alpha = -\mu_1\beta$. However, all eigenvalues involved in this equation are positive, so $\lambda_1\alpha = 0 = \mu_1\beta$. If $\lambda_1 = \mu_1 = 0$ we are done. If $\alpha = \beta = 0$, then $U_pv = 0$ and $U_{e-p}v = 0$. If $v \neq 0$ this means that either $U_pq = 0$ or $U_p(e-q) = 0$, and $U_{e-p}q = 0$ or $U_{e-p}(e-q) = 0$ but $v \geq 0$, so this contradicts [4, Proposition 1.38], which states that, for any positive $a \in V$ and idempotent c, $U_ca = 0$ if and only if $U_{e-c} = a$. Thus if both u and v are non-zero we are left with the case $\lambda_1, \lambda_2\beta > 0$ and $\alpha = 0 = \mu_1$, so

$$u = \lambda_1 p, \text{ and } v = \beta(e-p) + z.$$

As $u \bullet v = 0$ this means that $p \bullet z = 0$. However, we can calculate that

$$p \bullet z = \left(0, \frac{1}{2}w - 2\langle w, x\rangle x\right),$$

so $w = \eta x$ for $\eta \in \mathbb{R}$. As $p = (\frac{1}{2}, x)$, Lemma 2.9.13 thus means that $p = q$ or $p = e - q$. As $\lambda_1 p \bullet v = 0$ and $\lambda_2 > 0$, we must have $q = e - p$, and indeed that $\mu_1 = \mu_2 = 0$. We have thus proven the forward implication of the lemma, and the converse follows simply by inspection. □

It is useful to have an analytical form for the inverse of elements in V_+°, which is provided in the following lemma.

Lemma 2.9.15. *If $w = (\gamma, y) \in V_+^\circ$, with inverse $w^{-1} = (\gamma', y')$, then*

(i)

$$\gamma' = \frac{\gamma}{\gamma^2 - \|y\|_{\mathcal{H}}^2}, \text{ and } y' = -\frac{1}{\gamma^2 - \|y\|_{\mathcal{H}}^2} y$$

(ii)

$$\frac{1}{\gamma'^2 - \|y'\|_{\mathcal{H}}^2} = \gamma^2 - \|y\|_{\mathcal{H}}^2$$

Proof. We require that $w \bullet w^{-1} = e = (1, 0)$. This means that

$$\gamma\gamma' + \langle y, y' \rangle = 1, \text{ and } \gamma' y + \gamma y' = 0.$$

Thus if $y = 0$ the claim follows trivially. If $y \neq 0$, this means that $y' = -\frac{\gamma'}{\gamma} y$, from which we deduce

$$\gamma' = \left(\frac{\gamma^2 - \|y\|_{\mathcal{H}}^2}{\gamma} \right)^{-1},$$

and the result follows. □

Chapter 3

Metric Compactifications

3.1 The Horofunction Boundary

Let (M, d) be any metric space. Consider $\mathbb{R}^M$, the space of all real valued functions on M, equipped with the topology of pointwise convergence. We fix some *basepoint* $b \in M$, and denote by $\mathrm{Lip}_b^1(M)$ the set of all $h \in \mathbb{R}^M$ with $h(b) = 0$ and h a 1-Lipschitz function, by which we mean that $|h(x) - h(y)| \leq d(x, y)$ for all $x, y \in M$.

Lemma 3.1.1. $\mathrm{Lip}_b^1(M)$ *is a closed subset of* $\mathbb{R}^M$.

Proof. As $\mathrm{Lip}_b^1(M)$ is not in general metrisable, we must use nets instead of sequences to characterise its boundary points. Let $(h_\alpha)_{\alpha \in A}$ be a net in $\mathrm{Lip}_b^1(M)$ converging to some $h \in \mathbb{R}^M$. We now note, for any $x, y \in M$ and $\alpha \in A$,

$$|h(x) - h(y)| \leq |h(x) - h_\alpha(x)| + |h_\alpha(y) - h(y)| + |h_\alpha(x) - h_\alpha(y)|,$$

meaning that $|h(x) - h(y)| \leq d(x, y) + 2\varepsilon$ for any $\varepsilon > 0$, proving that $h \in \mathrm{Lip}_b^1(M)$. □

We also see, for any $h \in \mathrm{Lip}_b^1(M)$ that

$$|h(x)| = |h(x) - h(b)| \leq d(x, b),$$

from which we deduce that

$$\mathrm{Lip}_b^1(M) \subseteq [-d(x,b), d(x,b)]^M = \prod_{x\in M} [-d(x,b), d(x,b)].$$

By Tychonoff's Theorem, this set is compact in the product topology, which is equivalent to the topology of pointwise convergence, and as $\mathrm{Lip}_b^1(M)$ is closed it follows that it too must be compact. For any $y \in M$ we associate the *internal metric functional* h_y, by defining, for any $y \in M$,

$$h_y(x) = d(x,y) - d(b,y). \tag{3.1.1}$$

We immediately see by the reverse triangle inequality that $h_y \in \mathrm{Lip}_b^1(M)$. Hence, the map $\iota\colon y \mapsto h_y$ maps M into $\mathrm{Lip}_b^1(M)$. This map is injective. We denote by $\overline{M}^h$ the closure of $\iota(M)$ in $\mathrm{Lip}_b^1(M)$, and call it the *horofunction compactification* of M. We define the *horofunction boundary* of M to be the set $\partial\overline{M}^h = \overline{M}^h \backslash \iota(M)$, and its elements are called *horofunctions*. We should immediately point out that the horofunction boundary of M is defined differently to the usual topological boundary of $\iota(M)$ in $\bar{M}^h$, which would be equal to $\overline{M}^h \backslash \iota(M)^\circ$. This difference is discussed more fully in Remark 3.1.9. Some authors instead equip $\mathrm{Lip}_b^1(M)$ with the topology of uniform convergence on compact sets, however this does not matter in light of the following lemma:

Lemma 3.1.2. *The closure of $\iota(M)$ in $\mathrm{Lip}_b^1(M)$ with respect to the topology of pointwise convergence is equal to its closure with respect to the topology of uniform convergence on compact sets.*

Proof. Recall that a collection of functions $\mathcal{F}$ from some metric space X to another metric space Y is called *equicontinuous* at a point $x \in X$, if for every $\varepsilon > 0$ there exists a $\delta > 0$, such that, if $y \in B(x;\delta)$, then $f(y) \in B(f(x);\varepsilon)$ for all $f \in \mathcal{F}$. If this collection is equicontinuous at every point, we call it an equicontinuous collection. We claim that the topology of pointwise convergence coincides with the topology of uniform convergence on compact sets if $\mathcal{F}$ is equicontinuous. Recall that, for $x \in X$ and $y \in Y$ and $r > 0$, the collection of sets

$$A_{x,y,r} = \{f \in \mathcal{F} : f(x) \in B(y; r)\}$$

forms a subbasis for the topology of pointwise convergence on $\mathcal{F}$. For compact $K \subseteq X$ and $y \in Y$, $r > 0$, the sets

$$B_{K,y,r} = \{f \in \mathcal{F} : f(x) \in B(y; r) \text{ for all } x \in K\}$$

form a subbasis for the topology of uniform convergence on compact sets on $\mathcal{F}$. It is immediately clear from this that the topology of pointwise convergence is coarser than the topology of uniform convergence on compact sets, because singletons are compact. Conversely, suppose we are given a compact $K \subseteq X$ and open ball $B(y; r) \subseteq Y$. As $\mathcal{F}$ is equicontinuous, for every $x \in K$ there exists a $\delta_x > 0$ such that $f(B(x; \delta_x)) \subseteq B(f(x); r/3)$ for all $f \in \mathcal{F}$. As K is compact, there exists points $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^n B(x_i; \delta_{x_i})$. We then define

$$A = \bigcap_{i=1}^{n} A_{x_i,y,r/3},$$

and fix an arbitrary $f \in A$. Now any $x \in K$ is an element of some $B(x_i; \delta_{x_i})$, so $d(f(x), y) \leq d(f(x), f(x_i)) + d(f(x_i), y) \leq r/2$, meaning $f \in A$. Thus the topology of pointwise convergence is also finer than the topology of uniform convergence on compact sets of $\mathcal{F}$, so these topologies are equivalent as claimed.

Now, $\mathrm{Lip}_b^1(M)$ is a collection of functions from M to $\mathbb{R}$, and if we fix some $x \in M$ and $\varepsilon > 0$, by the definition of $\mathrm{Lip}_b^1(M)$ as a collection of 1-Lipschitz functions means that choosing $\delta = \varepsilon$ shows by definition that $\mathrm{Lip}_b^1(M)$ is equicontinuous at x. Thus $\mathrm{Lip}_b^1(M)$ is an equicontinuous collection, so by the above paragraph we are done. □

The choice of basepoint is not important, as the horofunction compactifications with a different basepoint are homeomorphic and horofunctions with respect to different basepoints agree up to a constant shift. It is thus convenient, if the metric space under consideration is a normed space, to set the base-point equal to 0. We shall do this throughout unless otherwise specified. For any metric space, the map ι is a continuous injection, which we prove in Proposition 3.1.3 below, but it may fail to have a continuous inverse. If, however, (M, d) is a proper geodesic metric space, then ι is a homeomorphism onto its image [67, Proposition 2.2], which we provide a proof for below.

Proposition 3.1.3. *For any metric space (M, d), the inclusion map $\iota\colon M \to \overline{M}^h$ is a continuous injection onto its image. If (M, d) is proper and geodesic, then it is a homeomorphism onto its image.*

Proof. We first prove injectivity. Suppose that $\iota(x) = \iota(z)$. Then, for all $y \in M$,

$$d(y, x) - d(b, x) = d(y, z) - d(b, z).$$

Plugging in $y = x$ and $y = z$ into the above equation shows that $d(b, x) = d(b, z)$, from which we see that $d(x, z) = d(x, x) = 0$, and indeed $x = z$. Suppose that $(x_\alpha) \subseteq M$ is a net converging to $x \in M$. Then, we can use the triangle and reverse triangle inequalities to show that

$$d(y, x) - d(x, x_\alpha) - (d(x, x_\alpha) + d(b, x)) \leq h_{x_\alpha}(y) \leq d(x, x_\alpha) + d(y, x) + d(x, x_\alpha) - d(b, x)$$

for any $y \in M$ and any α. Taking the limit in α in the above inequality therefore shows that $\lim_\alpha h_{x_\alpha}(y) = h_x(y)$ for all $y \in M$, so indeed ι is continuous.

Let us now assume that (M, d) is proper and geodesic. We note that as (M, d) is proper, $\mathrm{Lip}_b^1(M)$ is metrisable by Lemma 3.1.4 proved below. It therefore suffices to show that ι^{-1} is sequentially continuous to prove that ι is a homeomorphism onto its image. Suppose that $(h_{x_n}) \subseteq \iota(M)$ converges to some $h_x \in \iota(M)$. If $(x_n) \subseteq M$ is bounded, there exists a closed ball $B(x; r)$ containing the sequence. As M is proper this ball is compact, so for every subsequence of (x_n) there exists a further subsequence (x_{n_k}), and a

$z \in B(x;r)$ such that $d(x_{n_k}, z) \to 0$. The same argument used to show the continuity of ι above shows that $h_{x_{n_k}}(y) \to h_z(y)$ for every $y \in M$. As $\lim h_{n_k}(y)$ also equals $h_x(y)$ for all $y \in M$, the bijectivity of ι means that $z = x$. Thus every subsequence of (x_n) has a further subsequence converging to x, so indeed $\lim_{n\to\infty} \iota^{-1}(h_{x_n}) = x = \iota^{-1}(h_x)$. Let us now suppose that $d(x_n, b) \to \infty$. As M is geodesic, for every $n \in \mathbb{N}$ there exists a geodesic path $\gamma_n \colon [0, d(x, x_n)] \to M$ from x to x_n. We fix some $r > 0$. As $d(x_n, b) \to \infty$, there must exist a $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \notin B(x;r)$. As the paths γ_n are continuous, for every $n \geq N$ there exists a $t_n \in [0, d(x, x_n))$ such that $d(\gamma(t_n), x) = r$. Thus, for any $n \geq N$,

$$\begin{aligned} h_{x_n}(\gamma_n(t_n)) - h_x(\gamma_n(t_n)) &= d(\gamma(t_n), x_n) - d(x_n, b) - d(\gamma(t_n), x) + d(x, b) \\ &= d(x, x_n) - d(x_n, b) - 2r + d(x, b) = h_{x_n}(x) - 2r + d(x, b). \end{aligned} \tag{3.1.2}$$

As M is proper, $B(x;r)$ is compact, and each $\gamma_n(t_n) \in B(x;r)$, our assumption means that $h_{x_n} \to h_x$ uniformly on $B(x;r)$. Thus we must have

$$\lim_{n\to\infty} h_{x_n}(\gamma_n(t_n)) - h_x(\gamma_n(t_n)) = 0,$$

but (3.1.2) above shows that

$$\lim_{n\to\infty} h_{x_n}(\gamma_n(t_n)) - h_x(\gamma_n(t_n)) = h_x(x) - 2r + d(x, b) = -2r.$$

We chose $r > 0$, so this is a contradiction. Thus if $h_{x_n} \to h_x$ in $\iota(M)$, the corresponding sequence $(x_n) = (\iota^{-1}(x_n)) \subseteq M$ has to be bounded, and $d(x_n, x) \to 0$, proving that ι^{-1} is continuous when M is proper and geodesic.

□

The following lemma is useful, as we saw in the above proof.

Lemma 3.1.4. *If (M, d) is separable, then $\mathrm{Lip}_b^1(M)$ equipped with the topology of pointwise convergence is metrisable.*

Proof. We first show that $\mathrm{Lip}_b^1(M)$ is second countable. Let $A \subseteq M$ be a countable dense subset. For each $x_a \in A$, $q, r \in \mathbb{Q}$ define,

$$S_{x_a,B(q,r_q)} = \{f \in \mathrm{Lip}_b^1(M) : f(x_n) \in B(q,r)\},$$

and let $\mathcal{Q}$ be the collection of all the $S_{x_a,B(q,r)}$, so $\mathcal{Q}$ is countable. For every open $U \subseteq \mathbb{R}$ and $x \in M$ let $S_{x,U}$ be as in (2.1.1), and let $\mathcal{C}$ be the collection consisting of all $S_{x,U}$. As $\mathcal{Q} \subseteq \mathcal{C}$, it follows that the topology generated by $\mathcal{Q}$ is coarser than the topology of pointwise convergence on $\mathrm{Lip}_b^1(M)$. Thus, if we can show that $\mathcal{C}$ is contained in the topology generated by $\mathcal{Q}$ we would be done, because then $\mathcal{Q}$ must generate the same topology that $\mathcal{C}$ generates, which is the topology of pointwise convergence on $\mathrm{Lip}_b^1(M)$.

To that end, let us fix some $S_{x,U} \in \mathcal{C}$. We can choose, for $i \in \mathbb{N}$, $0 < r_i \in \mathbb{Q}$ and $q_i \in \mathbb{Q}$ such that $U = \cup_{i=1}^{\infty} B(q_i, r_i)$. Furthermore, as $\mathbb{R}$ is linearly ordered, we can make this choice in such a way that every interval overlaps with precisely two other intervals, but the centre of each interval is not contained in any other interval. Precisely, for each $i \in \mathbb{N}$, there exists a and $k_i^+, k_i^- \in \mathbb{N}$ such that $q_{k_i^-} + r_i < q_i < q_{k_i^+} - r_i$, and $q_{k_i^-} + r_{k_i^-} < q_i < q_{k_i^+} - r_{k_i^+}$, and $B(q_i; r_i) \cap B(q_j; r_j) = \emptyset$ for all $j \notin \{i, k_i^+, k_i^-\}$, but the length of the interval $B(q_i; r_i) \cap B(q_{k_i^+}; r_{k_i^+})$ is equal to some $0 < \delta_i^+ < r_{k_i^+}/2$, and the length of the interval $B(q_i; r_i) \cap B(q_{k_i^-}; r_{k_i^-})$ is equal to some $0 < \delta_i^- < r_{k_i^-}/2$. Clearly $\delta_i^+ = \delta_{k_i^+}^-$. The density of A means that for each $i \in \mathbb{N}$ we can choose an $x_i \in A$ such that $d(x, x_i) < \frac{1}{3}\min\{\delta_i^+, \delta_i^-\}$. Let us now fix some $f \in S_{x,U}$. There must be some $i \in \mathbb{N}$ such that $f(x) \in B(q_i; r_i)$. Now,

$$|f(x_i) - q_i| \leq |f(x_i) - f(x)| + |f(x) - q_i| \leq d(x_i, x) + |f(x) - q_i|.$$

If $f(x) \geq q_i$, and $f(x) - q_i < r_i - d(x_i, x)$ then $f \in S_{x_i,B(q_i,r_i)}$. However, if $f(x) - q_i \geq$

$r_i - d(x_i, x)$, then $q_{k_i^+} - f(x) \le r_{k_i^+} - \frac{2}{3}\delta_i^+$, but then

$$\begin{aligned}|f(x_{k_i^+}) - q_{k_i^+}| &\le |f(x) - q_{k_i^+}| + |f(x) - f(x_{k_i^+})| \\ &\le r_{k_i^+} - \frac{2}{3}\delta_i^+ + d(x, x_{k_i^+}) \\ &\le r_{k_i^+} - \frac{2}{3}\delta_i^+ + \frac{1}{3}\delta_{k_i^+}^- \\ &= r_{k_i^+} - \frac{2}{3}\delta_i^+ + \frac{1}{3}\delta_i^+,\end{aligned}$$

meaning that $f(x_{k_i}^+) \in S_{x_{k_i^+}, B(q_{k_i^+}; r_{k_i^+})}$. A symmetrical argument shows that if $f(x) < q_i$ then either $f \in S_{x_i, B(q_i, r_i)}$ or $f(x_{k_i}^+) \in S_{x_{k_i^+}, B(q_{k_i^+}; r_{k_i^+})}$. Thus, $S_{x,U} \subseteq \cup_{i=1}^{\infty} S_{x_i, B(q_i, r_i)}$. Conversely, suppose $f \in \cup_{i=1}^{\infty} S_{x_i, B(q_i, r_i)}$, so $f \in S_{x_j, B(q_j, r_j)}$ for some $j \in \mathbb{N}$. Then,

$$\begin{aligned}|f(x) - q_j| &\le |f(x) - f(x_j)| + |f(x_j) - q_j| \\ &< d(x, x_j) + r_j \\ &< \min\{\delta_j^+, \delta_j^-\} + r_j,\end{aligned}$$

meaning that $f(x) \in B(q_{k_j^-}, r_{k_j^-}) \cup B(q_j, r_j) \cup B(q_{k_j^+}, r_{k_j^+}) \subseteq U$. Therefore we can conclude that $S_{x,U} = \cup_{i=1}^{\infty} S_{x_i, B(q_i, r_i)}$, so $\mathcal{C}$ is indeed contained in the topology generated by $\mathcal{Q}$. Any regular second countable topological space is metrisable by Urysohn's metrisation theorem [54, Theorem 34.1], and we have just shown that $\mathrm{Lip}_b^1(M)$ is second countable. Furthermore, $\mathrm{Lip}_b^1(M)$ is a subspace of a product of intervals, so it is regular [54, Theorem 33.2], so indeed $\mathrm{Lip}_b^1(M)$ is metrisable. □

Lemma 3.1.5 and Proposition 3.1.3 imply the following useful and well known proposition, (see for example [43, Lemma 2.1]), but for which we provide our own proof building on what we have already established for the convenience of the reader.

Proposition 3.1.5. *If (M, d) is a proper geodesic metric space, a function $h \in \mathrm{Lip}_b^1(M)$ is a horofunction if and only if there exists a sequence $(x_n) \subseteq M$, such that $d(b, x_n) \to \infty$ and $h_{x_n} \to h$.*

Proof. First assume that $h \in \partial\overline{M}^h$ is a horofunction. As M is separable by Lemma 3.1.4, this means that there must exist a sequence $(x_n) \subseteq M$ such that $h_{x_n} \to h$. Assume by way of contradiction that (x_n) is bounded. There must then exist a closed ball $B(b;r)$ such that $x_n \in B(b;r)$ for all $n \in \mathbb{N}$. As M is proper, this ball is compact. There thus exists a subsequence (x_{n_k}) and a $x \in B(b;r)$ such that $\lim_{k\to\infty} d(x_{n_k}, x) = 0$. For any $y \in M$, and any $k \in \mathbb{N}$, $h_{x_{n_k}}(y) = d(y, x_{n_k}) - d(b, x_{n_k})$, so we can use the triangle inequality and reverse triangle inequality to bound

$$d(y,x) - d(x,x_{n_k}) - (d(x,x_{n_k}) + d(b,x)) \leq h_{x_{n_k}}(y) \leq d(x,x_{n_k}) + d(y,x) + d(x,x_{n_k}) - d(b,x).$$

The squeeze theorem applied to the above inequality means that $\lim_{k\to\infty} h_{x_{n_k}}(y) = h_x(y)$. However, we assumed that h was not a internal metric functional, a contradiction. Thus $d(b, x_n)$ is unbounded. Conversely, suppose there exists a $h \in \mathrm{Lip}^1_b(M)$ and a sequence $(x_n) \subseteq M$ with $d(b, x_n) \to \infty$ such that $h_{x_n} \to h$. This immediately implies that $h \in \overline{M}^h$. Assume $h = h_x \in \iota(M)$, where we recall $\iota\colon M \to \mathrm{Lip}^1_b(M)$ is the map $x \mapsto h_x$, then $\lim_{n\to\infty} h_{x_n} = h_x$. However, $d(b, x_n) \to \infty$, so the sequence $\iota^{-1}(h_{x_n}) = x_n$ does not converge to x in M, contradicting Proposition 3.1.3. □

As a corollary to Proposition 3.1.5 and the proof of Proposition 3.1.3, we have:

Corollary 3.1.6. *Any horofunction of a proper geodesic metric space is unbounded below.*

Proof. Let $h \in \partial\overline{M}^h$ be a horofunction for a proper geodesic metric space (M, d). Lemma 3.1.5 means there exists a sequence $(x_n) \subseteq M$ such that $d(b, x_n) \to \infty$ and $h_{x_n} \to h$. Now, following the construction in the latter half of the proof of Proposition 3.1.3, fix a $r > 0$ and consider the closed ball $B(b;r)$. The unboundedness of (x_n) means that there exists a $N_r \in \mathbb{N}$, such that for all $n \geq N_r$, $x_n \notin B(b;r)$. Following the same reasoning as in the proof of Proposition 3.1.3, because M is proper and geodesic, for every $n \geq N_r$, there exists a z_n in the sphere $S(b;r)$ such that $d(b, x_n) = d(b, z_n) + d(z_n, x_n)$, and

$\lim_{n\to\infty} \sup_{y\in B(b;r)} |h(y) - h_{x_n}(y)| = 0$. Furthermore, for any $n \geq N_r$

$$h(z_n) - h_{x_n}(z_n) = h(z_n) - d(x_n, z_n) + d(b, x_n) = h(z_n) + r.$$

Therefore, for every $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $h(z_N) \leq -r + \varepsilon$. As $r > 0$ is arbitrary, the result follows. □

For general metric spaces, Proposition 3.1.5 need not hold. All that can be said is that $h \in \mathrm{Lip}_b^1(M)$ is in $\overline{M}^h$ if and only if there exists a net $(y^\alpha) \subseteq M$ such that $h_{y^\alpha} \to h$, which is a consequence of Example 2.3.1.

Example 3.1.7. Consider the Banach space $X = \ell^1(\mathbb{N})$, the space of all real-valued, absolutely summable sequences, equipped with the norm $\|x\|_1 = \sum_{i=1}^\infty |x_i|$ for all $x = (x_1, x_2, \dots) \in X$. This is a well-known infinite dimensional Banach space, so it is geodesic, but not proper. Consider the sequence $(y^n) \subseteq X$, defined by $y_i^n = 0$ if $i \neq n$, and $y_n^n = n$. Clearly $\|y^n\|_1 \to \infty$, but, for any $x \in X$ and $n \in \mathbb{N}$

$$h_{y^n}(x) = \|x - y^n\|_1 - \|y^n\|_1 = \sum_{i\neq n} |x_i| + |n - x_n| - n.$$

Thus by taking n large enough, because $x_n \to 0$, we have

$$h_{y^n}(x) = \sum_{i\neq n} |x_i| - x_n \to \sum_{i=1}^\infty |x_i| = \|x\|_1 = h_0(x).$$

This example shows that Proposition 3.1.5 can fail if the space is not proper. This further shows that $\partial\overline{X}^h$ is not a compactification in the traditional sense, because this example shows the inverse of ι on $\iota(M)$ is not continuous.

For non-proper metric spaces, the state of Proposition 3.1.5 is even worse than the above example shows, as the next example shows it can fail in both directions.

Example 3.1.8. Consider the Banach space $X = \ell^2(\mathbb{N})$, the space of all real-valued, square summable sequences, equipped with the norm $\|x\|_2 = \left(\sum_{i=1}^\infty |x_i|^2\right)^{1/2}$ for all $x =$

$(x_1, x_2, \dots) \in X$. This is a well-known infinite dimensional Banach space, so it is geodesic, but not proper. In fact X is a Hilbert space, with norm arising from inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$. We consider the sequence (e_n), where $e_n(n) = 1$ and $e_n(j) = 0$ if $j \neq n$. For any $x \in X$,

$$h_{e_n}(x) = \sqrt{\|x\|_2^2 - 2\langle e_n, x\rangle + 1} - 1 \to \sqrt{\|x\|_2^2 + 1} - 1.$$

Thus, h_{e_n} converges to some $h \in \overline{X}^h$, but h is bounded below by 0, so the only internal metric functional h could be is h_0, but h is clearly not h_0. Therefore, there is an element in $\partial\overline{X}^h$ arising from a bounded sequence.

Remark 3.1.9. Example 3.1.8 shows that, in the non-proper case, there can exist horofunctions which are bounded below, showing that Corollary 3.1.6 does not hold true for non-proper metric spaces. This leads to the question, if (M, d) is a metric space possessing horofunctions that are bounded below, are there internal metric functionals that are the limit of a sequence of bounded horofunctions in $\overline{M}^h$? We will see in Section 7.1 that, in fact, for an infinite dimensional Hilbert space $\mathcal{H}$, the collection of horofunctions bounded below is dense in $\overline{\mathcal{H}}^h$. Furthermore, $\overline{\mathcal{H}}^h$ is a proper compactification, in the sense that $\iota\colon \mathcal{H} \to \overline{\mathcal{H}}^h$ is a homeomorphism onto its image. This illustrates why we define $\partial\overline{M}^h$ as $\overline{M}^h \backslash \iota(M)$, and not as the topological boundary of $\iota(M)$ in $\overline{M}^h$, which would be $\overline{M}^h \backslash \iota(M)^\circ$, where $\iota(M)^\circ$ is the interior of $\iota(M)$ considered as a subset of $\overline{M}^h$. In the infinite dimensional Hilbert space case, $\iota(\mathcal{H})^\circ$ is empty, so if we chose to define horofunctions as elements of the usual topological boundary of $\iota(\mathcal{H})$, every internal metric functional would be classed as a horofunction, which would not be useful nomenclature. A subtle point this discussion highlights, is that just because ι is a homeomorphism onto its image for a metric space M, it does not mean that $\iota(M)$ has to be open in $\overline{M}^h$ when considered as a subset of $\overline{M}^h$. All that it means is that when $\iota(M)$ is considered as topological subspace of $\overline{M}^h$, then $\iota\colon M \to \iota(M)$ maps open and closed sets respectively to open and closed sets in $\iota(M)$ equipped with the subspace topology, where we recall that a set $A \subseteq \iota(M)$ is open (closed) if and only if there exists an open (closed) $B \subset \overline{M}^h$ such that $A = \iota(M) \cap B$. It says nothing about whether $\iota(M)$ is open in $\overline{M}^h$ or not.

Remark 3.1.9 would be unnecessary if we were only dealing with proper geodesic metric spaces, as we prove in the following proposition:

Proposition 3.1.10. *The horofunction boundary of a proper geodesic metric, (M, d), is closed in $\overline{M}^h$.*

Proof. Fix a basepoint b. To show that $\partial\overline{M}^h$ is closed in $\overline{M}^h$ it suffices, by Lemma 3.1.4 because M is proper, to show that, if $(h_n) \subseteq \partial\overline{M}^h$ converges to some $h \in \overline{M}^h$, then $h \in \partial\overline{M}^h$. Suppose by way of contradiction that $h = h_x$ for some $x \in M$. We set $r = 2 + d(b; x)$. As M is proper and geodesic, Proposition 3.1.5 means that for each $n \in \mathbb{N}$ there exists a sequence $(x_m^{(n)})_{m\in\mathbb{N}}$ converging to infinity, such that $\lim_{m\to\infty} h_{x_m^{(n)}} = h_n$ uniformly on closed balls. Thus, for every $n \in \mathbb{N}$, there exists a point $z_n \in M$ such that $d(z_n, b) > r$, and

$$\sup_{y\in B(b;r)} |h_{z_n}(y) - h_n(y)| < \frac{1}{2^n}.$$

Borrowing the same argument as in the second half of the proof of Proposition 3.1.3, the fact that M is geodesic means that, for each $n \in \mathbb{N}$, there exists a $w_n \in S(b; r)$ such that $d(b, z_n) = d(b, w_n) + d(w_n, z_n)$. Now, for any $n \in \mathbb{N}$,

$$\begin{aligned}\sup_{y\in B(b;r)} |h_{z_n}(y) - h(y)| &\le \sup_{y\in B(b;r)} |h_{z_n}(y) - h_n(y)| + \sup_{y\in B(b;r)} |h_n(y) - h(y)| \\ &< \frac{1}{2^n} + \sup_{y\in B(b;r)} |h_n(y) - h(y)| \to 0.\end{aligned}$$

This implies that there exists a $N \in \mathbb{N}$ such that $h(w_N) - h_{z_N}(w_N) \le 1$. However, we can calculate

$$h(w_N) - h_{z_N}(w_N) = h(w_N) - d(w_N, z_N) + d(b, z_N) = h(w_N) + d(b, w_N) = h(w_N) + r.$$

Thus $h_x(w_N) \le -1 - d(b, x)$, contradicting the fact that h_x is bounded below by $-d(b, x)$. Therefore $h \in \partial\overline{M}^h$.

□

Proposition 3.1.10 immediately implies that if (M, d) is a proper geodesic metric space, $\iota(M)$ is open in $\overline{M}^h$, so equal to its interior. The horofunction boundary of proper geodesic metric spaces is therefore equal to the topological boundary of $\iota(M)$ in $\overline{M}^h$.

The failure of Proposition 3.1.5 to hold in the non-proper case has lead some authors (see [27] and [37] for example) to introduce some specific notation for the horofunction compactification of non-proper metric spaces. For a general metric space M we can partition $\overline{M}^h$ into the disjoint union $\overline{M}^h = \overline{M}^{h,\infty} \cup \overline{M}^{h,e}$, where we identify M with $i(M)$, and

$$\overline{M}^{h,\infty} = \{h \in \partial\overline{M}^h : \inf_M h(y) = -\infty\} \tag{3.1.3}$$

$$\overline{M}^{h,e} = \{h \in \partial\overline{M}^h : \inf_M h(y) > -\infty\}. \tag{3.1.4}$$

Thus $\overline{M}^{h,\infty}$, the horofunctions (or metric functionals at infinity according to some authors), consists solely of those horofunctions that can only be achieved as the limits of norm unbounded nets. On the other-hand the set of *exotic metric functionals* $\overline{M}^{h,e}$ consists of all those elements of $\overline{M}^h$ which are bounded below, but are not internal metric functionals.

We should also note here that the term horofunction can be used to describe slightly different functions than what we have defined horofunctions to be, and some authors may refer to what we refer to as horofunctions as metric functionals instead, (see for example [28, 37]). This disparity can be traced back to Gromov's 1978 paper [25], where he introduced horofunctions in order to study hyperbolic manifolds and groups. His construction was identical to ours, except he equipped $\mathrm{Lip}_b^1(M)$ with the topology of uniform convergence on bounded sets, instead of uniform convergence on compact sets, and he called horofunctions the elements in the boundary of the closure of $\iota(M)$ with respect to the topology of uniform convergence on bounded sets. If M is proper, then Gromov's definition is equivalent to the one we use, but for non-proper spaces they can be vastly different. We have chosen to follow the naming conventions of Rieffel [60] instead.

3.2 Busemann Points and Parts of the Boundary

3.2.1 Almost-Geodesics and Busemann Points

In a proper geodesic metric space (M, d), it has long been known (see for example [10, Lemma 8.18]) if $\gamma\colon [a,\infty) \to M$ is a geodesic, $\lim_{t\to\infty} h_{\gamma(t)}(z)$ exists for every $z \in M$. Thus every unbounded geodesic gives rise to a horofunction. In [60], Rieffel introduced the notion of an almost-geodesic, which also always give rise to a horofunction [60, Lemma 4.5]. An almost-geodesic is a path $\gamma\colon [0,\infty) \to M$, for which there exists, for every $\varepsilon > 0$, an $N \in \mathbb{N}$, such that for all $t \geq s \geq N$,

$$|d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t| < \varepsilon \tag{3.2.1}$$

By choosing $s = t$ we can see that this definition means an almost-geodesic is ”almost” parametrised by arc-length, i.e. for every $\varepsilon > 0$, $|d(\gamma(t), \gamma(0)) - t| < \varepsilon$ for all sufficiently large t. This condition can actually be relaxed, and we define an *almost-geodesic sequence* as a sequence $(x^n) \subseteq M$ such that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$, where for all $n \geq m \geq N$,

$$d(x^n, x^m) + d(x^m, x^0) - d(x^n, x^0) < \varepsilon. \tag{3.2.2}$$

This definition thus allows for bounded almost-geodesic sequences, and the rate that an unbounded almost-geodesic sequence escapes to infinity is not constrained. As we shall see later this will prove useful for more general constructions. The following lemma shows that these two different notions of almost-geodesics give rise to the same horofunctions. We should note that this result was already proved in [2, Proposition 7.12], but in a setting quite removed from ours, and using different constructions than what we will use.

Lemma 3.2.1. *If (M,d) is a proper geodesic metric space, and γ is an almost-geodesic such that $\lim_{t\to\infty} h_{\gamma(t)} = h \in \partial\overline{M}^h$, then there must exist a corresponding almost-geodesic sequence in the sense of (3.2.2), (x^n), such that $\lim_{n\to\infty} h_{x^n} = h$ in the topology of pointwise convergence. Conversely, if (x^n) is an unbounded almost-geodesic sequence giving rise to a horofunction h, then there must exist an almost-geodesic giving rise to h.*

Proof. First let γ be an almost-geodesic such that $\lim_{t\to\infty} h_{\gamma(t)} = h \in \partial\overline{M}^h$. Define a sequence $(x^n) \subseteq M$ by $x^n = \gamma(n)$ for all $n \in \mathbb{N}$. Fix $\varepsilon > 0$. As γ is an almost-geodesic there exists an $N \in \mathbb{N}$ such that, for all $n \geq m \geq N$,

$$|d(\gamma(n),\gamma(m)) + d(\gamma(m),\gamma(0)) - n| < \varepsilon.$$

Thus, for any $n \geq m \geq N$

$$d(x^n,x^m) + d(x^m,x^0) - d(x^n,x^0) \leq |d(x^n,x^m) + d(x^m,x^0) - n| + |n - d(x^n,x^0)| < 2\varepsilon.$$

There (x^n) is an almost-geodesic sequence, and because $\lim_{t\to\infty} h_{\gamma(t)}(y) = h(y)$ for all $y \in M$, we must have $\lim_{n\to\infty} h_{\gamma(n)}(y) = h(y)$ for all $y \in M$. Conversely suppose that (x^n) is an unbounded almost-geodesic sequence where $\lim_{n\to\infty} h_{x^n}(y) = h(y)$ for all $y \in M$, with $h \in \partial\overline{M}^h$. As (x^n) is unbounded, $d(x^n,x^0) \to \infty$. By choosing subsequences and relabelling we can thus assume that $d(x^{n+1},x^0) > d(x^n,x^0)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define $t_n = d(x^n,x^0)$. As M is a geodesic metric space, we can define, for each $n \in \mathbb{N}$, a geodesic path $c_n\colon [0, d(x^{n+1},x^n)] \to M$ such that $c_n(0) = x^n$ and $c_n(d(x^{n+1},x^n)) = x^{n+1}$. If we now define, for each $n \in \mathbb{N}$, $\delta_n = t_{n+1} - t_n$, we can define $\gamma_n\colon [t_n, t_{n+1}] \to M$ by

$$\gamma_n(t) = c_n\left((t-t_n)\frac{d(x^{n+1},x^n)}{\delta_n}\right).$$

Now, for any $n \in \mathbb{N}$,

$$\gamma_n(t_{n+1}) = c_n(d(x^{n+1},x^n)) = x^{n+1} = c_{n+1}(0) = \gamma_{n+1}(t_{n+1}),$$

so the γ_n's agree on the endpoints of their domains. We can thus define $\gamma\colon [0,\infty) \to M$ by $\gamma(t) = \gamma_n(t)$ for $t \in [t_n, t_{n+1}]$, which is well-defined by the preceding remark. We claim that γ is an almost-geodesic. To do so, we first show that for any $\varepsilon > 0$, there exists a $N^1_\varepsilon \in \mathbb{N}$ such that for all $t \geq N^1_\varepsilon$,

$$|d(\gamma(t), \gamma(0)) - t| < \varepsilon. \tag{3.2.3}$$

As (x^n) is an almost-geodesic, we can choose $N^1_\varepsilon \in \mathbb{N}$ such that (3.2.2) holds for all $n \geq m \geq N^1_\varepsilon \in \mathbb{N}$. Now suppose that $t \in [t_n, t_{n+1}]$ for $t_n \geq N^1_\varepsilon$. Then

$$\begin{aligned}
d(\gamma(t), \gamma(0)) - t &\leq d(\gamma(t), \gamma(t_n)) + d(\gamma(t_n), \gamma(0)) - (t - t_n + t_n) \\
&= d(\gamma(t), \gamma(t_n)) - (t - t_n) \\
&= d(c_n\left((t - t_n)\frac{d(x^{n+1}, x^n)}{\delta_n}\right), c_n(0)) - (t - t_n) \\
&= (t - t_n)\frac{d(x^{n+1}, x^n)}{\delta_n} - (t - t_n) \\
&= \frac{t - t_n}{\delta_n}(d(x^{n+1}, x^n) - \delta_n) \\
&\leq d(x^{n+1}, x^n) + d(x^n, x^0) - d(x^{n+1}, x^0) < \varepsilon.
\end{aligned}$$

Note for t as above, we can write $t = t_n + d(\gamma(s), x^n)$ for $s \in [t_n, t_{n+1}]$. So $d(\gamma(s), x^n) = t - t_n$, and from above we know that

$$d(\gamma(s), x^n) = (s - t_n)\frac{d(x^{n+1}, x^n)}{\delta_n}.$$

The reverse triangle inequality means that $\frac{d(x^{n+1}, x^n)}{\delta_n} \geq 1$, from which we conclude that $s \leq t$. Thus $d(\gamma(s), x^n) \leq d(\gamma(t), x^n)$. Also note that

$$\begin{aligned}
d(\gamma(t), x^{n+1}) &= d(c_n\left((t - t_n)\frac{d(x^{n+1}, x^n)}{\delta_n}\right), c_n(d(x^{n+1}, x^n))) \\
&= d(x^{n+1}, x^n) - (t - t_n)\frac{d(x^{n+1}, x^n)}{\delta^n},
\end{aligned}$$

meaning that $d(\gamma(t), x^n) + d(x^{n+1}, \gamma(t)) = d(x^{n+1}, x^n)$. Combining all of the above:

$$\begin{aligned} t - d(\gamma(t), \gamma(0)) &= d(\gamma(s), x^n) + d(x^n, x^0) - d(\gamma(t), x^0) \\ &\leq d(\gamma(t), x^n) + d(x^{n+1}, \gamma(t)) + d(x^n, x^0) - d(\gamma(t), x^{n+1}) - d(\gamma(t), x^0) \\ &\leq d(x^{n+1}, x^n) + d(x^n, x^0) - d(x^{n+1}, x^n) < \varepsilon. \end{aligned}$$

Thus indeed for any $\varepsilon > 0$, there exists a $N^1_\varepsilon \in \mathbb{N}$ such that for all $t \geq N^1_\varepsilon$, $|d(\gamma(t), \gamma(0)) - t| < \varepsilon$. We now need to show that, for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that, for all $t \geq s \geq N$,

$$|d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t| < \varepsilon.$$

Let us fix $\varepsilon > 0$ and $t \geq s \geq N^1_\varepsilon$. We then know from above that

$$-\left(d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t\right) \leq t - d(\gamma(t), \gamma(0)) < \varepsilon.$$

Now suppose that $t, s \in [t_n, t_{n+1}]$ for some $t_n \geq N^1_\varepsilon$. Then, using (3.2.3) and what we have seen above,

$$\begin{aligned} d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t &\leq d(\gamma(t), \gamma(s)) - (t - s) + \varepsilon \\ &= (t - t_n)\frac{d(x^{n+1}, x^n)}{\delta_n} - (s - t_n)\frac{d(x^{n+1}, x^n)}{\delta_n} - (t - s) + \varepsilon \\ &= (t - s)\frac{d(x^{n+1}, x^n)}{\delta_n} - (t - s) + \varepsilon. \end{aligned}$$

However, as (x^n) is an almost-geodesic sequence, and how N^1_ε is defined, we know that $d(x^{n+1}, x^n) < \varepsilon + d(x^{n+1}, x^0) - d(x^n, x^0)$, meaning that

$$\begin{aligned} (t - s)\frac{d(x^{n+1}, x^n)}{\delta_n} - (t - s) + \varepsilon &< (t - s)\left(\frac{\varepsilon}{\delta_n} + 1\right) - (t - s) + \varepsilon \\ &= \varepsilon\frac{t - s}{\delta_n} + \varepsilon \leq 2\varepsilon. \end{aligned}$$

Finally, let us suppose that $t \in [t_n, t_{n+1}]$ and $s \in [t_m, t_{m+1}]$ for $n > m$, and $t_m \geq N^1_\varepsilon$. Then,

$$\begin{aligned}
&d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t \\
\leq &d(\gamma(t), x^n) + d(x^n, x^{m+1}) + d(x^{m+1}, \gamma(s)) + d(\gamma(s), x^m) + d(x^m, x^0) - t \\
= &d(x^{n+1}, x^n) + d(x^n, x^{m+1}) + d(x^{m+1}, x^m) + d(x^m, x^0) - t - d(x^{n+1}, \gamma(t)) \\
= &d(x^{n+1}, x^n) + t_n + d(x^n, x^{m+1}) + t_{m+1} - t_n + d(x^{m+1}, x^m) + d(x^m, x^0) - t_{m+1} \\
&- t - d(x^{n+1}, \gamma(t)) \\
< &d(x^{n+1}, x^n) + d(x^n, x^0) - t - d(x^{n+1}, \gamma(t)) + 2\varepsilon \\
= &(x^{n+1}, x^n) + d(x^n, x^0) - t - (t_{n+1} - t)\frac{d(x^{n+1}, x^n)}{\delta_n} + 2\varepsilon \\
\leq &(x^{n+1}, x^n) + d(x^n, x^0) - d(x^{n+1}, x^0) + 2\varepsilon < 3\varepsilon.
\end{aligned}$$

We can finally conclude that, indeed, γ is an almost-geodesic path. Thus, $h_{\gamma(t)}$ converges pointwise to some $h' \in \partial\overline{M}^h$ [60, Lemma 4.5], but because $h_{\gamma(t_n)} = h_{x^n} \to h$, and $\overline{M}^h$ is Hausdorff, we must have $\lim_{t\to\infty} h_{\gamma(t)} = h$. □

Following the lead of Rieffel, in the context of proper geodesic metric spaces we call a horofunction $h \in \partial\overline{M}^h$ a *Busemann point* if there exists an almost-geodesic $\gamma\colon [0, \infty) \to M$ such that, for all $z \in M$,

$$h(z) = \lim_{t\to\infty} h_{\gamma(t)}(z).$$

We shall use the notation $\partial_B\overline{M}^h$ to represent the Busemann points of M. Lemma 3.2.1 means that the Busemann points are the sets of all horofunctions h, such that there exists an unbounded almost-geodesic sequence (x^n), such that, for all $z \in M$,

$$h(z) = \lim_{n\to\infty} h_{x^n}(z).$$

As mentioned, it is well known that every almost-geodesic sequence gives rise to a horofunction, but the converse, even in proper geodesic metric spaces, is not true. Walsh gives an example in [66] of a finite dimensional normed space where not every horofunction is a Busemann point. When the metric space under consideration is not proper, one needs to consider, instead of almost-geodesic sequences, the so called almost-geodesic nets. An almost-geodesic net starting from a point $x_0 \in M$ is a net $(x^\alpha) \subseteq M$, such that, for all $\varepsilon > 0$ there exists an $\eta \in A$, the indexing set, such that for all $\beta \geq \alpha \geq \eta$,

$$d(x^\beta, x^\alpha) + d(x^\alpha, x_0) - d(x^\beta, x_0) < \varepsilon.$$

Just like for almost-geodesics and almost-geodesic sequences, almost-geodesic nets always give rise to an element in the horofunction compactification. We prove this below. Therefore, in the context of non-proper metric spaces, we define a Busemann point to be a horofunction $h \in \partial\overline{M}^h$ such that there exists an almost-geodesic net (x^α) such that $h = \lim_\alpha h_{x^\alpha}$ in the topology of pointwise convergence.

Lemma 3.2.2. *If (M, d) is any metric space, with given basepoint b, and (x^α) is an almost-geodesic net in M starting at some point $x_0 \in X$, then $h(z) = \lim h_{x^\alpha}(z)$ exists for all $z \in M$. Furthermore, if $d(x_0, x_\alpha) \to \infty$, then h is a horofunction.*

Proof. Fix an arbitrary $\varepsilon > 0$ and $z \in M$. Define $\varphi_{x^\alpha}(z) = d(z, x^\alpha) - d(x_0, x^\alpha)$. The net $\varphi_{x^\alpha}(z)$ is clearly bounded in α. By definition there exists an ζ so that for all $\xi \geq \zeta$, $\varphi_{x^\xi}(z) \leq \limsup_\alpha \varphi_{x^\alpha}(z) + \varepsilon$. Via the definition of an almost-geodesic sequence there exists an $\eta \geq \zeta$, so that for all $\beta \geq \alpha \geq \eta$,

$$\varphi_{x^\beta}(z) - \varphi_{x^\alpha}(z) \leq d(x^\alpha, x^\beta) - d(x_0, x^\beta) + d(x_0, x^\alpha) < \varepsilon,$$

where we made use of the reverse triangle-inequality. As this is true for all $\beta \geq \alpha \geq \eta$ it must follow that

$$\limsup_\beta \varphi_{x^\beta}(z) - \varepsilon - \varphi_{x^\alpha}(z) < \varepsilon,$$

from which we deduce that, for all $\alpha \geq \eta$,

$$\limsup_\beta \varphi_{x^\beta}(z) - 2\varepsilon \leq \varphi_{x^\alpha}(z) \leq \limsup_\beta \varphi_{x^\beta}(z) + \varepsilon.$$

As ε is arbitrary, it follows that $\lim_\beta \varphi_{x^\beta}(z)$ exists for any $z \in M$. We can write, for any α,

$$h_{x^\alpha}(z) = d(z, x^\alpha) - d(x_0, x^\alpha) + d(x_0, x^\alpha) - d(b, x^\alpha) = \varphi_{x^\alpha}(z) - \varphi_{x^\alpha}(b),$$

but we have just proved that $\lim_\alpha \varphi_{x^\alpha}(z) - \varphi_{x^\alpha}(b)$ exists, which proves the claim. Now suppose $d(x_0, x^\alpha) \to \infty$. Then, there exists a γ such that, for all $\beta \geq \alpha \geq \gamma$,

$$\begin{aligned} h_{x^\beta}(x^\alpha) &= \varphi_{x^\beta}(x^\alpha) - \varphi_{x^\beta}(b) \\ &\leq d(x^\alpha, x^\beta) - d(x_0, x_\beta) + d(x_0, b) \\ &\leq 1 + d(x_0, b) - d(x_0, x^\alpha) \end{aligned}$$

As this is true for all $\beta \geq \alpha$, by passing to the limit it follows that $h(x^\alpha) \leq 1 + d(x_0, b) - d(x_0, x^\alpha)$ for all $\alpha \geq \gamma$. Thus $h \in \overline{M}^h$, and $\inf_{x \in M} h(x) = -\infty$, which means $h \in \partial \overline{M}^h$, because all internal metric functionals are bounded below. □

In [69, Proposition 2.5], Walsh proves

Lemma 3.2.3. *If (x^α) is an almost-geodesic in a complete metric space M, then (x^α) converges to some $x \in M$.*

Thus, if (x^α) is a bounded almost-geodesic net in a complete metric space, then h_{x^α} converges to an internal metric point. As a corollary to this and the proof of Lemma 3.2.2:

Corollary 3.2.4. *Any Busemann point in a complete metric space is not bounded below. If $h \in \overline{M}^h$ is the limit of an unbounded almost-geodesic net, it is a Busemann point.*

As we only deal with complete metric spaces in this thesis, this corollary will prove useful.

3.2.2 Parts of the Boundary

The Busemann points allow us to put a geometric structure on the horofunction boundary of a metric space (M, d). We do so by constructing an extended metric on $\partial_B \overline{M}^h(M)$.

Recall that an extended metric is a function satisfying all the conditions of a metric, but its range is $[0, \infty]$ instead of $[0, \infty)$. Let $\mathcal{N}_h$ denote the set of all neighbourhoods of the horofunction h in $\overline{M}^h$. We then define the *detour cost*, $H \colon \partial\overline{M}^h \times \partial\overline{M}^h \to [0, \infty]$.

$$H(h, h') = \sup_{U \in \mathcal{N}_{h'}} \{ \inf_{x : \iota(x) \in U} d(b, x) + h'(x) \}.$$

This allows us to define the *detour distance* $\delta \colon \partial\overline{M}^h \times \partial\overline{M}^h \to \mathbb{R}^+ \cup \{\infty\}$, by

$$\delta(h_1, h_2) = H(h_1, h_2) + H(h_2, h_1).$$

The detour distance can be quite difficult to compute in practice, which makes the following lemma very useful [69, Lemma 2.6].

Lemma 3.2.5. *If h, h' are Busemann points, and (x^α) is an almost-geodesic net such that (h_{x^α}) converges to h, then*

$$H(h, h') = \lim_\alpha d(b, x^\alpha) + h'(x^\alpha).$$

We should note that in [69], Walsh restricts his attention to almost-geodesic nets starting from the basepoint b, and not ones starting from an arbitrary point x_0. This is not a problem, as the proof of [49, Lemma 3.1], which deals with almost-geodesics starting at an arbitrary point, can be adjusted for almost-geodesic nets starting at an arbitrary point. For the convenience of the reader, we prove the first step of the adjusted proof in the following lemma, which is part one of [49, Lemma 3.1]:

Lemma 3.2.6. *If h is a Busemann point, and (x^α) is an almost-geodesic net such that (h_{x^α}) converges to h, then*

$$\lim_\alpha d(b, x^\alpha) + h(x^\alpha) = 0$$

Proof. Fix some $\varepsilon > 0$. As $h_{x^\alpha}(x_0) \to h(x_0)$, there exists an η so that for all $\alpha \geq \eta$,

$$|d(x_0, x^\alpha) - d(b, x^\alpha) - h(x_0)| \leq \varepsilon. \tag{3.2.4}$$

Due to the fact that (x^α) is an almost-geodesic net, there must also exist a $\eta_1 \geq \eta$ such that, for all $\beta \geq \alpha \geq \eta_1$,

$$d(x^\beta, x^\alpha) - d(x_0, x^\beta) + d(x_0, x^\alpha) \leq \varepsilon.$$

As this is true for all $\beta \geq \alpha$, we can take the limit in β, so

$$h(x^\alpha) - h(x_0) + d(x_0, x^\alpha) \leq \varepsilon, \tag{3.2.5}$$

where we have used the fact that $h_{x^\beta} \to h$, and

$$d(x^\beta, x^\alpha) - d(x_0, x^\beta) = d(x^\beta, x^\alpha) - d(b, x^\beta) - (d(x_0, x^\beta) + d(b, x^\beta)).$$

Inequalities (3.2.4) and (3.2.5) thus mean that for all $\alpha \geq \eta_1$,

$$d(b, x^\alpha) + h(x^\alpha) \leq 2\varepsilon.$$

The triangle inequality means that, for all $\beta \geq \alpha$,

$$d(x^\beta, x^\alpha) - d(b, x^\beta) + d(b, x^\alpha) \geq 0,$$

so by once again taking the limit in β, $d(b, x^\alpha) + h(x^\alpha) \geq 0$ for all α, proving the result. □

Using Lemma 3.2.5, we can mimic the proof of [49] to prove

Lemma 3.2.7. *For any metric space (M, d), the detour distance δ is an extended metric when restricted to $\partial_B \overline{M}^h \times \partial_B \overline{M}^h$.*

The following theorem, [69, Theorem 2.8], shows why we need to restrict δ to Busemann points in order for it to be an extended metric.

Theorem 3.2.8. *A horofunction h is a Busemann point if and only if $H(h, h) = 0$.*

The detour distance allows us to partition $\partial_B \overline{M}^h$ into different equivalence classes, where $h_1, h_2 \in \partial \overline{M}^h$ if and only if $\delta(h_1, h_2) < \infty$. These classes are known as the *parts* of the boundary. There is a, perhaps more natural, way of dividing the full horofunction boundary into equivalence classes by defining the relation $\sim$, where $h \sim g$ if and only if $\sup_{x \in M} |h(x) - g(x)| < \infty$. If the horofunction boundary consists only of Busemann points, then these two partitions coincide.

3.3 Examples

The aim of this section is to show how the horofunction compactification of some well known metric spaces is calculated, to help the reader build intuition. The simplest example is $(\mathbb{R}, |\cdot|)$, the real line with the absolute value. We claim that $\overline{\mathbb{R}}^h = \mathbb{R} \cup \{h^\varepsilon\}$, where $\varepsilon \in \{-1, 1\}$, and $h^\varepsilon(x) = -\varepsilon x$ for all $x \in \mathbb{R}$. We prove this claim by first showing that any element of $\overline{\mathbb{R}}^h$ is of this form. To do so, consider a sequence $(x_n) \subseteq \mathbb{R}$, such that $h_{x_n} \to h \in \overline{\mathbb{R}}^h$. We consider two cases. If (x_n) is bounded, there exists a subsequence (x_{n_k}) and an $x \in \mathbb{R}$ such that $|x_{n_k} - x| \to 0$, which means for any $y \in \mathbb{R}$, $h_{x_{n_k}}(y) = |x_{n_k} - y| - |x_{n_k}| \to h_x(y)$, which means that $h = h_x$ by the uniqueness of limits in a Hausdorff space. If (x_n) is unbounded, then there must exist a subsequence (x_{n_k}) such that $x_{n_k} \to \pm\infty$, where the sign corresponds to that of ε, and (x_{n_k}) is monotone. Thus, for any $y \in \mathbb{R}$, there exists a $K \in \mathbb{N}$, such that for all $k \geq K$

$$h_{x_{n_k}}(y) = |x_{n_k} - y| - |x_{n_k}| = \varepsilon(x_{n_k} - y) - \varepsilon x_{n_k} = h^\varepsilon(y).$$

Thus $h_{x_n} \to h^\varepsilon$. Therefore we have shown that $\overline{\mathbb{R}}^h \subseteq \mathbb{R} \cup \{h^\varepsilon\}$. To prove the reverse equality, we just need to show that $h^\varepsilon \in \overline{\mathbb{R}}^h$ for $\varepsilon \in \{-1, 1\}$, as the internal metric functionals are always included in the horofunction compactification by definition. To do this, all we need to do is find a sequence (x_n) such that $h_{x_n} \to h^\varepsilon$, and the sequence εn achieves this. It is easy to show that the sequences (εn) are almost-geodesics, meaning that the h^ε are Busemann points. Using Lemma 3.2.5, we can then calculate that

$$\delta(h^1, h^{-1}) = \lim_{n\to\infty} n + n + \lim_{n\to\infty} n + n = \infty,$$

so the parts of the boundary are singletons. It is interesting to note that the map given by $h_x \mapsto \tanh(x)$ and $h^\varepsilon \mapsto \varepsilon$ is a homeomorphism between $\overline{\mathbb{R}}^h$ and $[-1, 1]$, which maps parts onto the relative interior of faces.

3.3.1 The Horofunction Compactification of $\mathbb{C}^m$.

Let us consider $\mathbb{C}^m$ as a normed space with the standard Euclidean norm $\|\cdot\|$, where $\|(z_1, \ldots, z_m)\| = \left(\sum_{i=1}^m |z_i|^2\right)^{1/2}$. This norm is generated by the complex inner product $\langle\cdot,\cdot\rangle$, defined by, for any $u, z \in \mathbb{C}^m$

$$\langle u, z\rangle = \sum_{i=1}^m u_i \overline{z_i}.$$

If $\operatorname{Re} z$ denotes the real part of a complex number $z \in \mathbb{C}$, it is clear to see that

$$\langle u, z\rangle + \langle z, u\rangle = 2 \operatorname{Re}\langle u, z\rangle,$$

which we will need in the proof of the following.

Theorem 3.3.1. *The horofunctions of $(\mathbb{C}^m, \|\cdot\|)$ consists entirely of functions h^z, where $z \in \mathbb{C}^m$ and $\|z\| = 1$, and for $u \in \mathbb{C}^m$,*

$$h^z(u) = -\operatorname{Re}\langle u, z\rangle.$$

Proof. Fix some $h \in \partial\overline{\mathbb{C}^m}^h$. As $\mathbb{C}^m$ is a proper geodesic metric space, there must exist a sequence $(z^n) \subseteq \mathbb{C}^m$ such that $\|z^n\| \to \infty$, and $h_{z^n} \to h$ by Proposition 3.1.5. Let us define, for each $n \in \mathbb{N}$, $w^n = z^n/\|z^n\|$. By the compactness of the unit sphere, there exists a subsequence, which we relabel (w^n), and $z \in S_{\mathbb{C}^m}$ such that $w^n \to z$. Now, for any $u \in \mathbb{C}^m$ and any $n \in \mathbb{N}$,

$$\begin{aligned} h_{z^n}(u) &= \|z^n - u\| - \|z^n\| \\ &= \frac{\|z^n - u\|^2 - \|z^n\|^2}{\|z^n - u\| + \|z^n\|} \\ &= \frac{\langle u, u \rangle - 2\operatorname{Re}\langle u, z^n \rangle}{\|z^n - u\| + \|z^n\|} \\ &= \frac{\frac{\|u\|^2}{\|z^n\|} - 2\operatorname{Re}\langle u, w^n \rangle}{\sqrt{1 - 2\operatorname{Re}\langle w^n, u/\|z^n\| \rangle + \frac{\|u\|^2}{\|z^n\|^2}} + 1} \\ &\to -\operatorname{Re}\langle u, z \rangle . \end{aligned}$$

As $\overline{\mathbb{C}^m}^h$ is Hausdorff, we must thus have $h = h^z$. This shows then that every $h \in \partial\overline{\mathbb{C}^m}^h$ is of the form h^z for some $z \in S_{\mathbb{C}^m}$. Conversely, suppose we are given some $z \in S_{\mathbb{C}^m}$. Let us define $z^n = nz$ for every $n \in \mathbb{N}$. Then, using the same method as in the above calculations we see that, for any $n \in \mathbb{N}$ and $u \in \mathbb{C}^m$

$$h_{z^n}(u) = \frac{\frac{\|u\|^2}{n} - 2\operatorname{Re}\langle u, z \rangle}{\sqrt{1 - 2\operatorname{Re}\langle z, u/n \rangle + \frac{\|u\|^2}{n^2}} + 1} \to -\operatorname{Re}\langle u, z \rangle .$$

Thus $h^z \in \overline{\mathbb{C}^m}^h$. To show that $h^z \in \partial\overline{\mathbb{C}^m}^h$ we just need to show that it is not an internal metric functional, but that is immediate because it is not bounded below. □

As a consequence of the proof of the above theorem, we can see that every h^z is a Busemann point, because the sequence (nz) is geodesic for every $z \in S_{\mathbb{C}^m}$. Using these geodesics, we can prove

Proposition 3.3.2. *The parts of the Busemann boundary of* $\overline{\mathbb{C}^m}^h$ *are all singletons.*

Proof. Consider two Busemann points $h^z, h^w \in \partial\overline{\mathbb{C}^m}^h$. From above we know that the sequences (nz) and (nw) are almost-geodesic sequences converging to each horofunction. Lemma 3.2.5 thus allows us to calculate that

$$\delta(h^z, h^w) = \lim_{n\to\infty} 2(n - n \operatorname{Re}\langle w, z\rangle),$$

but by the Cauchy-Schwarz inequality $\operatorname{Re}\langle w, z\rangle < 1$ if $w \neq z$, in which case $\delta(h^z, h^w) = \infty$. □

Just like in the case of the reals, it is interesting to note that the map from $\overline{\mathbb{C}^m}^h$ to $B_{\mathbb{C}^m}$ given by $h_z \mapsto \frac{\tanh(\|z\|)}{\|z\|} z$ and $h^z \mapsto z$ is a continuous bijection, and hence homeomorphism. As all Busemann points lie in singleton parts, and the relative interior of the faces of $B_{\mathbb{C}^m}$ are precisely $B^\circ_{\mathbb{C}^m}$ and $\{z\}$ for $z \in S_{\mathbb{C}^m}$, this homeomorphism maps parts of the boundary bijectively onto the relative interior of faces of the dual ball.

3.3.2 Finite Dimensional Smooth and Strictly Convex Normed Spaces

The above example is actually a special example of a finite dimensional uniformly smooth and convex Banach space. In this section let us fix a real finite dimensional smooth and strictly convex Banach space, X with norm $\|\cdot\|$ and dual norm $\|\cdot\|_*$. We refer the reader to Section 2.6 for a recap of smoothness and convexity of normed spaces. As X is finite dimensional, it is automatically uniformly convex [50, Proposition 5.2.14], so X^* is uniformly smooth by Theorem 2.6.5. Furthermore, as X is smooth, X^* is strictly convex by Proposition 2.6.4, so X^* is uniformly convex [50, Proposition 5.2.14], which by Theorem 2.6.5, means X is uniformly smooth.

Theorem 3.3.3. *The horofunctions of X are precisely those functions h^ψ for $\psi \in S_{X^*}$, where for all $x \in X$*

$$h^\psi(x) = -\psi(x).$$

Furthermore, every horofunction is a Busemann point.

Proof. Let (y_n) be an unbounded sequence such that $h_{y_n} \to h \in \partial\overline{X}^h$. Define $z_n = y_n/\|y_n\| \in S_X$. If we define the sequence $(t_n) = (1/\|y_n\|)$, we can write, for any $x \in X$,

$$h_{y_n}(x) = \frac{\|z_n - t_n x\| - \|z_n\|}{t_n}.$$

As X is smooth, we know for each $n \in \mathbb{N}$ there exists unique norming functionals $z_n^* \in S_{X^*}$ and $\varphi_n^x \in S_{X^*}$, such that $z_n^*(z_n) = 1$ and $\varphi_n^x(z_n - t_n x) = 1$. The compactness of the unit sphere means there must exist $\psi, \varphi^x \in S_{X^*}$ and subsequence (n_k) such that $z_{n_k}^* \to \psi$ and $\varphi_{n_k}^x \to \varphi^x$. Thus, for any $k \in \mathbb{N}$:

$$\begin{aligned} -\psi_{n_k}(x) = \frac{z_{n_k}^*(z_{n_k} - t_{n_k}x) - z_{n_k}^*(z_{n_k})}{t_{n_k}} &\leq \frac{\|z_{n_k} - t_{n_k}x\| - \|z_{n_k}\|}{t_{n_k}} \\ &\leq \frac{\varphi_{n_k}^x(z_{n_k} - t_{n_k}x) - \varphi_{n_k}^x(z_{n_k})}{t_{n_k}} = -\varphi_{n_k}^x(x). \end{aligned} \tag{3.3.1}$$

We now note that, for all $k \in \mathbb{N}$

$$2 \geq \|\varphi_{n_k}^x + z_{n_k}^*\|_* \geq |\varphi_{n_k}^x(z_{n_k}) + z_{n_k}^*(z_{n_k})| \geq |\varphi_{n_k}^x(z_{n_k} - t_{n_k}x) + z_{n_k}^*(z_{n_k})| - |t_{n_k}\varphi_{n_k}^x(x)|.$$

However, $t_{n_k}\varphi_{n_k}^x(x) \to 0$, so by the squeeze theorem $\|\varphi_{n_k}^x + z_{n_k}^*\|_* \to 2$. As X^* is uniformly convex, this means that $\|\varphi_{n_k}^x - z_{n_k}^*\|_* \to 0$. As $z_{n_k}^* \to \psi$ and $\varphi_{n_k}^x \to \varphi^x$ we must thus have that $\psi = \varphi^x$. Taking the limit as $k \to \infty$ in inequality (3.3.1) thus shows that, for any $x \in X$, $\lim_{n_k} h_{y_{n_k}}(x) = -\psi(x)$.

Conversely, suppose we are given some $\psi \in S_{X^*}$. As X^* is uniformly convex and smooth, and X is reflexive, there exists a unique $z \in S_X$ such that $\psi(z) = 1$. Define the sequence $y_n = nz$. The exact same argument used above shows that $\lim_{n\to\infty} h_{y_n} = h^\psi$. As nz is a sequence lying on a straight line through the origin, it is trivially an almost-geodesic, so h^ψ is a Busemann point. □

Proposition 3.3.4. *The parts of the Busemann boundary of* $\overline{X}^h$ *are all singletons.*

Proof. Consider two Busemann points $h^\psi, h^\phi \in \partial\overline{X}^h$. From above we know that there exists points $z, w \in S_X$ such that $h_{nz} \to h^\psi$ and $h_{nw} \to h^\phi$. The sequences (nz) and (nw) are almost-geodesic sequences. Lemma 3.2.5 thus allows us to calculate that

$$H(h^\psi, h^\phi) = \lim_{n\to\infty} n - n\phi(z),$$

but X is smooth, so $\phi(z) = 1$ if and only if $\phi = \psi$, meaning that $\delta(h^\psi, h^\phi) = \infty$ if $\psi \neq \phi$. □

In her thesis [64, Theorem 3.3.10], Schilling proves the following theorem.

Theorem 3.3.5. $\overline{X}^h$ *is homeomorphic to* B_{X^*}*, the closed dual unit ball.*

We provide here an alternative proof, which we think is slightly shorter. Let us define $g\colon \overline{X}^h \to B_{X^*}$ by

$$g(x) = -\tanh(\|x\|)x^*, \text{ if } x \in X, \text{ and } \qquad g(h) = h \text{ if } h \in \partial\overline{X}^h, \tag{3.3.2}$$

where we define $0^* = 0$. As always, note that here we are identifying X with the embedding $i(X)$, where $i(x) = h_x$, the internal metric functional associated with x.

We shall prove Theorem 3.3.5 with a sequence of lemmas.

Lemma 3.3.6. *g is a bijection onto* B_{X^*}.

Proof. First let us assume that $g(x) = g(y)$. Without loss of generality we can assume $y \neq 0 \neq x$. Then $\|x\| = x^*(x) = y^*(\frac{\tanh(\|y\|)}{\tanh(\|x\|)}x)$. However, $|y^*(\frac{\tanh(\|y\|)}{\tanh(\|x\|)}x)| \leq \frac{\tanh(\|y\|)}{\tanh(\|x\|)}\|x\|$, meaning that $\frac{\tanh(\|y\|)}{\tanh(\|x\|)} \geq 1$. Symmetry gives the opposite inequality, so $\tanh(\|x\|) = \tanh(\|y\|)$, which means that $\|x\| = \|y\|$, as tanh is strictly monotone increasing. Thus $x^* = y^*$. Lemma 2.6.2 thus implies $g|_X$ is an injection.

Now fix a $0 \neq f \in \operatorname{int}(B_{X^*})$. By compactness there exists an $x \in S_X$ such that $-\frac{f}{\|f\|_*}(x) = 1$, meaning that $-\frac{f}{\|f\|_*}$ is the unique norming functional for x. Thus $g(x) =$

$\tanh(\|x\|)\frac{f}{\|f\|_*}$. As $\tanh\colon [0,\infty) \to [0,1)$ is bijective, there exists some $c > 0$ such that $\tanh(\|cx\|) = \|f\|_*$. Furthermore $(cx)^* = x^* = -\frac{f}{\|f\|_*}$. Thus $g(cx) = f$, and so $g|_X$ is a bijection onto $\operatorname{int}(B_{X^*})$.

Clearly $g|_X(X) \subseteq \operatorname{int}(B_{X^*})$, so if we can show that $g|_{\partial\overline{X}^h}$ is a bijection onto S_{X^*} then g will map $\overline{X}^h$ bijectively onto B_{X^*}, but this is exactly [26, Remark 4.4]. $\square$

Lemma 3.3.7. *g is continuous.*

Proof. Recall that $\overline{X}^h$ is equipped with the topology of uniform convergence on compact sets, which is metrisable, and X^* equipped with the norm topology is metrisable, so we only need to prove sequential continuity. Suppose $h_{x_n} \to h_x \in X$ for $(h_{x_n}) \subseteq X$. $(\|x_n\|)$ must be bounded, so there exists a compact set of X, say K, containing all the x_n and also x. This means that

$$\sup_{y\in K}(\|x_n - y\| - \|x_n\|) - (\|x - y\| - \|x\|) \to 0,$$

from which we see that $\|x_n - x\| - \|x_n\| \to \|x\|$, and $-\|x_n - x\| - \|x_n\| \to \|x\|$, allowing us to deduce that $x_n \to x$ in norm. Furthermore we note that if $x_n \to x$ in norm then $x_n^* \to x^*$ in the dual norm. Indeed, any subsequence of (x_n^*) must have a further subsequence $(x_{n_k}^*)$ converging to some $y^* \in S_{X^*}$. As $x_{n_k}^* x \to \|x\|$ it follows that $y^* = x^*$. Thus indeed $g(h_{x_n}) \to g(h_x)$.

Now we need to consider a sequence (h_{x_n}) converging to some $h \in \partial\overline{X}^h$. As X is finite dimensional we must have that every subsequence $(x_{n_k}) \to \infty$, and from [26] we know that there must exist an $x \in S_X$ and a further subsequence, relabelled x_{n_k}, such that $x_{n_k}/\|x_{n_k}\| \to x$ in norm, $x_{n_k}^* \to x^*$ in the dual norm, and $h_{x_{n_k}} \to -x^*$ pointwise. Thus we must have $-x^* = h$. Furthermore, this means that $g(h_{x_{n_k}}) \to -x^* = h$ in norm, from which we conclude that $g(h_{x_n}) \to g(h)$.

If $(h_n) \subseteq \partial\overline{X}^h$ converges to some $h \in \partial\overline{X}^h$, then $g(h_n) \to g(h)$, as the dual norm topology and topology of uniform convergence on compact sets are equal on finite dimensional spaces (because the closed unit ball is compact). Note that any $(h_n) \subseteq \partial\overline{X}^h$ cannot converge to some $h_x \in X$ by Proposition 3.1.10. We now consider an arbitrary sequence $(h_n) \subseteq \overline{X}^h$ converging to some $h \in \overline{X}^h$. We can partition (h_n) into two subsequences $(h_{n_k}) \subseteq \partial\overline{X}^h$ and $(h_{n_j}) \subseteq X$ (if not then we are just in one of the above cases), and g applied to each subsequence converges to $g(h)$ it follows that $g(h_n) \to g(h)$ as well. □

The above lemmas show that g is a continuous bijection between compact Hausdorff spaces, so g is a homeomorphism. As every horofunction forms a singleton part of the boundary, and these are the only parts of the boundary, and the faces of S_{X^*} are all extreme points by Proposition 2.6.7, g then also maps parts of the boundary bijectively onto the relative interiors of faces of the dual ball. Theorem 3.3.5 thus follows.

3.3.3 The Horofunction Compactification of $\ell^1(\mathcal{N})$.

Fix an integer $n \geq 2$. Borrowing the notation of [26], we define $\mathcal{N} = \{1, \ldots, n\}$, and use $\ell^1(\mathcal{N})$ to denote the space $\mathbb{R}^n$ equipped with the ℓ^1 norm, $\|x\|_1 = \sum_{i=1}^n |x_i|$. In the same paper, Gutiérrez proves the following [26]:

Theorem 3.3.8. *All horofunctions in $\partial\overline{\ell^1(\mathcal{N})}^h$ are of the form $h^{\mathcal{I}}_{\epsilon,\mu}$, where $\emptyset \neq \mathcal{I} \subseteq \mathcal{N}$, $\epsilon \subseteq \{-1,1\}^{\mathcal{I}}$, $\mu \in \mathbb{R}^{\mathcal{N}\setminus I}$, and for every $x \in \ell^1(\mathcal{N})$,*

$$h^{\mathcal{I}}_{\epsilon,\mu}(x) = \sum_{i\in\mathcal{I}} -\epsilon_i x_i + \sum_{j\in\mathcal{N}\setminus I} |x_j - \mu_j| - |\mu_j|.$$

Proof. The complete proof is given by Lemma 3.1 and Theorem 3.2 in [26], so we shall not recreate it fully here, but give an overview. If $h \in \partial\overline{\ell^1(\mathcal{N})}^h$, there exists, by Proposition 3.1.5, an unbounded sequence (y^n) such that $h_{y^n} \to h$. By choosing subsequences and relabelling, there must exist a $\emptyset \neq \mathcal{I} \subseteq \mathcal{N}$ such that $|y^n_i| \to \infty$ if and only if $i \in \mathcal{I}$. We define $\epsilon \in \{-1,1\}^{\mathcal{I}}$ by setting $\epsilon_i = 1$ if $\lim_n y^n_i = \infty$ and $\epsilon_i = -1$ if $\lim_n y^n_i = -\infty$. The fact that $\ell^1(\mathcal{N})$ is proper means we can choose further subnets, and $\mu_i \in \mathbb{R}$ for all $i \notin \mathcal{I}$

such that, after relabelling, $y_i^n \to \mu_i$. Thus, using the same reasoning as we did for the horofunction compactification of $\mathbb{R}$, for any $x \in \ell^1(\mathcal{N})$ we can choose n large enough such that

$$\begin{aligned} h_{y^n}(x) &= \sum_{i\in\mathcal{I}} |x_i - y_i^n| - |y_i^n| + \sum_{j\in\mathcal{N}\setminus\mathcal{I}} |x_j - y_j^n| - |y_j^n| \\ &= \sum_{i\in\mathcal{I}} -\epsilon_i x_i + \sum_{j\in\mathcal{N}\setminus\mathcal{I}} |x_j - y_j^n| - |y_j^n| \\ &\to \sum_{i\in\mathcal{I}} -\epsilon_i x_i + \sum_{j\in\mathcal{N}\setminus\mathcal{I}} |x_j - \mu_j| - |\mu_j| \\ &= h^{\mathcal{I}}_{\epsilon,\mu}(x). \end{aligned}$$

Thus, every horofunction is of the form $h^{\mathcal{I}}_{\epsilon,\mu}$. Conversely, if we are given some $h^{\mathcal{I}}_{\epsilon,\mu}$, we can define the sequence (y^n) by setting $y_i^n = \epsilon_i n$ for every $i \in \mathcal{I}$ and $n \in \mathbb{N}$, and $y_i^n = \mu_i$ for every $i \notin \mathcal{I}$. The above calculation can then be used to show that $h_{y^n} \to h^{\mathcal{I}}_{\epsilon,\mu}$. As $h^{\mathcal{I}}_{\epsilon,\mu}$ is not bounded below it cannot be an internal metric functional. □

With this characterization, it is simple to characterize the Busemann points of $\ell^1(\mathcal{N})$.

Lemma 3.3.9. *All horofunctions of $\ell^1(\mathcal{N})$ are Busemann points.*

Proof. Fix some $h^{\mathcal{I}}_{\epsilon,\mu} \in \partial\overline{\ell^1(\mathcal{N})}^h$. To prove that it is a Busemann point it suffices to show the existence of an almost-geodesic $\gamma\colon [0,\infty) \to \ell^1(\mathcal{N})$ satisfying $\|x-\gamma(t)\|_1 - \|\gamma(t)\|_1 \to h^{\mathcal{I}}_{\epsilon,\mu}(x)$ as $t \to \infty$ for all $x \in \ell^1(\mathcal{N})$. To that end we define a path γ component-wise as follows:

$$\gamma_i(t) = \begin{cases} \frac{-\epsilon_i t}{|\mathcal{I}|} & i \in \mathcal{I} \\ \mu_i & i \notin \mathcal{I} \end{cases}. \tag{3.3.3}$$

For any $0 \le s < t$

$$\|\gamma(t) - \gamma(s)\|_1 = \sum_{i\in\mathcal{N}\setminus\mathcal{I}} |\mu_i - \mu_i| + \sum_{j\in\mathcal{I}} \frac{1}{|\mathcal{I}|}|(t-s)| = |t-s|,$$

proving that γ is a geodesic. We can also calculate that, for any $t \in [0, \infty)$ and $x \in \ell^1(\mathcal{N})$:

$$\|x - \gamma(t)\|_1 - \|\gamma(t)\|_1 = \sum_{i \in \mathcal{I}} \left| x_i + \frac{\epsilon_i t}{|\mathcal{I}|} \right| + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} |x_i - \mu_i| - \sum_{i \in \mathcal{I}} \frac{|\epsilon_i| t}{|\mathcal{I}|} - \sum_{i \in \mathcal{N} \setminus \mathcal{I}} |\mu_i|.$$

For all sufficiently large t we have

$$\sum_{i \in \mathcal{I}} \left| x_i + \frac{\epsilon_i t}{|\mathcal{I}|} \right| = \sum_{i \in \mathcal{I}} \epsilon_i x_i + \frac{|\epsilon_i| t}{|\mathcal{I}|},$$

which means that

$$\lim_{t \to \infty} \|x - \gamma(t)\|_1 - \|\gamma(t)\|_1 = \sum_{i \in \mathcal{I}} \epsilon_i x_i + \sum_{i \in \mathcal{N} \setminus \mathcal{I}} |x_i - \mu_i| - |\mu_i|,$$

proving that $\gamma \to h^{\mathcal{I}}_{\epsilon,\mu}$ pointwise. □

Using the geodesic γ defined in the above proof we can prove the following:

Theorem 3.3.10. *If δ denotes the detour distance on $\partial\overline{\ell^1(\mathcal{N})}^h_B$, then $\delta(h^{\mathcal{I}}_{\epsilon,\mu}, h^{\mathcal{I}'}_{\epsilon',\mu'}) < \infty$ if and only if $\mathcal{I} = \mathcal{I}'$ and $\epsilon = \epsilon'$. Furthermore, if $\mathcal{I} = \mathcal{I}'$ and $\epsilon = \epsilon'$, then*

$$\delta(h^{\mathcal{I}}_{\epsilon,\mu}, h^{\mathcal{I}'}_{\epsilon',\mu'}) = 2 \sum_{i \in \mathcal{N} \setminus \mathcal{I}} |\mu_i - \mu'_i|.$$

Proof. For convenience we define $h_1 = h^{\mathcal{I}}_{\epsilon,\mu}$, and $h_2 = h^{\mathcal{I}'}_{\epsilon',\mu'}$. Define geodesics γ_1, γ_2 associated to h_1 and h_2 respectively as in the proof of the above lemma. Lemma 3.2.5 allows us to calculate the detour distance between h_1 and h_2 using γ_1 and γ_2:

$$\begin{aligned}
\delta(h_1, h_2) &= H(h_1, h_2) + H(h_2, h_1) \\
&= \lim_{t \to \infty} \|\gamma_1(t)\|_1 + h_2(\gamma_1(t)) + \lim_{t \to \infty} \|\gamma_2(t)\|_1 + h_1(\gamma_2(t)) \\
&= \lim_{t \to \infty} t + P(t) + C + \lim_{t \to \infty} t + P'(t) + C',
\end{aligned}$$

where we have defined

$$P(t) = \sum_{i \in \mathcal{I} \cap \mathcal{I}'^c} \left| \frac{-\epsilon_i t}{|\mathcal{I}|} - \mu_i' \right| - |\mu_i'| + \sum_{j \in \mathcal{I} \cap \mathcal{I}'} -\epsilon_j \epsilon_j' \frac{t}{|\mathcal{I}|},$$

$$C = \sum_{i \in \mathcal{I}^c} |\mu_i| + \sum_{j \in \mathcal{I}^c \cap \mathcal{I}'^c} |\mu_j - \mu_j'| - |\mu_j'| + \sum_{k \in \mathcal{I}^c \cap \mathcal{I}'} \epsilon_k' \mu_k,$$

and similarly

$$P'(t) = \sum_{i \in \mathcal{I}^c \cap \mathcal{I}'} \left| \frac{-\epsilon_i' t}{|\mathcal{I}'|} - \mu_i \right| - |\mu_i| + \sum_{j \in \mathcal{I} \cap \mathcal{I}'} -\epsilon_j \epsilon_j' \frac{t}{|\mathcal{I}'|},$$

$$C' = \sum_{i \in \mathcal{I}'^c} |\mu_i'| + \sum_{j \in \mathcal{I}^c \cap \mathcal{I}'^c} |\mu_j - \mu_j'| - |\mu_j| + \sum_{k \in \mathcal{I} \cap \mathcal{I}'^c} \epsilon_k \mu_k'.$$

We can see that $H(h_1, h_2)$ and $H(h_2, h_1)$ are finite if and only if P and P' behave asymptotically like $-t$. From the definition above we can see that P can behave asymptotically like $-t$ if and only if for large t

$$\sum_{i \in \mathcal{I}' \cap \mathcal{I}^c} \frac{t}{|\mathcal{I}^c|} + \sum_{j \in \mathcal{I}^c \cap \mathcal{I}'^c} -\epsilon_j \epsilon_j' \frac{t}{|\mathcal{I}^c|} = -t,$$

which only occurs when $\mathcal{I} = \mathcal{I}'$ and $\epsilon_j = \epsilon_j'$ for all $j \in \mathcal{I}$. By symmetry we see the same phenomenon for $P'(t)$. This proves the first statement. Finally, if $\mathcal{I} = \mathcal{I}'$ and $\epsilon_j = \epsilon_j'$ for all $j \in \mathcal{I}$ we calculate that

$$\begin{aligned} \delta(h_1, h_2) &= C + C' \\ &= \sum_{i \in \mathcal{I}^c} |\mu_i| + \sum_{j \in \mathcal{I}^c} |\mu_j - \mu_j'| - |\mu_j'| + \sum_{i \in \mathcal{I}^c} |\mu_i'| + \sum_{j \in \mathcal{I}^c} |\mu_j - \mu_j'| - |\mu_j| \\ &= 2 \sum_{i \in \mathcal{I}^c} |\mu_i - \mu_i'|. \end{aligned}$$

□

Theorem 3.3.11. *The horofunction compactification $\overline{\ell^1(\mathcal{N})}^h$ is homeomorphic to the dual unit ball of $\ell^1(\mathcal{N})$, and the homeomorphism maps parts of the boundary of $\overline{\ell^1(\mathcal{N})}^h$ bijectively* *onto the relative interiors of faces of the dual ball.*

Proof. It is well known that the dual of $\ell^1(\mathcal{N})$ is $\ell^\infty(\mathcal{N})$, the set of $\mathbb{R}^n$ equipped with the max norm $\|x\|_\infty = \max_{i\in\mathcal{N}} |x_i|$. Thus

$$B_{\ell^\infty(\mathcal{N})} = \{x \in \mathbb{R}^n : |x_i| \leq 1 \text{ for all } i \in \mathcal{N}\}, \text{ and} \tag{3.3.4}$$

$$S_{\ell^\infty(\mathcal{N})} = \{x \in B_{\ell^\infty(\mathcal{N})} : \exists i \in \mathcal{N} \text{ such that } |x_i| = 1\}. \tag{3.3.5}$$

The boundary faces of $B_{\ell^\infty(\mathcal{N})}$ are thus the sets $F^\epsilon_\mathcal{I}$, where $\emptyset \neq \mathcal{I} \subseteq \mathbb{N}$ and $\epsilon \in \{-1,1\}^\mathcal{I}$, and

$$F^\epsilon_\mathcal{I} = \{x \in S_{\ell^\infty(\mathcal{N})} : x_i = \epsilon_i \text{ for all } i \in \mathcal{N} \text{ and } |x_j| \leq 1 \text{ for all } j \in \mathcal{N}\backslash\mathcal{I}\},$$

meaning that

$$\mathrm{ri}(F^\epsilon_\mathcal{I}) = \{x \in S_{\ell^\infty(\mathcal{N})} : x_i = \epsilon_i \text{ for all } i \in \mathcal{N} \text{ and } |x_j| < 1 \text{ for all } j \in \mathcal{N}\backslash\mathcal{I}\}. \tag{3.3.6}$$

We can now define the candidate homeomorphism $\varphi\colon \overline{\ell^1(\mathcal{N})}^h \to B_{\ell^\infty(\mathcal{N})}$, if $\{e_1, \ldots, e_n\}$ is the standard set of basis vectors for $\mathbb{R}^n$, by

$$\varphi(h^\mathcal{I}_{\epsilon,\mu}) = \sum_{i\in\mathcal{I}} \epsilon_i e_i + \sum_{j\in\mathcal{N}\backslash\mathcal{I}} \tanh(\mu_j) e_j.$$

We should note here that in this definition of $h^\mathcal{I}_{\epsilon,\mu}$ we allow $\mathcal{I}$ to be empty, because $h^\emptyset_{\emptyset,\mu}$ is precisely the internal metric functional associated to $\mu \in \ell^1(\mathcal{N})$. By (3.3.4) it is clear that φ is bijective because tanh is bijective. If $h_{x^n} \to h_y$ in $\overline{\ell^1(\mathcal{N})}^h$, we can see by evaluating at multiples of basis vectors that $x^n \to y$ in $\ell^1(\mathcal{N})$, from which it is fairly simple to deduce that φ is also continuous, making it a homeomorphism as it is a map between compact Hausdorff spaces. Furthermore, Theorem 3.3.10 in combination with (3.3.6) shows that φ maps parts of the boundary bijectively onto the relative interiors of faces of the dual ball.

□

Remark 3.3.12. It is interesting to note that Theorem 3.3.10 shows that each part of the boundary of $\overline{\ell^1(\mathcal{N})}^h$ inherits a natural normed space structure. Indeed, each part of the boundary is given by a choice of $\mathcal{I}$ and $\epsilon \in \{-1,1\}^{\mathcal{I}}$, so on each part the map ϕ defined by $h^{\mathcal{I}}_{\epsilon,\mu} \mapsto \frac{1}{2}\mu$ is a bijection from $\overline{\ell^1(\mathcal{N})}^h$ onto $\mathbb{R}^{\mathcal{N}\setminus\mathcal{I}}$. The fact that the detour distance is simply twice the ℓ^1 norm on $\mathbb{R}^{\mathcal{N}\setminus\mathcal{I}}$ means that ϕ is actually an isometry, so we can consider each part of the boundary to be a normed space. In all the previous examples we've seen, each part of the boundary is a singleton, so can trivially be seen as the trivial normed space. This observation motivates the next example.

3.3.4 The Horofunction Compactification of $\ell^1(\mathcal{N}) \oplus \ell^1(\mathcal{N})$

In an attempt to investigate whether Remark 3.3.12 above hints at a genuine phenomenon, we now consider $X = \ell^1(\mathcal{N}) \oplus \ell^1(\mathcal{N})$, which we equip with the norm $\|(x_1, x_2)\| = \max\{\|x_1\|_1, \|x_2\|_1\}$. We want to calculate the parts of its horofunction boundary, and see whether they inherit a natural norm structure too. Theorem 2.3 and Proposition 2.8 in [41] in conjunction with Theorem 3.3.8 and Lemma 3.3.9 gives the following proposition for free:

Proposition 3.3.13. *A function h on X is in $\partial\overline{X}^h$ if and only if there exist a non-empty $J \subseteq \{1,2\}$, $\alpha \in \mathbb{R}^J$ with $\min_j \alpha_j = 0$, and $\{h^{\mathcal{I}^j}_{\epsilon^j,\mu^j}\}_{j\in J} \subseteq \partial\overline{\ell^1(\mathcal{N})}^h$ such that*

$$h(x) = \max_{j\in J}\{h^{\mathcal{I}^j}_{\epsilon^j,\mu^j}(x_j) - \alpha_j\} \quad \textit{for all } x = (x_1, x_2) \in X.$$

This proposition immediately allows us to prove the following lemma:

Lemma 3.3.14. *Every horofunction in $\partial\overline{X}^h$ is a Busemann point.*

Proof. Fix some $h \in \partial\overline{X}^h$. There are two cases to consider: either $|J| = 1$ or $|J| = 2$, where J is as in Proposition 3.3.13. If $|J| = 1$, then, without loss of generality, we can assume that $h(x) = h^I_{\epsilon,\mu}(x_1)$ for all $x \in X$ and some $\epsilon, \mu, \mathcal{I}$ as defined in Theorem 3.3.8. We can then define a path $\gamma\colon [0,\infty) \to X$ by $\gamma(t) = (\gamma_1(t), 0)$, with $\gamma_1\colon [0,\infty) \to \ell^1(\mathcal{N})$ defined as it was in (3.3.3). As $\|\gamma(t)-\gamma(s)\| = \|\gamma_1(t)-\gamma_1(s)\|_1$ and $\|x-\gamma(t)\| - \|\gamma(t)\| = \|x_1-\gamma_1(t)\| - \|\gamma_1(t)\|_1$

for all $t \geq s \geq 0$, we have already shown in the proof of Lemma 3.3.9 that γ is a geodesic converging to $h^I_{\epsilon,\mu}$, meaning that $h^I_{\epsilon,\mu}$ is a Busemann point.

If $|J| = 2$, we can assume without loss of generality that there exists some $\alpha > 0$ such that

$$h(x) = \max\{h^{\mathcal{I}^1}_{\epsilon^1,\mu^1}(x_1) - \alpha, h^{\mathcal{I}^2}_{\epsilon^2,\mu^2}(x_2)\} \quad \text{for all } x = (x_1, x_2) \in X.$$

We now define $\gamma = (\gamma_1, \gamma_2)\colon [0, \infty) \to X$ by first setting $\beta = -\sum_{i\in\mathcal{I}^2} \mu_i^2 + \alpha$, and then defining

$$\pi_i\gamma_1(t) = \begin{cases} \frac{-\epsilon_i^1 t}{|\mathcal{I}^1|} & i \in \mathcal{I}^1 \\ \mu_i^1 & i \notin \mathcal{I}^1 \end{cases}, \text{ and } \quad \pi_i\gamma_2(t) = \begin{cases} -\epsilon_i^2 \left(\frac{\|\gamma_1(t)\|_1+\beta}{|\mathcal{I}^2|}\right) & i \in \mathcal{I}^2 \\ \mu_i^2 & i \notin \mathcal{I}^2 \end{cases}. \tag{3.3.7}$$

To show that γ is a geodesic we fix some $t \geq s \geq 0$ and calculate

$$\begin{aligned} \|\gamma(t) - \gamma(s)\| &= \max\left\{\sum_{i\in\mathcal{I}^1} \frac{|\epsilon_i^1(t-s)|}{|\mathcal{I}^1|}, \sum_{i\in\mathcal{I}^2} \frac{|-\epsilon_i^2(\|\gamma_1(t)\|_1+\beta) + \epsilon_1^2(\|\gamma_1(s)\|_1+\beta)|}{|\mathcal{I}^2|}\right\} \\ &= \max\{|t-s|, |\|\gamma_1(t)\|_1 - \|\gamma_1(s)\|_1|\} \\ &= \max\left\{|t-s|, \left|\sum_{i\in\mathcal{I}_1} \frac{|t|}{|\mathcal{I}^1|} + \sum_{j\in(\mathcal{I}^1)^c} \mu_j^1 - \left(\sum_{i\in\mathcal{I}_1} \frac{|s|}{|\mathcal{I}^1|} + \sum_{j\in(\mathcal{I}^1)^c} \mu_j^1\right)\right|\right\} \\ &= |t-s|, \end{aligned}$$

showing that indeed γ is a geodesic. Finally, for any $x = ((x_{1,1}, x_{1,2}), (x_{2,1}, x_{2,2})) \in X$ and

$t \geq 0$,

$$\begin{aligned}\|x-\gamma(t)\| - \|\gamma(t)\| &= \max\{\|x_1-\gamma_1(t)\|_1, \|x_2-\gamma_2(t)\|\} - \max\{\|\gamma_1(t)\|_1, \|\gamma_2(t)\|_1\} \\ &= \max\left\{\sum_{i\in\mathcal{I}^1}\left|x_{1,i}+\frac{\epsilon_i^1 t}{|\mathcal{I}^1|}\right| + \sum_{j\in(\mathcal{I}^1)^c}|x_{1,j}-\mu_j^1|,\right. \\ &\qquad \left.\sum_{i\in\mathcal{I}^2}\left|x_{2,i}+\frac{\epsilon_i^2(\|\gamma_1(t)\|_1+\beta)}{|\mathcal{I}^2|}\right| + \sum_{j\in(\mathcal{I}^2)^c}|x_{2,j}-\mu_j^2|\right\} - \\ &\qquad \max\left\{\|\gamma_1(t)\|_1, \sum_{i\in\mathcal{I}^2}\frac{|\epsilon_i^2(\|\gamma_1(t)\|_1+\beta)|}{|\mathcal{I}^2|} + \sum_{j\in(\mathcal{I}^2)^c}|\mu_j^2|\right\}.\end{aligned}$$

For sufficiently large t this then becomes:

$$\begin{aligned}\|x-\gamma(t)\| - \|\gamma(t)\| &= \max\left\{t + \sum_{i\in\mathcal{I}^1}\epsilon_i^1 x_{1,i} + \sum_{j\in(\mathcal{I}^1)^c}|x_{1,j}-\mu_j^1|,\right. \\ &\qquad \left.\|\gamma_1(t)\|_1 + \beta + \sum_{i\in\mathcal{I}^2}\epsilon_i^2 x_{2,i} + \sum_{j\in(\mathcal{I}^2)^c}|x_{2,j}-\mu_j^2|\right\} - \\ &\qquad \left(\|\gamma_1(t)\|_1 + \beta + \sum_{j\in(\mathcal{I}^2)^c}|\mu_j^2|\right) \\ &= \max\left\{t + \sum_{i\in\mathcal{I}^1}\epsilon_i^1 x_{1,i} + \sum_{j\in(\mathcal{I}^1)^c}|x_{1,j}-\mu_j^1| - (\|\gamma_1(t)\|_1 + \beta + \sum_{j\in(\mathcal{I}^2)^c}|\mu_j^2|),\right. \\ &\qquad \left. h^{\mathcal{I}^2}_{\epsilon^2,\mu^2}(x_2)\right\} \\ &= \max\left\{h^{\mathcal{I}^1}_{\epsilon^1,\mu^1}(x_1) - \alpha, h^{\mathcal{I}^2}_{\epsilon^2,\mu^2}(x_2)\right\}.\end{aligned}$$

Thus $\gamma \to h$ pointwise, and so indeed h is a Busemann point. □

Theorem 3.3.15. *$\delta(h_1, h_2) < \infty$ for $h_1, h_2 \in \partial_B \overline{X}^h$ if and only if h_1, h_2 are of the following form:*

(i) There exists a $\emptyset \neq \mathcal{I} \subseteq \mathcal{N}$ and $\epsilon \in \{-1,1\}^{\mathcal{I}}$ and some fixed $i \in \{1,2\}$ such that $h_1(x) = h^{\mathcal{I}}_{\epsilon,\mu^1}(x_i)$ and $h_2(x) = h^{\mathcal{I}}_{\epsilon,\mu^2}(x_i)$ for all $x \in X$.

(ii) *There exist* $\emptyset \neq \mathcal{I}^1, \mathcal{I}^2 \subseteq \mathcal{N}$ *and* $\epsilon^1 \in \{-1,1\}^{\mathcal{I}^1}$ *and* $\epsilon^2 \in \{-1,1\}^{\mathcal{I}^2}$ *such that* $h_1(x) = \max_{j\in\{1,2\}}\{h^{\mathcal{I}^j}_{\epsilon^j,\mu^j}(x_j) - \alpha_j\}$ *and* $h_2(x) = \max_{j\in\{1,2\}}\{h^{\mathcal{I}^j}_{\epsilon^j,\nu^j}(x_j) - \theta_j\}$ *for all* $x \in X$, *where* $\mu^j, \nu^j \in \mathbb{R}^{\mathcal{N}\setminus\mathcal{I}^j}$, *and* $\alpha, \theta \in \mathbb{R}^{\{1,2\}}$ *with* $\min_j \alpha_j = 0 = \min_j \theta_j$.

Proof. First consider the case where $h_1(x) = h^{\mathcal{I}^1}_{\epsilon^1,\mu^1}(x_1)$ and $h_2(x) = h^{\mathcal{I}^2}_{\epsilon^2,\mu^2}(x_2)$. We know that there exists paths $\gamma^1 = (\gamma^1_1, 0)$ and $\gamma^2 = (0, \gamma^2_2)$, where γ^j_j are as in (3.3.3). Lemma 3.2.5 allows us to calculate

$$H(h_1,h_2) = \lim_{t\to\infty} \|\gamma^1_1(t)\|_1 + h^{\mathcal{I}^2}_{\epsilon^2,\mu^2}(0) = \infty.$$

Thus $\delta(h_1,h_2) = \infty$, and the only remaining case to consider where h_1, h_2 both depend on only one coordinate is where $h_1(x) = h^{\mathcal{I}^1}_{\epsilon^1,\mu^1}(x_i)$ and $h_2(x) = h^{\mathcal{I}^2}_{\epsilon^2,\mu^2}(x_i)$. Without loss of generality we can assume $i = 1$. Now set $\gamma^1 = (\gamma^1_1, 0)$ and $\gamma^2 = (\gamma^2_1, 0)$, where the coordinate geodesics are again as in (3.3.3). Again by Lemma 3.2.5 we calculate

$$\delta(h_1,h_2) = \lim_{t\to\infty} \|\gamma^1_1(t)\|_1 + h^{\mathcal{I}^2}_{\epsilon^2,\mu^2}(\gamma^1_1(t)) + \lim_{t\to\infty} \|\gamma^2_1(t)\|_1 + h^{\mathcal{I}^1}_{\epsilon^1,\mu^1}(\gamma^2_1(t)).$$

This is exactly the equality encountered in the proof of Theorem 3.3.10, in which we show that $\delta(h_1,h_2) < \infty$ if and only if $\mathcal{I}^1 = \mathcal{I}^2$ and $\epsilon^1 = \epsilon^2$, in which case $\delta(h_1,h_2) = 2\sum_{i\in\mathcal{N}^1\setminus\mathcal{I}^1} |\mu^1_i - \mu^2_i|$.

The next case to consider is when $h_1(x) = \max_{j\in\{1,2\}}\{h^{\mathcal{I}^j}_{\epsilon^j,\mu^j}(x_j) - \alpha_j\}$, and $h_2(x) = h^{\mathcal{I}^3}_{\epsilon^3,\mu^3}(x_i)$. Without loss of generality we again assume that $i = 1$. Thus we once again have the characterizing geodesic $\gamma^2(t) = (\gamma^2_1(t), 0)$, with $\gamma^2_1(t)$ as in (3.3.3). Thus Lemma 3.2.5 allows us to immediately see

$$\begin{aligned} H(h_2,h_1) &= \lim_{t\to\infty} \|\gamma^2_1(t)\|_1 + \max\{h^{\mathcal{I}^1}_{\epsilon^1,\mu^1}(\gamma^2_1(t)) - \alpha_1, h^{\mathcal{I}^2}_{\epsilon^2,\mu^2}(0) - \alpha_2\} \\ &\geq \lim_{t\to\infty} \|\gamma^2_1(t)\|_1 + h^{\mathcal{I}^2}_{\epsilon^2,\mu^2}(0) - \alpha_2 = \infty, \end{aligned}$$

from which we can conclude that $\delta(h_1,h_2) = \infty$.

Finally we consider the case where, for all $x \in X$, $h_1(x) = \max_{j\in\{1,2\}}\{h^{\mathcal{I}^j}_{\epsilon^j,\mu^j}(x_j) - \alpha_j\}$ and $h_2(x) = \max_{j\in\{1,2\}}\{h^{\mathcal{J}^j}_{\delta^j,\nu^j}(x_j) - \theta_j\}$ for some $\emptyset \neq \mathcal{I}^j, \mathcal{J}^j \subseteq \mathcal{N}$, $\mu^j \in \mathbb{R}^{\mathcal{N}\setminus\mathcal{I}^j}$, $\nu^j \in \mathbb{R}^{\mathcal{N}\setminus\mathcal{J}^j}$, $\epsilon^j \in \{-1,1\}^{\mathcal{I}^j}$, $\delta^j \in \{-1,1\}^{\mathcal{J}^j}$, and $\alpha, \theta \in \mathbb{R}^{\{1,2\}}$ with $\min_j \alpha_j = 0 = \min_j \theta_j$. There are two sub-cases to consider: either $\alpha_j = \theta_j = 0$ for the same j, or if $\alpha_j = 0$ then $\theta_j \neq 0$. Let us first consider the former, and suppose without loss of generality that $\alpha_2 = \theta_2 = 0$. Now let γ^1 and γ^2 be geodesics as in (3.3.7). Once again, Lemma 3.2.5 allows us to calculate:

$$\begin{aligned}
H(h_1,h_2) &= \lim_{t\to\infty} \|\gamma^1(t)\| + \max\{h^{\mathcal{J}^1}_{\delta^1,\nu^1}(\gamma^1_1(t)) - \alpha_1, h^{\mathcal{J}^2}_{\delta^2,\nu^2}(\gamma^1_2(t))\} \\
&= \lim_{t\to\infty} \|\gamma^1_1(t)\|_1 + \beta^1 + \sum_{i\in(\mathcal{I}^2)^c} |\mu^2_i| \\
&\quad + \max\Bigg\{ \sum_{i\in\mathcal{J}^1\cap\mathcal{I}^1} -\delta^1_i\epsilon^1_i \frac{t}{|\mathcal{I}^1|} + \sum_{i\in\mathcal{I}^3\cap(\mathcal{I}^1)^c} \epsilon^3_i\mu^1_i + \sum_{i\in(\mathcal{J}^1)^c\cap\mathcal{I}^1} \left|\nu^1_i + \frac{\epsilon^1_i t}{|\mathcal{I}^1|}\right| - |\nu^1_i| \\
&\qquad + \sum_{i\in(\mathcal{J}^1)^c\cap(\mathcal{I}^1)^c} |\nu^1_i - \mu^1_i| - |\nu^1_i| - \theta_1, \sum_{i\in\mathcal{J}^2\cap\mathcal{I}^2} -\delta^2_i\epsilon^2_i \frac{\|\gamma^1_1(t)\|_1 + \beta^1}{|\mathcal{I}^2|} + \sum_{i\in\mathcal{J}^2\cap(\mathcal{I}^2)^c} \delta^2_i\mu^2_i \\
&\qquad + \sum_{i\in(\mathcal{J}^2)^c\cap\mathcal{I}^2} \left|\nu^2_i + \frac{\epsilon^2_i(\|\gamma^1_1(t)\|_1 + \beta^1)}{|\mathcal{I}^2|}\right| - |\nu^2_i| + \sum_{i\in(\mathcal{J}^2)^c\cap(\mathcal{I}^2)^c} |\nu^2_i - \mu^2_i| - |\nu^2_i| \Bigg\}.
\end{aligned}$$

As $\sup_t |\|\gamma^1_1(t)\|_1 - t| < \infty$ we can use the same analysis as in the proof of Theorem 3.3.10 to conclude that $H(h_1,h_2) < \infty$ if and only if $\mathcal{I}^j = \mathcal{J}^j$ and $\epsilon^j = \delta^j$, for $j \in \{1,2\}$. In this case we can further the above calculations:

$$\begin{aligned}
H(h_1,h_2) = \max\Bigg\{ &\sum_{i\in(\mathcal{I}^1)^c} |\nu^1_i - \mu^1_i| - |\nu^1_i| + |\mu^1_i| + (\alpha_1 - \theta_1), \\
&\sum_{i\in(\mathcal{I}^2)^c} |\nu^2_i - \mu^2_i| - |\nu^2_i| + |\mu^2_i| \Bigg\}
\end{aligned}$$

Symmetry also allows us to calculate that

$$H(h_2,h_1) = \max\Big\{ \sum_{i\in(\mathcal{I}^1)^c} |\nu_i^1 - \mu_i^1| + |\nu_i^1| - |\mu_i^1| + (\theta_1 - \alpha_1), \\ \sum_{i\in(\mathcal{I}^2)^c} |\nu_i^2 - \mu_i^2| + |\nu_i^2| - |\mu_i^2| \Big\}$$

Finally we consider the case where α and θ are zero at different indices. Without loss of generality we assume that $\alpha_2 = 0$ and $\theta_1 = 0$. We once again let γ^1 be the geodesic converging to h_1 as in (3.3.7). We set $\beta^2 = -\sum_{i\in(\mathcal{J}^1)^c} |\nu_i^1| + \theta_2$ and define $\gamma^2\colon [0,\infty) \to X$ by

$$\pi_i\gamma_2^2(t) = \begin{cases} \frac{-\delta_i^2 t}{|\mathcal{J}^2|} & i \in \mathcal{J}^2 \\ \nu_i^2 & i \notin \mathcal{J}^2 \end{cases}, \text{ and } \quad \pi_i\gamma_1^2(t) = \begin{cases} -\delta_i^1 \left(\frac{\|\gamma_2^2(t)\|_1 + \beta^2}{|\mathcal{J}^1|}\right) & i \in \mathcal{J}^1 \\ \nu_i^1 & i \notin \mathcal{J}^1 \end{cases}. \tag{3.3.8}$$

Thus we calculate,

$$\begin{aligned} H(h_1,h_2) &= \lim_{t\to\infty} \|\gamma^1(t)\| + \max\{h^{\mathcal{J}^1}_{\delta^1,\nu^1}(\gamma_1^1(t)), h^{\mathcal{J}^2}_{\delta^2,\nu^2}(\gamma_2^1(t)) - \theta_2\} \\ &= \lim_{t\to\infty} \|\gamma_1^1(t)\|_1 + \beta^1 + \sum_{i\in(\mathcal{I}^2)^c} |\mu_i^2| \\ &\quad + \max\Big\{ \sum_{i\in\mathcal{J}^1\cap\mathcal{I}^1} -\delta_i^1\epsilon_i^1 \frac{t}{|\mathcal{I}^1|} + \sum_{i\in\mathcal{J}^1\cap(\mathcal{I}^1)^c} \delta_i^1\mu_i^1 + \sum_{i\in(\mathcal{J}^1)^c\cap\mathcal{I}^1} \left|\nu_i^1 + \frac{\epsilon_i^1 t}{|\mathcal{I}^1|}\right| - |\nu_i^1| \\ &\qquad + \sum_{i\in(\mathcal{J}^1)^c\cap(\mathcal{I}^1)^c} |\nu_i^1 - \mu_i^1| - |\nu_i^1|, \sum_{i\in\mathcal{J}^2\cap\mathcal{I}^2} -\delta_i^2\epsilon_i^2 \frac{\|\gamma_1^1(t)\|_1 + \beta^1}{|\mathcal{I}^2|} + \sum_{i\in\mathcal{J}^2\cap(\mathcal{I}^2)^c} \delta_i^2\mu_i^2 \\ &\qquad + \sum_{i\in(\mathcal{J}^2)^c\cap\mathcal{I}^2} \left|\nu_i^2 + \frac{\epsilon_i^2(\|\gamma_1^1(t)\|_1 + \beta^1)}{|\mathcal{I}^2|}\right| - |\nu_i^2| + \sum_{i\in(\mathcal{J}^2)^c\cap(\mathcal{I}^2)^c} |\nu_i^2 - \mu_i^2| - |\nu_i^2| - \theta_2 \Big\}. \end{aligned}$$

The same analysis as above shows that $H(h_1,h_2) < \infty$ if and only if $\mathcal{I}^j = \mathcal{J}^j$ and $\epsilon^j = \delta^j$, for $j \in \{1,2\}$. In this case we see that

$$H(h_1,h_2) = \max\Bigg\{ \sum_{i\in(\mathcal{I}^1)^c} |\nu_i^1 - \mu_i^1| - |\nu_i^1| + |\mu_i^1| + \alpha_1, \\ \sum_{i\in(\mathcal{I}^2)^c} |\nu_i^2 - \mu_i^2| - |\nu_i^2| + |\mu_i^2| - \theta_2 \Bigg\},$$

and we calculate that

$$\begin{aligned} H(h_2,h_1) &= \lim_{t\to\infty} \|\gamma^2(t)\| + \max\{h^{\mathcal{I}^1}_{\epsilon^1,\mu^1}(\gamma_1^2(t)) - \alpha_1, h^{\mathcal{I}^2}_{\epsilon^2,\mu^2}(\gamma_2^2(t))\} \\ &= \lim_{t\to\infty} \|\gamma_2^2(t)\|_1 + \beta^2 + \sum_{i\in(\mathcal{I}^1)^c} |\nu_i^1| \\ &\quad + \max\Bigg\{ \sum_{i\in\mathcal{I}^1} -\frac{\|\gamma_2^2(t)\|_1 + \beta^2}{|\mathcal{I}^1|} + \sum_{i\in(\mathcal{I}^1)^c} |\mu_i^1 - \nu_i^1| - |\mu_i^1| - \alpha_1, \\ &\qquad \sum_{i\in\mathcal{I}^2} -\frac{t}{|\mathcal{I}^2|} + \sum_{i\in(\mathcal{I}^2)^c} |\mu_i^2 - \nu_i^2| - |\mu_i^2| \Bigg\} \\ &= \max\Bigg\{ \sum_{i\in(\mathcal{I}^1)^c} |\mu_i^1 - \nu_i^1| - |\mu_i^1| + |\nu_i^1| - \alpha_1, \\ &\qquad \sum_{i\in(\mathcal{I}^2)^c} |\mu_i^2 - \nu_i^2| - |\mu_i^2| + |\nu_i^2| + \theta_2 \Bigg\}. \end{aligned}$$

□

The proof of the above theorem gives us the following corollary:

Corollary 3.3.16. *If* $h_1, h_2 \in \partial_B \overline{X}^h$ *are of the form, for all* $x \in X$, $h_1(x) = \max_{j\in\{1,2\}}\{h^{\mathcal{I}^j}_{\epsilon^j,\mu^j}(x_j) - \alpha_j\}$ *and* $h_2(x) = \max_{j\in\{1,2\}}\{h^{\mathcal{I}^j}_{\epsilon^j,\nu^j}(x_j) - \theta_j\}$ *where* $\mu^j, \nu^j \in \mathbb{R}^{\mathcal{N}\setminus\mathcal{I}^j}$, *and* $\alpha, \theta \in \mathbb{R}^{\{1,2\}}$ *with*

$\min_j \alpha_j = 0 = \min_j \theta_j$, *then*

$$\delta(h_1,h_2) = \max_{j\in\{1,2\}}\left\{\sum_{i\in(\mathcal{I}^j)^c} |\mu_i^j - \nu_i^j| + |\mu_i^j| - |\nu_i^j| - (\theta_j - \alpha_j)\right\} + \max_{j\in\{1,2\}}\left\{\sum_{i\in(\mathcal{I}^j)^c} |\mu_i^j - \nu_i^j| - |\mu_i^j| + |\nu_i^j| + (\theta_j - \alpha_j)\right\}.$$

As in Remark 3.3.12 if h is of the form $h(x) = \max_{j\in\{1,2\}}\{h^{\mathcal{I}^j}_{\epsilon^j,\mu^j}(x_j) - \alpha_j\}$,then there is a natural identification between the part that h belongs to and a vector space. Indeed we can treat $\mathcal{P}(h)$ as a vector space isomorphic to the vector space

$$V^h = \mathbb{R}^{\mathcal{N}\setminus\mathcal{I}^1} \bigoplus \mathbb{R}^{\mathcal{N}\setminus\mathcal{I}^2} \bigoplus \mathbb{R}^2/\operatorname{Sp}(\mathbf{1}),$$

where $\operatorname{Sp}(\mathbf{1}) = \operatorname{span}\{(1,1)\}$. V^h and $\mathcal{P}(h)$ are also isometric if we equip V^h with the detour distance in the natural way. The following example shows that the detour distance on V^h is not always induced by a norm:

Example 3.3.17. Let $h \in \partial_B \overline{X}^h$ be of the form $h(x) = \max_{j\in\{1,2\}}\{h^{\mathcal{I}^j}_{\epsilon^j,\mu^j}(x_j) - \alpha_j^1\}$ where $\mathcal{I}^1 = \mathcal{I}^2 = \mathcal{I} \neq \mathcal{N}$ and $\epsilon^1 = \epsilon^2$. We know that norms are translation invariant, so show that the detour distance on $V^h = \mathbb{R}^{\mathcal{N}\setminus\mathcal{I}} \bigoplus \mathbb{R}^{\mathcal{N}\setminus\mathcal{I}} \bigoplus \mathbb{R}^2/\operatorname{Sp}(\mathbf{1})$ is not induced by a norm we need to find $v_1, v_2, w \in V^h$ such that $\delta(v_1,v_2) \neq \delta(v_1+w, v_2+w)$. Let us define $v_1 = (\mu^1,\mu^2,0)$, $v_2 = (\nu^1,\nu^2,0)$, and $w = (-\nu^1,0,0)$, where for all $i \in \mathcal{N}\setminus\mathcal{I}$ we have

$$\mu_i^1 = 5,\ \mu_i^2 = 5,\ \nu_i^1 = -1,\ \text{and } \nu_i^2 = -4.$$

With these values we calculate

$$\delta(v_1,v_2) = \max\left\{\sum_{(i\in\mathcal{I})^c} 10, \sum_{(i\in\mathcal{I})^c} 10\right\} + \max\left\{\sum_{(i\in\mathcal{I})^c} 2, \sum_{(i\in\mathcal{I})^c} 8\right\} = 18|\mathcal{I}^c|$$

While

$$\begin{aligned}\delta(v_1+w, v_2+w) = &\max\left\{\sum_{i\in(\mathcal{I})^c} 2|\mu_i^1-\nu_j^1|, \sum_{i\in(\mathcal{I})^c} |\mu_i^2-\nu_i^2|+|\mu_i^2|-|\nu_i^2|\right\}+\\ &\max\left\{\sum_{i\in(\mathcal{I})^c} 0, \sum_{i\in(\mathcal{I})^c} |\mu_i^2-\nu_i^2|-|\mu_i^2|+|\nu_i^2|\right\}\\ =&\max\left\{\sum_{i\in(\mathcal{I})^c} 12, \sum_{i\in(\mathcal{I})^c} 10\right\}+\max\left\{0, \sum_{i\in(\mathcal{I})^c} 8\right\}\\ =& \ 20|\mathcal{I}^c|.\end{aligned}$$

Thus indeed δ is not translation invariant on V^h, and so cannot be induced by a norm.

The above example of course does not serve as a proof that there exists no isometry between $\mathcal{P}(h)$ and some very convoluted normed space, but it does show that the natural identification between $\mathcal{P}(h)$ and V^h is not an isometry if we equip V^h with a norm, unlike in Remark 3.3.12.

3.4 Metric Compactification of ℓ^1 Metric Spaces

In Subsection 3.3.3, we showed that the horofunctions of $\ell^1(\mathcal{N})$ are precisely those functions $h^{\mathcal{I}}_{\epsilon,\mu}$, where, for $x=(x_1,\ldots,x_n)\in\ell^1(\mathcal{N})$,

$$h^{\mathcal{I}}_{\epsilon,\mu}(x)=\sum_{i\in\mathcal{I}}-\epsilon_i x_i+\sum_{j\in\mathcal{N}\setminus I}|x_j-\mu_j|-|\mu_j|.$$

In the beginning of Section 3.3 we also showed that the horofunctions of $(\mathbb{R},|\cdot|)$ are simply the functions h^ϵ for $\epsilon\in\{-1,1\}$, where

$$h^\epsilon(x)=-\epsilon x.$$

We can thus write

$$h^{\mathcal{I}}_{\epsilon,\mu}(x) = \sum_{i\in\mathcal{I}} h^{\epsilon}(x_i) + \sum_{j\in\mathcal{N}\setminus I} h_{\mu_j}(x_j).$$

There is thus a natural identification between $\overline{\ell^1(\mathcal{N})}^h$ and $\prod_{i=1}^n \overline{\mathbb{R}}^h$. In this section we show that this observation can be extended to classify the horofunction compactification of the ℓ^1 product of a finite collection of arbitrary metric spaces. A similar classification for the horofunction compactification of the ℓ^1 sum of proper geodesic metric spaces can be found in joint work with Lemmens and Milliken in [45].

We let $(M_i, d_i)_{i=1}^m$ be a finite collection of complete metric spaces. We define (M, d) by $M = \prod_{i=1}^m M_i$ and $d((x_i),(y_i)) = \sum_{i=1}^m d_i(x_i, y_i)$. We let $b = (b_i)_{i=1}^m$ be an arbitrary basepoint. Recall for any metric space X, we have the split $\partial\overline{X}^h = \overline{X}^{h,\infty} \cup \overline{X}^{h,e}$, where

$$\overline{X}^{h,\infty} = \{h \in \partial\overline{X}^h : \inf_X h = -\infty\}, \text{ and } \overline{X}^{h,e} = \{h \in \partial\overline{X}^h : \inf_X h > -\infty\}.$$

If $f_i \in \mathbb{R}^{M_i}$ for all $i \in \{1,\ldots,m\}$, we define $\sum_{i=1}^m f_i \in \mathbb{R}^M$ in the natural way, by,

$$\sum_{i=1}^m f_i(x) = \sum_{i=1}^m f_i(x_i),$$

for all $x = (x_1,\ldots,x_m) \in M$. With this notation we can introduce the main theorem of this section.

Theorem 3.4.1. *A function $h \in \mathbb{R}^M$ is in $\overline{M}^h$ if and only if for each $i \in \{1,\ldots,m\}$ there exists a $h_i \in \overline{M}_i^h$ such that*

$$h = \sum_{i=1}^m h_i. \tag{3.4.1}$$

Furthermore, $h \in \partial\overline{M}^h$ if and only if $h_j \in \partial\overline{M}_j^h$ for some $j \in \{1,\ldots,m\}$, and if $h_j \in \overline{M}_j^{h,\infty}$, then $h \in \overline{M}^{h,\infty}$. Finally, $h \in \partial\overline{M}^h$ is a Busemann point if and only if $h_i \in \partial\overline{M}_i^h$ means that h_i is a Busemann point, for all $i \in \{1,\ldots,m\}$.

Proof. Lemmas 3.4.2, 3.4.3 and 3.4.4 proved below prove the theorem. □

Lemma 3.4.2. *For any $h \in \overline{M}^h$ there exists, for each $j \in \{1, \ldots, m\}$, a $h_j \in \overline{M}_j^h$, so that for all $y = (y_i) \in M$*

$$h(y) = \sum_{i=1}^{m} h_j(y_j).$$

Furthermore, if $h \in \overline{M}^{h,\infty}$ then there exists some $j_0 \in \{1, \ldots, m\}$, so that $h_{j_0} \in \overline{M}_{j_0}^{h,\infty}$, and if instead $h \in \overline{M}^{h,e}$ then there exists some $j_0 \in \{1, \ldots, m\}$, so that $h_{j_0} \in \overline{M}_{j_0}^{h,e}$.

Proof. There must exist a net $(x^\alpha) = ((x_i^\alpha)_{i=1}^m) \subseteq M$ such that $h_{x^\alpha} \to h$ pointwise. As $\overline{M}_i^h$ is compact in the topology of pointwise convergence, by taking subnets we can assume that $h_{x_i^\alpha}$ converges pointwise to some $h_i \in \overline{M}_i$ for each $i \in \{1, \ldots, m\}$. Thus for every $y \in M$

$$\begin{aligned} h(y) &= \lim_\alpha d(x^\alpha, y) - d(x^\alpha, b) = \lim_\alpha \sum_{j=1}^{m} d_j(x_j^\alpha, y_j) - \sum_{i=1}^{m} d_i(x_i^\alpha, b_i) \\ &= \lim_\alpha \sum_{j=1}^{m} h_{x_j^\alpha}(y_j) = \sum_{j=1}^{m} h_j(y_j). \end{aligned}$$

By way of contradiction, assume that $h_j \notin \partial\overline{M}_j^h$ for all $j \in \{1, \ldots, m\}$. There must thus exist an $x_j \in M_j$ for all $j \in \{1, \ldots, m\}$ such that $h_j = h_{x_j}$. Set $x = (x_1, \ldots, x_m) \in M$. Thus for any $y \in M$, by the above calculation:

$$h(y) = \sum_{j=1}^{m} h_{x_j}(y_j) = \sum_{j=1}^{m} d_j(x_j, y_j) - d(x_j, b_j) = d(x, y) - d(x, b) = h_x(y),$$

which contradicts the fact that h is not an internal metric functional. Now assume that $h \in \overline{M}^{h,\infty}$, but $h_i \notin \overline{M}_i^{h,\infty}$ for all $i \in \{1, \ldots, m\}$. This would then mean that

$$h(y) \geq m \min_{i \in \{1,\ldots,m\}} \inf_{y_i \in M_i} h_i(y_i) > -\infty,$$

a contradiction. Finally assume that $h \in \overline{M}^{h,e}$, but $h_i \notin \overline{M}_i^{h,e}$ for all $i \in \{1, \ldots, m\}$. As we know there must exist some $h_{j_0} \in \partial\overline{M}_{j_0}^h$, our assumption forces h_{j_0} to be an element of $\overline{M}_{j_0}^{h,\infty}$, which means that there exists a sequence $(y_{j_0}^n)$ in M_{j_0} such that $h_{j_0}(y_{j_0}^n) \to -\infty$, else $h_{j_0} \in \overline{M}_{j_0}^{h,e}$ but this means that $h(b_1, \ldots, y_{j_0}^n, \ldots, b_m) \to -\infty$, a contradiction.

□

We thus see that every horofunction on M is of form (3.4.1). The converse is also true.

Lemma 3.4.3. *Suppose we are given some function h on M where*

$$h = \sum_{i=1}^{m} h_i$$

and each $h_i \in \overline{M}^h$. If there exists a $j_0 \in \{1, \ldots, m\}$ with $h_{j_0} \in \overline{M}_{j_0}^{h,\infty}$, then $h \in \overline{M}^{h,\infty}$. If, instead, there exists a $j_0 \in \{1, \ldots, m\}$ with $h_{j_0} \in \overline{M}_{j_0}^{h,e}$ and $h_i \notin \overline{M}_i^{h,\infty}$ for any $i \neq j_0$, then $h \in \overline{M}^{h,e}$

Proof. As $h_i \in \overline{M}_i^h$ for each $i \in \{1, \ldots, m\}$, there must exist nets $(x_i^{\alpha_i}) \subseteq M_i$ such that $h_{x_i^{\alpha_i}} \to h_i$. Let $(A_i, \leq_i)$ be the directed set underlying the net $(x_i^{\alpha_i})$. We then consider the set $\Gamma = \prod_{i=1}^m A_i$, and equip it with the order $\leq$ defined by $(\alpha_1, \ldots, \alpha_m) \leq (\beta_1, \ldots, \beta_m)$ if and only if $\alpha_i \leq \beta_i$ for all $i \in \{1, \ldots, m\}$. This order makes Γ directed. For each $\gamma = (\alpha_1, \ldots, \alpha_m) \in \Gamma$ we define $x^\gamma = (x^{\alpha_1}, \ldots, x^{\alpha_m}) \in M$, so $(x^\gamma) \subseteq M$ is a net. For any $y \in M$ and $\gamma = (\alpha_1, \ldots, \alpha_m) \in \Gamma$ we have

$$h_{x^\gamma}(y) = \sum_{j=1}^{m} d_j(x_j^{\alpha_j}, y_j) - d_j(x_j^{\alpha_j}, b_j) = \sum_{j=1}^{m} h_{x_j^{\alpha_j}}(y_j).$$

Now, for every $\varepsilon > 0$ there exists $\alpha_j' \in A_j$ for each $j \in \{1, \ldots, m\}$, such that $|h_{x^{\alpha_j}}(y_j) - h_j(y_j)| < \varepsilon$ for all $\alpha_j \geq \alpha_j'$. Thus, if we define $\gamma' = (\alpha_1', \ldots, \alpha_m')$, the above equation means that for all $\gamma \geq \gamma'$

$$|h(y) - h_{x^\gamma}(y)| \leq \sum_{j=1}^{m} |h_{x_j^{\gamma(j)}}(y_j) - h_j(y_j)| < m\varepsilon.$$

so $\lim_\gamma h_{x^\gamma}(y) = h(y)$. Therefore $h \in \overline{M}^h$. Now, if there exists a $j_0 \in \{1, \ldots, m\}$ such that ${}_{j_0} \in \overline{M}^{h,\infty}_{j_0}$ there exists a sequence $(y^n_{j_0}) \in M_{j_0}$ such that $h_{j_0}(y^n_{j_0}) \to -\infty$. If we then evaluate h on the sequence $(b_1, \ldots, y^n_{j_0}, \ldots, b_m)$ it follows that $\inf_M h(y) = -\infty$, so indeed $h \in \overline{M}^{h,\infty}$. If instead there exists a $j_0 \in \{1, \ldots, m\}$ with $h_{j_0} \in \overline{M}^{h,e}_{j_0}$ and $h_i \notin \overline{M}^{h,\infty}_i$ for any $i \neq j_0$ then $\inf_M h > -\infty$. Furthermore there cannot exist any $x_{j_0} \in M_{j_0}$ such that $h_{j_0}(y) = h_{x_{j_0}}(y)$ for all $y \in M_{j_0}$. Now suppose by way of contradiction that there exists some $x \in M$ such that $h = h_x$. This would then mean that for every $y \in M$, $h(y) = h_x(y)$. In particular this would mean that, for every $y \in M_{j_0}$,

$$h((b_1, \ldots, y, \ldots, b_m)) = h_x((b_1, \ldots, y, \ldots, b_m)) = h_{x_{j_0}}(y_{j_0}),$$

which implies that $h_{j_0}(y) = h_{x_{j_0}}(y)$ for all $y \in M_{j_0}$, a contradiction. Thus $h \notin M$. Our assumption also means that $\inf_{M_j} h_j > -\infty$ for all j, which means that $h \notin \overline{M}^{h,\infty}$. As $h \in \overline{M}^h$ this must mean that $h \in \overline{M}^{h,e}$. □

We end the proof of Theorem 3.4.1 with:

Lemma 3.4.4. *The horofunction $h = \sum_{i=1}^m h_i \in M$ is a Busemann point of M if and only if each h_i is either an internal metric functional or Busemann point of M_i, for $i \in \{1, \ldots, m\}$, and there exists at least one $j \in \{1, \ldots, m\}$ such that h_j is a Busemann point.*

Proof. First assume we are given, for each $i \in \{1, \ldots, m\}$, an almost-geodesic net $(x_i^{\alpha_i})$ in M_i with corresponding internal metric functionals converging to h_i, and there exists a j such that h_j is a Busemann point. Just as in the proof of Lemma 3.4.3 above, we can construct the net $(x^\gamma) \subset M$ such that $h_{x^\gamma} \to h$. The linearity of the ℓ^1 metric d means that (x^γ) is an almost-geodesic net. Conversely, suppose that h is a Busemann point, and that $h_{x^\alpha} \to h$ pointwise, where $(x^\alpha) = ((x_i^\alpha)_{i=1}^m)$ is an almost-geodesic net. Fix an $\varepsilon > 0$. By definition there exists a α', so that for all $\beta \geq \alpha \geq \alpha'$

$$\sum_{i=1}^m d_i(b_i, x_i^\beta) \geq \sum_{i=1}^m d_i(b_i, x_i^\alpha) + d_i(x_i^\beta, x_i^\alpha) - \varepsilon.$$

For any $j \in \{1, \dots, m\}$, we can subtract $\sum_{i \neq j} d_i(b_i, x_i^\beta)$ from both sides in the above in equality to calculate

$$\begin{aligned} d_j(b_j, x_j^\beta) &\geq d_j(b_j, x_j^\alpha) + d_j(x_j^\beta, x_j^\alpha) + \sum_{i \neq j}^{m} \Big[d_i(b_i, x_i^\alpha) + d_i(x_i^\beta, x_i^\alpha) - d_i(b_i, x_i^\beta) \Big] - \varepsilon \\ &\geq d_j(b_j, x_j^\beta) + d_j(x_j^\beta, x_j^\alpha) - \varepsilon, \end{aligned}$$

where in the last line we applied the triangle inequality to each $d_i(b_i, x_i^{\alpha'})$ for $i \neq j$. This then shows that each $j \in \{1, \dots, m\}$, the net (x_j^α) is an almost-geodesic. Lemma 3.2.2 tells us that, for each $j \in \{1, \dots, m\}$, there exists $h_j \in \overline{M}^h$ which $h_{x_j^\alpha}$ converges to. If, for any $j \in \{1, \dots, m\}$, the net (x_j^α) is bounded, h_j is an internal metric functional, say h_{x_j} by Proposition 2.5 in [69]. Thus, if all (x_j^α) are bounded, $h = \sum_{j=1}^m h_{x_j}$ is also an internal metric functional, a contradiction. □

The above lemma means that for any $h \in \partial_B \overline{M}^h$ there exists a $\emptyset \neq \mathcal{I} \subseteq \{1, \dots, m\}$ and a decomposition

$$h = \sum_{i \in \mathcal{I}} h_i + \sum_{j \in \mathcal{I}^c} h_{x_j},$$

where for each $i \in \mathcal{I}$, h_i is a Busemann point.

Theorem 3.4.5. *Two Busemann points*

$$h = \sum_{i \in \mathcal{I}} h_i + \sum_{j \in \mathcal{I}^c} h_{x_j}, \quad h' = \sum_{i \in \mathcal{I}'} h'_i + \sum_{j \in \mathcal{I}'^c} h_{y_j}$$

are in the same part of the Busemann boundary if and only if $\mathcal{I}' = \mathcal{I}$ and each $h_i \in \partial_B \overline{M}_i^h$ is in the same part as $h'_i \in \partial_B \overline{M}_i^h$. If h and h' are in the same part, then

$$\delta(h, h') = \sum_{i \in \mathcal{I}} \delta(h_i, h'_i) + 2 \sum_{j \notin \mathcal{I}} d(y_j, x_j).$$

Proof. First assume we are given two Busemann points

$$h = \sum_{i \in \mathcal{I}} h_i + \sum_{j \in \mathcal{I}^c} h_{x_j}, \quad h' = \sum_{i \in \mathcal{I}'} h'_i + \sum_{j \in \mathcal{I}'^c}, h_{y_j} \tag{3.4.2}$$

where $\mathcal{I}' = \mathcal{I}$ and each $h_i \in \partial_B \overline{M}_i^h$ is in the same part as $h'_i \in \partial_B \overline{M}_i^h$. Let (x^α) be an almost-geodesic net such that $h_{x^\alpha} \to h$, and (y^β) an almost-geodesic net such that $h_{y^\beta} \to h'$. The proof of Lemma 3.4.4 shows that, for all $i \in \{1, \dots, m\}$, the nets (x_i^α) and (y_i^β) are also almost-geodesic nets in M_i. By Lemma 3.2.5 we know that

$$\begin{aligned} H(h, h') &= \lim_\alpha d(b, x^\alpha) + h'(x^\alpha) \\ &= \lim_\alpha \sum_{i \in \mathcal{I}} d_i(b_i, x_i^\alpha) + h'_i(x_i^\alpha) + \sum_{j \notin \mathcal{I}} d_j(b_j, x_j^\alpha) + h_{y_j}(x_\alpha^j) \\ &= \sum_{i \in \mathcal{I}} H(h_i, h'_i) + \sum_{j \notin \mathcal{I}} d_j(b_j, x_j) + h_{y_j}(x_j) < \infty, \end{aligned}$$

where the last inequality follows from the assumption that each h_i and h'_i are in the same part of the boundary for each $i \in \mathcal{I}$. A symmetrical argument shows similarly that $H(h', h) < \infty$, so we conclude that h and h' lie in the same part of the boundary.

Conversely, assume that h and h' are of the form (3.4.2) and lie in the same part of $\partial_B \overline{M}^h$. Thus $H(h, h') < \infty$ and $H(h', h) < \infty$. We can assume that $h_{x^\alpha} \to h$ and $h_{y^\beta} \to h'$ for almost-geodesic nets (x^α) and (y^α). The proof of Lemma 3.4.4 shows that each coordinate net (x_i^α) and (y_i^β) for $i \in \{1, \dots, m\}$ is also an almost-geodesic. Thanks to Lemma 3.2.5 we can write

$$\begin{aligned} H(h, h') = \lim_\alpha & \sum_{i \in \mathcal{I} \cap \mathcal{I}'} d_i(b_i, x_i^\alpha) + h'_i(x_i^\alpha) + \sum_{j \in \mathcal{I} \setminus \mathcal{I}'} d_j(b_j, x_j^\alpha) + h_{y_j}(x_j^\alpha) \\ & + \sum_{i \in \mathcal{I}' \setminus \mathcal{I}} d_i(b_i, x_i^\alpha) + h'_i(x_i^\alpha) + \sum_{j \notin \mathcal{I} \cup \mathcal{I}'} d_j(b_j, x_j^\alpha) + h_{y_j}(x_j^\alpha). \end{aligned}$$

The sum of the two constituent terms in each of the above four sums is non-negative for all indices, so for $H(h, h')$ to be finite it is required that the limit of each sum of two terms within the sums is finite. For $j \notin \mathcal{I}$, (x_j^α) is bounded, so we need only be concerned with instances where $j \in \mathcal{I}$. In particular we require that

$$\sum_{j \in \mathcal{I} \setminus \mathcal{I}'} d_j(b_j, x_j^\alpha) + h_{y_j}(x_j^\alpha) < \infty,$$

but $h_{y_j}(x_\alpha^j) \geq d_j(b_j, y_j)$ for each $j \in \mathcal{I} \setminus \mathcal{I}'$, whereas $d_j(b_j, x_j^\alpha) \xrightarrow{\alpha} \infty$, so this is only possible if $\mathcal{I} \subseteq \mathcal{I}'$. A symmetrical argument applied to $H(h', h)$ shows that $\mathcal{I}' \subseteq \mathcal{I}$, meaning that $\mathcal{I} = \mathcal{I}'$. Thus,

$$H(h, h') = \lim_\alpha \sum_{i \in \mathcal{I}} d_i(b_i, x_i^\alpha) + h_i'(x_i^\alpha) + \sum_{j \notin \mathcal{I}} d_j(b_j, x_j^\alpha) + h_{y_j}(x_j^\alpha), \quad \text{and}$$
$$H(h', h) = \lim_\beta \sum_{i \in \mathcal{I}} d_i(b_i, y_i^\beta) + h_i(y_i^\beta) + \sum_{j \notin \mathcal{I}} d_j(b_j, y_j^\beta) + h_{x_j}(y_j^\beta).$$

Lemma 2.4.2 guarantees the existence of subnets (x^γ) and (y^γ) such that

$$\begin{aligned}\delta(h, h') = &\lim_\gamma \sum_{i \in \mathcal{I}} d_i(b_i, x_i^\gamma) + h_i'(x_i^\gamma) + d_i(b_i, y_i^\gamma) + h_i(y_i^\gamma) \\ &+ \sum_{j \notin \mathcal{I}} d_j(b_j, x_j^\gamma) + h_{y_j}(x_j^\gamma) + d_j(b_j, y_j^\gamma) + h_{x_j}(y_j^\gamma).\end{aligned}$$

Furthermore, the construction of the subnet in the proof of Lemma 2.4.2 makes it easy to check that both (x^γ) and (y^γ) are also almost-geodesic nets. Thus, we can once again use Lemma 3.2.5 in conjunction with the above equality to deduce that

$$\delta(h, h') = \sum_{i \in \mathcal{I}} \delta(h_i, h_i') + \sum_{j \notin \mathcal{I}} d_j(b_j, x_j) + h_{y_j}(x_j) + d_j(b_j, y_j) + h_{x_j}(y_j),$$

which can be finite only if $\delta(h_i, h_i')$ is finite for each $i \in \mathcal{I}$. It also immediately follows that if h and h' are in the same part, then

$$\delta(h, h') = \sum_{i \in \mathcal{I}} \delta(h_i, h'_i) + 2 \sum_{j \notin \mathcal{I}} d(y_j, x_j).$$

$\square$

Chapter 4

Product Domains

This chapter, along with Chapters 5 and 6, consists of sections in [47], which was joint work with Bas Lemmens. The reader may notice some content in the preliminary sections of each of these chapters 4, 5, and 6 which has already been discussed in some form in Chapters 2 and 3 above. The author decided to keep this content in these chapters for the convenience of the reader, and to maintain consistency between these chapters and the published journal article [47].

In this chapter we analyse the geometry and topology of the horofunction compactification of bounded symmetric domains of the form $B^\circ = B_1^\circ \times \cdots \times B_r^\circ$, where $B_i^\circ = \{z \in \mathbb{C}^{n_i} \colon |z_1|^2 + \cdots + |z_{n_i}|^2 < 1\}$, under the Kobayashi distance. In fact, we shall consider slightly more general product domains where each B_i° is the open unit ball of a norm on $\mathbb{C}^{n_i}$ with a strongly convex C^3-boundary. Even though these domains no longer correspond to noncompact type symmetric spaces we shall see that there still exists a homeomorphism between the horofunction compactification and the closed dual unit ball of the Finsler metric at the origin. We will start by recalling some basic concepts.

4.1 Product domains and Kobayashi distance

On a convex domain $D \subseteq \mathbb{C}^n$ the *Kobayashi distance* is given by

$$k_D(z,w) = \inf\{\rho(\zeta,\eta)\colon\ \exists f\colon \Delta \to D \text{ holomorphic with } f(\zeta) = z \text{ and } f(\eta) = w\}$$

for all $z, w \in D$, where

$$\rho(z,w) = \log \frac{1 + \left|\frac{w-z}{1-\bar{z}w}\right|}{1 - \left|\frac{w-z}{1-\bar{z}w}\right|} = 2\tanh^{-1}\left(1 - \frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^{1/2}$$

is the *hyperbolic distance* on the open disc $\Delta = \{z \in \mathbb{C}\colon |z| < 1\}$.

It is known, see [1, Proposition 2.3.10], that if $D \subset \mathbb{C}^n$ is bounded convex domain, then (D, k_D) is a proper metric space, whose topology coincides with the usual topology on $\mathbb{C}^n$. Moreover, (D, k_D) is a geodesic metric space containing geodesic rays, see [1, Theorem 2.6.19] or [39, Theorem 4.8.6].

For the Euclidean ball $B_n^\circ = \{(z_1, \dots, z_n) \in \mathbb{C}^n\colon \|z\|^2 < 1\}$, where $\|z\|^2 = \sum_i |z_i|^2$, the Kobayashi distance satisfies

$$k_{B_n}(z,w) = 2\tanh^{-1}\left(1 - \frac{(1-\|w\|^2)(1-\|z\|^2)}{|1-\langle z,w\rangle|^2}\right)^{1/2}$$

for all $z, w \in B_n^\circ$, see [1, Chapters 2.2 and 2.3].

In our setting we will consider product domains $B^\circ = \prod_{i=1}^r B_i^\circ$, where each B_i° is an open unit ball of a norm in $\mathbb{C}^{n_i}$, and we will use the product property of k_B, which says that

$$k_B(z,w) = \max_{i=1,\dots,r} k_i(z_i, w_i),$$

where k_i is the Kobayashi distance on B_i°, see [39, Theorem 3.1.9]. So for the polydisc

$\Delta^r = \{(z_1, \dots, z_r) \in \mathbb{C}^r \colon \max_i |z_i| < 1\}$, the Kobayashi distance satisfies

$$k_{\Delta^r}(z, w) = \max_i \rho(z_i, w_i) \quad \text{for all } w = (w_1, \dots, w_r), z = (z_1, \dots, z_r) \in \Delta^r.$$

For the Euclidean ball, B_n°, it is well known that the horofunctions of (B_n°, k_{B_n}), with basepoint $b = 0$, are given by

$$h_\xi(z) = \log \frac{|1 - \langle z, \xi \rangle|^2}{1 - \|z\|^2} \quad \text{for all } z \in B_n^\circ, \tag{4.1.1}$$

where $\xi \in \partial B_n^\circ$. Moreover, each horofunction h_ξ is a Busemann point, as it is the limit induced by the geodesic ray $t \mapsto \frac{e^t - 1}{e^t + 1}\xi$, for $0 \le t < \infty$.

Moreover, if B° is a product of Euclidean balls, then the horofunctions are known, see [1, Proposition 2.4.12] and [41, Corollary 3.2]. Indeed, for a product of Euclidean balls $B^\circ = B_{n_1}^\circ \times \cdots \times B_{n_r}^\circ$ the Kobayashi distance horofunctions with basepoint $b = 0$ are precisely the functions of the form

$$h(z) = \max_{j \in J} \left(h_{\xi_j}(z_j) - \alpha_j \right),$$

where $J \subseteq \{1, \dots, r\}$ nonempty, $\xi_j \in \partial B_{n_j}^\circ$ for $j \in J$, and $\min_{j \in J} \alpha_j = 0$. Moreover, each horofunction is a Busemann point.

The form of the horofunctions of the product of Euclidean balls is essentially due to the product property of the Kobayashi distance and the smoothness and convexity properties of the balls. Indeed, more generally, the following result holds, see [41, Section 2 and Lemma 3.3].

Theorem 4.1.1. *If $D_i \subset \mathbb{C}^{n_i}$ is a bounded strongly convex domain with C^3-boundary, then for each $\xi_i \in \partial D_i$ there exists a unique horofunction h_{ξ_i} which is the limit of a geodesic γ from the basepoint $b_i \in D_i$ to ξ_i. Moreover, these are all the horofunctions. If $D = \prod_{i=1}^r D_i$, where each D_i is a bounded strongly convex domain with C^3-boundary, then each* *horofunction h of (D, k_D) with respect to the basepoint $b = (b_1, \dots, b_r)$ is of the form*

$$h(z) = \max_{j\in J} \left(h_{\xi_j}(z_j) - \alpha_j\right), \tag{4.1.2}$$

where $J \subseteq \{1, \dots, r\}$ nonempty, $\xi_j \in \partial D_j$ for $j \in J$, and $\min_{j\in J} \alpha_j = 0$. Furthermore, each horofunction is a Busemann point, and the part of h, where h is given by (4.1.2), consists of those horofunctions h' of the form,

$$h'(z) = \max_{j\in J} \left(h_{\xi_j}(z_j) - \beta_j\right),$$

with $\min_{j\in J} \beta_j = 0$.

Now let $D = \prod_{i=1}^r D_i$, where each D_i is a bounded strongly convex domain with C^3-boundary. Given $J \subseteq \{1, \dots, r\}$ nonempty, $\xi_j \in \partial D_j$ for $j \in J$, and $\alpha_j \geq 0$ for $j \in J$ with $\min_{j\in J} \alpha_j = 0$, we can find geodesic paths $\gamma_j \colon [0, \infty) \to D_j$ from b_j to ξ_j, and form the path $\gamma \colon [0, \infty) \to D$, where

$$\gamma(t)_j = \begin{bmatrix} \gamma_j(t - \alpha_j) & \text{for all } j \in J \text{ and } t \geq \alpha_j \\ b_j & \text{otherwise.} \end{bmatrix} \tag{4.1.3}$$

Lemma 4.1.2. *The path $\gamma \colon [0, \infty) \to D$ in (4.1.3) is a geodesic path, and $h_{\gamma(t)} \to h$ where h is given by (4.1.2).*

Proof. Let k_i denote the Kobayashi distance on D_i. By the product property we have that

$$k_D(\gamma(s), \gamma(t)) = \max_i k_i(\gamma(s)_i, \gamma(t)_i)$$

for all $s \geq t \geq 0$. By construction $k_i(\gamma(s)_i, \gamma(t)_i) \leq k_i(\gamma_i(s), \gamma_i(t)) = s - t$ for all i and $s \geq t \geq 0$. For $j \in J$ with $\alpha_j = 0$ we have that $k_j(\gamma(s)_j, \gamma(t)_j) = k_j(\gamma_j(s), \gamma_j(t)) = s - t$ for all $s \geq t \geq 0$, and hence

$$k_D(\gamma(s), \gamma(t)) = \max_i k_i(\gamma(s)_i, \gamma(t)_i) = s - t$$

for all $s \geq t \geq 0$.

Note that for $z \in D$ we have

$$\begin{aligned}
\lim_{t\to\infty} h_{\gamma(t)}(z) &= \lim_{t\to\infty} k_D(z, \gamma(t)) - k_D(\gamma(t), b) \\
&= \lim_{t\to\infty} \max_i (k_i(z_i, \gamma(t)_i) - t) \\
&= \lim_{t\to\infty} \max_{j\in J} (k_j(z_j, \gamma(t)_j) - t) \\
&= \lim_{t\to\infty} \max_{j\in J} (k_j(z_j, \gamma_j(t - \alpha_j)) - k_j(\gamma_j(t - \alpha_j), b_j) - \alpha_j) \\
&= \max_{j\in J} \left(h_{\xi_j}(z_j) - \alpha_j\right),
\end{aligned}$$

which shows that $h_{\gamma(t)} \to h$. □

Consider $B^\circ = \prod_{i=1}^r B^\circ_{n_i} \subseteq \mathbb{C}^n$, where each $B^\circ_{n_i}$ is an open unit ball of a norm in $\mathbb{C}^{n_i}$. Then B° is the open unit ball of the norm $\|\cdot\|_{B^\circ}$ on $\mathbb{C}^n$. In fact,

$$\|w\|_{B^\circ} = \max_{i=1,\dots,r} \|w_i\|_{B_i^\circ},$$

where $\|\cdot\|_{B_i^\circ}$ is the norm on $\mathbb{C}^{n_i}$ with open unit ball B_i°.

To analyse the dual norm of $\|\cdot\|_{B^\circ}$ we identify the dual space of $\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r}$ with itself using the standard inner-product

$$\langle x, y\rangle = \sum_{i=1}^r \langle x_i, y_i\rangle \quad \text{for } x = (x_1, \dots, x_r), y = (y_1, \dots, y_r) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r}.$$

So, $y \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r} \mapsto \langle \cdot, y\rangle \in (\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r})^*$. Note that the dual norm $\|\cdot\|^*_{B^\circ}$

satisfies

$$\|y\|_{B^\circ}^* = \sup_{\|x\|_{B^\circ}=1} \mathrm{Re}\langle x,y\rangle = \sup_{\|x\|_{B^\circ}=1} \sum_{i=1}^r \mathrm{Re}\langle x_i,y_i\rangle = \sum_{i=1}^r \|y_i\|_{B_i^\circ}^* \qquad \text{for } y=(y_1,\ldots,y_r)\in\mathbb{C}^{n_1}\times\cdots\times\mathbb{C}^{n_r},$$

as $\|x\|_{B^\circ} = \max_i \|x_i\|_{B_i^\circ}$. So we see that the closed dual unit ball is given by

$$B^* = \{y\in\mathbb{C}^{n_1}\times\cdots\times\mathbb{C}^{n_r} \colon \mathrm{Re}\langle x,y\rangle \leq 1 \text{ for all } x\in B\} = \{y\in\mathbb{C}^{n_1}\times\cdots\times\mathbb{C}^{n_r}\colon \sum_{i=1}^r \|y_i\|_{B_i^\circ}^* \leq 1\}.$$

Now suppose that each B_i° is strictly convex and smooth. The closed ball B^* has extreme points $p(\xi_i^*) = (0,\ldots,0,\xi_i^*,0,\ldots,0)$, where $\xi_i^*\in\mathbb{C}^{n_i}$ is the unique supporting functional at $\xi_i\in\partial B_i^\circ$, i.e., $\mathrm{Re}\langle\xi_i,\xi_i^*\rangle = 1$ and $\mathrm{Re}\langle w_i,\xi_i^*\rangle < 1$ for $w_i\in B_i$ with $w_i\neq\xi_i$.

The relatively open faces of B^* are the sets of the form

$$F(\{\xi_j\in\partial B_j^\circ\colon j\in J\}) = \left\{\sum_{j\in J}\lambda_j p(\xi_j^*)\colon \sum_{j\in J}\lambda_j = 1 \text{ and } \lambda_j>0 \text{ for all } j\in J\right\},$$

where $J\subseteq\{1,\ldots,r\}$ is nonempty and $\xi_j\in\partial B_j^\circ$ for $j\in J$ are fixed. Here the relative topology is taken with respect to the affine span of $\{p(\xi_j^*)\colon j\in J\}$.

On B° the Kobayashi distance has a Finsler structure in terms of the infinitesimal Kobayashi metric, see e.g., [1, Chapter 2.3]. Indeed, we have that

$$k_B(z,w) = \inf_\gamma L(\gamma),$$

where the infimum is taken over all piecewise C^1-smooth paths $\gamma\colon[0,1]\to B^\circ$ with $\gamma(0)=z$ and $\gamma(1)=w$, and

$$L(\gamma) = \int_0^1 \kappa_B(\gamma(t),\gamma'(t))\mathrm{d}t,$$

with

$$\kappa_B(u,v) = \inf\{|\xi|\colon \exists\phi\in\mathrm{Hol}(\Delta,B^\circ) \text{ such that } \phi(0)=u \text{ and } (\mathrm{D}\phi)_0(\xi)=v\}.$$

Proposition 4.1.3. *[1, Proposition 2.3.24] If B° is the open unit ball of a norm on $\mathbb{C}^n$, then*

$$\kappa_B(0,v) = \|v\|_{B^\circ} \quad \textit{for all } v \in \mathbb{C}^n.$$

For $z \in B^\circ$ and $i = 1,\dots,r$, if $z_i \neq 0$, then we let $z_i' = \|z_i\|_{B_i^\circ}^{-1} z_i \in \partial B_i^\circ$ and we write $p(z_i^*) = (0,\dots,0,z_i^*,0,\dots,0)$, where z_i^* is the unique supporting functional at $z_i' \in \partial B_i^\circ$. If $z_i = 0$, we set $p(z_i^*) = 0$.

We now define a map $\phi_B \colon \overline{B^\circ}^h \to B^*$ and show in the remainder of this section that it is a homeomorphism. For $z \in B^\circ = B_1^\circ \times \cdots \times B_r^\circ$ let

$$\phi_B(z) = \frac{1}{\sum_{i=1}^r e^{k_i(z_i,0)} + e^{-k_i(z_i,0)}} \left(\sum_{i=1}^r (e^{k_i(z_i,0)} - e^{-k_i(z_i,0)}) p(z_i^*) \right),$$

where k_i is the Kobayashi distance on B_i°. For a horofunction h given by (4.1.2) we define

$$\phi_B(h) = \frac{1}{\sum_{j\in J} e^{-\alpha_j}} \left(\sum_{j \in J} e^{-\alpha_j} p(\xi_j^*) \right).$$

More precisely, we prove the following theorem.

Theorem 4.1.4. *If $B^\circ = \prod_{i=1}^r B_i^\circ$, where each B_i° is the open unit ball of a norm on $\mathbb{C}^{n_i}$ which is strongly convex and has a C^3-boundary, then $\phi_B \colon \overline{B^\circ}^h \to B^*$ is a homeomorphism, which maps each part of $\partial \overline{B^\circ}^h$ onto the relative interior of a boundary face of B^*.*

4.2 The map ϕ_B: injectivity and surjectivity

Throughout the remainder of this section we assume that $B^\circ = \prod_{i=1}^r B_i^\circ$ and each B_i° is the open unit ball of a norm on $\mathbb{C}^{n_i}$, which is strongly convex and has a C^3-boundary. So for each $\xi_i \in \partial B_i^\circ$ there exists a unique $\xi_i^* \in \mathbb{C}^{n_i}$ such that

$$\mathrm{Re}\langle \xi_i, \xi_i^* \rangle = 1 \text{ and } \mathrm{Re}\langle w_i, \xi_i^* \rangle < 1 \text{ for all } w_i \in B_i \text{ with } w_i \neq \xi_i,$$

as B_i is strictly convex and smooth.

We start with the following basic observation.

Lemma 4.2.1. *For each $z \in B^\circ$ we have that $\phi_B(z) \in \operatorname{int} B^*$, and $\phi_B(h) \in \partial B^*$ for all $h \in \partial \overline{B^\circ}^h$.*

Proof. Note that for $z \in B^\circ$ and $w \in B$ we have that

$$\begin{aligned}
\operatorname{Re}\langle w, \phi_B(z)\rangle &= \frac{1}{\sum_{i=1}^r e^{k_j(z_i,0)} + e^{-k_i(z_i,0)}} \left(\sum_{i=1}^r (e^{k_i(z_i,0)} - e^{-k_i(z_i,0)}) \operatorname{Re}\langle w_i, z_i^*\rangle\right) \\
&\leq \frac{1}{\sum_{i=1}^r e^{k_i(z_i,0)} + e^{-k_i(z_i,0)}} \left(\sum_{i=1}^r e^{k_i(z_i,0)} - e^{-k_i(z_i,0)}\right) \\
&< 1 - \delta
\end{aligned}$$

for some $0 < \delta < 1$, which is independent of w. Thus, $\sup_{w \in B} \operatorname{Re}\langle w, \phi_B(z)\rangle < 1 - \delta < 1$, hence $\phi_B(z) \in \operatorname{int} B^*$.

To see that $\phi_B(h) \in \partial B^*$, note that for $w = \sum_{j \in J} p(\xi_j) \in B$, where $p(\xi_j) = (0, \ldots, 0, \xi_j, 0, \ldots, 0)$, we have that $\operatorname{Re}\langle w, \phi_B(h)\rangle = 1$. $\square$

To show that ϕ_B is injective on B°, we need the following basic calculus fact, which can be found in [32, Section 4].

Lemma 4.2.2. *If $\mu\colon \mathbb{R}^r \to \mathbb{R}$ is given by $\mu(x_1, \ldots, x_r) = \sum_{i=1}^r e^{x_i} + e^{-x_i}$, then $x \mapsto \nabla \log \mu(x)$ is injective on $\mathbb{R}^r$.*

Note that

$$(\nabla \log \mu(x))_j = \frac{e^{x_j} - e^{-x_j}}{\sum_{i=1}^r e^{x_i} + e^{-x_i}} \quad \text{for all } j.$$

Lemma 4.2.3. *The map ϕ_B is a continuous bijection from B° onto $\operatorname{int} B^*$.*

Proof. Cleary ϕ_B is continuous on B° and $\phi_B(z) = 0$ if and only if $z = 0$. Suppose that $z, w \in B^\circ \setminus \{0\}$ are such that $\phi_B(z) = \phi_B(w)$. For simplicity write

$$\alpha_j = \frac{e^{k_j(z_j,0)} - e^{-k_j(z_j,0)}}{\sum_{i=1}^r e^{k_i(z_i,0)} + e^{-k_i(z_i,0)}} \geq 0 \quad \text{and} \quad \beta_j = \frac{e^{k_j(w_j,0)} - e^{-k_j(w_j,0)}}{\sum_{i=1}^r e^{k_i(w_i,0)} + e^{-k_i(w_i,0)}} \geq 0.$$

Note that $\alpha_j p(z_j^*) = 0$ if and only if $z_j = 0$, and $\beta_j p(w_j^*) = 0$ if and only if $w_j = 0$. Thus, $z_j = 0$ if and only if $w_j = 0$. Now suppose that $z_j \neq 0$, so $w_j \neq 0$. Then $\langle p(v_j), \phi_B(z)\rangle = \langle p(v_j), \phi_B(w)\rangle$ for each $v_j \in B_j^\circ$. This implies that

$$\alpha_j \langle v_j, z_j^*\rangle = \beta_j \langle v_j, w_j^*\rangle \quad \text{for all } v_j \in B_j^\circ,$$

hence $\alpha_j z_j^* = \beta_j w_j^*$. It follows that $\alpha_j = \beta_j$ and $z_j^* = w_j^*$. Thus $z_j = \mu_j w_j$ for some $\mu_j > 0$. As $\alpha_i = \beta_i$ for all $i \in \{1, \dots, r\}$, we know by Lemma 4.2.2 that $k_j(z_j, 0) = k_j(w_j, 0)$, hence $z_j = w_j$ by [1, Proposition 2.3.5]. So $z = w$, which shows that ϕ_B is injective.

As ϕ_B is injective and continuous on B°, it follows from Brouwer's domain invariance theorem that $\phi_B(B^\circ)$ is an open subset of $\operatorname{int} B^*$ by Lemma 4.2.1. Suppose, by way of contradiction, that $\phi_B(B^\circ) \neq \operatorname{int} B^*$. Then $\partial\phi_B(B^\circ) \cap \operatorname{int} B^*$ is nonempty, as otherwise $\phi_B(B^\circ)$ is closed and open in $\operatorname{int} B^*$, which would imply that $\operatorname{int} B^*$ is the disjoint union of the nonempty open sets $\phi_B(B^\circ)$ and its complement contradicting the connectedness of $\operatorname{int} B^*$. So let $w \in \partial\phi_B(B^\circ) \cap \operatorname{int} B^*$ and (z^n) be a sequence in B° such that $\phi_B(z^n) \to w$. As ϕ_B is continuous on B°, we have that $k_B(z^n, 0) \to \infty$.

Using the product property, $k_B(z^n, 0) = \max_i k_i(z_i^n, 0)$, we may assume after taking subsequences that $\alpha_i^n = k_B(z^n, 0) - k_i(z_i^n, 0) \to \alpha_i \in [0, \infty]$ and $z_i^n \to \zeta_i \in B_i$ for all i. Let $I = \{i \colon \alpha_i < \infty\}$, and note that for each $i \in I$, $\zeta_i \in \partial B_i^\circ$, as $k_i(z_i^n, 0) \to \infty$. Then

$$\begin{aligned}\phi_B(z^n) &= \frac{1}{\sum_{i=1}^r e^{k_i(z_i^n,0)} + e^{-k_i(z_i^n,0)}} \left(\sum_{i=1}^r (e^{k_i(z_i^n,0)} - e^{-k_i(z_i^n,0)}) p((z_i^n)^*)\right) \\ &= \frac{1}{\sum_{i=1}^r e^{-\alpha_i^n} + e^{-k_B(z^n,0)-k_i(z_i^n,0)}} \left(\sum_{i=1}^r (e^{-\alpha_i^n} - e^{-k_B(z^n,0)-k_i(z_i^n,0)}) p((z_i^n)^*)\right).\end{aligned}$$

Letting $n \to \infty$, the righthand side converges to

$$\frac{1}{\sum_{i\in I} e^{-\alpha_i}} \left(\sum_{i\in I} e^{-\alpha_i} p(\zeta_i^*)\right) = w.$$

But this implies that $w \in \partial B^*$, as $\operatorname{Re}\langle \sum_{i\in I} p(\zeta_i), w\rangle = 1$ and $\sum_{i\in I} p(\zeta_i) \in B$, where

$p(\zeta_i) = (0, \ldots, 0, \zeta_i, 0, \ldots, 0)$. This is impossible and hence $\phi_B(B^\circ) = \operatorname{int} B^*$. □

We now analyse ϕ_B on $\partial\overline{B^\circ}^h$.

Lemma 4.2.4. *The map ϕ_B maps $\partial\overline{B^\circ}^h$ bijectively onto ∂B^*. Moreover, the part $\mathcal{P}_h$, where h is given by (4.1.2), is mapped onto the relative open boundary face*

$$F(\{\xi_j \in \partial B_j^\circ \colon j \in J\}) = \left\{ \sum_{j\in J} \lambda_j p(\xi_j^*) \colon \sum_{j\in J} \lambda_j = 1 \text{ and } \lambda_j > 0 \text{ for all } j \in J \right\}.$$

Proof. We know from Lemma 4.2.1 that ϕ_B maps $\partial\overline{B^\circ}^h$ into ∂B^*. To show that it is onto we let $w \in \partial B^*$. As B^* is the disjoint union of its relative open faces (see [61, Theorem 18.2]), there exist $J \subseteq \{1, \ldots, r\}$, extreme points $p(\xi_j^*)$ of B^*, and $0 < \lambda_j \leq 1$ for $j \in J$ with $\sum_{j\in J} \lambda_j = 1$ such that $w = \sum_{j\in J} \lambda_j p(\xi_j^*)$. Let $\mu_j = -\log \lambda_j$ and $\mu^* = \min_{j\in J} \mu_j$. Now set $\alpha_j = \mu_j - \mu^*$ for $j \in J$. Then $\alpha_j \geq 0$ for $j \in J$ and $\min_{j\in J} \alpha_j = 0$.

Let $h \in \partial\overline{B^\circ}^h$ be given by $h(z) = \max_{j\in J}(h_{\xi_j}(z_j) - \alpha_j)$. Then

$$\phi_B(h) = \frac{\sum_{j\in J} e^{-\alpha_j} p(\xi_j^*)}{\sum_{j\in J} e^{-\alpha_j}} = \frac{\sum_{j\in J} e^{-\mu_j} p(\xi_j^*)}{\sum_{j\in J} e^{-\mu_j}} = \frac{\sum_{j\in J} \lambda_j p(\xi_j^*)}{\sum_{j\in J} \lambda_j} = w.$$

To prove injectivity let $h, h' \in \partial\overline{B^\circ}^h$, where h is as in (4.1.2) and

$$h'(z) = \max_{j\in J'}(h_{\eta_j}(z_j) - \beta_j) \tag{4.2.1}$$

for $z \in B^\circ$. Suppose that $\phi_B(h) = \phi_B(h')$, so

$$\phi_B(h) = \frac{\sum_{j\in J} e^{-\alpha_j} p(\xi_j^*)}{\sum_{j\in J} e^{-\alpha_j}} = \frac{\sum_{j\in J'} e^{-\beta_j} p(\eta_j^*)}{\sum_{j\in J'} e^{-\beta_j}} = \phi_B(h').$$

We have that $J = J'$. Indeed, if $k \in J$ and $k \notin J'$, then

$$0 = \operatorname{Re}\langle p(\xi_k), \phi_B(h')\rangle = \operatorname{Re}\langle p(\xi_k), \phi_B(h)\rangle > 0,$$

which is impossible. For the other case a contradiction can be derived in the same way.

Now suppose there exists $k \in J$ such that $\xi_k \neq \eta_k$. If

$$\frac{e^{-\alpha_k}}{\sum_{j\in J} e^{-\alpha_j}} \leq \frac{e^{-\beta_k}}{\sum_{j\in J} e^{-\beta_j}},$$

then

$$\mathrm{Re}\langle p(\eta_k), \phi_B(h)\rangle = \frac{e^{-\alpha_k}}{\sum_{j\in J} e^{-\alpha_j}}\mathrm{Re}\langle \eta_k, \xi_k^*\rangle < \frac{e^{-\alpha_k}}{\sum_{j\in J} e^{-\alpha_j}} \leq \frac{e^{-\beta_k}}{\sum_{j\in J} e^{-\beta_j}} = \mathrm{Re}\langle p(\eta_k), \phi_B(h')\rangle,$$

as B_k° is smooth and strictly convex, which contradicts $\phi_B(h) = \phi_B(h')$. The other case goes in the same way. Thus, $J = J'$ and $\xi_j = \eta_j$ for all $j \in J$.

It follows that

$$\frac{e^{-\alpha_k}}{\sum_{j\in J} e^{-\alpha_j}} = \mathrm{Re}\langle p(\xi_k), \phi_B(h)\rangle = \mathrm{Re}\langle p(\eta_k), \phi_B(h')\rangle = \frac{e^{-\beta_k}}{\sum_{j\in J} e^{-\beta_j}}$$

for all $k \in J$. To show that $\alpha_k = \beta_k$ for all $k \in J$ let $\nu\colon \mathbb{R}^J \to \mathbb{R}$ be given by $\nu(x) = \sum_{j\in J} e^{-x_j}$. Then for $x, y \in \mathbb{R}^J$ and $0 < t < 1$ we have that

$$\nu(tx + (1-t)y) \leq \nu(x)^t \nu(y)^{1-t},$$

and we have equality if and only if there exists a constant c such that $x_k = y_k + c$ for all $k \in J$. So, if $x \neq y + (c, \ldots, c)$ for all c, then $-\nabla \log \nu(x) \neq -\nabla \log \nu(y)$.

As $\min_{j\in J} \alpha_j = 0 = \min_{j\in J} \beta_j$, we can conclude that $\alpha_k = \beta_k$ for all $k \in J$. This shows that $h = h'$ and hence ϕ_B is injective on $\partial \overline{B^\circ}^h$.

To complete the proof, note that $\phi_B(h)$ is in the relative open boundary face $F(\{\xi_j \in \partial B_j^\circ \colon j \in J\})$ of B^*. Moreover, h' given by (4.2.1) is in the same part as h if, and only if, $J = J'$ and $\xi_j = \eta_j$ for all $j \in J$ by [41, Propositions 2.8 and 2.9]. So, $\phi_B(h')$ lies in $F(\{\xi_j \in \partial B_j^\circ \colon j \in J\})$ if and only if h' lies in the same part as h. $\square$

4.3 Continuity and the proof of Theorem 4.1.4

We now show that ϕ_B is continuous on $\overline{B^\circ}^h$.

Proposition 4.3.1. *The map* $\phi_B \colon \overline{B^\circ}^h \to B^*$ *is continuous.*

Proof. Clearly ϕ_B is continuous on B°. Suppose that (z^n) is sequence in B° converging to $h \in \partial\overline{B^\circ}^h$, where h is given by (4.1.2). To show that $\phi_B(z^n) \to \phi_B(h)$ we show that every subsequence of $(\phi_B(z^n))$ has a subsequence converging to $\phi_B(h)$. So, let $(\phi_B(z^{n_k}))$ be a subsequence. We can take a further subsequence $(z^{n_{k,m}})$ such that

1. $$\beta_j^m = k_B(z^{n_{k,m}}, 0) - k_j(z_j^{n_{k,m}}, 0) \to \beta_j \in [0, \infty] \quad \text{for all } j \in \{1, \dots, r\}.$$

2. There exists j_0 such that $\beta_{j_0}^m = 0$ for all $m \geq 1$.

3. $(z_j^{n_{k,m}})$ converges to $\eta_j \in B_j$ and $h_{z^{n_{k,m}}} \to h_{\eta_j}$ for all $j \in \{1, \dots, r\}$.

Let $J' = \{j \colon \beta_j < \infty\}$. Then $h_{z^{n_{k,m}}} \to h'$, where $h'(z) = \max_{j \in J'}(h_{\eta_j}(z_j) - \beta_j)$ for $z \in B^\circ$, as

$$\lim_{m\to\infty} k_B(z, z^{n_{k,m}}) - k_B(z^{n_{k,m}}, 0) = \lim_{m\to\infty} \max_j (k_j(z_j, z_j^{n_{k,m}}) - k_j(z_j^{n_{k,m}}, 0) - \beta_j^m) = \max_{j\in J'}(h_{\eta_j}(z_j) - \beta_j),$$

by the product property of k_B.

As $h = h'$, we know by [41, Propositions 2.8 and 2.9] that $J = J'$, $\xi_j = \eta_j$ and $\alpha_j = \beta_j$ for all $j \in J$. We also know by Lemma 3.1.5 that $k_B(z^{n_{k,m}}, 0) \to \infty$, as h is a horofunction. So,

$$\phi_B(z^{n_{k,m}}) = \frac{\sum_{i=1}^r (e^{-\beta_i^m} - e^{-k_B(z^{n_{k,m}},0) - k_i(z_i^{n_{k,m}},0)}) p((z^{n_{k,m}})^*)}{\sum_{i=1}^r e^{-\beta_i^m} - e^{-k_B(z^{n_{k,m}},0) - k_i(z_i^{n_{k,m}},0)}} \to \frac{\sum_{j\in J} e^{-\beta_j} p(\eta_j^*)}{\sum_{j \in J} e^{-\beta_j}} = \phi_B(h),$$

which shows that $\phi_B(z^n) \to \phi_B(h)$.

We know from Lemma 4.2.1 that $\phi_B(B^\circ) \subseteq \operatorname{int} B^*$ and $\phi_B(\partial\overline{B^\circ}^h) \subseteq \partial B^*$. So, to complete the proof it remains to show that if (h_n) in $\partial\overline{B^\circ}^h$ converges to $h \in \partial\overline{B^\circ}^h$, where h is as in (4.1.2), then $\phi_B(h_n) \to \phi_B(h)$. For $n \geq 1$ let h_n be given by

$$h_n(z) = \max_{j\in J_n}(h_{\eta_j^n}(z_j) - \beta_j^n) \qquad \text{for } z \in B^\circ.$$

Again we show that every subsequence of $(\phi_B(h_n))$ has a convergent subsequence with limit $\phi_B(h)$.

Let $(\phi_B(h_{n_k}))$ be a subsequence. Taking a further subsequence we may assume that

1. There exists $J_0 \subseteq \{1,\dots,r\}$ such that $J_{n_k} = J_0$ for all k.

2. There exists $j_0 \in J_0$ such that $\beta_{j_0}^{n_k} = 0$ for all k.

3. $\beta_j^{n_k} \to \beta_j \in [0,\infty]$ for all $j \in J_0$.

4. $\eta_j^{n_k} \to \eta_j$ for all $j \in J_0$.

Note that for each $j \in J_0$ we have that $h_{\eta_j^{n_k}} \to h_{\eta_j}$ in $\overline{B^\circ}_j^h$, as the identity map on B_j, that is $\xi_j \in B_j \to h_{\xi_j} \in \overline{B^\circ}_j^h$, is a homeomorphism by [5, Theorem 1.2].

Let $J' = \{j \in J_0 \colon \beta_j < \infty\}$ and note that $j_0 \in J'$. Then for each $z \in B^\circ$ we have that

$$\lim_{m\to\infty} h_{n_k}(z) = \lim_{k\to\infty}\max_{j\in J_0}(h_{\eta_j^{n_k}}(z_j) - \beta_j^{n_k}) = \lim_{k\to\infty}\max_{j\in J'}(h_{\eta_j^{n_k}}(z_j) - \beta_j^{n_k}) = \max_{j\in J'}(h_{\eta_j}(z_j) - \beta_j).$$

So, if we let $h'(z) = \max_{j\in J'}(h_{\eta_j}(z_j) - \beta_j)$ for $z \in B^\circ$, then h' is a horofunction by Theorem 4.1.1 and $h_{n_k} \to h'$ in $\overline{B^\circ}^h$. As $h_n \to h$, we conclude that $h' = h$. This implies that $J' = J$ and $\eta_j = \xi_j$ and $\beta_j = \alpha_j$ for all $j \in J$, as otherwise $\delta(h, h') \neq 0$ by [41, Proposition 2.9 and Lemma 3.3]. This implies that $\beta_j^{k_m} \to \alpha_j$ and $\eta_j^{k_m} \to \xi_j$ for all $j \in J'$. Moreover, by definition $\beta_j^{n_k} \to \infty$ for all $j \in J_0 \setminus J'$. Thus,

$$\phi_B(h_{n_k}) = \frac{\sum_{j\in J_0} e^{-\beta_j^{n_k}} p((\eta_j^{n_k})^*)}{\sum_{j\in J_0} e^{-\beta_j^{n_k}}} \to \frac{\sum_{j\in J} e^{-\alpha_j} p(\xi_j^*)}{\sum_{j\in J} e^{-\alpha_j}} = \phi_B(h),$$

which completes the proof. □

The proof of Theorem 4.1.4 is now straightforward.

Proof of Theorem 4.1.4. It follows from Lemmas 4.2.3 and 4.2.4 and Proposition 4.3.1 that $\phi_B \colon \overline{B^\circ}^h \to B^*$ is a continuous bijection. As $\overline{B^\circ}^h$ is compact and B^* is Hausdorff, we conclude that ϕ_B is a homeomorphism. Moreover, ϕ_B maps each part of $\partial\overline{B^\circ}^h$ onto the relative interior of a boundary face of B^* by Lemma 4.2.4. □

Chapter 5

Finite Dimensional JB-algebras

Every finite dimensional normed space $(V, \|\cdot\|)$ has a Finsler structure. Indeed, if we let

$$L(\gamma) = \int_0^1 \|\gamma'(t)\| \mathrm{d}t$$

be the length of a piecewise C^1-smooth path $\gamma\colon [0,1] \to V$, then

$$\|x - y\| = \inf_{\gamma} L(\gamma),$$

where the infimum is taken over all C^1-smooth paths $\gamma\colon [0,1] \to V$ with $\gamma(0) = x$ and $\gamma(1) = y$. So, for normed spaces V the unit ball in the tangent space $T_b V$ is the same for all $b \in V$.

In this section we analyse the problem posed by Kapovich and Leeb [34, Question 6.18] concerning the existence of a natural homeomorphism between the horofunction compactification of a finite dimensional normed space V and the closed dual unit ball of V in the setting of Euclidean Jordan algebras equipped with the spectral norm. So we consider the Euclidean Jordan algebra not as inner-product space, but as an order-unit space, which makes it a finite dimensional (formally real) JB-algebra, see [4, Theorem 1.11]. We will give an explicit description of the horofunctions of these normed spaces and identify the parts and the detour distance. In our analysis we make frequent use of the theory of Jordan algebras and order-unit spaces. For the reader's convenience we will recall some of the basic concepts. Throughout the chapter we will follow the terminology used in [3, 4, 20].

5.1 Preliminaries

Order-unit spaces A *cone* V_+ in a real vector space V is a convex subset of V with $\lambda V_+ \subseteq V_+$ for all $\lambda \geq 0$ and $V_+ \cap -V_+ = \{0\}$. The cone V_+ induces a partial ordering $\leq$ on V by $x \leq y$ if $y - x \in V_+$. We write $x < y$ if $x \leq y$ and $x \neq y$. The cone V_+ is said to be *Archimedean* if for each $x \in V$ and $y \in V_+$ with $nx \leq y$ for all $n \geq 1$ we have that $x \leq 0$. An element u of V_+ is called an *order-unit* if for each $x \in V$ there exists $\lambda \geq 0$ such that $-\lambda u \leq x \leq \lambda u$. The triple (V, V_+, u), where V_+ is an Archimedean cone and u is an order-unit, is called an *order-unit space*. An order-unit space admits a norm

$$\|x\|_u = \inf\{\lambda \geq 0\colon -\lambda u \leq x \leq \lambda u\},$$

which is called the *order-unit norm*, and we have that $-\|x\|_u u \leq x \leq \|x\|_u u$ for all $x \in V$. The cone V_+ is closed under the order-unit norm and $u \in \operatorname{int} V_+$.

A linear functional ϕ on an order-unit space is said to be *positive* if $\phi(x) \geq 0$ for all $x \in V_+$. It is called a *state* if it is positive and $\phi(u) = 1$. The set of all states is denoted by $S(V)$ and is called the *state space*, which is a convex set. In our case, the order-unit space is finite dimensional, hence $S(V)$ is compact. The extreme points of $S(V)$ are called the *pure states*.

The dual space V^* of an order-unit space V is a *base norm space*, see [3, Theorem 1.19]. More specifically, V^* is an ordered normed vector space with cone $V^*_+ = \{\phi \in V^*\colon \phi \text{ is positive}\}$, $V^*_+ - V^*_+ = V^*$, and the unit ball of the norm of V^* is given by

$$B_{V^*} = \operatorname{conv}(S(V) \cup -S(V)).$$

Jordan algebras Important examples of order-unit spaces come from Jordan algebras. A *Jordan algebra* (over $\mathbb{R}$) is a real vector space V equipped with a commutative bilinear product $\bullet$ that satisfies the identity

$$x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x \quad \text{for all } x, y \in V.$$

A basic example is the space $\mathrm{Herm}_n(\mathbb{C})$ consisting of $n \times n$ Hermitian matrices with Jordan product $A \bullet B = (AB + BA)/2$.

Throughout the chapter we will assume that V has a *unit*, denoted u. For $x \in V$ we let L_x be the linear map on V given by $L_x y = x \bullet y$. A finite dimensional Jordan algebra is said to be *Euclidean* if there exists an inner-product $(\cdot|\cdot)$ on V such that

$$(L_x y|z) = (y|L_x z) \quad \text{for all } x, y, z \in V.$$

A Euclidean Jordan algebra has a cone $V_+ = \{x^2 \colon x \in V\}$. The interior of V_+ is a *symmetric cone*, i.e., it is self-dual and $\mathrm{Aut}(V_+) = \{A \in \mathrm{GL}(V) \colon A(V_+) = V_+\}$ acts transitively on the interior of V_+. In fact, the Euclidean Jordan algebras are in one-to-one correspondence with the symmetric cones by the Koecher-Vinberg theorem, see for example [20].

The algebraic unit u of a Euclidean Jordan algebra is an order-unit for the cone V_+, so the triple (V, V_+, u) is an order-unit space. We will consider the Euclidean Jordan algebras as an order-unit space equipped with the order-unit norm. These are precisely the finite dimensional formally real JB-algebras, see [4, Theorem 1.11]. In the analysis, however, the inner-product structure on V will be exploited to identify V^* with V.

Throughout we will fix the rank of the Euclidean Jordan algebra V to be r. In a Euclidean Jordan algebra each x can be written in a unique way as $x = x^+ - x^-$, where x^+ and x^- are orthogonal element x^+ and x^- in V_+, see [4, Proposition 1.28]. This is called the *orthogonal decomposition of* x.

Given x in a Euclidean Jordan algebra V, the *spectrum* of x is given by $\sigma(x) = \{\lambda \in \mathbb{R} \colon \lambda u - x \text{ is not invertible}\}$, and we have that $V_+ = \{x \in V \colon \sigma(x) \subset [0, \infty)\}$. We write

$\Lambda(x) = \inf\{\lambda\colon x \leq \lambda u\}$ and note that $\Lambda(x) = \max\{\lambda\colon \lambda \in \sigma(x)\}$, so that

$$\|x\|_u = \max\{\Lambda(x), \Lambda(-x)\} = \max\{|\lambda|\colon \lambda \in \sigma(x)\}$$

for all $x \in V$. We also note that

$$\Lambda(x + \mu u) = \Lambda(x) + \mu$$

for all $x \in V$ and $\mu \in \mathbb{R}$. Moreover, if $x \leq y$, then $\Lambda(x) \leq \Lambda(y)$.

Recall that $p \in V$ is an *idempotent* if $p^2 = p$. If, in addition, p is non-zero and cannot be written as the sum of two non-zero idempotents, then it is said to be a *primitive* idempotent. The set of all primitive idempotent is denoted $\mathcal{J}_1(V)$ and is known to be a compact set [31]. Two idempotents p and q are said to be orthogonal if $p \bullet q = 0$, which is equivalent to $(p|q) = 0$. According to the spectral theorem [20, Theorem III.1.2], each x has a *spectral decomposition*, $x = \sum_{i=1}^r \lambda_i p_i$, where each p_i is a primitive idempotent, the λ_i's are the eigenvalues of x (including multiplicities), and $p_1, \ldots, p_r$ is a Jordan frame, i.e., the p_i's are mutually orthogonal and $p_1 + \cdots + p_r = u$.

Throughout the chapter we will fix the inner-product on V to be

$$(x|y) = \operatorname{tr}(x \bullet y),$$

where $\operatorname{tr}(x) = \sum_{i=1}^r \lambda_i$ and $x = \sum_{i=1}^r \lambda_i p_i$ is the spectral decomposition of x.

For $x \in V$ we denote the *quadratic representation* by $U_x\colon V \to V$, which is the linear map,

$$U_x y = 2x \bullet (x \bullet y) - x^2 \bullet y = 2L_x(L_x y) - L_{x^2} y.$$

In case of a Euclidean Jordan algebra U_x is self-adjoint, i.e. $(U_x y|z) = (y|U_x z)$.

We identify V^* with V using the inner-product. So, $S(V) = \{w \in V_+\colon (u|w) = 1\}$, which is a compact convex set, as V is finite dimensional. Moreover, the extreme points of $S(V)$ are the primitive idempotents, see [20, Proposition IV.3.2]. The dual space $(V, \|\cdot\|_u^*)$

is a base norm space with norm,

$$\|z\|_u^* = \sup\{(x|z)\colon x \in V \text{ with } \|x\|_u = 1\}.$$

If V is a Euclidean Jordan algebra, it is known that the (closed) boundary faces of the dual ball $B_{V^*} = \mathrm{conv}(S(V) \cup -S(V))$ are precisely the sets of the form,

$$\mathrm{conv}\left((U_p(V) \cap S(V)) \cup (U_q(V) \cap -S(V))\right), \tag{5.1.1}$$

where p and q are orthogonal idempotents not both zero, see [18, Theorem 4.4].

5.2 Summary of results

To conveniently describe the horofunction compactification $\overline{V}^h$ of $(V, \|\cdot\|_u)$, where V is a Euclidean Jordan algebra, we need some additional notation. Throughout this section we will fix the basepoint $b \in V$ to be 0.

Let $p_1, \ldots, p_r$ be a Jordan frame in V. Given $I \subseteq \{1, \ldots, r\}$ nonempty, we write $p_I = \sum_{i \in I} p_i$ and we let $V(p_I) = U_{p_I}(V)$. For convenience we set $p_\emptyset = 0$, so $V(p_\emptyset) = U_0(V) = \{0\}$.

Recall that $V(p_I)$ is the Peirce 1-space of the idempotent p_I:

$$V(p_I) = \{x \in V \colon p_I \bullet x = x\},$$

which is a subalgebra, see [20, Theorem IV.1.1]. Given $z \in V(p_I)$, we write $\Lambda_{V(p_I)}(z)$ to denote the maximal eigenvalue of z in the subalgebra $V(p_I)$.

The following theorem characterises the horofunctions in $\overline{V}^h$.

Theorem 5.2.1. *Let $p_1, \ldots, p_r$ be a Jordan frame, $I, J \subseteq \{1, \ldots, r\}$, with $I \cap J = \emptyset$ and $I \cup J$ nonempty, and $\alpha \in \mathbb{R}^{I \cup J}$ such that* $\min\{\alpha_i \colon i \in I \cup J\} = 0$. *The function $h \colon V \to \mathbb{R}$,* *given by*

$$h(x) = \max\left\{\Lambda_{V(p_I)}\left(-U_{p_I}x - \sum_{i\in I}\alpha_i p_i\right), \Lambda_{V(p_J)}\left(U_{p_J}x - \sum_{j\in J}\alpha_j p_j\right)\right\} \quad \text{for } x \in V, \tag{5.2.1}$$

is a horofunction, where we use the convention that if I or J is empty, the corresponding term is omitted from the maximum. Each horofunction in $\overline{V}^h$ is of the form (5.2.1) and a Busemann point.

To conveniently describe the parts and the detour distance we introduce the following notation. Given orthogonal idempotents p_I and p_J we let $V(p_I, p_J) = V(p_I) + V(p_J)$, which is a subalgebra of V with unit $p_{IJ} = p_I + p_J$. The subspace $V(p_I, p_J)$ can be equipped with the *variation norm*,

$$\|x\|_{\mathrm{var}} = \Lambda_{V(p_I,p_J)}(x) + \Lambda_{V(p_I,p_J)}(-x) = \operatorname{diam}\sigma_{V(p_I,p_J)}(x),$$

which is a semi-norm on $V(p_I, p_J)$. The variation norm is, however, a norm on the quotient space $V(p_I, p_J)/\mathbb{R}p_{IJ}$.

Theorem 5.2.2. *Given horofunctions h and h', where*

$$h(x) = \max\left\{\Lambda_{V(p_I)}\left(-U_{p_I}x - \sum_{i\in I}\alpha_i p_i\right), \Lambda_{V(p_J)}\left(U_{p_J}x - \sum_{j\in J}\alpha_j p_j\right)\right\} \tag{5.2.2}$$

and

$$h'(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i\in I'}\beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j\in J'}\beta_j q_j\right)\right\}, \tag{5.2.3}$$

we have that

1. *h and h' are in the same part if and only if $p_I = q_{I'}$ and $p_J = q_{J'}$.*

2. *If h and h' are in the same part, then $\delta(h, h') = \|a - b\|_{\mathrm{var}}$, where $a = \sum_{i\in I}\alpha_i p_i +$*

$\sum_{j\in J}\alpha_j p_j$ *and* $b=\sum_{i\in I'}\beta_i q_i+\sum_{j\in J'}\beta_j q_j$ *in* $V(p_I,p_J)$.

3. *The part* $(\mathcal{P}_h,\delta)$ *is isometric to* $(V(p_I,p_J)/\mathbb{R}p_{IJ},\|\cdot\|_{\mathrm{var}})$.

Remark 5.2.3. A basic example is $(\mathbb{R}^n,\|\cdot\|_\infty)$, where $\|z\|_\infty=\max_i|z_i|$, which is an associative Euclidean Jordan algebra. In that case every horofunction is a Busemann points and of the form

$$h(x)=\max\{\max_{i\in I}(-x_i-\alpha_i),\max_{j\in J}(x_j-\alpha_i)\},$$

where $I,J\subseteq\{1,\dots,n\}$ are disjoint, $I\cup J$ is nonempty and $\alpha\in\mathbb{R}^{I\cup J}$ with $\min_{k\in I\cup J}\alpha_k=0$, (see [26, Theorem 5.2] and [41]). Moreover, $(\mathcal{P}_h,\delta)$ is isometric to $(\mathbb{R}^{I\cup J}/\mathbb{R}\mathbf{1},\|\cdot\|_{\mathrm{var}})$, where $\mathbf{1}=(1,\dots,1)\in\mathbb{R}^{I\cup J}$.

We will show that the following map is a homeomorphism from $\overline{V}^h$ onto B_{V^*}. Let $\phi\colon\overline{V}^h\to B_{V^*}$ be given by

$$\phi(x)=\frac{e^x-e^{-x}}{(e^x+e^{-x}|u)}=\frac{1}{\sum_{i=1}^r e^{\lambda_i}+e^{-\lambda_i}}\left(\sum_{i=1}^r(e^{\lambda_i}-e^{-\lambda_i})p_i\right)\tag{5.2.4}$$

for $x=\sum_{i=1}^r\lambda_i p_i\in V$, and

$$\phi(h)=\frac{1}{\sum_{i\in I}e^{-\alpha_i}+\sum_{j\in J}e^{-\alpha_j}}\left(\sum_{i\in I}e^{-\alpha_i}p_i-\sum_{j\in J}e^{-\alpha_j}p_j\right)\tag{5.2.5}$$

for $h\in\partial\overline{V}^h$ given by (5.2.1).

We should note that ϕ is well defined. To verify this assume that the horofunction h given by (5.2.1) is represented as

$$h(x)=\max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x-\sum_{i\in I'}\beta_i q_i\right),\Lambda_{V(q_{J'})}\left(U_{q_{J'}}x-\sum_{j\in J'}\beta_j q_j\right)\right\}$$

for $x\in V$. Then it follows from Theorem 5.2.2 that $p_I=q_{I'}$ and $p_J=q_{J'}$. Moreover, as

$\delta(h,h) = 0$, we have that $a = \sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j = \sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j = b$, as $\min\{\alpha_i \colon I\cup J\} = 0 = \min\{\beta_i \colon I\cup J\}$. This implies that $U_{p_I}a = U_{q_{I'}}b$ and $U_{p_J}a = U_{q_{J'}}b$, so that

$$\sum_{i\in I}\alpha_i p_i = \sum_{i\in I'}\beta_i q_i \quad \text{and} \quad \sum_{j\in J}\alpha_j p_j = \sum_{j\in J'}\beta_j q_j.$$

Using the map $v\in V\mapsto e^{-v}$ we deduce that $\sum_{i\in I}e^{-\alpha_i}p_i + (u-p_I) = \sum_{i\in I'}e^{-\beta_i}q_i + (u-q_{I'})$, and hence $\sum_{i\in I}e^{-\alpha_i}p_i = \sum_{i\in I'}e^{-\beta_i}q_i$. Likewise $\sum_{j\in J}e^{-\alpha_j}p_j = \sum_{j\in J'}e^{-\beta_j}q_j$. We also find that

$$\sum_{i\in I}e^{-\alpha_i}+\sum_{j\in J}e^{-\alpha_j} = (\sum_{i\in I}e^{-\alpha_i}p_i+\sum_{j\in J}e^{-\alpha_j}p_j|u) = (\sum_{i\in I'}e^{-\beta_i}q_i+\sum_{j\in J'}e^{-\beta_j}q_j|u) = \sum_{i\in I'}e^{-\beta_i}+\sum_{j\in J'}e^{-\beta_j},$$

so $\phi(h)$ is well defined.

We will also show that ϕ maps each part of the horofunction boundary onto the relative interior of a boundary face of the dual unit ball. Recall that the relative interior of a face F of B_{V^*} is the interior of F as a subset of the affine span of F.

Theorem 5.2.4. *Given a Euclidean Jordan algebra $(V,\|\cdot\|_u)$, the map $\phi\colon \overline{V}^h \to B_{V^*}$ is a homeomorphism. Moreover, the part $\mathcal{P}_h$, with h given by (5.2.1), is mapped onto the relative interior of the closed boundary face*

$$\mathrm{conv}\,(U_{p_I}(V)\cap S(V))\cup(U_{p_J}(V)\cap -S(V))).$$

5.3 Horofunctions

In this section we will prove Theorem 5.2.1. We first make some preliminary observations. Note that $x\le\lambda u$ if and only if $0\le\lambda u - x$, which by the Hahn-Banach separation theorem is equivalent to $(\lambda u - x|w)\ge 0$ for all $w\in S(V)$. As the state space is compact, we have for each $x\in V$ that

$$\Lambda(x) = \max_{w\in S(V)}(x|w). \tag{5.3.1}$$

As $\|\cdot\|_u$ is the JB-algebra norm, $\|x \bullet y\|_u \leq \|x\|_u \|y\|_u$, see [4, Theorem 1.11]. It follows that if $x^n \to x$ and $y^n \to y$ in $(V, \|\cdot\|_u)$, then $x^n \bullet y^n \to x \bullet y$. Thus, we have the following lemma.

Lemma 5.3.1. *If $x^n \to x$ and $y^n \to y$ in $(V, \|\cdot\|_u)$, then $U_{x^n} y^n \to U_x y$.*

We will also use the following technical lemma several times.

Lemma 5.3.2. *For $n \geq 1$, let $p_1^n, \ldots, p_r^n$ be a Jordan frame in V and $I \subseteq \{1, \ldots, r\}$ nonempty. Suppose that*

1. *$p_i^n \to p_i$ for all $i \in I$.*

2. *$x^n \in V(p_I^n)$ with $x^n \to x \in V(p_I)$.*

3. *$\beta_i^n \geq 0$ with $\beta_i^n \to \beta_i \in [0, \infty]$ for all $i \in I$.*

If $I' = \{i \in I : \beta_i < \infty\}$ is nonempty, then

$$\lim_{n \to \infty} \Lambda_{V(p_I^n)}(x^n - \sum_{i \in I} \beta_i^n p_i^n) = \Lambda_{V(p_{I'})}(U_{p_{I'}} x - \sum_{i \in I'} \beta_i p_i).$$

Proof. We will show that every subsequence of $(\Lambda_{V(p_I^n)}(x^n - \sum_{i \in I} \beta_i^n p_i^n))$ has a convergent subsequence with limit $\Lambda_{V(p_{I'})}(U_{p_{I'}} x - \sum_{i \in I'} \beta_i p_i)$. So let $(\Lambda_{V(p_I^{n_k})}(x^{n_k} - \sum_{i \in I} \beta_i^{n_k} p_i^{n_k}))$ be a subsequence. By (5.3.1) there exists $d^{n_k} \in S(V(p_I^{n_k}))$ with

$$\Lambda_{V(p_I^{n_k})}(x^{n_k} - \sum_{i \in I} \beta_i^{n_k} p_i^{n_k}) = (x^{n_k} - \sum_{i \in I} \beta_i^{n_k} p_i^{n_k} | d^{n_k}).$$

By taking subsequences we may assume that $d^{n_k} \to d \in S(V(p_I))$.

Using the Peirce decomposition with respect to the Jordan frame $p_i^{n_k}$, $i \in I$, in $V(p_I^{n_k})$, we can write

$$d^{n_k} = \sum_{i \in I} \mu_i^{n_k} p_i^{n_k} + \sum_{i < j \in I} d_{ij}^{n_k}.$$

Note that as $d^{n_k} \geq 0$, we have that $\mu_i^{n_k} = (d^{n_k} | p_i^{n_k}) \geq 0$ for all $i \in I$.

We claim that for each $i \in I \setminus I'$ we have that $\mu_i^{n_k} \to 0$. Indeed, as I' is nonempty, there exist $l \in I'$ and a constant $C > 0$ such that

$$(x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k} | d^{n_k}) \geq (x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k} | p_l^{n_k}) = (x^{n_k} | p_l^{n_k}) - \beta_l^{n_k} \geq -\|x^{n_k}\|_u - \beta_l^{n_k} > -C$$

for all k, since $(x^{n_k}|p_l^{n_k}) \leq \|x^{n_k}\|_u$. Moreover,

$$(x^{n_k} - \sum_{i\in I} \beta_i^{n_k} p_i^{n_k} | d^{n_k}) = (x^{n_k}|d^{m_k}) - \sum_{i\in I} \beta_i^{n_k} \mu_i^{n_k} \leq \|x^{n_k}\|_u - \sum_{i\in I'} \beta_i^{n_k} \mu_i^{n_k} - \sum_{i\in I\setminus I'} \beta_i^{n_k} \mu_i^{n_k}.$$

As $\beta_i^{n_k}, \mu_i^{n_k} \geq 0$ for all $i \in I$ and $\beta_i^{n_k} \to \infty$ for all $i \in I \setminus I'$, we conclude from the previous two inequalities that $\mu_i^{n_k} \to 0$ for all $i \in I \setminus I'$.

Using the Peirce decomposition with respect to the Jordan frame p_i, $i \in I$, we write

$$d = \sum_{i\in I} \mu_i p_i + \sum_{i<j\in I} d_{ij}.$$

We now show that

$$d = \sum_{i\in I'} \mu_i p_i + \sum_{i<j\in I'} d_{ij}, \tag{5.3.2}$$

and hence $d \in V(p_{I'})$. Note that

$$\mu_i - \mu_i^{n_k} = (d|p_i) - (d^{n_k}|p_i^{n_k}) = (d - d^{n_k}|p_i) + (d^{n_k}|p_i - p_i^{n_k}) \to 0.$$

We conclude that $\mu_i^{n_k} \to \mu_i$ for all $i \in I$, and hence $(d|p_j) = \mu_j = 0$ for all $j \in I \setminus I'$. This implies by [20, III, Exercise 3] that $d \bullet p_j = 0$ for all $j \in I \setminus I'$. So,

$$0 = d \bullet p_j = \frac{1}{2}\left(\sum_{l<j} d_{lj} + \sum_{j<m} d_{jm}\right),$$

which shows that $d_{lj} = 0 = d_{jm}$ for all $l < j < m$, as they are all orthogonal. This gives (5.3.2).

Next we show that $\lim_{k\to\infty}\Lambda_{V(p_I^{n_k})}(x^{n_k}-\sum_{i\in I}\beta_i^{n_k}p_i^{n_k})=(U_{p_{I'}}x-\sum_{i\in I'}\beta_i p_i|d)$. First note that

$$\begin{aligned}\Lambda_{V(p_I^{n_k})}(x^{n_k}-\sum_{i\in I}\beta_i^{n_k}p_i^{n_k}) &= (x^{n_k}-\sum_{i\in I'}\beta_i^{n_k}p_i^{n_k}|d^{n_k})-\sum_{i\in I\setminus I'}(\beta_i^{n_k}p_i^{n_k}|d^{n_k})\\ &= (x^{n_k}-\sum_{i\in I'}\beta_i^{n_k}p_i^{n_k}|d^{n_k})-\sum_{i\in I\setminus I'}\beta_i^{n_k}\mu_i^{n_k}\\ &\leq (x^{n_k}-\sum_{i\in I'}\beta_i^{n_k}p_i^{n_k}|d^{n_k})\end{aligned}$$

as $\beta_i^{n_k},\mu_i^{n_k}\geq 0$ for all i and k. This implies that

$$\limsup_{k\to\infty}\Lambda_{V(p_I^{n_k})}(x^{n_k}-\sum_{i\in I}\beta_i^{n_k}p_i^{n_k})\leq\lim_{k\to\infty}(x^{n_k}-\sum_{i\in I'}\beta_i^{n_k}p_i^{n_k}|d^{n_k})=(x-\sum_{i\in I'}\beta_i p_i|d)$$

As $U_{p_{I'}}d=d$ and $U_{p_{I'}}$ is self-adjoint, we find that

$$(x-\sum_{i\in I'}\beta_i p_i|d)=(x-\sum_{i\in I'}\beta_i p_i|U_{p_{I'}}d)=(U_{p_{I'}}x-\sum_{i\in I'}\beta_i p_i|d),$$

so that

$$\limsup_{k\to\infty}\Lambda_{V(p_I^{n_k})}(x^{n_k}-\sum_{i\in I}\beta_i^{n_k}p_i^{n_k})\leq(U_{p_{I'}}x-\sum_{i\in I'}\beta_i p_i|d). \tag{5.3.3}$$

Now let $p_{I'}^{n_k}=\sum_{i\in I'}p_i^{n_k}$. As $p_{I'}^{n_k}\to p_{I'}$, it follows from Lemma 5.3.1 that $U_{p_{I'}^{n_k}}d\to U_{p_{I'}}d=d$. This implies that

$$(x^{n_k}-\sum_{i\in I}\beta_i^{n_k}p_i^{n_k}|U_{p_{I'}^{n_k}}d)(U_{p_{I'}^{n_k}}d|p_I^{n_k})^{-1}\leq\Lambda_{V(p_I^{n_k})}(x^{n_k}-\sum_{i\in I}\beta_i^{n_k}p_i^{n_k})$$

for all k large, as $(U_{p_{I'}^{n_k}}d|p_I^{n_k})\to(U_{p_{I'}}d|p_I)=(d|U_{p_{I'}}p_I)=(d|p_{I'})=(d|p_I)=1$. Moreover,

$$\begin{aligned}\lim_{k\to\infty}(x^{n_k}-\sum_{i\in I}\beta_i^{n_k}p_i^{n_k}|U_{p_{I'}^{n_k}}d)(U_{p_{I'}^{n_k}}d|p_I^{n_k})^{-1} &= \lim_{k\to\infty}(U_{p_{I'}^{n_k}}x^{n_k}-\sum_{i\in I'}\beta_i^{n_k}p_i^{n_k}|d)(U_{p_{I'}^{n_k}}d|p_I^{n_k})^{-1}\\ &= (U_{p_{I'}}x-\sum_{i\in I'}\beta_i p_i|d).\end{aligned}$$

This shows that $(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i|d) \leq \liminf_{k\to\infty}\Lambda_{V(p_I^{n_k})}(x^{n_k} - \sum_{i\in I}\beta_i^{n_k}p_i^{n_k})$. From (5.3.3) we conclude that

$$(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i|d) = \lim_{k\to\infty}\Lambda_{V(p_I^{n_k})}(x^{n_k} - \sum_{i\in I}\beta_i^{n_k}p_i^{n_k}). \qquad (5.3.4)$$

To complete the proof we show that

$$(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i|d) = \Lambda_{V(p_{I'})}(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i). \qquad (5.3.5)$$

As $(d|p_{I'}) = (d|p_I) = 1$, we know that $d \in S(V_{p_{I'}})$. So, we get from (5.3.1) that

$$(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i|d) \leq \sup_{z\in S(V(p_{I'}))}(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i|z) = \Lambda_{V(p_{I'})}(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i).$$

On the other hand, if $w \in S(V(p_{I'}))$ is such that

$$(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i|w) = \sup_{z\in S(V(p_{I'}))}(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i|z) = \Lambda_{V(p_{I'})}(U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i),$$

then by definition of d^{n_k} we get for all k large that

$$(x - \sum_{i\in I}\beta_i^{n_k}p_i^{n_k}|d^{n_k}) \geq (x - \sum_{i\in I}\beta_i^{n_k}p_i^{n_k}|U_{p_{I'}^{n_k}}w)(U_{p_{I'}^{n_k}}w|p_I^{n_k})^{-1} = (U_{p_{I'}^{n_k}}x - \sum_{i\in I'}\beta_i^{n_k}p_i^{n_k}|w)(U_{p_{I'}^{n_k}}w|p_I^{n_k})^{-1},$$

as $(U_{p_{I'}^{n_k}}w|p_I^{n_k}) \to (U_{p_{I'}}w|p_I) = (w|p_{I'}) = 1$. This implies that

$$\lim_{k\to\infty}\Lambda_{V(p_I^{n_k})}(x^{n_k} - \sum_{i\in I}\beta_i^{n_k}p_i^{n_k}) \geq \lim_{k\to\infty}(U_{p_{I'}^{n_k}}x - \sum_{i\in I'}\beta_i^{n_k}p_i^{n_k}|w)(U_{p_{I'}^{n_k}}w|p_I^{n_k})^{-1} = (U_{p_{I'}}x - \sum_{i\in I'}\beta_i p_i|w),$$

and hence (5.3.5) holds by (5.3.4). □

To prove that all horofunctions in $\overline{V}^h$ are of the form (5.2.1), we first establish the following proposition by using the previous lemma.

Proposition 5.3.3. *Let (y^n) be a sequence in V, with $y^n = \sum_{i=1}^r \lambda_i^n p_i^n$. Suppose that $h_{y^n} \to h \in \partial\overline{V}^h$ and (y^n) satisfies the following properties:*

1. *There exists $1 \le s \le r$ such that $|\lambda_s^n| = r^n$ for all n, where $r^n = \|y^n\|_u$.*
2. *$p_k^n \to p_k$ for all $1 \le k \le r$.*
3. *There exist $I, J \subseteq \{1, \dots, r\}$ disjoint with $I \cup J$ nonempty, and $\alpha \in \mathbb{R}^{I\cup J}$ with $\min\{\alpha_i \colon i \in I \cup J\} = 0$ such that $r^n - \lambda_i^n \to \alpha_i$ for all $i \in I$, $r^n + \lambda_j^n \to \alpha_j$ for all $j \in J$, and $r^n - |\lambda_k^n| \to \infty$ for all $k \notin I \cup J$.*

Then h satisfies (5.2.1).

Proof. Take $x \in V$ fixed. Note that for all $n \ge 1$,

$$\|x-y^n\|_u-\|y^n\|_u = \max\{\Lambda(x-y^n), \Lambda(-x+y^n)\}-r^n = \max\{\Lambda(x-y^n-r^n u), \Lambda(-x+y^n-r^n u)\}.$$

As h is a horofunction, $\|y^n\|_u = r^n \to \infty$ by Lemma 3.1.5. Thus, $\lambda_i^n \to \infty$ for all $i \in I$ and $\lambda_j^n \to -\infty$ for all $j \in J$. Now suppose that J is nonempty. Then $r^n + \lambda_k^n \ge r^n - |\lambda_k^n| \to \infty$ for all $k \notin J$. As

$$\Lambda(x - y^n - r^n u) = \Lambda(x - \sum_{j\in J}(r^n + \lambda_j^n)p_j^n - \sum_{k\notin J}(r^n + \lambda_k^n)p_k^n),$$

it follows that

$$\lim_{n\to\infty} \Lambda(x - y^n - r^n u) = \Lambda_{V(p_J)}(U_{p_J} x - \sum_{j\in J} \alpha_j p_j)$$

by Lemma 5.3.2. Likewise, if I is nonempty, then

$$\lim_{n\to\infty} \Lambda(-x + y^n - r^n u) = \Lambda_{V(p_I)}(-U_{p_I} x - \sum_{i\in I} \alpha_i p_i)$$

by Lemma 5.3.2. We conclude that if I and J are both nonempty, then

$$\begin{aligned} h(x) = \lim_{n\to\infty} \|x - y^n\|_u - \|y^n\|_u &= \lim_{n\to\infty} \max\{\Lambda(-x + y^n - r^n u), \Lambda(x - y^n - r^n u)\} \\ &= \max\{\Lambda_{V(p_I)}(-U_{p_I} x - \sum_{i\in I} \alpha_i p_i), \Lambda_{V(p_J)}(U_{p_J} x - \sum_{j\in J} \alpha_j p_j)\}. \end{aligned}$$

To complete the proof it remains to show that $\lim_{n\to\infty} \|x-y^n\|_u - \|y^n\|_u = \lim_{n\to\infty} \Lambda(-x+y^n - r^n u)$ if J is empty, and $\lim_{n\to\infty} \|x - y^n\|_u - \|y^n\|_u = \lim_{n\to\infty} \Lambda(x - y^n - r^n u)$ if I is empty. Suppose that I is empty, so J is nonempty. Then for each $i \in \{1, \dots, r\}$ we have that $r^n - \lambda_i^n \to \infty$. Note that

$$-x + y^n - r^n u = -x - \sum_i (r^n - \lambda_i^n) p_i^n \leq -x - \min_i (r^n - \lambda_i^n) u \leq (\|x\|_u - \min_i (r^n - \lambda_i^n)) u.$$

Thus, $\Lambda(-x + y^n - r^n u) \leq \Lambda((\|x\|_u - \min_i(r^n - \lambda_i^n))u) = \|x\|_u - \min_i(r^n - \lambda_i^n)$ for all n, hence $\Lambda(-x + y^n - r^n u) \to -\infty$. As

$$\max\{\Lambda(x - y^n - r^n u), \Lambda(-x + y^n - r^n u)\} = \|x - y^n\|_u - \|y^n\|_u \geq -\|x\|_u > -\infty,$$

we conclude that $\|x - y^n\|_u - \|y^n\|_u = \Lambda(x - y^n - r^n u)$ for all n sufficiently large, hence

$$h(x) = \lim_{n\to\infty} \Lambda(x - y^n - r^n u) = \Lambda_{V(p_J)}(U_{p_J} x - \sum_{j\in J} \alpha_j p_j).$$

The argument for the case where J is empty goes in the same way. □

The following corollary shows that each horofunction is of the form (5.2.1).

Corollary 5.3.4. *If h is a horofunction in $\overline{V}^h$, then there exist a Jordan frame $p_1, \dots, p_r$ in V, disjoint subsets $I, J \subseteq \{1, \dots, r\}$, with $I \cup J$ nonempty, and $\alpha \in \mathbb{R}^{I\cup J}$ with* $\min\{\alpha_i \colon i \in I \cup J\} = 0$*, such that $h \colon V \to \mathbb{R}$ satisfies (5.2.1) for all $x \in V$.*

Proof. Suppose that (y^n) is a sequence in V with $h_{y^n} \to h$ in $\overline{V}^h$. Then for each $x \in V$ we have that

$$\lim_{n\to\infty} \|x - y^n\|_u - \|y^n\|_u = h(x)$$

and $\|y^n\|_u \to \infty$ by Lemma 3.1.5.

To show that the limit is equal to (5.2.1) it suffices to show that we can take a subsequences of (y^n) that satisfies the conditions in Proposition 5.3.3. First we note that by the spectral theorem [20, Theorem III.1.2], there exist for each $n \geq 1$ a Jordan frame $p_1^n, \ldots, p_r^n$ in V and $\lambda_1^n, \ldots, \lambda_r^n \in \mathbb{R}$ such that

$$y^n = \lambda_1^n p_1^n + \cdots + \lambda_r^n p_r^n,$$

where r is the rank of V. Denote $r^n = \|y^n\|_u = \max_i |\lambda_i^n|$.

Now by taking subsequences we may assume that there exist $I_+ \subseteq \{1, \ldots, r\}$ and $1 \leq s \leq r$ such that for each $n \geq 1$ we have $r^n = |\lambda_s^n|$ and

$$\lambda_i^n > 0 \text{ for all } i \in I_+ \quad \text{and} \quad \lambda_i^n \leq 0 \text{ for all } i \notin I_+.$$

Now for each $i \in \{1, \ldots, r\}$ and $n \geq 1$ define

$$\alpha_i^n = \begin{bmatrix} r^n - \lambda_i^n & \text{for } i \in I_+ \\ r^n + \lambda_i^n & \text{for } i \notin I_+. \end{bmatrix}$$

Note that $\alpha_i^n \in [0, \infty)$ for all i. Again by taking subsequences we may assume that $\alpha_i^n \to \alpha_i \in [0, \infty]$ as $n \to \infty$, for all i. Recall that $\alpha_s^n = 0$ for all n, so $\alpha_s = 0$. Furthermore, we may assume that $p_i^n \to p_i$ in $\mathcal{J}_1(V)$ for all i, as it is a compact set [31]. Note that $p_1, \ldots, p_r$ is a Jordan frame in V.

Now let

$$I = \{i \colon \alpha_i < \infty \text{ and } i \in I_+\} \quad \text{and} \quad J = \{j \colon \alpha_j < \infty \text{ and } j \notin I_+\}.$$

So, $I \cap J$ is empty, $s \in I \cup J$ and $\min\{\alpha_i : i \in I \cup J\} = \alpha_s = 0$. Then the subsequence of (y^n) satisfies the conditions in Proposition 5.3.3, hence h is a horofunction of the form (5.2.1). □

The next proposition shows that each function of the form (5.2.1) can be realised as a horofunction, and is a Busemann point.

Proposition 5.3.5. *Let $p_1, \dots, p_r$ be a Jordan frame in V. Suppose that $I, J \subseteq \{1, \dots, r\}$ with $I \cap J = \emptyset$ and $I \cup J$ nonempty, and $\alpha \in \mathbb{R}^{I \cup J}$ with $\min\{\alpha_i : i \in I \cup J\} = 0$. If for $n \geq 1$ we let $y^n = \lambda_1^n p_1 + \dots + \lambda_r^n p_r$, where*

$$\lambda_i^n = \begin{bmatrix} n - \alpha_i & \text{if } i \in I \\ -n + \alpha_i & \text{if } i \in J \\ 0 & \text{otherwise,} \end{bmatrix}$$

then (y^n) is an almost geodesic sequence and $h_{y^n} \to h$, where h satisfies (5.2.1) for all $x \in V$. In particular, h is a Busemann point in $\overline{V}^h$.

Proof. Let $k \geq \max\{\alpha_i : i \in I \cup J\}$ and note that for $n \geq k$ we have that $r^n = \|y^n\|_u = n$, as $\min\{\alpha_i : i \in I \cup J\} = 0$. The sequence (y^n), where $n \geq k$, satisfies the conditions in Proposition 5.3.3. Indeed, for $n \geq k$ we have that $r^n - \lambda_i^n = \alpha_i$ for all $i \in I$, $r^n + \lambda_i^n = \alpha_i$ for all $i \in J$, and $r^n - \lambda_i^n = n$ otherwise. Also for s with $\alpha_s = 0$, we have that $|\lambda_s^n| = n = \|y^n\|_u$.

Finally to see that (h_{y^n}) converges, we note that if we define $z = \sum_{i \in I} -\alpha_i p_i + \sum_{j \in J} \alpha_j p_j$ and $w = \sum_{i \in I} p_i - \sum_{j \in J} p_j$, then $y^n = nw + z$, which lies on the straight-line $t \mapsto tw + z$. Hence (y^n) is an almost geodesic sequence, so

$$h(x) = \lim_{n \to \infty} \|x - y^n\|_u - \|y^n\|_u$$

exists for all $x \in V$. Thus, we can apply Proposition 5.3.3 and conclude that h satisfies (5.2.1), and h is a Busemann point in the horofunction boundary. □

Combining the results so far we now prove Theorem 5.2.1.

Proof of Theorem 5.2.1. Corollary 5.3.4 shows that each horofunction in $\overline{V}^h$ is of the form (5.2.1). It follows from Proposition 5.3.5 that any function of the form (5.2.1) is a horofunction and by the second part of that proposition each horofunction is a Busemann point. □

5.4 Parts and the detour metric

In this section we will identify the parts in the horofunction boundary of $\overline{V}^h$, derive a formula for the detour distance, and establish Theorem 5.2.2. We begin by proving the following proposition.

Proposition 5.4.1. *If*

$$h(x) = \max\left\{\Lambda_{V(p_I)}\left(-U_{p_I}x - \sum_{i\in I}\alpha_i p_i\right), \Lambda_{V(p_J)}\left(U_{p_J}x - \sum_{j\in J}\alpha_j p_j\right)\right\}, \tag{5.4.1}$$

and

$$h'(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i\in I'}\beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j\in J'}\beta_j q_j\right)\right\} \tag{5.4.2}$$

are horofunctions with $p_I = q_{I'}$ and $p_J = q_{J'}$, then h and h' are in the same part and

$$\delta(h,h') = \|a-b\|_{\text{var}} = \Lambda_{V(p_I,p_J)}(a-b) + \Lambda_{V(p_I,p_J)}(b-a),$$

where $a = \sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j$ and $b = \sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j$ in $V(p_I,p_J) = V(p_I) + V(p_J)$.

Proof. As in Proposition 5.3.5, for $n \geq 1$ let $y^n = \lambda_1^n p_1 + \cdots + \lambda_r^n p_r$, where

$$\lambda_i^n = \begin{cases} n - \alpha_i & \text{if } i \in I \\ -n + \alpha_i & \text{if } i \in J \\ 0 & \text{otherwise,} \end{cases}$$

and let $w^n = \mu_1^n q_1 + \cdots + \mu_r^n q_r$, where

$$\mu_i^n = \begin{cases} n - \beta_i & \text{if } i \in I' \\ -n + \beta_i & \text{if } i \in J' \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 5.3.5 we know that (y^n) and (w^n) are almost geodesic sequences with $h_{y^n} \to h$ and $h_{w^n} \to h'$. Note that

$$U_{p_I} w^m = U_{q_{I'}} w^m = \sum_{i \in I'} \mu_i^m U_{q_{I'}} q_i = \sum_{i \in I'} \mu_i^m q_i$$

for all m, so

$$\begin{aligned} \Lambda_{V(p_I)}(-U_{p_I} w^m - \sum_{i \in I} \alpha_i p_i + \|w^m\|_u p_I) &= \Lambda_{V(p_I)}(-U_{q_{I'}} w^m - \sum_{i \in I} \alpha_i p_i + \|w^m\|_u q_{I'}) \\ &= \Lambda_{V(p_I)}(\sum_{i \in I'} (\|w^m\|_u - \mu_i^m) q_i - \sum_{i \in I} \alpha_i p_i). \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{m \to \infty} \Lambda_{V(p_I)}(-U_{p_I} w^m - \sum_{i \in I} \alpha_i p_i + \|w^m\|_u p_I) &= \lim_{m \to \infty} \Lambda_{V(p_I)}(\sum_{i \in I'} (\|w^m\|_u - \mu_i^m) q_i - \sum_{i \in I} \alpha_i p_i) \\ &= \Lambda_{V(p_I)}(\sum_{i \in I'} \beta_i q_i - \sum_{i \in I} \alpha_i p_i) \\ &= \Lambda_{V(p_I)}(b - a). \end{aligned}$$

In the same way it can be shown that

$$\lim_{m\to\infty} \Lambda_{V(p_J)}(U_{p_J}w^m - \sum_{j\in J}\alpha_j p_i + \|w^m\|_u p_J) = \Lambda_{V(p_J)}(\sum_{j\in J'}\beta_j q_j - \sum_{j\in J}\alpha_j p_j) = \Lambda_{V(p_J)}(b-a).$$

So, it follows from Lemma 3.2.5 that

$$\begin{aligned} H(h,h') &= \lim_{m\to\infty} \|w^m\|_u + \max\{\Lambda_{V(p_I)}(-U_{p_I}w^m - \sum_{i\in I}\alpha_i p_i), \Lambda_{V(p_J)}(U_{p_J}w^m - \sum_{j\in J}\alpha_j p_j)\} \\ &= \lim_{m\to\infty} \max\{\Lambda_{V(p_I)}(-U_{p_I}w^m - \sum_{i\in I}\alpha_i p_i + \|w^m\|_u p_I), \Lambda_{V(p_J)}(U_{p_J}w^m - \sum_{j\in J}\alpha_j p_j + \|w^m\|_u p_J)\} \\ &= \max\{\Lambda_{V(p_I)}(\sum_{i\in I'}\beta_i q_i - \sum_{i\in I}\alpha_i p_i), \Lambda_{V(p_J)}(\sum_{j\in J'}\beta_j q_j - \sum_{j\in J}\alpha_j p_j)\} \\ &= \Lambda_{V(p_I,p_J)}(b-a). \end{aligned}$$

Interchanging the roles of h and h' gives $H(h',h) = \Lambda_{V(p_I,p_J)}(a-b)$, hence $\delta(h,h') = \|a-b\|_{\mathrm{var}}$. □

To show that h and h' are in different part if $p_I \neq q_{I'}$ or $p_J \neq q_{J'}$, we need the following lemma.

Lemma 5.4.2. *If p and q are idempotents in V with $p \not\leq q$, then $U_p q < p$.*

Proof. We have that $U_p q \leq U_p u = p$. In fact, $U_p q < p$. Indeed, if $U_p q = p$, then

$$p = U_p u = U_p(u-q) + U_p q = U_p(u-q) + p,$$

and hence $U_p(u-q) = 0$. This implies that $p + (u-q) \leq u$ by [30, Lemma 4.2.2], so that $p \leq q$. This is impossible, as $p \not\leq q$, and hence $U_p q < p$. □

Proposition 5.4.3. *If h and h' are horofunctions given by (5.4.1) and (5.4.2), respectively, and $p_I \neq q_{I'}$ or $p_J \neq q_{J'}$, then*

$$\delta(h,h') = \infty.$$

Proof. Suppose that $p_I \neq q_{I'}$. Then $p_I \not\leq q_{I'}$ or $q_{I'} \not\leq p_I$. Without loss of generality assume that $p_I \not\leq q_{I'}$. Let (y^n) in $V(p_I)$ and (w^n) in $V(q_{I'})$ be as in Proposition 5.3.5, so $h_{y^n} \to h$ and $h_{w^m} \to h'$. To prove the statement in this case, we use Lemma 3.2.5 and show that

$$H(h', h) = \lim_{m\to\infty} \|w^m\|_u + h(w^m) = \infty. \tag{5.4.3}$$

Note that

$$\|w^m\|_u + h(w^m) \geq \|w^m\|_u + \Lambda_{V(p_I)}(-U_{p_I}w^m - \sum_{i\in I}\alpha_i p_i) = \Lambda_{V(p_I)}(-U_{p_I}w^m - \sum_{i\in I}\alpha_i p_i + \|w^m\|_u p_I).$$

As $w^m \leq \|w^m\|_u q_{I'}$ for all m large, we have that $U_{p_I}w^m \leq \|w^m\|_u U_{p_I}q_{I'}$ for all m large. Thus,

$$\begin{aligned} -U_{p_I}w^m - \sum_{i\in I}\alpha_i p_i + \|w^m\|_u p_I &\geq -\|w^m\|_u U_{p_I}q_{I'} - \sum_{i\in I}\alpha_i p_i + \|w^m\|_u p_I \\ &= \|w^m\|_u (p_I - U_{p_I}q_{I'}) - \sum_{i\in I}\alpha_i p_i \end{aligned}$$

for all m large.

We know from Lemma 5.4.2 that $p_I - U_{p_I}q_{I'} > 0$. As $p_I - U_{p_I}q_{I'} \in V(p_I)$ we also have that $p_I - U_{p_I}q_{I'} = \sum_{j=1}^s \gamma_j r_j$, where $\gamma_j > 0$ for all j and the r_j's are orthogonal idempotents in $V(p_I)$. It now follows that for all m large,

$$\begin{aligned} \Lambda_{V(p_I)}(-U_{p_I}w^m - \sum_{i\in I}\alpha_i p_i + \|w^m\|_u p_I) &\geq (\|w^m\|_u \sum_{j=1}^s \gamma_j r_j - \sum_{i\in I}\alpha_i p_i | r_1)(p_I|r_1)^{-1} \\ &= (\|w^m\|_u \gamma_1 - (\sum_{i\in I}\alpha_i p_i | r_1))(p_I|r_1)^{-1}. \end{aligned}$$

The right-hand side goes to ∞ as $m \to \infty$, and hence (5.4.3) holds.

For the case $p_J \neq q_{J'}$ a similar argument can be used. □

We now prove Theorem 5.2.2.

Proof of Theorem 5.2.2. Parts (i) and (ii) follow directly from Propositions 5.4.1 and 5.4.3. Clearly the map $\rho\colon \mathcal{P}_h \to V(p_I, p_J)/\mathbb{R}p_{IJ}$ given by $\rho(h') = [b]$, where

$$h'(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i\in I'}\beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j\in J'}\beta_j q_j\right)\right\},$$

and $b = \sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j \in V(p_I, p_J)$ with $\min_{i\in I\cup J}\beta_i = 0$, is a bijection. So, by Proposition 5.4.1, ρ is an isometry from $(\mathcal{P}_h, \delta)$ onto $(V(p_I, p_J)/\mathbb{R}p_{IJ}, \|\cdot\|_{\text{var}})$. □

5.5 The homeomorphism onto the dual unit ball

In this section we prove Theorem 5.2.4. To start we prove a lemma that will be useful in the sequel.

Lemma 5.5.1. *If $q \leq p$ are idempotents in V and $z \in V(p)$, then $\Lambda_{V(q)}(U_q z) \leq \Lambda_{V(p)}(z)$.*

Proof. If $\lambda = \Lambda_{V(p)}(z)$, then $0 \leq \lambda p - z$, so that $0 \leq \lambda U_q p - U_q z$. As $q = U_q q \leq U_q p \leq U_q u = q^2 = q$, we find that $0 \leq \lambda U_q p - U_q z = \lambda q - U_q z$, hence $\Lambda_{V(q)}(U_q z) \leq \lambda$. □

We will show that ϕ given by (5.2.4) and (5.2.5) is a continuous bijection from $\overline{V}^h$ onto B_{V^*}. As $\overline{V}^h$ is compact and B_{V^*} is Hausdorff, we can then conclude that ϕ is a homeomorphism. We begin by showing that ϕ maps V into the interior of B_{V^*}.

Lemma 5.5.2. *For each $x \in V$ we have that $\phi(x) \in \operatorname{int} B_{V^*}$.*

Proof. For $x \in V$ there exists $y \in V$ with $\|y\|_u = 1$, such that

$$\|\phi(x)\|_u^* = \sup_{w\in V\colon \|w\|_u\leq 1} |(w|\phi(x))| = (y|\phi(x)),$$

where $(v|w) = \operatorname{tr}(v \bullet w)$. So, if x has spectral decomposition $x = \sum_{i=1}^r \lambda_i p_i$, then we can consider the Peirce decomposition of y,

$$y = \sum_{i=1}^r \mu_i p_i + \sum_{i<j} y_{ij},$$

to find that

$$\|\phi(x)\|_u^* = (\phi(x)|y) = \frac{1}{\sum_{i=1}^r e^{\lambda_i} + e^{-\lambda_i}} (\sum_{i=1}^r (e^{\lambda_i} - e^{-\lambda_i}) p_i | y) \leq \frac{\sum_{i=1}^r (e^{\lambda_i} - e^{-\lambda_i}) |\mu_i|}{\sum_{i=1}^r e^{\lambda_i} + e^{-\lambda_i}} < 1,$$

as $\mu_i = (y|p_i) \leq (u|p_i) = 1$ and $\mu_i = (y|p_i) \geq (-u|p_i) = -1$. □

Lemma 5.5.3. *The map ϕ is injective on V.*

Proof. Suppose that $x, y \in V$ with $x = \sum_{i=1}^r \sigma_i p_i$ and $y = \sum_{i=1}^r \tau_i q_i$, where $\sigma_1 \leq \ldots \leq \sigma_r$ and $\tau_1 \leq \ldots \leq \tau_r$, satisfy $\phi(x) = \phi(y)$. Then $\phi(x) = \sum_{i=1}^r \alpha_i p_i = \sum_{i=1}^r \beta_i q_i = \phi(y)$. where

$$\alpha_j = \frac{e^{\sigma_j} - e^{-\sigma_j}}{\sum_{i=1}^r e^{\sigma_i} + e^{-\sigma_i}} \quad \text{and} \quad \beta_j = \frac{e^{\tau_j} - e^{-\tau_j}}{\sum_{i=1}^r e^{\tau_i} + e^{-\tau_i}} \qquad \text{for all } j.$$

As $\alpha_1 \leq \ldots \leq \alpha_r$ and $\beta_1 \leq \ldots \leq \beta_r$, it follows from the spectral theorem (version 2) [20, Theorem III.1.2] that $\alpha_j = \beta_j$ for all j. Lemma 4.2.2 now implies that $\sigma = (\sigma_1, \ldots, \sigma_r) = (\tau_1, \ldots, \tau_r) = \tau$, as

$$(\alpha_1, \ldots, \alpha_r) = \nabla \log \mu(\sigma) \quad \text{and} \quad (\beta_1, \ldots, \beta_r) = \nabla \log \mu(\tau).$$

Note that $\alpha_i = \alpha_j$ if and only if $\sigma_i = \sigma_j$, and $\beta_i = \beta_j$ if and only if $\tau_i = \tau_j$, as $\nabla \log \mu(x)$ is injective. It now follows from the spectral theorem (version 1) [20, Theorem III.1.1] that $x = y$. □

Lemma 5.5.4. *The map ϕ maps V onto* $\operatorname{int} B_{V^*}$.

Proof. As ϕ is continuous on V and $\phi(V) \subseteq \operatorname{int} B_{V^*}$, it follows from Brouwer's domain invariance theorem that $\phi(V)$ is open in $\operatorname{int} B_{V^*}$. Suppose, for the sake of contradiction, that $\phi(V) \neq \operatorname{int} B_{V^*}$. So, we can find a $z \in \partial\phi(V) \cap \operatorname{int} B_{V^*}$. Let (y^n) in V be such that $\phi(y^n) \to z$ and write $y^n = \sum_{i=1}^r \lambda_i^n p_i^n$. As ϕ is continuous on V, we may assume that $r^n = \|y^n\|_u \to \infty$. Furthermore, after taking a subsequence, we may assume that (y^n) satisfies the conditions in Proposition 5.3.3. So, using the notation as in Proposition 5.3.3, we get that

$$\phi(y^n) = \frac{\sum_{i=1}^r (e^{\lambda_i^n} - e^{-\lambda_i^n}) p_i^n}{\sum_{i=1}^r e^{\lambda_i^n} + e^{-\lambda_i^n}} = \frac{\sum_{i=1}^r (e^{-r^n+\lambda_i^n} - e^{-r^n-\lambda_i^n}) p_i^n}{\sum_{i=1}^r e^{-r^n+\lambda_i^n} + e^{-r^n-\lambda_i^n}}.$$

The right-hand side converges to

$$\frac{1}{\sum_{i\in I} e^{-\alpha_i} + \sum_{j\in J} e^{-\alpha_j}} \left(\sum_{i\in I} e^{-\alpha_i} p_i - \sum_{j\in J} e^{-\alpha_j} p_j \right) = z.$$

But this implies that $z \in \partial B_{V^*}$, which is impossible. Indeed, if we let $p_I = \sum_{i\in I} p_i$ and $p_J = \sum_{j\in J} p_j$, then $1 \geq \|z\|_u^* \geq (z|p_I - p_J) = 1$, as $-u \leq p_I - p_J \leq u$. □

For simplicity we denote the (closed) boundary faces of B_{V^*} by

$$F_{p,q} = \operatorname{conv}\left((U_p(V) \cap S(V)) \cup (U_q(V) \cap -S(V))\right)$$

where p and q are orthogonal idempotents in V not both zero, see [18, Theorem 4.4].

Lemma 5.5.5. *If h is a horofunction given by (5.2.1), then ϕ maps $\mathcal{P}_h$ into* relint F_{p_I,p_J}.

Proof. Clearly, $\phi(h) \in F_{p_I,p_J}$ if h is given by (5.2.1). So, ϕ maps $\mathcal{P}_h$ into F_{p_I,p_J} by Theorem 5.2.2(i). To show that ϕ maps $\mathcal{P}_h$ into relint F_{p_I,p_J}, it suffices to show that $\phi(h) \in \operatorname{relint} F_{p_I,p_J}$.

To do this we first consider $w = (|I| + |J|)^{-1}(p_I - p_J) \in F_{p_I,q_J}$ and show that $w \in \operatorname{relint} F_{p_I,q_J}$. Let $c \in F_{p_I,p_J}$ be arbitrary. Note that we can write $c = \sum_{i\in I'} \lambda_i q_i - \sum_{j\in J'} \lambda_j q_j$, where $\sum_{i\in I'} q_i = p_I$, $\sum_{j\in J'} q_j = p_J$, and $\sum_{i\in I'} \lambda_i + \sum_{j\in J'} \lambda_j = 1$ with $0 \leq \lambda_i, \lambda_j \leq 1$ for all i and j. We see that $w + \epsilon(w - c) = (1+\epsilon)w - \epsilon c \in F_{p_I,p_J}$ for all $\epsilon > 0$ small, so $w \in \operatorname{relint} F_{p_I,p_J}$ by [61, Theorem 6.4].

To complete the proof we argue by contradiction. So suppose that $\phi(h) \notin \operatorname{relint} F_{p_I,p_J}$. Then $\phi(h)$ is in the (relative) boundary of F_{p_I,p_J}, hence

$$z_\epsilon = (1+\epsilon)\phi(h) - \epsilon w \notin F_{p_I,p_J}$$

for all $\epsilon > 0$, as $w \in \mathrm{relint} F_{p_I,p_J}$ and F_{p_I,p_J} is convex. However, for each $i \in I$ we have that the coefficient of p_i in z_ϵ,

$$\frac{(1+\epsilon)e^{-\alpha_i}}{\sum_{i\in I} e^{-\alpha_i} + \sum_{j\in J} e^{-\alpha_j}} - \frac{\epsilon}{|I|+|J|},$$

is strictly positive for all $\epsilon > 0$ sufficiently small. Likewise, for each $j \in J$ we have that the coefficient of $-p_j$ in z_ϵ,

$$\frac{(1+\epsilon)e^{-\alpha_j}}{\sum_{i\in I} e^{-\alpha_i} + \sum_{j\in J} e^{-\alpha_j}} - \frac{\epsilon}{|I|+|J|},$$

is strictly positive for all $\epsilon > 0$ sufficiently small. This implies that $z_\epsilon \in F_{p_I,p_J}$ for all $\epsilon > 0$ small, which is impossible. This completes the proof. □

Using the previous results we now show that ϕ is injective on $\overline{V}^h$.

Corollary 5.5.6. *The map* $\phi\colon \overline{V}^h \to B_{V^*}$ *is injective.*

Proof. We already saw in Lemmas 5.5.2 and 5.5.3 that ϕ maps V into $\mathrm{int}\, B_{V^*}$ and is injective on V. So by the previous lemma, it suffices to show that if $\phi(h) = \phi(h')$ for horofunctions $h \sim h'$, then $h = h'$. Let h be given by (5.2.1) and suppose that h' is given by

$$h'(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i\in I'} \beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j\in J'} \beta_j q_j\right)\right\}.$$

Then

$$\frac{\sum_{i\in I} e^{-\alpha_i}p_i - \sum_{j\in J} e^{-\alpha_j}p_j}{\sum_{i\in I} e^{-\alpha_i} + \sum_{j\in J} e^{-\alpha_j}} = \frac{\sum_{i\in I'} e^{-\beta_i}q_i - \sum_{j\in J'} e^{-\beta_j}q_j}{\sum_{i\in I'} e^{-\beta_i} + \sum_{j\in J'} e^{-\beta_j}}.$$

As $\min_k \alpha_k = 0 = \min_k \beta_k$, it follows from the spectral theorem [20, Theorem III.1.2] that

$$\frac{1}{\sum_{i\in I} e^{-\alpha_i} + \sum_{j\in J} e^{-\alpha_j}} = \|\phi(h)\|_u = \|\phi(h')\|_u = \frac{1}{\sum_{i\in I'} e^{-\beta_i} + \sum_{j\in J'} e^{-\beta_j}},$$

so that

$$\sum_{i\in I} e^{-\alpha_i}p_i - \sum_{j\in J} e^{-\alpha_j}p_j = \sum_{i\in I'} e^{-\beta_i}q_i - \sum_{j\in J'} e^{-\beta_j}q_j.$$

As each $x \in V$ can be written in a unique way as $x = x^+ - x^-$, where x^+ and x^- are orthogonal elements in V_+, see [4, Proposition 1.28], we find that $\sum_{i\in I} e^{-\alpha_i}p_i = \sum_{i\in I'} e^{-\beta_i}q_i$ and $\sum_{j\in J} e^{-\alpha_j}p_i = \sum_{j\in J'} e^{-\beta_j}q_j$. This implies that

$$\sum_{i\in I}\alpha_i p_i = -\log(\sum_{i\in I} e^{-\alpha_i}p_i + (u - p_I)) = -\log\sum_{i\in I'}(e^{-\beta_i}q_i + (u - q_{I'})) = \sum_{i\in I'}\beta_i q_i$$

and

$$\sum_{j\in J}\alpha_j p_i = -\log(\sum_{j\in J} e^{-\alpha_j}p_i + (u - p_J)) = -\log(\sum_{j\in J'} e^{-\beta_j}q_j + (u - q_{J'})) = \sum_{j\in J'}\beta_j q_j,$$

and hence $h = h'$. □

The next result shows that ϕ is continuous on $\partial\overline{V}^h$.

Theorem 5.5.7. *The map $\phi\colon \overline{V}^h \to B_{V^*}$ is continuous.*

Proof. Clearly ϕ is continuous on V. Suppose (y^n) is a sequence in V with $h_{y^n} \to h \in \partial\overline{V}^h$. We wish to show that $\phi(y^n) \to \phi(h)$. Let $(\phi(y^{n_k}))$ be a subsequence. We will show that it has a subsequence which converges to $\phi(h)$.

As h is a horofunction, we know that $r^n = \|y^{n_k}\|_u \to \infty$ by Lemma 3.1.5. For each k there exists a Jordan frame $q_1^{n_k}, \dots, q_r^{n_k}$ in V and $\lambda_1^{n_k}, \dots, \lambda_r^{n_k} \in \mathbb{R}$ such that

$$y^{n_k} = \sum_{i=1}^r \lambda_i^{n_k} q_i^{n_k}.$$

By taking a subsequence we may assume that there exist $I_+ \subseteq \{1, \dots, r\}$ and $1 \leq s \leq r$ such that for each k, $r^{n_k} = \|y^{n_k}\|_u = |\lambda_s^{n_k}|$, and $\lambda_i^{n_k} > 0$ if and only if $i \in I_+$.

For each k, let $\beta_i^{n_k} = r^{n_k} - \lambda_i^{n_k}$ for $i \in I_+$, and $\beta_i^{n_k} = r^{n_k} + \lambda_i^{n_k}$ for $i \notin I_+$. Note that $\beta^{n_k} \geq 0$ for all i and k, and $\beta_s^{n_k} = 0$ for all k. By taking a further subsequence we may assume that $\beta_i^{n_k} \to \beta_i \in [0,\infty]$ and $q_i^{n_k} \to q_i$ for all i. Let $I' = \{i \in I_+ \colon \beta_i < \infty\}$ and $J' = \{j \notin I_+ \colon \beta_j < \infty\}$. Note that $s \in I' \cup J'$ and we can apply Proposition 5.3.3 to conclude that $h_{y^{n_k}} \to h' \in \partial\overline{V}^h$, where

$$h'(x) = \max\{\Lambda_{V(q_{I'})}(-U_{q_{I'}}x - \sum_{i\in I'}\beta_i q_i), \Lambda_{V(q_{J'})}(U_{q_{J'}}x - \sum_{j\in J'}\beta_j q_j)\}.$$

As $h_{y^{n_k}} \to h$, we know that $h = h'$ and hence $\delta(h,h') = 0$. This implies that $p_I = q_{I'}$ and $p_J = q_{J'}$ by Theorem 5.2.2. Moreover,

$$\sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j = \sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j.$$

It follows that

$$\sum_{i\in I}\alpha_i p_i = U_{p_I}(\sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j) = U_{q_{I'}}(\sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j) = \sum_{i\in I'}\beta_i q_i$$

and

$$\sum_{j\in J}\alpha_j p_j = U_{p_J}(\sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j) = U_{q_{J'}}(\sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j) = \sum_{j\in J'}\beta_j q_j,$$

so that $\sum_{i\in I} e^{\alpha_i}p_i = \sum_{i\in I'} e^{\beta_i}q_i$ and $\sum_{j\in J} e^{\alpha_j}p_j = \sum_{j\in J'} e^{\beta_j}q_j$. We conclude that

$$\lim_{k\to\infty}\phi(y^{n_k}) = \lim_{k\to\infty}\frac{\sum_{i=1}^r (e^{-r^{n_k}+\lambda_i^{n_k}} - e^{-r^{n_k}-\lambda_i^{n_k}})q_i^{n_k}}{\sum_{i=1}^r (e^{-r^{n_k}+\lambda_i^{n_k}} + e^{-r^{n_k}-\lambda_i^{n_k}})} = \frac{\sum_{i\in I'} e^{-\beta_i}q_i - \sum_{j\in J'} e^{-\beta_j}q_j}{\sum_{i\in I'} e^{-\beta_i} + \sum_{j\in J'} e^{-\beta_j}} = \phi(h).$$

From Lemmas 5.5.2 and 5.5.5 we know that ϕ maps V into int B_{V^*} and $\partial\overline{V}^h$ into ∂B_{V^*}. So to complete the proof it remains to show that if (h_n) in $\partial\overline{V}^h$ converges to $h \in \partial\overline{V}^h$, then $\phi(h_n) \to \phi(h)$. Suppose h is given by (5.2.1) and for each n the horofunction h_n is given by

$$h_n(x) = \max\left\{\Lambda_{V(q^n_{I_n})}\left(-U_{q^n_{I_n}}x - \sum_{i\in I_n}\beta^n_i q^n_i\right), \Lambda_{V(q^n_{J_n})}\left(U_{q^n_{J_n}}x - \sum_{j\in J_n}\beta^n_j q^n_j\right)\right\} \quad \text{for } x\in V, \tag{5.5.1}$$

where $I_n, J_n \subseteq \{1,\ldots,r\}$ are disjoint, $I_n \cup J_n$ is nonempty, and $\min\{\beta^n_k \colon k \in I_n \cup J_n\} = 0$.

To prove the assertion we show that each subsequence of $(\phi(h_n))$ has a convergent subsequence with limit $\phi(h)$. Let $(\phi(h_{n_k}))$ be a subsequence. By taking subsequences we may assume that

1. There exist $I_0, J_0 \subseteq \{1,\ldots,r\}$ disjoint with $I_0 \cup J_0$ nonempty, such that $I_{n_k} = I_0$ and $J_{n_k} = J_0$ for all k.
2. $\beta^{n_k}_i \to \beta_i \in [0,\infty]$ and $q^{n_k}_i \to q_i$ for all $i \in I_0 \cup J_0$.
3. There exists $i^* \in I_0 \cup J_0$ such that $\beta^{n_k}_{i^*} = 0$ for all k.

Let $I' = \{i \in I_0 \colon \beta_i < \infty\}$ and $J' = \{j \in J_0 \colon \beta_j < \infty\}$, and note that $i^* \in I' \cup J'$.

Using Lemma 5.3.2 we now show that $h_{n_k} \to h'$, where

$$h'(x) = \max\left\{\Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i\in I'}\beta_i q_i\right), \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j\in J'}\beta_j q_j\right)\right\}. \tag{5.5.2}$$

Note that if I' is nonempty, then by Lemma 5.3.2 we have that

$$\lim_{k\to\infty}\Lambda_{V(q^{n_k}_{I_0})}\left(-U_{q^{n_k}_{I_0}}x - \sum_{i\in I_0}\beta^{n_k}_i q^{n_k}_i\right) = \Lambda_{V(q_{I'})}\left(-U_{q_{I'}}x - \sum_{i\in I'}\beta_i q_i\right),$$

as $U_{q^{n_k}_{I_0}}x \to U_{q_{I_0}}x$ by Lemma 5.3.1 and $U_{q_{I'}}(U_{q_{I_0}}x) = U_{q_{I'}}x$ by [4, Proposition 2.26]. Likewise if J' is nonempty, we have that

$$\lim_{k\to\infty}\Lambda_{V(q^{n_k}_{J_0})}\left(U_{q^{n_k}_{J_0}}x - \sum_{j\in J_0}\beta^{n_k}_j q^{n_k}_j\right) = \Lambda_{V(q_{J'})}\left(U_{q_{J'}}x - \sum_{j\in J'}\beta_j q_j\right).$$

Thus, if I' and J' are both nonempty (5.5.2) holds.

Now suppose that I' is empty, so J' is nonempty. As $-x \leq \|x\|_u u$, we get that

$$-U_{q_{I_0}^{n_k}} x \leq \|x\|_u U_{q_{I_0}^{n_k}} u = \|x\|_u q_{I_0}^{n_k}.$$

This implies that $-U_{q_{I_0}^{n_k}} x - \sum_{i \in I_0} \beta_i^{n_k} q_i^{n_k} \leq \sum_{i \in I_0} (\|x\|_u - \beta_i^{n_k}) q_i^{n_k}$, hence

$$\Lambda_{V(q_{I_0}^{n_k})} \left(-U_{q_{I_0}^{n_k}} x - \sum_{i \in I_0} \beta_i^{n_k} q_i^{n_k} \right) \leq \max_{i \in I_0} (\|x\|_u - \beta_i^{n_k}) \to -\infty.$$

On the other hand, $h_{n_k}(x) \geq -\|x\|_u$ for all k. Thus, for all k sufficiently large, we have that

$$h_{n_k}(x) = \Lambda_{V(q_{J_0}^{n_k})} \left(U_{q_{J_0}^{n_k}} x - \sum_{j \in J_0} \beta_j^{n_k} q_j^{n_k} \right),$$

which implies that (5.5.2) holds if I' is empty. In the same way it can be shown that (5.5.2) holds if J' is empty.

As $h_n \to h$, we know that $h' = h$, so $\delta(h, h') = 0$. It follows from Theorem 5.2.2 that $p_I = q_{I'}$, $p_J = q_{J'}$, and $\sum_{i \in I} \alpha_i p_i + \sum_{j \in J} \alpha_j p_j = \sum_{i \in I'} \beta_i q_i + \sum_{j \in J'} \beta_j q_j$. This implies that

$$\sum_{i \in I} \alpha_i p_i = \sum_{i \in I'} \beta_i q_i \quad \text{and} \quad \sum_{j \in J} \alpha_j p_j = \sum_{j \in J'} \beta_j q_j,$$

so that $\sum_{i \in I} e^{-\alpha_i} p_i = \sum_{i \in I'} e^{-\beta_i} q_i$ and $\sum_{j \in J} e^{-\alpha_j} p_j = \sum_{j \in J'} e^{-\beta_j} q_j$. Thus,

$$\lim_{k \to \infty} \phi(h_{n_k}) = \lim_{k \to \infty} \frac{\sum_{i \in I_0} e^{-\beta_i^{n_k}} q_i^{n_k} - \sum_{j \in J_0} e^{-\beta_j^{n_k}} q_j^{n_k}}{\sum_{i \in I_0} e^{-\beta_i^{n_k}} + \sum_{j \in J_0} e^{-\beta_j^{n_k}}} = \frac{\sum_{i \in I'} e^{-\beta_i} q_i - \sum_{j \in J'} e^{-\beta_j} q_j}{\sum_{i \in I'} e^{-\beta_i} + \sum_{j \in J'} e^{-\beta_j}} = \phi(h),$$

which completes the proof. □

Theorem 5.5.8. *The map* $\phi \colon \overline{V}^h \to B_{V^*}$ *is onto.*

Proof. From Lemma 5.5.4 we know that $\phi(V) = \operatorname{int} B_{V^*}$. Let $z \in \partial B_{V^*}$. As B_{V^*} is the disjoint union of the relative interiors of its faces, see [61, Theorem 18.2], we know that there exist orthogonal idempotents p_I and p_J such that $z \in \mathrm{relint} F_{p_I,p_J}$. Thus, we can write

$$z = \sum_{i\in I} \lambda_i p_i - \sum_{j\in J} \lambda_j p_j,$$

where $p_I = \sum_{i\in I} p_i$, $q_J = \sum_{j\in J} q_j$, $0 < \lambda_k \leq 1$ for all $k \in I \cup J$, and $\sum_{k\in I\cup J} \lambda_k = 1$.

Define $\mu_k = -\log \lambda_k$ for $k \in I \cup J$. So, $\mu_k \geq 0$. Let $\mu^* = \min\{\mu_k \colon k \in I \cup J\}$ and set $\alpha_k = \mu_k - \mu^* \geq 0$. Note that $\min\{\alpha_k \colon k \in I \cup J\} = 0$.

Then h, given by

$$h(x) = \max\left\{ \Lambda_{V(p_I)}\left(-U_{p_I} x - \sum_{i\in I} \alpha_i p_i \right), \Lambda_{V(p_J)}\left(U_{p_J} x - \sum_{j\in J} \alpha_j p_j \right) \right\}$$

for $x \in V$, is a horofunction by Proposition 5.3.5. Moreover,

$$\frac{1}{\sum_{i\in I} e^{-\mu_i} + \sum_{j\in J} e^{-\mu_j}} \left(\sum_{i\in I} e^{-\mu_i} p_i - \sum_{j\in J} e^{-\mu_j} p_j \right) = \frac{1}{\sum_{i\in I} \lambda_i + \sum_{j\in J} \lambda_j} \left(\sum_{i\in I} \lambda_i p_i - \sum_{j\in J} \lambda_j p_j \right),$$

hence $\phi(h) = z$, which completes the proof. □

The proof of Theorem 5.2.4 is now straightforward.

Proof of Theorem 5.2.4. It follows from Theorems 5.5.7 and 5.5.8 and Corollary 5.5.6 that $\phi \colon \overline{V}^h \to B_{V^*}$ is a continuous bijection. As $\overline{V}^h$ is compact and B_{V^*} is Hausdorff, we conclude that ϕ is a homeomorphism. It follows from Lemma 5.5.5 that ϕ maps each part onto the relative interior of a boundary face of B_{V^*}. □

Remark 5.5.9. It is interesting to note that a similar idea can be used to show that the horofunction compactification of a finite dimensional normed space $(V, \|\cdot\|)$ with a smooth and strictly convex norm is homeomorphic to the closed dual unit ball. Indeed, in that case the horofunctions are given by $h \colon z \mapsto -x^*(z)$, where $x^* \in V^*$ has norm 1, see for example [27, Lemma 5.3]. Moreover, for (y^n) in V we have that $h_{y^n} \to h$ if and only if $y^n/\|y^n\| \to x$ and $\|y^n\| \to \infty$.

In this case we define a map $\psi\colon \overline{V}^h \to B_{V^*}$ as follows. For $x \in V$ with $x \neq 0$, let

$$\psi(x) = -\left(\frac{e^{\|x\|} - e^{-\|x\|}}{e^{\|x\|} + e^{-\|x\|}}\right) x^*,$$

where $x^* \in V^*$ is the unique functional with $x^*(x) = \|x\|$ and $\|x^*\| = 1$, and let $\psi(0) = 0$. For $h \in \partial\overline{V}^h$ with $h\colon z \mapsto -x^*(z)$ let

$$\psi(h) = -x^*.$$

It is straightforward to check that ψ is a bijection from $\overline{V}^h$ onto B_{V^*}, and ψ is continuous on $\operatorname{int} B_{V^*}$. To show continuity on $\partial\overline{V}^h$, we assume, by way of contradiction, that (h_n) is a sequence of horofunctions with $h_n \to h$ and $h_n(z) = -x_n^*(z)$ for all $z \in V$, and there exists a neighbourhood U of $\psi(h)$ in B_{V^*} such that $\psi(h_n) \notin U$ for all n. Then, for each $z^* \in \partial B_{V^*}$ with $z^* \notin U$ we have that $z^*(x) < 1$. So, by compactness, $\delta = \max\{1 - z^*(x)\colon z^* \in \partial B_{V^*} \setminus U\} > 0$. It now follows that

$$h_n(x) - h(x) = -x_n^*(x) + x^*(x) = 1 - x_n^*(x) \geq \delta > 0$$

for all n, which contradicts $h_n \to h$. This shows that ψ is a continuous bijection, and hence a homeomorphism, as $\overline{V}^h$ is compact and B_{V^*} is Hausdorff.

More generally, one can consider product spaces $V = \prod_{i=1}^r V_i$ with norm $\|x\|_V = \max_{i=1}^r \|v_i\|_i$, where each $(V_i, \|\cdot\|_i)$ is a finite dimensional normed space with a smooth and strictly convex norm. In that case we have by [41, Theorem 2.10] that the horofunctions of V are given by

$$h(v) = \max_{j\in J}(h_{\xi_j^*}(v_j) - \alpha_j), \tag{5.5.3}$$

where $J \subseteq \{1, \ldots, r\}$ nonempty, $\min_{j\in J} \alpha_j = 0$, $\xi_j^* \in V_j^*$ with $\|\xi_j^*\| = 1$, and $h_{\xi_j^*}(v_j) = -\xi_j^*(v_j)$. One can use similar ideas as the ones in Section 3 to show that the horofunction compactification is homeomorphic to the closed unit dual ball of V. Indeed, one can define a map $\phi_V : \overline{V}^h \to B_{V^*}$ by

$$\phi_V(v) = \frac{1}{\sum_{i=1}^r e^{\|v_i\|_i} + e^{-\|v_i\|_i}} \left(\sum_{i=1}^r (e^{\|v_i\|_i} - e^{-\|v_i\|_i}) p(v_i^*) \right) \qquad \text{for } v \in V \setminus \{0\}$$

and $\phi_V(0) = 0$. Here $p(v_i^*) = (0, \ldots, 0, v_i^*, 0, \ldots, 0)$ and v_i^* is the unique functional such that $v_i^*(v_i) = \|v_i\|_i$ and $\|v_i^*\|_i = 1$ if $v_i \neq 0$, and we set $p(v_i^*) = 0$, if $v_i = 0$. For a horofunction h given by (5.5.3) we define

$$\phi_V(h) = \frac{1}{\sum_{j \in J} e^{-\alpha_j}} \left(\sum_{j \in J} e^{-\alpha_j} p(\xi_j^*) \right).$$

Following the same line of reasoning as in Section 3 one can prove that ϕ_V is a homeomorphism.

Remark 5.5.10. The connection between the geometry of the horofunction compactification and the dual unit ball seems hard to establish for general finite dimensional normed spaces, and might not even hold. For the normed spaces discussed in this chapter and in [32, 33] all horofunctions are Busemann points, but there are normed spaces with horofunctions that are not Busemann, see [66]. It could well be the case that the horofunction compactification of these spaces is not naturally homeomorphic to the closed dual unit ball, but no counter example is known at present.

Chapter 6

Hilbert Geometries

In this chapter we study the global topology and geometry of the horofunction compactification of symmetric cones under the Hilbert distance. Recall that the Hilbert distance is defined as follows. Let A be a real finite dimensional affine space. Consider a bounded, open, convex set $\Omega \subseteq A$. For $x, y \in \Omega$, let ℓ_{xy} be the straight-line through x and y in A, and denote the points of intersection of ℓ_{xy} and $\partial\Omega$ by x' and y', where x is between x' and y, and y is between x and y'. On Ω the *Hilbert distance* is then defined by

$$\rho_H(x, y) = \log\left(\frac{|x' - y|}{|x' - x|}\frac{|y' - x|}{|y' - y|}\right) \tag{6.0.1}$$

for all $x \neq y$ in Ω, and $\rho_H(x, x) = 0$ for all $x \in \Omega$. The metric space (Ω, ρ_H) is called the *Hilbert geometry* on Ω.

These metric spaces generalise Klein's model of hyperbolic space and have a Finsler structure, see [55, 57]. In our analysis we will work with Birkhoff's version of the Hilbert metric, which is defined on a cone in an order-unit space in terms of its partial ordering. This provides a convenient way to work with the Hilbert distance and its Finsler structure. In the next section we will recall the basic concepts involved in our analysis. Throughout we will follow the terminology used in [46, Chapter 2], which contains a detailed discussion of Hilbert geometries and some of their applications. We refer the reader to [57] for a comprehensive account of the theory of Hilbert geometries.

6.1 Preliminaries and Finsler structure

Let (V, V_+, u) be a finite dimensional order-unit space. So, V_+ is a closed cone in V with $u \in \operatorname{int} V_+$. Recall that the cone V_+ induces a partial ordering on V by $x \leq y$ if $y - x \in V_+$, see Section 4.1. For $x \in V$ and $y \in V_+$, we say that y *dominates* x if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha y \leq x \leq \beta y$. In that case, we write

$$M(x/y) = \inf\{\beta \in \mathbb{R} \colon x \leq \beta y\} \quad \text{and} \quad m(x/y) = \sup\{\alpha \in \mathbb{R} \colon \alpha y \leq x\}.$$

By the Hahn-Banach theorem, $x \leq y$ if and only if $\psi(x) \leq \psi(y)$ for all $\psi \in V_+^* = \{\phi \in V^* \colon \phi \text{ positive}\}$, which is equivalent to $\psi(x) \leq \psi(y)$ for all $\psi \in S(V)$. Using this fact we see that for each $x \in V$ and $y \in \operatorname{int} V_+$,

$$M(x/y) = \sup_{\psi \in S(V)} \frac{\psi(x)}{\psi(y)} \quad \text{and} \quad m(x/y) = \inf_{\psi \in S(V)} \frac{\psi(x)}{\psi(y)}.$$

We also note that if $A \in \mathrm{GL}(V)$ is a linear automorphism of V_+, i.e., $A(V_+) = V_+$, then $x \leq \beta y$ if, and only if, $Ax \leq \beta Ay$. It follows that $M(Ax/Ay) = M(x/y)$ and $m(x/y) = m(Ax/Ay)$.

If $w \in \operatorname{int} V_+$, then w dominates each $x \in V$, and we define

$$|x|_w = M(x/w) - m(x/w).$$

One can verify that $|\cdot|_w$ is a semi-norm on V, see [46, Lemma A.1.1], and a genuine norm on the quotient space $V/\mathbb{R}w$, as $|x|_w = 0$ if and only if $x = \lambda w$ for some $\lambda \in \mathbb{R}$.

Clearly, if $x, y \in V$ are such that $y = 0$ and y dominates x, then $x = 0$, as V_+ is a cone. On the other hand, if $y \in V_+ \setminus \{0\}$, and y dominates x, then $M(x/y) \geq m(x/y)$. The domination relation yields an equivalence relation on V_+ by $x \sim y$ if y dominates x and x

dominates y. The equivalence classes are called the *parts* of V_+. As V_+ is closed, one can check that $\{0\}$ and $\operatorname{int} V_+$ are parts of V_+. The parts of a finite dimensional cone are closely related to its faces. Indeed, if V_+ is the cone of a finite dimensional order-unit space, then it can be shown that the parts correspond to the relative interiors of the faces of V_+, see [46, Lemma 1.2.2]. Recall that a *face* of a convex set $S \subseteq V$ is a subset F of S with the property that if $x, y \in S$ and $\lambda x + (1-\lambda)y \in F$ for some $0 < \lambda < 1$, then $x, y \in F$.

It is easy to verify that if $x, y \in V_+ \setminus \{0\}$, then $x \sim y$ if, and only if, there exist $0 < \alpha \leq \beta$ such that $\alpha y \leq x \leq \beta y$. Furthermore, if $x \sim y$, then

$$m(x/y) = \sup\{\alpha > 0 \colon y \leq \alpha^{-1}x\} = M(y/x)^{-1}. \tag{6.1.1}$$

Birkhoff's version of the Hilbert distance on V_+ is defined as follows:

$$d_H(x, y) = \log\left(\frac{M(x/y)}{m(x/y)}\right) = \log M(x/y) + \log M(y/x) \tag{6.1.2}$$

for all $x \sim y$ with $y \neq 0$, $d_H(0, 0) = 0$, and $d_H(x, y) = \infty$ otherwise.

Note that $d_H(\lambda x, \mu y) = d_H(x, y)$ for all $x, y \in V_+$ and $\lambda, \mu > 0$, so d_H is not a distance on V_+. It is, however, a distance between pairs of rays in each part of V_+. In particular, if $\phi\colon V \to \mathbb{R}$ is a linear functional such that $\phi(x) > 0$ for all $x \in V_+ \setminus \{0\}$, then d_H is a distance on

$$\Omega_V = \{x \in \operatorname{int} V_+ \colon \phi(x) = 1\},$$

which is a (relatively) open, bounded, convex set, see [46, Lemma 1.2.4]. Moreover, the following holds, see [46, Proposition 2.1.1 and Theorem 2.1.2].

Theorem 6.1.1. (Ω_V, d_H) *is a metric space and* $d_H = \rho_H$ *on* Ω_V*.*

It is worth noting that any Hilbert geometry can be realised as (Ω_V, d_H) for some order-unit space V and strictly positive linear functional ϕ.

A Hilbert geometry (Ω_V, d_H) has a Finsler structure, with Finsler metric $F : T\Omega_V \to [0, \infty)$ given by $F(w, x) = |x|_w$ for $w \in \Omega_V$ and $x \in T_w$, see [55]. It should be noted that in the case of Hilbert geometries the unit ball $\{x \in V/\mathbb{R}w \colon |x|_w \leq 1\}$ in the tangent space at $w \in \Omega_V$ may have a different facial structure for different w. This phenomenon appears frequently in the case where Ω_V is a polytope, but does not appear in the Hilbert geometries we will be considering here.

For the remainder of this chapter, let (V, V_+, u) be an order-unit space, where V is a Euclidean Jordan algebra of rank r, V_+ is the cone of squares, and u is the algebraic unit. So, $\operatorname{int} V_+$ is a symmetric cone and $\operatorname{Isom}(\Omega_V)$ acts transitively on Ω_V.

Throughout we will take $\phi \colon V \to \mathbb{R}$ with $\phi(x) = \frac{1}{r}\operatorname{tr}(x)$, which is a state, and

$$\Omega_V = \{x \in \operatorname{int} V_+ \colon \phi(x) = 1\} = \{x \in \operatorname{int} V_+ \colon \operatorname{tr}(x) = r\}.$$

In this setting we shall call (Ω_V, d_H) a *symmetric* Hilbert geometry. A prime example is

$$\Omega_V = \{A \in \operatorname{Herm}_n(\mathbb{C}) \colon \operatorname{tr}(A) = n \text{ and } A \text{ positive definite}\}.$$

These spaces are important examples of noncompact type symmetric spaces with an invariant Finsler metric, see [59]. In particular, the example above corresponds to the symmetric space $\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$. For completeness, we now provide a proof that symmetric Hilbert geometries have a Finsler structure.

Proposition 6.1.2. *If (Ω_V, d_H) is a symmetric Hilbert geometry, then for each $x, y \in \Omega_V$ we have that $d_H(x, y) = \inf L(\gamma)$, where the infimum is taken over all piecewise C^1-smooth paths $\gamma \colon [0,1] \to \Omega_V$ with $\gamma(0) = x$ and $\gamma(1) = y$, and*

$$L(\gamma) = \int_0^1 |\gamma'(t)|_{\gamma(t)} \mathrm{d}t.$$

Proof. Let $\gamma\colon [0,1]\to\Omega_V$ be a piecewise C^1-path with $\gamma(0)=x$ and $\gamma(1)=y$. We have

$$\begin{aligned}
d_H(x,y) &= \log M(y/x)-\log m(y/x)\\
&= \max_{\psi\in S(V)}\log\frac{\psi(y)}{\psi(x)}-\min_{\psi\in S(V)}\log\frac{\psi(y)}{\psi(x)}\\
&= \max_{\psi\in S(V)}\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}\log\psi(\gamma(t))\mathrm{d}t-\min_{\psi\in S(V)}\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}\log\psi(\gamma(t))\mathrm{d}t\\
&= \max_{\psi\in S(V)}\int_0^1\frac{\psi(\gamma'(t))}{\psi(\gamma(t))}\mathrm{d}t-\min_{\psi\in S(V)}\int_0^1\frac{\psi(\gamma'(t))}{\psi(\gamma(t))}\mathrm{d}t\\
&\leq \int_0^1\max_{\psi\in S(V)}\frac{\psi(\gamma'(t))}{\psi(\gamma(t))}\mathrm{d}t-\int_0^1\min_{\psi\in S(V)}\frac{\psi(\gamma'(t))}{\psi(\gamma(t))}\mathrm{d}t\\
&= \int_0^1 M(\gamma'(t)/\gamma(t))-m(\gamma'(t)/\gamma(t))\mathrm{d}t\\
&= \int_0^1|\gamma'(t)|_{\gamma(t)}\mathrm{d}t.
\end{aligned}$$

Now let $x,y\in\Omega_V$ and consider the C^1-smooth path σ in C° given by,

$$\sigma(t)=U_{x^{1/2}}(U_{x^{-1/2}}y)^t\quad\text{for } 0\leq t\leq 1.$$

Note that $\sigma(0)=U_{x^{1/2}}u=x$ and $\sigma(1)=y$. Define

$$\mu(t)=\frac{\sigma(t)}{\phi(\sigma(t))}\quad\text{for all } 0\leq t\leq 1.$$

So, μ is a C^1-smooth path connecting x and y in Ω_V. A direct calculation gives

$$\mu'(t)=\frac{\sigma'(t)}{\phi(\sigma(t))}-\frac{\phi(\sigma'(t))}{\phi(\sigma(t))^2}\sigma(t)\quad\text{for } 0\leq t\leq 1.$$

We also have that $U_{\mu(t)^{-1/2}}=\phi(\sigma(t))U_{\sigma(t)^{-1/2}}$ for $0\leq t\leq 1$, which implies

$$U_{\mu(t)^{-1/2}}\mu'(t)=U_{\sigma(t)^{-1/2}}\sigma'(t)-\frac{\phi(\sigma'(t))}{\phi(\sigma(t))}u. \tag{6.1.3}$$

Furthermore

$$\sigma'(t) = U_{x^{1/2}}((U_{x^{-1/2}}y)^t \log(U_{x^{-1/2}}y)) \quad \text{for } 0 \leq t \leq 1.$$

Write $z = U_{x^{-1/2}}y$ and let $z = \sum_{i=1}^r \lambda_i p_i$ be the spectral decomposition of z. As $z \in \operatorname{int} V_+$, we have that $z^t = \sum_{i=1}^r \lambda_i^t p_i$ and $\log z = \sum_{i=1}^r (\log \lambda_i) p_i$, and hence

$$z^t \log z = \sum_{i=1}^r (\lambda_i^t \log \lambda_i) p_i. \quad \text{and} \quad U_{z^{-t/2}}(z^t \log z) = \log z.$$

From (6.1.3) we get that

$$\begin{aligned} M(\mu'(t)/\mu(t)) - m(\mu'(t)/\mu(t)) &= M(U_{\mu(t)^{-1/2}}\mu'(t)/u) - m(U_{\mu(t)^{-1/2}}\mu'(t)/u) \\ &= M(U_{\sigma(t)^{-1/2}}\sigma'(t)/u) - m(U_{\sigma(t)^{-1/2}}\sigma'(t)/u). \end{aligned}$$

It follows that

$$\begin{aligned} M(\mu'(t)/\mu(t)) - m(\mu'(t)/\mu(t)) &= M(\sigma'(t)/\sigma(t)) - m(\sigma'(t)/\sigma(t)) \\ &= M(U_{x^{-1/2}}\sigma'(t)/U_{x^{-1/2}}\sigma(t)) - m(U_{x^{-1/2}}\sigma'(t)/U_{x^{-1/2}}\sigma(t)) \\ &= M(z^t \log z/z^t) - m(z^t \log z/z^t) \\ &= M(\log z/u) - m(\log z/u) \\ &= \log M(U_{x^{-1/2}}y/u) - \log m(U_{x^{-1/2}}y/u) \\ &= \log M(y/x) - \log m(y/x). \end{aligned}$$

We conclude that

$$L(\mu) = \int_0^1 \log M(y/x) - \log m(y/x) \mathrm{d}t = d_H(x, y),$$

which completes the proof. □

In a symmetric Hilbert geometry the distance can be expressed in terms of the spectrum. Indeed, we know that for $x \in V$ invertible, the quadratic representation $U_x\colon V \to V$ is a linear automorphism of V_+, see [20, Proposition III.2.2]. Moreover, $U_x^{-1} = U_{x^{-1}}$ and $U_{x^{-1/2}}x = u$. Furthermore, for $x \in V$ we have that

$$M(x/u) = \inf\{\lambda\colon x \leq \lambda u\} = \max \sigma(x) \quad \text{and} \quad m(x/u) = \sup\{\lambda\colon \lambda u \leq x\} = \min \sigma(x),$$

so that $|x|_u = \max \sigma(x) - \min \sigma(x)$. Also for $x, y \in \operatorname{int} V_+$ we have that

$$\log M(x/y) = \max \sigma(\log U_{y^{-1/2}}x) \quad \text{and} \quad \log M(y/x) = -\min \sigma(\log U_{y^{-1/2}}x).$$

It follows that

$$d_H(x,y) = \log M(x/y) + \log M(y/x) = |\log U_{y^{-1/2}}x|_u = \operatorname{diam} \sigma(\log U_{y^{-1/2}}x) \quad \text{for all } x, y \in \operatorname{int} V_+.$$

Moreover, for each $w \in \Omega_V$ we have that

$$|x|_w = M(x/w) - m(x/w) = M(U_{w^{-1/2}}x/u) - m(U_{w^{-1/2}}x/u) = |U_{w^{-1/2}}x|_u \quad \text{for all } x \in V,$$

which shows that the facial structure of the unit ball in each tangent space is identical, as $U_{w^{-1/2}}$ is an invertible linear map.

6.2 Horofunctions of symmetric Hilbert geometries

The main objective is to show for symmetric Hilbert geometries (Ω_V, d_H) that there exists a natural homeomorphism between $\overline{\Omega}_V^h$ and the closed dual unit ball of the Finsler metric $|\cdot|_u$ in the tangent space $V/\mathbb{R}u$ at the unit u. To describe the homeomorphism, we recall the description of the horofunction compactification of symmetric Hilbert geometries given in [44, Theorem 5.6].

Theorem 6.2.1. *The horofunctions of a symmetric Hilbert geometry (Ω_V, d_H) are precisely the functions $h\colon \Omega_V \to \mathbb{R}$ of the form*

$$h(x) = \log M(y/x) + \log M(z/x^{-1}) \quad \text{for } x \in \Omega_V, \tag{6.2.1}$$

where $y, z \in \partial V_+$ are such that $\|y\|_u = \|z\|_u = 1$ and $(y|z) = 0$.

It follows from the proof of [44, Theorem 5.6] that all horofunctions are in fact Busemann points. Indeed, if y and z have spectral decompositions

$$y = \sum_{i\in I} \lambda_i p_i \quad \text{and} \quad z = \sum_{j\in J} \mu_j p_j,$$

where $I, J \subset \{1, \dots, r\}$ are nonempty and disjoint, and $p_1, \dots, p_r$ is a Jordan frame, then the sequence $(y_n) \in \operatorname{int} V_+$ given by

$$y_n = \sum_{i\in I} \lambda_i p_i + \sum_{j\in J} \frac{1}{n^2\mu_j} p_j + \sum_{k\notin I\cup J} \frac{1}{n} p_k$$

has the property that $y_n \to y$, $y_n^{-1}/\|y_n^{-1}\|_u \to z$ and $h_{y_n} \to h$, where h is as in (6.2.1). Note that if we let $v_n = y_n/\phi(y_n) \in \Omega_V$, then $h_{v_n}(z) = h_{y_n}(z)$ for all $z \in \Omega_V$, so $h_{v_n} \to h$.

Also note that for $n, m \geq 1$,

$$U_{y_n^{-1/2}} y_m = \sum_{i\in I} p_i + \sum_{j\in J} \frac{n^2}{m^2} p_j + \sum_{k\notin I\cup J} \frac{n}{m} p_k.$$

This implies that for each $n \geq m \geq 1$,

$$M(y_m/y_n) = M(U_{y_n^{-1/2}} y_m/u) = \|U_{y_n^{-1/2}} y_m\|_u = n^2/m^2,$$

so that $\log M(y_m/y_n) = 2\log n - 2\log m$. Moreover, $\log M(y_n/y_m) = \log 1 = 0$ for all

$n \geq m \geq 1$. It follows that

$$d_H(v_n, v_m) + d_H(v_m, v_1) = d_H(y_n, y_m) + d_H(y_m, y_1) = d_H(y_n, y_1) = d_H(v_n, v_1)$$

for all $n \geq m \geq 1$. Thus, (v_n) is an almost geodesic sequence in Ω_V, and hence each horofunction in $\overline{\Omega}_V^h$ is a Busemann point.

Before we identify the parts in $\partial\overline{\Omega}_V^h$ and the detour distance, it is useful to recall the following fact:

$$M(x/y) = M(y^{-1}/x^{-1}) \quad \text{for all } x, y \in \operatorname{int} V_+,$$

if $\operatorname{int} V_+$ is a symmetric cone, see [48, Section 2.4].

Proposition 6.2.2. *Let (Ω_V, d_H) be a symmetric Hilbert geometry and $h, h' \in \partial\overline{\Omega}_V^h$ with*

$$h(x) = \log M(y/x) + \log M(z/x^{-1}) \quad \textit{and} \quad h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$$

for $x \in \Omega_V$. The following assertions hold:

1. *h and h' are in the same part if and only if $y \sim y'$ and $z \sim z'$.*

2. *If h and h' are in the same part, then $\delta(h, h') = d_H(y, y') + d_H(z, z')$.*

Proof. Consider the spectral decompositions: $y = \sum_{i\in I} \lambda_i p_i$, $z = \sum_{j\in J} \mu_j p_j$, $y' = \sum_{i\in I'} \alpha_i q_i$, and $z' = \sum_{j\in J'} \beta_j q_j$. Set

$$y_n = \sum_{i\in I} \lambda_i p_i + \sum_{j\in J} \frac{1}{n^2\mu_j} p_j + \sum_{k\notin I\cup J} \frac{1}{n} p_k \quad \text{and} \quad w_n = \sum_{i\in I'} \alpha_i q_i + \sum_{j\in J'} \frac{1}{n^2\beta_j} q_j + \sum_{k\notin I'\cup J'} \frac{1}{n} q_k.$$

Then $h_{y_n} \to h$ and $h_{w_n} \to h'$ by the proof of [44, Theorem 5.6].

For all $n \geq 1$ large we have that $\|w_n\|_u = \|y'\|_u = 1$, so that

$$d_H(w_n, u) = \log M(w_n/u) + \log M(u/w_n) = \log \|w_n\|_u + \log M(w_n^{-1}/u) = \log \|w_n^{-1}\|_u.$$

Now set $v_n = w_n^{-1}/\|w_n^{-1}\|_u$ and note that by Lemma 3.2.5,

$$\begin{aligned} H(h',h) &= \lim_{n\to\infty} d_H(w_n,u) + h(w_n) \\ &= \lim_{n\to\infty} \log\|w_n^{-1}\|_u + \log M(y/w_n) + \log M(z/w_n^{-1}) \\ &= \lim_{n\to\infty} \log M(y/w_n) + \log M(z/v_n^{-1}). \end{aligned}$$

Clearly $w_{n+1} \leq w_n$ and $w_n \to y'$. Also,

$$w_n^{-1} = \sum_{i\in I'} \alpha_i^{-1} q_i + \sum_{j\in J'} n^2 \beta_j q_j + \sum_{k\notin I'\cup J'} n q_k.$$

So, for all $n \geq 1$ large, we have that $\|w_n^{-1}\|_u = n^2$, as $\max_{j\in J} \beta_j = \|z'\|_u = 1$. It follows that

$$v_n = \sum_{i\in I'} \frac{1}{n^2\alpha_i} q_i + \sum_{j\in J'} \beta_j q_j + \sum_{k\notin I'\cup J'} \frac{1}{n} q_k$$

for all $n \geq 1$ large. So, $v_{n+1} \leq v_n$ for all $n \geq 1$ large and $v_n \to z'$. It now follows from Lemma 2.7.9 that $H(h',h) = \infty$ if y' does not dominate y, or, z' does not dominate z. Moreover, if y' dominates y, and, z' dominates z, then $H(h',h) = \log M(y/y') + \log M(z/z')$.

Interchanging the roles between h and h' we find that $H(h,h') = \infty$ if y does not dominate y', or, z does not dominate z', and $H(h,h') = \log M(y'/y) + \log M(z'/z)$, otherwise. Thus, $\delta(h,h') = d_H(y,y') + d_H(z,z')$ if and only if $y \sim y'$ and $z \sim z'$, and $\delta(h,h') = \infty$ otherwise. □

The characterisation of the parts and the detour distance is a more explicit description of the general one one given in [49, Theorem 4.9] in the case of symmetric Hilbert geometries.

6.3 The homeomorphism

Let us now define a map $\varphi_H\colon \overline{\Omega}_V^h \to B_1^*$, where B_1^* is the unit ball of the dual norm of $|\cdot|_u$ on $V/\mathbb{R}u$. For $x \in \Omega_V$ let

$$\varphi_H(x) = \frac{x}{\mathrm{tr}(x)} - \frac{x^{-1}}{\mathrm{tr}(x^{-1})},$$

and for $h \in \partial\overline{\Omega}_V^h$ given by (6.2.1) let

$$\varphi_H(h) = \frac{y}{\mathrm{tr}(y)} - \frac{z}{\mathrm{tr}(z)}.$$

We note that $\varphi_H(h)$ is well-defined by Proposition 6.2.2.

We will prove the following theorem in the sequel.

Theorem 6.3.1. *If (Ω_V, d_H) is a symmetric Hilbert geometry, then the map $\varphi_H\colon \overline{\Omega}_V^h \to B_1^*$ is a homeomorphism which maps each part of $\partial\overline{\Omega}_V^h$ onto the relative interior of a boundary face of B_1^*.*

We first analyse the dual unit ball B_1^* of $|\cdot|_u$ and its facial structure. The following fact, which can be found in [48, Section 2.2], will be useful.

Lemma 6.3.2. *Given an order-unit space (V, V_+, u), the norm $|\cdot|_u$ on $V/\mathbb{R}u$ coincides with the quotient norm of $2\|\cdot\|_u$ on $V/\mathbb{R}u$.*

Recall that in a Euclidean Jordan algebra V each x has a unique orthogonal decomposition $x = x^+ - x^-$, where x^+ and x^- are orthogonal elements in V_+, see [4, Proposition 1.28]. Let

$$\mathbb{R}u^\perp = \{x \in V\colon (u|x) = 0\} = \{x \in V\colon \mathrm{tr}(x^+) = \mathrm{tr}(x^-)\}.$$

It follows from Lemma 6.3.2 that

$$(V/\mathbb{R}u, |\cdot|_u)^* = (\mathbb{R}u^\perp, \frac{1}{2}\|\cdot\|_u^*).$$

So the dual unit ball B_1^* in $\mathbb{R}u^\perp$ is given by

$$B_1^* = 2\text{conv}(S(V) \cup -S(V)) \cap \mathbb{R}u^\perp,$$

see [3, Theorem 1.19], and its (closed) boundary faces are precisely the nonempty sets of the form

$$A_{p,q} = 2\text{conv}\left((U_p(V) \cap S(V)) \cup (U_q(V) \cap -S(V))\right) \cap \mathbb{R}u^\perp,$$

where p and q are orthogonal idempotents, see [18, Theorem 4.4].

To prove Theorem 6.3.1 we collect a number of preliminary results.

Lemma 6.3.3. *For each $x \in \Omega_V$ we have that $\varphi_H(x) \in \text{int}\, B_1^*$, and for each $h \in \partial\overline{\Omega}_V^h$ we have that $\varphi_H(h) \in \partial B_1^*$.*

Proof. Let $x = \sum_{i=1}^r \lambda_i p_i \in \Omega_V$, so $\lambda_i > 0$ for all i. Note that $(u|\varphi_H(x)) = 1 - 1 = 0$ and hence $\varphi_H(x) \in \mathbb{R}u^\perp$. Given $-u \leq z \leq u$, we have the Peirce decomposition of z with respect to the frame $p_1, \ldots, p_r$,

$$z = \sum_{i=1}^r \sigma_i p_i + \sum_{i<j} z_{ij},$$

with $-1 = -(u|p_i) \leq \sigma_i = (z|p_i) \leq (u|p_i) = 1$. As this is an orthogonal decomposition we have that

$$(z|\varphi_H(x)) = \sum_{i=1}^r \sigma_i \left(\frac{\lambda_i}{\sum_{j=1}^r \lambda_j} - \frac{\lambda_i^{-1}}{\sum_{j=1}^r \lambda_j^{-1}} \right) < \sum_{i=1}^r \left(\frac{\lambda_i}{\sum_{j=1}^r \lambda_j} \right) + \sum_{i=1}^r \left(\frac{\lambda_i^{-1}}{\sum_{j=1}^r \lambda_j^{-1}} \right) = 2.$$

This implies that $\frac{1}{2}\|\varphi_H(x)\|_u^* = \frac{1}{2}\sup_{-u \leq z \leq u}(z|\varphi_H(x)) < 1$, hence $\varphi_H(x) \in \text{int}\, B_1^*$.

To prove the second assertion let h be a horofunction given by $h(x) = \log M(y/x) + \log M(z/x^{-1})$, where $\|y\|_u = \|z\|_u = 1$ and $(y|z) = 0$. Write $y = \sum_{i \in I} \alpha_i q_i$ and $z = \sum_{j \in J} \beta_j q_j$. If we now let $q_I = \sum_{i \in I} q_i$ and $q_J = \sum_{j \in J} q_j$, then $-u \leq q_I - q_J \leq u$ and

$$\|\varphi_H(h)\|_u^* \geq \frac{1}{2}(q_I - q_J|\varphi_H(h)) = (1+1)/2 = 1.$$

Moreover, for each $-u \leq w \leq u$ we have that

$$|(w|\varphi_H(h))| \leq |(w|y/\mathrm{tr}(y))| + |(w|z/\mathrm{tr}(z))| \leq (u|y/\mathrm{tr}(y)) + (u|z/\mathrm{tr}(z)) = 2.$$

Combining the inequalities shows that $\varphi_H(h) \in \partial B_1^*$. $\square$

To prove injectivity of φ_H on Ω_V we need the following lemma, which has a proof similar to the one of Lemma 4.2.2 given in [32, Section 4].

Lemma 6.3.4. *Let $\mu_i\colon \mathbb{R}^r \to \mathbb{R}$, for $i = 1, 2$, be given by $\mu_1(x) = \sum_{i=1}^r e^{x_i}$ and $\mu_2(x) = \sum_{i=1}^r e^{-x_i}$ for $x \in \mathbb{R}^r$, and let $g\colon x \mapsto \log \mu_1(x) + \log \mu_2(x)$. If $x, y \in \mathbb{R}^r$ are such that $y \neq x + c(1, \dots, 1)$ for all $c \in \mathbb{R}$, then $\nabla g(x) \neq \nabla g(y)$.*

Lemma 6.3.5. *The map φ_H is injective on Ω_V.*

Proof. Suppose that $\varphi_H(x) = \varphi_H(y)$, where $x = \sum_{i=1}^r \lambda_i p_i$ and $y = \sum_{i=1}^r \mu_i q_i$ in Ω_V. Note that $0 < \lambda_i, \mu_i$ for all i and $(x|u) = \mathrm{tr}(x) = r = \mathrm{tr}(y) = (y|u)$. After possibly relabelling we can write

$$\varphi_H(x) = \sum_{i=1}^r \left(\frac{\lambda_i}{\sum_{j=1}^r \lambda_j} - \frac{\lambda_i^{-1}}{\sum_{j=1}^r \lambda_j^{-1}} \right) p_i = \sum_{i=1}^r \alpha_i p_i$$

and

$$\varphi_H(y) = \sum_{i=1}^r \left(\frac{\mu_i}{\sum_{j=1}^r \mu_j} - \frac{\mu_i^{-1}}{\sum_{j=1}^r \mu_j^{-1}} \right) q_i = \sum_{i=1}^r \beta_i q_i,$$

where $\alpha_1 \leq \ldots \leq \alpha_r$ and $\beta_1 \leq \ldots \leq \beta_r$. By the spectral theorem (version 2) [20] we conclude that $\alpha_i = \beta_i$ for all i.

Consider the injective map $\mathrm{Log}\colon \mathrm{int}\,\mathbb{R}^r_+ \to \mathbb{R}^r$ given by $\mathrm{Log}(\gamma) = (\log \gamma_1, \dots, \log \gamma_r)$. Let $\Delta = \{\gamma \in \mathrm{int}\,\mathbb{R}^r_+ \colon \sum_{i=1}^r \gamma_i = r\}$. The map $(\nabla g) \circ \mathrm{Log}$ is injective on Δ by Lemma 6.3.4 and

$$\nabla g(\mathrm{Log}(\gamma)) = \left(\frac{\gamma_1}{\sum_{i=1}^r \gamma_i} - \frac{\gamma_1^{-1}}{\sum_{i=1}^r \gamma_i^{-1}}, \dots, \frac{\gamma_r}{\sum_{i=1}^r \gamma_i} - \frac{\gamma_r^{-1}}{\sum_{i=1}^r \gamma_i^{-1}} \right).$$

Writing $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_r)$, we have that $\lambda, \mu \in \Delta$ and

$$\nabla g(\operatorname{Log}(\lambda)) = (\alpha_1, \dots, \alpha_r) = (\beta_1, \dots, \beta_r) = \nabla g(\operatorname{Log}(\mu)),$$

so that $\lambda = \mu$.

As $(\nabla g) \circ \operatorname{Log}$ is injective on Δ, we also know that $\alpha_k = \alpha_{k+1}$ if and only if $\lambda_k = \lambda_{k+1}$. Likewise, $\beta_k = \beta_{k+1}$ if and only if $\mu_k = \mu_{k+1}$. From the spectral theorem (version 1) [20] we now conclude that $x = y$. □

In the next couple of lemmas we show that φ_H is onto.

Lemma 6.3.6. *The map φ_H maps Ω_V onto* $\operatorname{int} B_1^*$.

Proof. Note that Ω_V is an open set of the affine space $\{x \in V : \operatorname{tr}(x) = r\}$, which has dimension $\dim V - 1$. Also $B_1^* \subset \mathbb{R}u^\perp$ has dimension $\dim V - 1$. As φ_H is a continuous injection from Ω_V into $\operatorname{int} B_1^*$ by Lemmas 6.3.3 and 6.3.5, we know that $\varphi_H(\Omega_V)$ is a open subset of $\operatorname{int} B_1^*$ by Brouwer's invariance of domain theorem. We now argue by contradiction. So, suppose that $\varphi_H(\Omega_V) \neq \operatorname{int} B_1^*$. There then exists a $w \in \partial\varphi_H(\Omega_V) \cap \operatorname{int} B_1^*$. Let (v_n) in Ω_V be such that $\varphi_H(v_n) \to w$.

As φ_H is continuous on Ω_V, we may assume that $d_H(v_n, u) \to \infty$. After taking a subsequence, we may also assume that $v_n \to v \in \partial\Omega_V$. Now let $y_n = v_n/\|v_n\|_u$ and set $y = v/\|v\|_u$. Furthermore, let $z_n = y_n^{-1}/\|y_n^{-1}\|_u$. After taking subsequences we may assume that $z_n \to z \in \partial V_+$ and $y_n \to y \in \partial V_+$, so $\|y\|_u = \|z\|_u = 1$. As $y_n \bullet z_n = u/\|y_n^{-1}\|_u \to 0$, we find that $y \bullet z = 0$, which implies that $(y|z) = 0$.

Using the spectral decomposition we write $y_n = \sum_{i=1}^r \lambda_i^n p_i^n$ and $y = \sum_{i \in I} \lambda_i p_i$, where $\lambda_i > 0$ for all $i \in I$. Likewise, we let $z_n = \sum_{i=1}^r \mu_i^n p_i^n$ and $z = \sum_{j \in J} \mu_j p_j$ with $\mu_j > 0$ for all $j \in J$. Note that $\mu_i^n = (\lambda_i^n)^{-1}/\|y_n^{-1}\|_u$.

Then

$$\begin{aligned}\varphi_H(v_n) &= \frac{\sum_{i=1}^r \lambda_i^n p_i^n}{\sum_{k=1}^r \lambda_k^n} - \frac{\sum_{i=1}^r (\lambda_i^n)^{-1} p_i^n}{\sum_{k=1}^r (\lambda_k^n)^{-1}} = \frac{\sum_{i=1}^r \lambda_i^n p_i^n}{\sum_{k=1}^r \lambda_k^n} - \frac{\sum_{i=1}^r \mu_i^n p_i^n}{\sum_{k=1}^r \mu_k^n} \\ &\to \frac{\sum_{i\in I} \lambda_i p_i}{\sum_{k\in I} \lambda_k} - \frac{\sum_{j\in J} \mu_j p_j}{\sum_{k\in J} \mu_j} = w.\end{aligned}$$

Now let $w^* = \sum_{i\in I} p_i - \sum_{j\in J} p_j$ and note that $-u \leq w^* \leq u$, as $(y|z) = 0$. We find that

$$\frac{1}{2}\|w\|_u^* \geq \frac{1}{2}(w|w^*) = (1+1)/2 = 1,$$

hence $w \in \partial B_1^*$, which is a contradiction. □

Lemma 6.3.7. *The map φ_H maps $\partial\overline{\Omega}_V^h$ onto ∂B_1^*.*

Proof. We know from Lemma 6.3.3 that φ_H maps $\partial\overline{\Omega}_V^h$ into ∂B_1^*. To prove that it is onto let $w \in \partial B_1^*$. Then there exists a face, say

$$A_{p,q} = 2\mathrm{conv}\left(\left(U_p(V) \cap S(V)\right) \cup \left(U_q(V) \cap -S(V)\right)\right) \cap \mathbb{R}u^\perp$$

where p and q are orthogonal idempotents, such that w is in the relative interior of $A_{p,q}$, as B_1^* is the disjoint union of the relative interiors of its faces [61, Theorem 18.2]. So,

$$w = \sum_{i\in I} \alpha_i p_i - \sum_{j\in J} \beta_j q_j,$$

where $\alpha_i > 0$ for all $i \in I$, $\beta_j > 0$ for all $j \in J$, and $\sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j = 2$. Moreover, $\sum_{i\in I} p_i = p$ and $\sum_{j\in J} q_j = q$.

As $w \in \mathbb{R}u^\perp$, we have that $0 = (u|w) = \sum_{i\in I} \alpha_i - \sum_{j\in J} \beta_j$, hence $\sum_{i\in I} \alpha_i = \sum_{j\in J} \beta_j = 1$.

Put $\alpha^* = \max_{i\in I} \alpha_i$ and $\beta^* = \max_{j\in J} \beta_j$. Furthermore, for $i \in I$ set $\lambda_i = \alpha_i/\alpha^*$ and for

$j \in J$ set $\mu_j = \beta_j/\beta^*$. Then

$$w = \left(\frac{\sum_{i\in I}\alpha_i p_i}{\sum_{k\in I}\alpha_k}\right) - \left(\frac{\sum_{j\in J}\beta_j q_j}{\sum_{k\in J}\beta_k}\right) = \left(\frac{\sum_{i\in I}\lambda_i p_i}{\sum_{k\in I}\lambda_k}\right) - \left(\frac{\sum_{j\in J}\mu_j q_j}{\sum_{k\in J}\mu_k}\right).$$

Note that $0 < \lambda_i \leq 1$ for all $i \in I$ and $\max_{i\in I}\lambda_i = 1$. Likewise, $0 < \mu_j \leq 1$ for all $j \in J$ and $\max_{j\in J}\mu_j = 1$.

Now let $y = \sum_{i\in I}\lambda_i p_i$ and $z = \sum_{j\in J}\mu_j q_j$. Then $\|y\|_u = \|z\|_u = 1$ and $(y|z) = 0$. Furthermore, if we let $h\colon \Omega_V \to \mathbb{R}$ be given by

$$h(x) = \log M(y/x) + \log M(z/x^{-1})$$

for $x \in \Omega_V$, then h is a horofunction by Theorem 6.2.1 and $\varphi_H(h) = w$, which completes the proof. $\square$

We already saw in Lemma 6.3.6 that φ_H is injective on Ω_V. The next lemma shows that φ_H is injective on $\overline{\Omega}_V^h$.

Lemma 6.3.8. *The map $\varphi_H\colon \overline{\Omega}_V^h \to B_1^*$ is injective.*

Proof. We know from Lemma 6.3.6 that φ_H is injective on Ω_V. So, it remains to show that if $h, h' \in \partial\overline{\Omega}_V^h$ and $\varphi_H(h) = \varphi_H(h')$, then $h = h'$.

Suppose $h(x) = \log M(y/x) + \log M(z/x^{-1})$ and $h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$ for all $x \in \Omega_V$. Then

$$\varphi_H(h) = \frac{y}{\mathrm{tr}(y)} - \frac{z}{\mathrm{tr}(z)} = \frac{y'}{\mathrm{tr}(y')} - \frac{z'}{\mathrm{tr}(z')} = \varphi_H(h').$$

Using the fact that the orthogonal decomposition of an element in V is unique, see [4, Proposition 1.26], we conclude that

$$\frac{y}{\mathrm{tr}(y)} = \frac{y'}{\mathrm{tr}(y')} \quad \text{and} \quad \frac{z}{\mathrm{tr}(z)} = \frac{z'}{\mathrm{tr}(z')}.$$

As $\|y\|_u = \|y'\|_u = 1$, we get that $\mathrm{tr}(y) = \mathrm{tr}(y')$, and hence $y = y'$. Likewise, $\|z\|_u = \|z'\|_u = 1$ implies that $z = z'$, hence $h = h'$. □

6.4 Proof of Theorem 6.3.1

Before we prove Theorem 6.3.1, we recall a fact from Jordan theory. For $x, z \in V$ let $[x, z] = \{y \in V : x \leq y \leq z\}$ be the order-interval. Given $y \in V_+$ we write

$$\mathrm{face}(y) = \{x \in V_+ : x \leq \lambda y \text{ for some } \lambda \geq 0\}.$$

In a Euclidean Jordan algebra V every idempotent p satisfies

$$\mathrm{face}(p) \cap [0, u] = [0, p],$$

see [4, Lemma 1.39]. Also note that $y \sim y'$ if and only if $\mathrm{face}(y) = \mathrm{face}(y')$.

Proof of Theorem 6.3.1. We know from the results in the previous section that $\varphi_H : \overline{\Omega}_V^h \to B_1^*$ is a bijection, which is continuous on Ω_V.

To prove continuity of φ_H on the whole of $\overline{\Omega}_V^h$ we first show that if (v_n) in Ω_V is such that $h_{v_n} \to h \in \partial\overline{\Omega}_V^h$, then $\varphi_H(v_n) \to \varphi_H(h)$. Let $h(x) = \log M(y/x) + \log M(z/x^{-1})$ for $x \in \Omega_V$, where $\|y\|_u = \|z\|_u = 1$ and $(y|z) = 0$. For $n \geq 1$ let $y_n = v_n/\|v_n\|_u$ and note that $\varphi_H(v_n) = \varphi_H(y_n)$ for all n. Let $w_k = \varphi_H(v_{n_k})$, $k \geq 1$ be a subsequence of $(\varphi_H(v_n))$. We need to show that (w_k) has a subsequence that converges to $\varphi_H(h)$.

As h is a horofunction and (Ω_V, d_H) is a proper metric space, $d_H(v_n, u) = d_H(y_n, u) \to \infty$ by Lemma 3.1.5. It follows that (y_{n_k}) has a subsequence (y_{k_m}) with $y_{k_m} \to y' \in \partial V_+$ and $z_{k_m} = y_{k_m}^{-1}/\|y_{k_m}^{-1}\|_u \to z' \in V_+$. Note that as $y \in \partial V_+$, we have that $\|y_{k_m}^{-1}\|_u \to \infty$. This implies that

$$y' \bullet z' = \lim_{m\to\infty} y_{k_m} \bullet \frac{y_{k_m}^{-1}}{\|y_{k_m}^{-1}\|_u} = \lim_{m\to\infty} \frac{u}{\|y_{k_m}^{-1}\|_u} = 0,$$

hence $(y'|z') = 0$ (see [20, III, Exercise 3.3]) and $z' \in \partial V_+$. For $x \in \Omega_V$,

$$\begin{aligned}
\lim_{m\to\infty} h_{y_{k_m}}(x) &= \lim_{m\to\infty} \log M(y_{k_m}/x) + \log M(x/y_{k_m}) - \log M(y_{k_m}/u) - \log M(u/y_{k_m}) \\
&= \lim_{m\to\infty} \log M(y_{k_m}/x) + \log M(y_{k_m}^{-1}/x^{-1}) - \log \|y_{k_m}^{-1}\|_u \\
&= \lim_{m\to\infty} \log M(y_{k_m}/x) + \log M(z_{k_m}/x^{-1}) \\
&= \log M(y'/x) + \log M(z'/x^{-1}).
\end{aligned}$$

So, if we let $h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$, then h' is a horofunction by Theorem 6.2.1 and $h_{y_{k_m}} \to h'$. As $h = h'$, we know that $\delta(h, h') = d_H(y, y') + d_H(z, z') = 0$, hence $y = y'$ and $z = z'$. It follows that

$$\varphi_H(v_{k_m}) = \varphi_H(y_{k_m}) = \frac{y_{k_m}}{\mathrm{tr}(y_{k_m})} - \frac{y_{k_m}^{-1}}{\mathrm{tr}(y_{k_m}^{-1})} \to \frac{y}{\mathrm{tr}(y)} - \frac{z}{\mathrm{tr}(z)} = \varphi_H(h).$$

Recall that φ_H maps Ω_V into int B_1^* and φ_H maps $\partial\overline{\Omega}_V^h$ into ∂B_1^* by Lemma 6.3.3. So, to prove the continuity of φ_H it remains to show that if (h_n) is a sequence in $\partial\overline{\Omega}_V^h$ converging to $h \in \partial\overline{\Omega}_V^h$, then $\varphi_H(h_n) \to \varphi_H(h)$.

Let $(\varphi_H(h_{n_k}))$ be a subsequence of $(\varphi_H(h_n))$. We show that it has a subsequence $(\varphi_H(h_{k_m}))$ converging to $\varphi_H(h)$. We know there exists $v_m, w_m \in \partial V_+$, with $\|v_m\|_u = \|w_m\|_u = 1$ and $(v_m|w_m) = 0$ such that

$$h_{k_m}(x) = \log M(v_m/x) + \log M(w_m/x^{-1})$$

for $x \in \Omega_V$. By taking a further subsequence we may assume that $v_m \to v \in \partial V_+$ and $w_m \to w \in \partial V_+$. Then $\|v\|_u = \|w\|_u = 1$ and $(v|w) = 0$. Moreover,

$$\log M(v_m/x) \to \log M(v/x) \quad \text{and} \quad \log M(w_m/x^{-1}) \to \log M(w/x^{-1})$$

for each $x \in \Omega_V$, as $y \mapsto M(y/x)$ is continuous on V, see [44, Lemma 2.2]. Thus, $h_{k_m} \to$

$h^* \in \partial\overline{\Omega}_V^h$, where

$$h^*(x) = \log M(v/x) + \log M(w/x^{-1}),$$

by Theorem 6.2.1. As $h_n \to h$, we have that $h = h^*$. This implies that $y = v$ and $z = w$ by Proposition 6.2.2. Thus, $v_m \to y$ and $w_m \to z$, hence

$$\varphi_H(h_{k_m}) = \frac{v_m}{\operatorname{tr}(v_m)} - \frac{w_m}{\operatorname{tr}(w_m)} \to \frac{y}{\operatorname{tr}(y)} - \frac{z}{\operatorname{tr}(z)} = \varphi_H(h).$$

This completes the proof of the continuity of φ_H.

As $\varphi_H \colon \overline{\Omega}_V^h \to B_1^*$ is a continuous bijection, $\overline{\Omega}_V^h$ is compact, and B_1^* is Hausdorff, we conclude that φ_H is a homeomorphism.

To prove the second assertion let $h(x) = \log M(y/x) + \log M(z/x^{-1})$ be a horofunction, where $y = \sum_{i\in I}\lambda_i p_i$ and $z = \sum_{j\in J}\mu_j p_j$ with $\lambda_i, \mu_j > 0$ for all $i \in I$ and $j \in J$. Let $p_I = \sum_{i\in I} p_i$ and $p_J = \sum_{j\in J} p_j$. As φ_H is surjective, it suffices to show that φ_H maps $\mathcal{P}_h$ into the relative interior of

$$A_{p_I,p_J} = 2\operatorname{conv}\left((U_{p_I}(V) \cap S(V)) \cup (U_{p_J}(V) \cap -S(V))\right) \cap \mathbb{R}u^\perp.$$

So, let $h' \in \mathcal{P}_h$ where $h'(x) = \log M(y'/x) + \log M(z'/x^{-1})$ for $x \in \Omega_V$. Then $p_I \sim y \sim y'$ and $p_J \sim z \sim z'$. Using the spectral decomposition write $y' = \sum_{i\in I'}\alpha_i q_i$ and $z' = \sum_{j\in J'}\beta_j q_j$, where $\alpha_i > 0$ for all $i \in I'$ and $\beta_j > 0$ for all $j \in J'$. Now let $q_{I'} = \sum_{i\in I'} q_i$ and $q_{J'} = \sum_{j\in J'} q_j$. It follows that $p_I \sim q_{I'}$ and $p_J \sim q_{J'}$. So, $\operatorname{face}(p_I) = \operatorname{face}(q_{I'})$ and $\operatorname{face}(p_J) = \operatorname{face}(q_{J'})$. As $\operatorname{face}(p_I) \cap [0,u] = [0,p_I]$ and $\operatorname{face}(q_{I'}) \cap [0,u] = [0,q_{I'}]$ by [4, Lemma 1.39], we conclude that $p_I = q_{I'}$. In the same way we get that $p_J = q_{J'}$. As $\alpha_i > 0$ for all $i \in I'$ and $\beta_j > 0$ for all $j \in J'$, we have that

$$\varphi_H(h') = \frac{y'}{\operatorname{tr}(y')} - \frac{z'}{\operatorname{tr}(z')}$$

is in the relative interior of $A_{q_{I'},q_{J'}} = A_{p_I,p_J}$. □

Chapter 7

Infinite Dimensional Normed Spaces

This chapter is a direct follow-up to the work of Gutiérrez in [27], where he provides a nice classification of the horofunctions of arbitrary infinite dimensional ℓ^p spaces, for $1 \leq p < \infty$. He does not study the global geometry and topology of the horofunction compactifications of these spaces, which is the goal of this chapter. We briefly recall the definition of arbitrary ℓ^p spaces. Fix $p \in [1, \infty)$, and a set J of arbitrary cardinality. We can equip J with its powerset as a σ-algebra. We can then equip the measurable space $(J, 2^J)$ with the counting measure $\mu \colon 2^J \to [0, \infty]$, defined by

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

The triple $(J, 2^J, \mu)$ is then a measure space, and we define $\ell^p(J)$ to be $L^p(J, 2^J, \mu)$, the space of all functions $f \colon J \to \mathbb{R}$ such that

$$\int_J |f|^p d\mu < \infty,$$

and we equip $\ell^p(J)$ with the standard L^p norm

$$\|f\| = \left(\int_J |f|^p d\mu \right)^{1/p}.$$

If $J = \mathbb{N}$, this definition means that $\ell^p(\mathbb{N})$ is simply the more familiar space of p summable sequences with norm

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$

If J is uncountable, it follows from this definition of $\ell^p(J)$ that $f \in \ell^p(J)$ if and only if there exists a countable $I \subseteq J$ such that $f(x) = 0$ for all $x \in J \backslash I$, and $\int_J |f|^p d\mu < \infty$. In this case [15, Example 1.9]

$$\|x\|^p = \sup \left\{ \sum_{i \in F} |x_i|^p : F \subseteq I \text{ is finite} \right\}.$$

This justifies the notation, for any $x \in \ell^p(J)$,

$$\|x\| = \left(\sum_{i \in J} |x_i|^p \right)^{1/p},$$

where the sum over J is understood to be taken as the supremum of the sum over finite subsets of J.

The space $\ell^2(J)$ is an important example of a Hilbert space, and in Section 7.1 we study the topology and geometry of the horofunction compactification of arbitrary infinite dimensional Hilbert spaces. For $1 < p < \infty$, ℓ^p spaces are important examples of uniformly convex and uniformly smooth normed spaces (see Section 2.6 for a more detailed discussion of smoothness and convexity), and in Section 7.2 we study the geometry and topology of the horofunction compactification of uniformly smooth and strictly convex infinite dimensional Banach spaces. We also include a subsection specifically on ℓ^p spaces, as more can be said in that setting. Finally, Section 7.3 deals with the study of the horofunction compactification of $\ell^1(J)$. The ℓ^1 norm is not uniformly convex or smooth, and the results are strikingly different.

7.1 Infinite Dimensional Hilbert Spaces

Let $\mathcal{H}$ be an infinite-dimensional real Hilbert space with norm $\|\cdot\|$ arising from the inner product. Recall from Example 2.6.8 that $\mathcal{H}$ is uniformly smooth and convex. In [27, Section 4], Gutiérrez explicitly calculated the horofunction compactification of $\mathcal{H}$, which we can divide into two parts- The so called exotic metric functionals $\partial\overline{\mathcal{H}}^{h,e}$, consisting of horofunctions that are bounded below, and $\partial\overline{\mathcal{H}}^{h,\infty}$, consisting of metric functionals which are unbounded below:

$$\begin{aligned} \partial\overline{\mathcal{H}}^{h,e} &= \{h^{z,c} : z \in \mathcal{H},\ c > \|z\|\} \cup \{h^0\} \\ \partial\overline{\mathcal{H}}^{h,\infty} &= \{h^z : 0 < \|z\| \leq 1\}, \end{aligned} \tag{7.1.1}$$

where for all $x \in \mathcal{H}$,

$$h^{z,c}(x) = \sqrt{\|x\|^2 - 2\langle x, z\rangle + c^2} - c, \text{ and, } h^z(x) = -\langle x, z\rangle. \tag{7.1.2}$$

It is interesting to note that for finite dimensional Hilbert spaces, the horofunction boundary consists entirely of functions h^z, where z has norm 1 (see Section 3.3.2).

Remark 7.1.1. Recall that Kapovich and Leeb defined the stratification of the horofunction boundary of a metric space (M, d) to be its partition under the equivalence relation $\sim$, where $h \sim h'$ if and only if $\sup_{x\in M} |h(x) - h'(x)| < \infty$. In [34] they asked the normed space version of Question 1.0.1: whether the horofunction compactification of any finite dimensional normed space is homeomorphic to the dual unit ball, under a homeomorphism which bijectively associates equivalence classes in the stratification with the relative interiors of faces of the dual unit ball. Gutiérrez's classification shows that this is impossible in the infinite dimensional Hilbert space case. Indeed, Lemma 7.1.2 proved below shows that, for any $z \in \mathcal{H}$ and $c > \|z\|$, the equivalence class in the stratification of the horoboundary containing $h^{z,c}$ also contains $h^{z,d}$ for all $d > \|z\|$. Proposition 2.6.7 means that the only faces of $B_{\mathcal{H}}$ are extreme points. It is thus impossible for a homeomorphism mapping $\overline{\mathcal{H}}^h$ onto $B_{\mathcal{H}}$ to map the equivalence class $[h^{z,c}]$ into a single face of the dual ball.

Lemma 7.1.2. *Fix $z \in \mathcal{H}$ and $c > d > \|z\|$. Then $\sup_{x\in\mathcal{H}} |h^{z,c}(x) - h^{z,d}(x)| < \infty$, where $h^{z,c}, h^{z,d} \in \partial\overline{\mathcal{H}}^{h,e}$ are as in* (7.1.2).

Proof. The Cauchy-Schwarz inequality shows that $\|x\|^2 - 2\langle x, z\rangle \geq \|x\|^2 - 2\|x\|\|z\|$ for all $x \in \mathcal{H}$, where equality is achieved if and only if $x = \alpha z$ for $\alpha \geq 0$. Thus, for all $x \in \mathcal{H}$,

$$\|x\|^2 - 2\langle x, z\rangle \geq \inf_{\alpha\geq 0} \|z\|^2(\alpha^2 - 2\alpha).$$

From this we deduce that

$$\|x\|^2 - 2\langle x, z\rangle \geq -\|z\|^2, \tag{7.1.3}$$

for all $x \in \mathcal{H}$. Let us now define the function $f\colon [-d^2, \infty) \to \mathbb{R}$ by

$$f(t) = \sqrt{t + c^2} - \sqrt{t + d^2}.$$

As $c > d$, a direct calculation shows that $f(-d^2) = \sqrt{c^2 - d^2}$, $f(t) \geq 0$ for all $t \geq -d^2$, and $f'(t) < 0$ for all $t > -d^2$. From this we conclude that $\sup_{t\geq -d^2} |f(t)| = \sqrt{c^2 - d^2}$. Combining this with (7.1.3) and the fact that $\|z\| < d < c$ allows us to conclude that

$$\sup_{x\in\mathcal{H}} |h^{z,c}(x) - h^{z,d}(x)| \leq \sqrt{c^2 - d^2} + c - d < \infty.$$

□

We can still investigate the geometry of the Busemann points of $\overline{\mathcal{H}}^h$. Corollary 3.2.4 immediately implies that all Busemann points lie in $\partial\overline{\mathcal{H}}^{h,\infty}$, as all elements in $\partial\overline{\mathcal{H}}^{h,e}$ are bounded below.

Proposition 7.1.3. *The Busemann points are precisely those elements* $h^z \in \partial\overline{\mathcal{H}}^{h,\infty}$ *where* $\|z\| = 1$.

Proof. In the proof of Theorem 4.4 in[27], Guttiérez shows that for any $h^z \in \partial\overline{\mathcal{H}}^{h,\infty}$ the sequence (h_{y_n}) converges to h^z, where (y_n) is defined by

$$y_n = \sqrt{1 - \|z\|^2} u_n + nz,$$

where (u_n) is an orthonormal sequence weakly converging to 0. If $\|z\| = 1$ the horofunction h^z is thus the limit of the almost-geodesic sequence (nz), so it is Busemann point.

Conversely, assume that $h = h^z \in \partial\overline{\mathcal{H}}^{h,\infty}$ is a Busemann point, with converging almost-geodesic net (x_α). Lemma 3.2.3 means that (x_α) must be unbounded, else (x_α) would contain a bounded almost-geodesic subnet converging to an internal metric functional, a contradiction. The proof of Lemma 4.3 in [27] means that, by taking subnets if necessary, we can assume $x_\alpha/\|x_\alpha\| \xrightarrow{w} z$. Suppose, by way of a contradiction, that $\|z\| < 1$. Fix some $\varepsilon > 0$. As (x_α) is an almost-geodesic net there exists an α' so that for all $\alpha \geq \beta \geq \alpha'$ we have

$$\|x_\alpha - x_\beta\| + \|x_\beta\| - \|x_\alpha\| < \varepsilon.$$

As this is true for all $\alpha \geq \beta$ we can take the limit in α to find

$$-\langle x_\beta, z\rangle + \|x_\beta\| < \varepsilon,$$

which by the Cauchy-Schwarz inequality means that

$$\|x_\beta\|(1 - \|z\|) < \varepsilon$$

for all $\beta \geq \alpha'$. However, $\|x_\beta\| \to \infty$, which results in a contradiction when $\|z\| < 1$. □

Proposition 7.1.4. *The Busemann boundary consists entirely of singleton parts.*

Proof. Fix Busemann points $h^z, h^w \in \partial\overline{\mathcal{H}}^{h,\infty}$, with $z \neq w$. By Proposition 7.1.3 we know $\|z\| = \|w\| = 1$. We know that the sequences (nz) and (nw) are geodesics converging to the respective Busemann points h^z and h^w. Lemma 3.2.5 means we can calculate the detour distance using these geodesics:

$$\begin{aligned}\delta(h^z, h^w) &= \lim_{n\to\infty} n + h^w(nz) + \lim_{m\to\infty} m + h^z(mw)\\ &= \lim_{n\to\infty} n(1 - \langle w, z\rangle) + \lim_{m\to\infty} m(1 - \langle w, z\rangle).\end{aligned}$$

As $\mathcal{H}$ is a Hilbert space it has unique norming functionals, meaning $\langle w, z\rangle < 1$ when w, z both lie on the unit sphere and are distinct. Thus $\delta(h^z, h^w) = \infty$. □

As in the case of finite dimensional smooth spaces (Section 3.3.2), we would like to homeomorphically identify $\partial\overline{\mathcal{H}}^h$ with the closed dual unit ball. However, in infinite dimensions the closed dual unit ball is not compact with respect to the norm topology, so we must consider it equipped with the weak topology. Another issue that arises, is the fact that there is no natural way to include the exotic metric functionals in the domain of the candidate homeomorphism. We can, however, prove a weaker result. Let us define the map $g\colon \mathcal{H} \cup \partial_B\overline{\mathcal{H}}^h \to B_{\mathcal{H}}$ by

$$g(x) = -\tanh(\|x\|)\left\langle \cdot, \frac{x}{\|x\|}\right\rangle \text{ for } x \in \mathcal{H}\backslash\{0\}, \quad g(0) = 0, \quad \text{and } g(h) = h \text{ for } h \in \partial_B\overline{\mathcal{H}}^h. \tag{7.1.4}$$

As $\mathcal{H}$ is self-dual by the Riesz Representation Theorem, $\langle \cdot, x/\|x\|\rangle = x^*$ for all $x \in \mathcal{H}\backslash\{0\}$, so this is equivalent to the definition of g in Section 3.3.2.

Theorem 7.1.5. *The map g defined above is a continuous bijection from $\mathcal{H} \cup \partial_B\overline{\mathcal{H}}^h$ to the dual unit ball of $\mathcal{H}$ equipped with the weak topology. Furthermore, $g|_{\partial_B\overline{\mathcal{H}}^h}$ is a homeomorphism onto $S_{\mathcal{H}^*}$, and the homeomorphism maps parts of the boundary to extreme points of the dual ball.*

We will prove Theorem 7.1.5 via a series of lemmas.

Lemma 7.1.6. *The map* $g\colon \mathcal{H} \cup \partial_B \overline{\mathcal{H}}^h \to B_{\mathcal{H}^*}$ *defined above is a bijection onto* $B_{\mathcal{H}^*}$*.*

Proof. The proof that g is injective is an exact replica of the corresponding part of the proof of Lemma 3.3.6, as it does not rely on finite dimensionality. We need to adjust that proof somewhat to prove surjectivity of g. Fix some $0 \neq f \in \operatorname{int}(B^*_{\mathcal{H}})$. We know that $f = \langle \cdot, z\rangle$ for $z \in \operatorname{int}(B_{\mathcal{H}})$. Set $y = z/\|z\| \in S_{\mathcal{H}}$. The function $\tanh\colon \mathbb{R} \to (-1, 1)$ is bijective, so there exists a $c > 0$ such that $\tanh(c) = \|z\|$, meaning $\tanh(\|cy\|) = \|z\|$. Therefore, $g(-cy) = \|z\| \langle \cdot, z/\|z\|\rangle = f$. Thus $g|_{\mathcal{H}}$ is a bijection onto $B^\circ_{\mathcal{H}^*}$, meaning $g\colon \mathcal{H} \cup \partial_B \overline{\mathcal{H}}^h \to B_{\mathcal{H}^*}$ is bijective, because g is a bijection on the boundary by its definition. □

Lemma 7.1.7. *If* $h_{x_\alpha} \to h_x$ *in* $\overline{\mathcal{H}}^h$ *for* $(x_\alpha) \subseteq \mathcal{H}$ *and* $x \in \mathcal{H}$*, then* $\|x - x_\alpha\| \to 0$*.*

Proof. The net (x_α) must be eventually bounded, else, as in the proof of [27, Lemma 4.3], there will exist a subnet such that $h_{x_\alpha} \to h^z$ for some $z \in B_{\mathcal{H}}$, a contradiction, because h^z is linear whereas h_x isn't. Therefore (x_α) must contain a bounded subnet, so by the Banach-Alaoglu theorem, there exists a subnet (x_β) converging weakly to some $y \in \mathcal{H}$, with $\|y\| \leq \liminf \|x_\beta\| = c$. By passing to a further subnet we can assume that $c = \lim_\beta \|x_\beta\|$. Thus, for any $z \in \mathcal{H}$, by the proof of [27, Lemma 4.1],

$$\lim_\beta h_{x_\beta}(z) = \sqrt{c^2 - 2\langle z, y\rangle + \|z\|^2} - c.$$

By assumption we thus have, for all $z \in \mathcal{H}$,

$$h^{y,c}(z) = \sqrt{c^2 - 2\langle z, y\rangle + \|z\|^2} - c = \|x - z\| - \|x\| = h_x(z).$$

We now note that, by the Cauchy-Schwarz inequality, for any $z \in \mathcal{H}$,

$$\sqrt{c^2 - 2\langle z, y\rangle + \|z\|^2} \geq \sqrt{c^2 - 2\|z\|\|y\| + \|z\|^2},$$

where equality holds if and only if $z = \alpha y$ for some $\alpha \in \mathbb{R}$. Furthermore, for any $\alpha \in \mathbb{R}$ we see that

$$h^{y,c}(\alpha z) = \sqrt{c^2 + \|y\|^2(\alpha^2 - 2\alpha)} - c.$$

which is at its global minimum when $\alpha = 1$. From this we conclude that

$$\min h^{y,c} = \sqrt{c^2 - \|y\|^2} - c.$$

If two functions are equal, their minimums must be equal too, so we immediately deduce that $\|x\| = c - \sqrt{c^2 - \|y\|^2}$. Furthermore, $h^{y,c}(y) = h_x(y)$, so

$$\|y - x\| = \sqrt{c^2 - \|y\|^2} - c + \|x\| = 0,$$

meaning $y = x$. Once again using the fact that $h^{y,c}(y) = h_x(y)$ along with the fact that $x = y$, we must have that

$$\sqrt{(c - \|y\|)(c + \|y\|)} = c - \|y\|.$$

Thus, if $c \neq \|y\|$, $c + \|y\| = c - \|y\|$, from which we deduce that $y = 0 = x$. This means that $h_x = h_0$, and that $h^{y,c}(z) = \sqrt{c^2 + \|z\|^2} - c$ for all $z \in \mathcal{H}$. Therefore, for all $z \in \mathcal{H}$,

$$\sqrt{c^2 + \|z\|^2} = \|z\| + c,$$

which is only possible if $c = 0$. Thus, in all cases, $c = \|y\| = \|x\|$, from which we deduce that

$$\lim_\beta \|x_\beta - x\|^2 = c^2 - 2\|x\|^2 + \|x\|^2 = 0.$$

This argument can be used to show that every subnet of (x_α) has a further subnet converging to x in norm, which proves the lemma. □

The proof of the above lemma can be simply adapted to prove the following:

Lemma 7.1.8. *Two exotic metric functionals, $h^{x,c}$ and $h^{x',c'}$, are equal if and only if $x = x'$*

and $c = c'$.

Proof. Just as in the proof of the above lemma, $h^{x,c}$ and $h^{x',c'}$, are equal if and only if they achieve their minimum at the same point, and exactly as above we see that the minimum of $h^{x,c}$ is achieved only at x, (because $\|x\| < c$), and similarly the minimum of $h^{x',c'}$ is achieved only at x'. Thus $x = x'$, which forces $c' = c$. $\square$

The proof of Lemma 7.1.7 also proves the following:

Corollary 7.1.9. *A function of the form $h^{x,c}$ is equal to an internal point if and only if $\|x\|_{\mathcal{H}} = c$, in which case $h^{x,c} = h_x$.*

We are now in a position to prove the continuity of g.

Lemma 7.1.10. *The function g is continuous when $B_{\mathcal{H}^*}$ is equipped with the weak topology.*

Proof. First we consider a net $(h_{x_\alpha}) \subseteq \mathcal{H}$ converging to some $h_x \in \mathcal{H}$. Lemma 7.1.7 means that $x_\alpha \to x$. From this we conclude that $x^*_\alpha \to x^*$ in the weak topology (where we use the convention that $0^* = 0$). Indeed, this follows immediately from applying the Cauchy-Schwarz inequality to the fact that, for any $y \in \mathcal{H}$,

$$|x^*_\alpha - x^*|(y) = \frac{1}{\|x_\alpha\|\|x\|} \langle y, \|x\|x_\alpha - \|x_\alpha\|x \rangle .$$

As $\|x_\alpha\| \to \|x\|$, the continuity of tanh means that $\tanh(\|x_\alpha\|) \to \tanh(\|x\|)$ in $\mathbb{R}$, so $g(x_\alpha) \to g(x)$.

Now consider some $(h_{x_\alpha}) \subseteq \mathcal{H}$ converging to some $h^z \in \partial_B \overline{\mathcal{H}}^h$. The proof of [27, Lemma 4.3] implies (x_α) is unbounded, because otherwise it would possess a bounded subnet converging to a function bounded below, a contradiction. By the Banach-Alaoglu theorem and the self-duality of Hilbert spaces there exists a subnet $(x_\beta/\|x_\beta\|)$ converging in the weak topology to some x in the dual unit ball. Furthermore, because $\|x_\beta\| \to \infty$, $\tanh(\|x_\beta\|) \to 1$. Thus

$$\lim_\beta g(h_{x_\beta}) = -\langle \cdot, x \rangle .$$

If we use the Taylor expansion of the real function $\lambda \mapsto \sqrt{1+\lambda}$, we get, as in the proof of [27, Lemma 4.3], that for any $y \in \mathcal{H}$ and large enough β,

$$h_{x_\beta}(y) = \frac{\|y\|^2}{2\|x_\beta\|} - \left\langle \frac{x_\beta}{\|x_\beta\|}, y \right\rangle + O(\frac{1}{\|x_\beta\|}),$$

from which we see that h_{x_β} converges pointwise to $-\langle \cdot, x\rangle$, meaning that $x = z$ by uniqueness of limits. This argument shows that all subnets of $(g(x_\alpha))$ must have a further subnet converging to $g(h^z)$.

Suppose $(h_\alpha) \subseteq \partial_B \overline{\mathcal{H}}^h$ converges to some $h \in \mathcal{H} \cup \partial_B \overline{\mathcal{H}}^h$. It is immediate from the definition of g that $g(h_\alpha) \to g(h)$ in the weak topology, because the weak topology on $B_{\mathcal{H}}$ is equal to the topology of pointwise convergence on $B_{\mathcal{H}}$. Finally consider an arbitrary net $(h_\alpha) \subseteq \mathcal{H} \cup \partial_B \overline{\mathcal{H}}^h$, converging to some $h \in \mathcal{H} \cup \partial_B \overline{\mathcal{H}}^h$. If $h = h_x$ for some $x \neq 0$, and (h_α) has a subnet (h_β) such that $h_\beta = h^{z_\beta}$ for all β in some tail, then $h(2x) = 0$ and $h(-2x) = 2\|x\|$, meaning that $\lim_\beta -2\langle x, z_\beta\rangle > 0$ but $\lim_\beta 2\langle x, z_\beta\rangle = 0$, an impossibility. If $x = 0$ then $h_0 \geq 0$, leading to similar sign contradictions. Thus in this case we must have that all subnets of (h_α) are eventually internal metric functionals. We have then already proven that $g(h_\alpha) \to g(h)$ above. If $h \in \partial_B \overline{\mathcal{H}}^h$, then every subnet of (h_α) must have a further subnet, say (h_β), lying entirely in $\mathcal{H}$ or in $\partial_\beta \overline{\mathcal{H}}^h$, but in either case we have already proven that $g(h_\beta) \to g(h)$. Thus we can conclude that g is continuous. □

Lemmas 7.1.10 and 7.1.6 show that g is a continuous bijection between $\mathcal{H} \cup \partial_B \overline{\mathcal{H}}^h$. Furthermore, because the topology of pointwise convergence on linear functionals on $\mathcal{H}$ is equal to the weak topology on $\mathcal{H}^* = \mathcal{H}$, it is clear from definition that $g^{-1}|S_{\mathcal{H}^*}$ is continuous, which proves Theorem 7.1.5.

Proposition 7.1.11. *The closure of $\partial\overline{\mathcal{H}}^{h,\infty}$ intersected with $\partial\overline{\mathcal{H}}^{h,e}$ is equal to the singleton $\{h^0\}$.*

Proof. Consider some $h \in \mathrm{cl}(\partial\overline{\mathcal{H}}^{h,\infty})$, and corresponding net $(h_\alpha) \subseteq \partial\overline{\mathcal{H}}^{h,\infty}$ converging to h, with $h_\alpha = h^{z_\alpha}$ for $z_\alpha \in B_{\mathcal{H}}$. Suppose by way of contradiction that $h = h^{y,c}$. Take a fixed

$x \in y^\perp$ with $\|x\|$ much larger than c. Then $h(x) > 0$. Thus $\lim_\alpha h_\alpha(x) > 0$, which means that no subnet of (z_α) can converge weakly to 0. There thus exists, by Banach-Alaoglu, some non-zero $z \in \mathcal{H}$ and a subnet (z_β) converging weakly to z. As $z \neq 0$ there exists some $u \in \mathcal{H}$ with $\langle u, z \rangle > 2c$. Thus $\lim_k h_\beta(u) = -\langle u, z \rangle < -2c$. However, $h^{y,c}(u) \geq -c$, contradicting the assumed convergence. Thus if $(h_\alpha) \subseteq \partial\overline{\mathcal{H}}^{h,\infty}$ converges to $h \in \partial\overline{\mathcal{H}}^{h,e}$, we must have $h = h^0$. If (e_n) is an orthonormal sequence in $\mathcal{H}$ converging weakly to 0, then $h^{e_n}(y) = -\langle e_n, y \rangle$ converges to 0 for every $y \in \mathcal{H}$. Thus, indeed, h^0 is in the closure of $\partial\overline{\mathcal{H}}^{h,\infty}$. □

Proposition 7.1.11 means that there cannot exist a homeomorphism from the whole horofunction compactification onto $B_\mathcal{H}$ equipped with the weak topology, which maps Busemann points bijectively onto the unit sphere, because $S_\mathcal{H}$ is dense in $B_\mathcal{H}$ under the weak topology.

7.2 Uniformly Smooth and Strictly Convex Banach Spaces

In this section we fix a uniformly smooth and strictly convex real Banach space $(X, \|\cdot\|)$. We recall from Section 2.6 that any $x \in X\backslash\{0\}$ has a unique norming functional, which we denote by $x^* \in S_{X^*}$. We know that X is reflexive by Theorem 2.6.6, which means that the weak topology on X^* is equal to the weak* topology on X^*. We also know that X^* is uniformly convex by Theorem 2.6.5, and smooth by 2.6.4. The classic example to keep in mind in this section is $\ell^p(\mathbb{N})$ for $1 < p < \infty$. This space is actually uniformly convex, which our proofs do not require. The following result is [27, Lemma 5.3], but we include a slightly different proof for convenience.

Lemma 7.2.1. *If (y_α) is an unbounded net, then there exists a subnet (y_β) and a $\psi \in B_{X^*}$*

such that $h_{y_\beta}(x) \to h^\psi(x)$ for all $x \in X$, where

$$h^\psi(x) = -\psi(x). \tag{7.2.1}$$

Proof. By taking subnets we can assume that $y_\alpha \neq 0$ for each α. We then define $z_\alpha = y_\alpha/\|y_\alpha\| \in S_X$. If we define the net $(t_\alpha) = (1/\|y_\alpha\|)$, we can write, for any $x \in X$,

$$h_{y_\alpha}(x) = \frac{\|z_\alpha - t_\alpha x\| - \|z_\alpha\|}{t_\alpha}.$$

We know, for each α, there exists functionals $\psi_\alpha, \varphi^x_\alpha \in S_{X^*}$, such that $\psi_\alpha(z_\alpha) = 1$ and $\varphi^x_\alpha(z_\alpha - t_\alpha x) = 1$. The Banach Alaoglu Theorem means there must exist $\psi, \varphi^x \in B_{X^*}$ and subnets (ψ_β) and (φ^x_β) (with the same directed set and defining monotone function), such that $\psi_\beta \xrightarrow{w} \psi$ and $\varphi^x_\beta \xrightarrow{w} \varphi^x$. Thus, for any β:

$$-\psi_\beta(x) = \frac{\psi_\beta(z_\beta - t_\beta x) - \psi_\beta(z_\beta)}{t_\beta} \leq \frac{\|z_\beta - t_\beta x\| - \|z_\beta\|}{t_\beta} \leq \frac{\varphi^x_\beta(z_\beta - t_\beta x) - \varphi^x_\beta(z_\beta)}{t_\beta} = -\varphi^x_\beta(x). \tag{7.2.2}$$

We now note that, for all β

$$2 \geq \|\varphi^x_\beta + \psi_\beta\|_* \geq |\varphi^x_\beta(z_\beta) + \psi_\beta(z_\beta)| \geq |\varphi^x_\beta(z_\beta - t_\beta x) + \psi_\beta(z_\beta)| - |t_\beta \varphi^x_\beta(x)|.$$

However, $t_\beta \varphi^x_\beta(x) \to 0$, so by the squeeze theorem, $\|\varphi^x_\beta + \psi_\beta\|_* \to 2$. As X^* is uniformly convex, Lemma 2.6.3 means that $\|\varphi^x_\beta - \psi_\beta\|_* \to 0$. As $\psi_\beta \xrightarrow{w} \psi$ and $\varphi^x_\beta \xrightarrow{w} \varphi^x$ we must thus have that $\psi = \varphi^x$. Taking the limit as $\beta \to \infty$ in inequality (7.2.2) thus shows that, for any $x \in X$, $\lim_\beta h_{y_\beta}(x) = -\psi(x)$.

□

The following lemma is the converse of Lemma 7.2.1. It is a generalisation of a part of [27, Theorem 5.4], where the result is only proved in the case of ℓ^p spaces.

Lemma 7.2.2. *For any $\psi \in B_{X^*}$, there exists an unbounded sequence $(y_n) \subseteq X$ such that $h_{y_n} \to h^\psi$, with h^ψ as in* (7.2.1).

Proof. We first note that because X^* is reflexive, there exists a sequence $(\phi_n) \subseteq S_{X^*}$ converging weakly to 0 in X^*. As X^* is smooth, and X is reflexive, there exists, for each $n \in \mathbb{N}$, a unique $x_n \in S_X$ such that $\phi_n(x_n) = 1$. We claim that $x_n \overset{w}{\rightharpoonup} 0$ in X. Indeed, suppose not. By the Banach-Alaoglu theorem, there exists a $0 \neq x \in B_X$, and a subsequence which we relabel (x_n), such that $x_n \overset{w}{\rightharpoonup} x$. As X is smooth there exists a unique norming functional $x^* \in S_{X^*}$. As $x_n \overset{w}{\rightharpoonup} x$, $x^*(x_n) \to 1$. Thus, for large enough n,

$$2 \geq \|\phi_n + x^*\|_* \geq |\phi_n(x_n) + x^*(x_n)| = 1 + x^*(x_n) \to 2.$$

The squeeze theorem means $\|\phi_n + x^*\|_* \to 2$, which implies by Lemma 2.6.3 that ϕ_n converges in norm to x^*, contradicting the fact that it weakly converges to 0. Thus indeed, (x_n) converges weakly to 0.

If $\psi = 0$, we define the sequence (y_n) by $y_n = nx_n$, with (x_n) as in the above paragraph. So, if we set, as in the proof of Lemma 7.2.1, $z_n = \|y_n\|^{-1} y_n \in S_X$, then $z_n = x_n$ for each $n \in \mathbb{N}$. The unique norming functional of each x_n is ϕ_n, and we know $\phi_n \overset{w}{\rightharpoonup} 0$. Thus the proof of Lemma 7.2.1 shows that $h^{y_n} \to h^\psi$. If $\psi \neq 0$, there exists, because X is reflexive and X^* is smooth by Proposition 2.6.4, a unique $x \in S_X$ such that $\psi(x) = 1$. If $\psi \in S_{X^*}$, we choose $y_n = nx$. Then, $z_n = \|y_n\|^{-1} y_n = x$, so the norming functional for z_n is ψ for all $n \in \mathbb{N}$, so again the proof of Lemma 7.2.1 shows that $h_{y_n} \to h^\psi$. Finally suppose that $0 < \|\psi\|_* < 1$. As ϕ_n weakly converges to 0, for large enough n we know that $c\phi_n \neq \psi$ for any $c \in \mathbb{R}$. Thus $1 - \|\psi\|_* < \|\phi_n + \psi\|_* < 2$. For any fixed $n \in \mathbb{N}$, the map $\lambda \mapsto \|\lambda \phi_n + \psi\|_*$ is continuous, and unbounded above. There thus exists, for every $n \in \mathbb{N}$, a $0 < \lambda_n < 2 - \|\psi\|_*$, such that $\varphi_n = \lambda_n \phi_n + \psi \in S_{X^*}$. As X is reflexive, and X^* is smooth, for every $n \in N$ there exists a unique $w_n \in S_X$ such that $\varphi_n(w_n) = 1$. Let us define $y_n = nw_n$. For every $n \in \mathbb{N}$, $z_n = \|y_n\|^{-1} y_n = w_n$, which means that φ_n is the norming functional for z_n for every $n \in \mathbb{N}$. As $\phi_n \xrightarrow{w} 0$, and (λ_n) is bounded, $\varphi_n \xrightarrow{w} \psi$. The proof of Lemma 7.2.1 thus means that $h_{y_n} \to h^{\psi}$. $\square$

Lemmas 7.2.1 and 7.2.2 completely characterise $\partial\overline{X}^{h,\infty}$, and show that $h \in \partial\overline{X}^{h,\infty}$ if and only if $h = h^{\psi}$ for some $\psi \in B_{X^*}\backslash\{0\}$ as in 7.2.1. To completely characterise $\partial\overline{X}^{h}$ we would need to classify the horofunctions that arise as limits of eventually bounded nets. Without knowing more about the norm, all that we can say at this time is that if (x_α) is eventually bounded in X, and $h = \lim_\alpha h_{x_\alpha}$, then $h \in \partial\overline{X}^{e,\infty}$. Indeed, by passing to a subnet if necessary we may assume that (x_α) is bounded, so $\liminf_\alpha \|x_\alpha\| = c < \infty$, meaning $h(x) \geq -c$ for all $x \in X$. As the Busemann points are all contained in $\partial\overline{X}^{h,\infty}$, we can still study the geometry and topology of the Busemann boundary without any further assumptions on the norm, which we now proceed to do.

Proposition 7.2.3. *The set of all Busemann points of X is precisely the set of all functions h^{ψ} where $\psi \in S_{X^*}$.*

Proof. Fix some arbitrary $\psi \in S_{X^*}$. In the proof of Lemma 7.2.2 we demonstrated a sequence of the form $(y_n) = (ny)$ for $y \in S_X$ such that $h_{y_n} \to h^{\psi}$. The sequence (y_n) is a geodesic sequence, so indeed h^{ψ} is a Busemann point. Conversely, let h be a Busemann point. Thus h is the pointwise limit of some almost-geodesic net, say (y_α). Lemma 3.2.3 means that (y_α) is unbounded, else h would be an internal metric functional. Lemma 7.2.1 means there is a subnet (y_β) such that $\lim_\beta h_{y_\beta} = h^{\psi}$ for some $\psi \in B_{X^*}$. As the metric compactification is Hausdorff this means that $h = h^{\psi}$. Let us assume, by way of contradiction, that $\|\psi\|_* < 1$. As (y_α) is an almost-geodesic net, for every $\varepsilon > 0$ there exists a γ_ε, such that for all $\alpha' \geq \alpha \geq \gamma_\varepsilon$,

$$\|y_{\alpha'}\| \geq \|y_\alpha\| + \|y_{\alpha'} - y_\alpha\| - \varepsilon.$$

As this is true for all $\alpha' \geq \alpha$, we can take the limit in α' to calculate

$$\|y_\alpha\| - \psi(y_\alpha) \leq \varepsilon.$$

However, $\|y_\alpha\| - \psi(y_\alpha) \geq \|y_\alpha\|(1 - \|\psi\|_*)$, which by assumption tends to infinity as α increases, a contradiction. Thus $\|\psi\|_* = 1$. □

Proposition 7.2.4. *The Busemann boundary of X consists entirely of singleton parts.*

Proof. Fix an arbitrary $\phi, \psi \in S_{X^*}$ with $\phi \neq \psi$. As X is reflexive and X^* is smooth, there exists unique $x, y \in S_X$ such that $\phi(x) = 1$ and $\psi(y) = 1$. From the proof of Proposition 7.2.3 we know that $h_{nx} \to h^\phi$ and $h_{ny} \to h^\psi$. Lemma 3.2.5 thus means that we can calculate $\delta(h^\psi, h^\phi)$ as

$$\delta(h^\psi, h^\phi) = \lim_{n\to\infty} n - \psi(nx) + \lim_{m\to\infty} m - \phi(my).$$

As both ψ and ϕ are norm 1, the only way that $\delta(h^\psi, h^\phi)$ can be finite is if $\psi(x) = 1 = \phi(x)$, and $\phi(y) = 1 = \psi(y)$. However, by the uniqueness of x and y as norming functionals for ϕ and ψ this means that $x = y$ but X^* is uniformly convex, so Lemma 2.6.2 thus implies that $\psi = \phi$, a contradiction. Therefore h^ϕ and h^ψ lie in different parts of the boundary. □

Having classified the Busemann boundary and its parts, we now prove the following.

Theorem 7.2.5. *The Busemann boundary $\partial_B \overline{X}^h$ is homeomorphic to S_{X^*} when equipped with the weak* topology, and the homeomorphism maps parts of the boundary onto the relative interiors of the faces of the dual ball.*

Proof. Proposition 7.2.3 combined with the fact that the weak* topology is equivalent to the topology of pointwise convergence on X^* shows that the map $h^\varphi \mapsto -\varphi$ is a homeomorphism. As X^* is uniformly convex, the only boundary faces of B_{X^*} are the singleton sets on the boundary, i.e. $F \subseteq B_{X^*}$ is a boundary face if and only if $F = \{x\}$ for $x \in S_{X^*}$. This in conjunction with Proposition 7.2.4 completes the proof. □

It would be interesting to see whether, as in the Hilbert space case, there exists a continuous bijection from $X \cup \partial_B \overline{X}^h$ to B_{X^*}, which is a homeomorphism onto the dual sphere when restricted to the Busemann boundary. It would seem natural to choose the analogue map of the Hilbert space homeomorphism (7.1.4), namely the function g defined by

$$g(x) = -\tanh(\|x\|)x^* \text{ for } x \in X\backslash\{0\}, \quad g(0) = 0, \quad \text{and } g(h) = h \text{ for } h \in \partial_B \overline{X}^h. \quad (7.2.3)$$

We can immediately prove that g still has some desirable properties:

Lemma 7.2.6. *The map g is a bijection onto B_{X^*}.*

Proof. Injectivity of $g|_X$ follows in exactly the same way as in the proof of Lemma 3.3.6, as finite dimensionality is not needed in that proof. Injectivity of $g|_{\partial_B \overline{X}^h}$ follows directly from Proposition 7.2.3 and the definition of $g|_{\partial_B \overline{X}^h}$. Thus g is injective. The proof needs to be adjusted somewhat for surjectivity. Fix some $\varphi \neq 0 \in \operatorname{int}(B_{X^*})$. As X is reflexive, there exists some $x \in S_X$ such that $\varphi(x) = \|\varphi\|_*$, meaning that $\varphi = \|\varphi\|_* x^*$. As $\tanh\colon [0, \infty) \to [0, 1)$ is monotone increasing and surjective, there exists a $\lambda > 0$ such that $\tanh(\|\lambda x\|) = \|\varphi\|_*$. Thus $g(-\lambda x) = \|\varphi\|_* x^* = \varphi$. Therefore $g|_X$ is surjective onto $\operatorname{int}(B_{X^*})$. Moreover, $g|_{\partial_B \overline{X}^h}$ is clearly surjective onto S_{X^*} in light of Proposition 7.2.3, so g is surjective. □

To prove continuity in the interior, we would need something akin to Lemma 7.1.7, but it is not clear why this would necessarily be true for the general smooth and strictly convex case. However, more can be said if we restrict our attention to ℓ^p.

7.2.1 The Topology of the Busemann Boundary of ℓ^p.

Let $X = \ell^p(J)$ for $1 < p < \infty$, and J an arbitrary index set. Recall that $\ell^p(J)$ is uniformly smooth and uniformly convex [14]. Also note that $X^* = \ell^q(J)$, where $1/p + 1/q = 1$ [50, Example 1.10.2]. In [27, Section 5], Guttiérez completely characterises the horofunction boundary of $\ell^p(J)$, and shows that

$$\begin{aligned}\partial\overline{\ell^p(J)}^{h,e} &= \{h^{z,c} : z \in \ell^p(J),\ c > \|z\|\} \cup \{h^0\} \\ \partial\overline{\ell^p(J)}^{h,\infty} &= \{h^\varphi : \varphi \in B_{\ell^q(J)} \backslash \{0\}\},\end{aligned} \tag{7.2.4}$$

where for all $y \in \ell^p(J)$,

$$h^{z,c}(y) = (\|z-y\|^p - \|z\|^p + c^p)^{1/p} - c, \text{ and, } h^\varphi(y) = -\varphi(y).$$

We should note here how to interpret elements of $\ell^q(J)$ as elements of X^*- any $\varphi \in \ell^q(J)$ acts on X by the map

$$y \mapsto \varphi(y) = \sum_{i \in J} y_i \varphi_i.$$

The horofunction boundary of $\ell^p(\mathcal{N})$ for finite $\mathcal{N}$ is markedly different, and consists only of functions h^φ for $\varphi \in S_{\ell^q(\mathcal{N})}$. The proof of Lemma 7.1.2 can be adapted almost verbatim to prove that, for any $z \in \ell^p(J)$, and any $c \geq d > \|z\|$,

$$\sup_{y \in \ell^p(J)} |h^{z,c}(y) - h^{z,d}(y)| < \infty.$$

The observations in Remark 7.1.1 thus hold true also for infinite dimensional ℓ^p spaces, and there exists no stratification preserving homeomorphism between the horofunction compactification of $\ell^p(J)$ and the closed unit ball in $\ell^q(J)$. We define the map $g\colon \overline{X}^h \cup \partial_B \overline{X}^h \to B_{X^*}$ exactly as (7.2.3) above.

Theorem 7.2.7. *The map g is a continuous bijection onto B_{X^*} equipped with the weak* topology, and $g|_{\partial_B \overline{X}^h}$ is a homeomorphism that maps parts of the boundary onto the relative interior of the faces of the dual ball.*

Before proving Theorem 7.2.7, we need, as in the Hilbert space case, the following lemma:

Lemma 7.2.8. *If $h_{x_\alpha} \to h_x$ in the topology of compact convergence for $(x_\alpha) \subseteq X$ and $x \in X$, then $\|x - x_\alpha\|_p \to 0$.*

Proof. Without loss of generality, we can assume that (x_α) is bounded, because if not then h_x cannot be bounded below, or $h_x = h^0$, a contradiction. Lemma 5.1 in [27] guarantees the existence of a subnet (x_β), a $z \in X$, a $c \geq 0$ with $c \geq \|z\|$ such that $\lim_\beta \|x_\beta\| = c$, and for every $y \in X$,

$$h_{x_\beta}(y) \xrightarrow{\beta} \left(\|y - z\|_p^p + c^p - \|z\|_p^p\right)^{\frac{1}{p}} - c = h^{z,c}(y).$$

It is clear that for all $y \in X$ that

$$h^{z,c}(y) \geq \left(c^p - \|z\|_p^p\right)^{\frac{1}{p}} - c,$$

where equality holds if and only if $y = z$. As $h^{z,c} = h_x$ it follows that $\inf h^{z,c} = \inf h_x$, and also that $\inf h_x = h_x(z)$, so $\|x - z\|_p = 0$ meaning that $x = z$.

We must now consider some cases. If $z = x = 0$, then $h_x = h_0$, so for all $y \in X$

$$(\|y\|_p + c)^p = \|y\|^p + c^p.$$

If $c \neq 0$ choosing y with $\|y\|_p = c$ would lead to a contradiction, so $z = x = c = 0$. If $x = z \neq 0$, then by choosing our argument to be x we require that $c^p - \|x\|_p^p = (c - \|x\|_p)^p$, which is only possible if $\|x\|_p = c$ or $\|x\|_p = 0$, which shows that $\|x\|_p = c$ in all cases. We can thus calculate that

$$\|x_\beta - x\|_p = h_{x_\beta}(x) + \|x_\beta\|_p \to (c^p - \|x\|_p^p)^{\frac{1}{p}} - c + c = 0.$$

This argument shows that every subnet of (x_α) has a further subnet converging to x, which completes the proof. □

We are now in a place to prove Theorem 7.2.7:

Proof. Thanks to Theorem 7.2.5 and Lemma 7.2.6, we only need to show two things: Namely that if $h_{x_\alpha} \to h_x$ for an interior metric functional h_x, then $g(h_{x_\alpha}) \to g(h_x)$, and if $h_{x_\alpha} \to h$ where h is a Busemann point, then $g(h_{x_\alpha}) \to g(h)$. If $h_{x_\alpha} \to h_x$, Lemma 7.2.8 immediately gives that $\tanh(\|x_\alpha\|) \to \tanh(\|x\|)$. By reflexivity of $\ell^p(J)$ and Lemma 7.2.8,

$$x_\alpha^*(y) = y^*(x_\alpha) \to y^*(x) = x^*(y),$$

meaning that x_α^* converges to x^* in the weak* topology, so indeed $g(h_{x_\alpha}) \to g(h_x)$. Finally, let $h_{x_\alpha} \to h \in \partial_B \overline{X}^h$. By definition $g(h_{x_\alpha}) = -\tanh(\|x_\alpha\|)x_\alpha^*$ for all α, and we know from the proof of Lemma 7.2.1 that x_α^* converges weakly to a $\psi \in B_{X^*}$ such that $h_{x_\alpha}(y) \to h^\psi(y)$ for every $y \in X$. Moreover, (x_α) cannot be bounded, else h would be bounded below. Therefore $\tanh(\|x_\alpha\|) \to 1$. Thus, indeed, $g(h_{x_\alpha}) \to h = g(h)$ by the uniqueness of limits. □

Proposition 7.2.9. *The closure of $\partial\overline{\ell^p(J)}^{h,\infty}$ intersected with $\partial\overline{\ell^p(J)}^{h,e}$ is equal to the singleton $\{h^0\}$.*

Proof. Consider some $h \in \mathrm{cl}(\partial\overline{\ell^p(J)}^{h,\infty})$, and corresponding net $(h_\alpha) \subseteq \partial\overline{\ell^p(J)}^{h,\infty}$ converging to h, with $h_\alpha = h^{z_\alpha}$ for $z_\alpha \in B_{\ell^q(j)}$. Suppose by way of contradiction that $h = h^{x,c}$. Take a fixed $y \in \ell^p(J)$ such that$\|y - x\| > c$, making $h(x) > 0$. Thus $\lim_\alpha h_\alpha(x) > 0$, which means that no subnet of (z_α) can converge weakly to 0. There thus exists, by Banach-Alaoglu, some non-zero $z \in \ell^q(J)$ and a subnet (z_β) converging weakly to z. As $z \neq 0$ there exists some $u \in \ell^p(I)$ with $\langle u, z\rangle > 2c$. Thus $\lim_\beta h_\beta(u) = -\langle u, z\rangle < -2c$. However, $h^{x,c}(u) \geq -c$, contradicting the assumed convergence. Thus if $(h_\alpha) \subseteq \partial\overline{\ell^p(J)}^{h,\infty}$ converges to $h \in \partial\overline{\ell^p(J)}^{h,e}$, we must have $h = 0$. If (e_n) is an orthonormal sequence in $\ell^q(J)$ converging weakly to 0, then $h^{e_n}(y) = -\langle e_n, y\rangle$ converges to 0 for every $y \in \ell^p(J)$. Thus, indeed, 0 is in the closure of $\partial\overline{\ell^p(J)}^{h,\infty}$. □

Just as in the Hilbert space case, Proposition 7.2.9 means that there cannot exist a homeomorphism from the whole horofunction compactification $\overline{\ell^p(J)}^h$ onto B_{X^*} equipped with the weak* topology, which maps Busemann points bijectively onto the unit sphere, because the unit sphere of $\ell^q(J)$ is dense in the closed unit ball under the weak* topology.

7.3 The Topology of the Horoboundary of ℓ^1.

Let J be an arbitrary set of infinite cardinality. In this section we study the horofunction compactification of $\ell^1(J)$. Before proceeding, we recall that, if J is countable, $\ell^1(J)$ can be identified with the sequence space $\ell^1(\mathbb{N})$, and it is well known [50, Example 1.10.3] that the dual of $\ell^1(\mathbb{N})$ is $\ell^\infty(\mathbb{N})$, the space of all bounded sequences equipped with the norm

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} x_n.$$

If J is uncountable, $\ell^\infty(J)$ is the collection of all bounded functions $x \in \mathbb{R}^J$ equipped with the norm

$$\|x\|_\infty = \sup_{j\in J} x(j).$$

The dual of $\ell^1(J)$ for uncountable J is still $\ell^\infty(J)$, but because the counting measure space $(J, 2^J, \mu)$ is not σ-finite, the usual proof must be adjusted, so we include a proof for convenience.

Lemma 7.3.1. *The dual of $\ell^1(J)$ is $\ell^\infty(J)$, where we identify $\ell^\infty(J)$ with $\ell^1(J)^*$ under the map $x \mapsto \varphi_x$, where, for any $y \in \ell^1(J)$,*

$$\varphi_x(y) = \int_J x(j)y(j)d\mu(j).$$

Proof. We first show that each φ_x is an element of $\ell^1(J)^*$. Note, for any $y \in\in \ell^1(J)$,

$$|\int_J x(j)y(j)d\mu(j)| \le \|x\|_\infty \int_J |y(j)|d\mu(j),$$

so $|\varphi_x(y)| \le \|x\|_\infty \|y\|_1$. The linearity of the integral also means that φ_x is linear, so indeed

φ_x is a bounded linear functional. Furthermore, $\|\varphi_x\|_* \leq \|x\|_\infty$. Moreover, there must exist a sequence $(\lambda_n) \subseteq x(J)$ such that $|\lambda_n| \to \|x\|_\infty$, so for every $n \in \mathbb{N}$ we define $y_n \in \ell^1(J)$ by $y_n(j) = 0$ if $j \notin x^{-1}(\lambda_n)$, and $y_n(j) = 1$ if $j \in x^{-1}(\lambda_n)$. Thus $|\varphi_x(y_n)| \to \|x\|_\infty$, so $\|\varphi_x\|_* = \|x\|_\infty$. Conversely, fix some $\phi \in \ell^1(J)^*$. For any $j \in J$ we can define $e_j \in \ell^1(J)$ by setting $e_j(j) = 1$ and $e_j(i) = 0$ if $i \neq j$. This allows us to define $x \in \mathbb{R}^J$ by $x(j) = \phi(e_j)$. If $x(j)$ was unbounded, there must exist a sequence $(j_n) \subseteq J$ such that $\phi(e_{j_n}) \to \infty$. However, ϕ is bounded by assumption, and each $e_j \in B_{\ell^1(J)}$, a contradiction. Thus $x \in \ell^\infty(J)$. Moreover, any $y \in \ell^1(J)$ is non-zero only on a countable subset $I \subseteq J$, meaning that we can order $I = \{i_n\}_{n\in\mathbb{N}}$ and write $y = \sum_{n=1}^\infty y(i_n)e_{i_n}$. Linearity and continuity thus mean that $\phi(y) = \sum_{n=1}^\infty y(i_n)\phi(e_{i_n})$. Moreover

$$\varphi_x(y) = \int_I \phi(e_j)y(j) = \sup\left\{\sum_{n\in F} y(i_n)\phi(e_{i_n}) : F \subseteq I \text{ is finite}\right\} = \phi(y).$$

Thus $x \mapsto \varphi_x$ is an isometric isomorphism between $\ell^\infty(J)$ and $\ell^1(J)^*$. □

We will also need the facial structure of the dual unit ball of $\ell^1(J)$. In particular we are interested in the weak* closed boundary faces.

Proposition 7.3.2. *The weak* closed boundary faces of $B_{\ell^\infty(J)}$ are precisely the sets F_ε^I, for $\emptyset \neq I \subseteq J$ and $\varepsilon \in \{-1, 1\}$, where*

$$F_\varepsilon^I = \{x \in B_{\ell^\infty(J)} : x(j) = \varepsilon(j) \textit{ for all } j \in I\}.$$

Proof. We first note that $\ell^\infty(J)$ is a unital Jordan algebra when equipped with pointwise multiplication, with unit $\mathbf{1}$ defined by $\mathbf{1}(j) = 1$ for all $j \in J$. Furthermore, $\|\cdot\|_\infty$ turns $\ell^\infty(J)$ into a JB-algebra, and $\|x^2 - y^2\|_\infty \leq \max\{\|x^2\|_\infty, \|y\|_\infty^2\}$. Thus, because $\ell^\infty(J)$ is the dual of $\ell^1(J)$, it is a JBW- algebra. In Section 2.9.4 we discussed how JB-algebras are also order unit spaces, and $\ell^\infty(J)$ is an order unit space with order unit $\mathbf{1}$. Recall that a symmetry in a JBW-algebra is an element x such that x^2 is the identity. We immediately see that $x \in \ell^\infty(J)$ is a symmetry if and only if $x(j) = \pm 1$ for all $j \in J$. For any $x, y \in \ell^\infty(J)$, recall the definition of the order interval

$$[x,y]_{\mathbf{1}} = \{z \in \ell^\infty(J) : z \le y,\ x \le z\}.$$

Therefore for any symmetries $x \le y$, $z \in [x,y]_{\mathbf{1}}$ if and only if $z(j) - x(j) \ge 0$ and $y(j) - z(j) \ge 0$ for all $j \in J$. This means that, if $z \in [x,y]_{\mathbf{1}}$ and $x(j) = y(j)$, we must have $z(j) = x(j) = y(j)$, and if $x(j) = -1$ and $y(j) = 1$ then $z(j) \in [-1,1]$. Thus, if we set $I = \{j \in J : y(j) = x(J)\}$ and define $\varepsilon \in \{-1,1\}^I$ by $\varepsilon(j) = y(j) = x(j)$ for all $j \in I$,

$$[x,y]_{\mathbf{1}} = F^I_\varepsilon.$$

Theorem 4.1 in [18] then proves the result, as it states that the weak* closed faces of the unit ball of a JBW-algebra are precisely the order intervals between symmetries. □

Gutiérrez showed in [27, Section 3] that the horofunction compactification of $\ell^1(J)$ has the form:

$$\overline{\ell^1(J)}^h = \{h^{I,\varepsilon,z} : \emptyset \subseteq I \subseteq J,\ \varepsilon \in \{-1,1\}^I,\ z \in \mathbb{R}^{J\setminus I}\}, \tag{7.3.1}$$

where

$$h^{I,\varepsilon,z}(x) = \sum_{i\in I} \varepsilon(i)x(i) + \sum_{j \in J\setminus I} |x(j) - z(j)| - |z(j)|.$$

As a consequence,

$$\partial\overline{\ell^1(J)}^h = \{h^{I,\varepsilon,z} : \emptyset \ne I \subseteq J,\ \varepsilon \in \{-1,1\}^I,\ z \in \mathbb{R}^{J\setminus I}\},$$

and, for any $x \in \ell^1(J)$, the internal metric functional h_x is equal to $h^{\emptyset,\emptyset,x}$. Unlike the case for other infinite dimensional ℓ^p spaces studied above, the horofunctions of infinite dimensional ℓ^1 are all of the same form as in finite dimensions. Indeed, if we just replace J with $\mathcal{N}$ in (7.3.1) above, we obtain the exact same horofunctions as described in Section 3.3.3. As a corollary to [27, Theorem 3.6], we find another startling difference:

Corollary 7.3.3. *Every horofunction in $\partial\overline{\ell^1(J)}^h$ is a Busemann point.*

Proof. In the proof of [27, Theorem 3.6], Gutiérrez constructs, for each horofunction h given by (7.3.1), a net $(y_\alpha) \subseteq \ell^1(J)$ such that $h_{x_\alpha} \to h$. If we can show that these nets are almost-geodesic nets, we will be done. For convenience of the reader we repeat the construction here. Fix some I, ε, and z as in (7.3.1). For every finite subset $F \subseteq J$ we can define $y_F \in \ell^1(J)$ by

$$y_F(j) = \begin{cases} -\varepsilon(j)|F| & \text{if } j \in F \cap I \\ z(j) & \text{if } j \in F \cap (J\backslash I) \\ 0 & \text{otherwise} \end{cases}. \tag{7.3.2}$$

This then defines a net $(y_F)_F$ indexed by the set of all finite subsets of J ordered by $\leq$, where $F \leq E$ if $F \subseteq E$. Gutiérrez shows in the proof of Theorem 3.6 in [27] that the net h_{y_F} converges to the horofunction $h^{I,\varepsilon,z}$. We now show that this net is an almost-geodesic. To this end, fix arbitrary $F \subseteq F' \subseteq J$, with both F and F' finite. Note that we can write

$$\|y_{F'}\|_1 = \sum_{i\in I\cap F'} |F'| + \sum_{j\in F'\cap(J\backslash I)} |z(j)|,$$

and

$$\|y_{F'} - y_F\|_1 = \sum_{i\in I\cap(F'\backslash F)} |F'| + \sum_{i\in I\cap F} (|F'| - |F|) + \sum_{j\in(F'\backslash F)\cap(J\backslash I)} |z(j)|.$$

Thus,

$$\|y'_F\|_1 - \|y_F\|_1 = \|y_{F'} - y_F\|_1.$$

As F' and F were arbitrarily chosen, this shows that $(y_F)_F$ is an almost-geodesic. □

Knowing this, we can calculate the parts of $\partial\overline{\ell^1(J)}^h$:

Proposition 7.3.4. *Two horofunctions, $h^{I_1,\varepsilon_1,z_1}$ and $h^{I_2,\varepsilon_2,z_2}$, are in the same part of the boundary $\partial\overline{\ell^1(J)}^h$ if and only if $I_1 = I_2 = I$, $\varepsilon_1 = \varepsilon_2$, and $z_1 - z_2 \in \ell^1(J\backslash I)$.*

Proof. Fix $h_1 = h^{I_1,\varepsilon_1,z_1}$ and $h_2 = h^{I_2,\varepsilon_2,z_2}$ in $\partial\overline{\ell^1(J)}^h$. Let (y_F^1) and (y_F^2) be the geodesic nets converging to h_1 and h_2 respectively, as in (7.3.2). Lemma 3.2.5 thus allows us to calculate the detour distance, $\delta(h_1, h_2)$

$$\delta(h_1, h_2) = H(h_1, h_2) + H(h_2, h_1) = \lim_F \|y_F^1\|_1 + h_2(y_F^1) + \lim_F \|y_F^2\|_1 + h_1(y_F^2).$$

Now, for any F,

$$\begin{aligned} \|y_F^1\|_1 + h_2(y_F^1) = &\sum_{i\in I_1\cap F} |F| + \sum_{j\in F\cap(J\setminus I_1)} |z_1(j)| + \sum_{i\in I_2\cap I_1\cap F} -\varepsilon_1(i)\varepsilon_2(i)|F| \\ &+ \sum_{i\in I_2\cap F\cap(J\setminus I_1)} \varepsilon_2(i)z_1(i) + \sum_{j\in(J\setminus I_2)\cap(I_1\cap F)} |-\varepsilon_1(j)|F| - z_2(j)| - |z_2(j)| \\ &+ \sum_{j\in(J\setminus I_2)\cap(J\setminus I_1)\cap F} |z_1(j) - z_2(j)| - |z_2(j)|, \end{aligned} \tag{7.3.3}$$

and

$$\begin{aligned} \|y_F^2\|_1 + h_1(y_F^2) = &\sum_{i\in I_2\cap F} |F| + \sum_{j\in F\cap(J\setminus I_2)} |z_2(j)| + \sum_{i\in I_2\cap I_1\cap F} -\varepsilon_1(i)\varepsilon_2(i)|F| \\ &+ \sum_{i\in I_1\cap F\cap(J\setminus I_2)} \varepsilon_1(i)z_2(i) + \sum_{j\in(J\setminus I_1)\cap(I_2\cap F)} |-\varepsilon_2(j)|F| - z_1(j)| - |z_1(j)| \\ &+ \sum_{j\in(J\setminus I_2)\cap(J\setminus I_1)\cap F} |z_1(j) - z_2(j)| - |z_1(j)|. \end{aligned} \tag{7.3.4}$$

If $\delta(h_1, h_2)$ is finite, then both (7.3.3) and (7.3.4) must be finite in the limit, as the detour cost is non-negative, which means so too must the limit of their sum. We should therefore evaluate:

$$\begin{aligned}
&\|y_F^1\|_1 + h_2(y_F^1) + \|y_F^2\|_1 + h_1(y_F^2) && (7.3.5)\\
&= \sum_{i\in I_2\cap I_1\cap\ F} 2|F|(1-\varepsilon_1(i)\varepsilon_2(i)) && (7.3.6)\\
&\quad + \sum_{j\in(J\backslash I_1)\cap(I_2\cap F)} |F| + |-\varepsilon_2(j)|F| - z_1(j)| + \varepsilon_2(j)z_1(j) && (7.3.7)\\
&\quad + \sum_{j\in(J\backslash I_2)\cap(I_1\cap F)} |F| + |-\varepsilon_1(j)|F| - z_2(j)| + \varepsilon_1(j)z_2(j) && (7.3.8)\\
&\quad + \sum_{j\in(J\backslash I_2)\cap(J\backslash I_1)\cap F} 2|z_1(j) - z_2(j)|.
\end{aligned}$$

We first note that, for any finite F and any $j \in J$, the reverse triangle inequality means that

$$|F| + |-\varepsilon_2(j)|F| - z_1(j)| + \varepsilon_2(j)z_1(j) \geq |z_1(j)| + \varepsilon_2(j)z_1(j) \geq 0,$$

and similarly we note that all terms in the sum (7.3.8) are non-negative, meaning that (7.3.5) consists of four sums of non-negative terms. Therefore, if $I_1 \neq I_2$, at least one of the sums (7.3.7) or (7.3.8) is non-empty, and is unbounded in $|F|$, meaning that $\delta(h_1, h_2) = \infty$ if $I_1 \neq I_2$. If $I_1 = I_2$, but $\varepsilon_1 \neq \varepsilon_2$, there exists a term in sum (7.3.6) of the form $2|F|$, and so the whole sum goes to infinity as $|F|$ goes to infinity. Thus for $\delta(h_1, h_2) < \infty$ it is necessary for $\varepsilon_1 = \varepsilon_2$. Finally, it is clear that we must also have $z_1 - z_2 \in \ell^1(J\backslash I)$, from which we can conclude that h_1 is in the same part of h_2 if and only if $I_1 = I_2$, $\varepsilon_1 = \varepsilon_2$, and $z_1 - z_2 \in \ell^1(J\backslash I)$. □

Remembering Proposition 7.3.2,

Theorem 7.3.5. *There exists a homeomorphism $\varphi\colon \overline{\ell^1(J)}^h \to B_{\ell^\infty(J)}$, where $\ell^\infty(J)$ is equipped with the weak* topology, which injects parts of the horofunction boundary onto the relative interiors of faces of $B_{\ell^\infty(J)}$.*

Proof. We first define, for each $j \in J$, the element $e_j \in \ell^\infty(J)$ as the usual coordinate basis vector. I.e. for $e_j(j) = 1$ and $e_j(i) = 0$ if $i \neq j$. We can then define the map

$\varphi\colon \overline{\ell^1(J)}^h \to B_{\ell^\infty(J)}$ by

$$\varphi(h^{I,\varepsilon,z})(j) = \begin{cases} -\varepsilon(j)e_j & j \in I \\ \tanh(z(j))e_j & j \notin I. \end{cases}.$$

It follows from routine calculation that this map is bijective, because tanh is bijective onto $(-1,1)$, and I can be any subset of J. Let us now show that it is continuous, and let $(h^{I_\alpha,\varepsilon_\alpha,z_\alpha}) \subseteq \overline{\ell^1(J)}^h$ be a net converging to some $h^{I,\varepsilon,z} \in \overline{\ell^1(J)}^h$. First we claim that, for each $i \in I$, there must exist an α_i, so that for all $\alpha \geq \alpha_i$, we have $i \in I_\alpha$. Indeed, suppose for the sake of contradiction that there exists $i \in I$ so that for all α there exists a $\beta \geq \alpha$ such that $i \notin I_\beta$. For any $\lambda \in \mathbb{R}$ we consider the element $f_i^\lambda = \lambda\varepsilon(i)e_i \in \ell^1(J)$. We then have, for any α,

$$h^{I_\alpha,\varepsilon_\alpha,z_\alpha}(f_i^\lambda) = \begin{cases} \varepsilon_\alpha(i)\varepsilon(i)\lambda & \text{if } i \in I_\alpha \\ |z_\alpha(i) - \varepsilon(i)\lambda| - |z_\alpha(i)| & \text{if } i \notin I_\alpha \end{cases}, \quad \text{and} \quad h^{I,\varepsilon,z}(f_i^\lambda) = \lambda.$$

By our assumption, there exists a subnet $h^{I_\beta,\varepsilon_\beta,z_\beta}$ such that $h^{I_\beta,\varepsilon_\beta,z_\beta}(f_i^\lambda) = |z_\beta(i) - \varepsilon(i)\lambda| - |z_\beta(i)|$ for all β. The only way this can converge to λ is if $z_\beta(i)$ tends to plus or minus infinity, depending on the sign of $\varepsilon(i)$. Therefore $\varepsilon(i)z_\beta(i) \to -\infty$. Now let us define, for $j \in J\backslash I$, and $\lambda \in \mathbb{R}$, $g_j^\lambda = \lambda e_j$. We thus have, for any β,

$$h^{I_\beta,\varepsilon_\beta,z_\beta}(g_j^\lambda) = \begin{cases} \lambda & \text{if } j \in I_\beta \\ |z_\beta(j) - \lambda| - |z_\beta(j)| & \text{if } j \notin I_\beta \end{cases}, \quad \text{and} \quad h^{I,\varepsilon,z}(g_j^\lambda) = |z(j) - \lambda| - |z(j)|.$$

It is clear that if $j \in I_\beta$ for infinitely many β, then $h^{I_\beta,\varepsilon_\beta,z_\beta}(g_j^\lambda)$ cannot converge to $h^{I,\varepsilon,z}(g_j^\lambda)$ for all values of λ, meaning that there must exist a β' such that $\beta \geq \beta'$ implies $j \notin I_\beta$. However, we know that $|z_\beta(j) - \lambda| - |z_\beta(j)| \to \varepsilon(i)\lambda$, which cannot equal $|z(j) - \lambda| - |z(j)|$ for all $\lambda \in \mathbb{R}$. This is a contradiction, so indeed, for each $i \in I$, there must exist an α_i, so that for all $\alpha \geq \alpha_i$ we have $i \in I_\alpha$. As a corollary from this proof we immediately see that

$\varepsilon_\alpha(i) \to \varepsilon(i)$ for all $i \in I$.

We can essentially repeat the above argument to show that for all $j \in J\backslash I$, there must exist an α_j such that $\alpha \geq \alpha_j$ implies that $j \in J\backslash I_\alpha$. Next we show that $z_\alpha \to z$ pointwise on $J\backslash I$. Using the previous remark, for any $j \in J\backslash I$ we know that for large enough α, $h^{I_\alpha,\varepsilon_\alpha,z_\alpha}(g_j^\lambda) = |z_\alpha(j)-\lambda|-|z_\alpha(j)|$, which by assumption must converge to $|z(j)-\lambda|-|z(j)|$. As previously noted, this must mean that $(z_\alpha(j))$ is bounded. Thus, every subnet of $(z_\alpha(j))$ must have a further subnet, say $(z_\beta(j))$, converging to some $y \in \mathbb{R}$. By choosing $\lambda = y$ and $\lambda = z(j)$ it is clear that $y = z(j)$. Thus, indeed, $z_\alpha(j) \to z(j)$.

With this information, we can now show that $\varphi(h^{I_\alpha,\varepsilon_\alpha,z_\alpha}) \to \varphi(h^{I,\varepsilon,z})$ in the weak* topology on $B_{\ell^\infty(J)}$. To that end, let $x \in \ell^1(J)$ be arbitrary. If we can show that $\varphi(h^{I_\alpha,\varepsilon_\alpha,z_\alpha})(x) \to \varphi(h^{I,\varepsilon,z})(x)$ we will be done. There exists a countable set $C \subseteq J$, such that $x(i) \neq 0$ if and only if $i \in C$. Thus, for any α,

$$\begin{aligned}|[\varphi(h^{I_\alpha,\varepsilon_\alpha,z_\alpha}) - \varphi(h^{I,\varepsilon,z})](x)| = \Bigg| &\sum_{i\in I_\alpha\cap C} -\varepsilon_\alpha(i)x(i) + \sum_{j\in(J\backslash I_\alpha)\cap C} x(j)\tanh(z_\alpha(j)) \\ &- \sum_{i\in I\cap C} -\varepsilon(i)x(i) - \sum_{j\in(J\backslash I)\cap C} x(j)\tanh(z(j))\Bigg|.\end{aligned}$$

Let us fix $\varepsilon > 0$. There must exist a finite $F_\varepsilon \subseteq C$ such that $\sum_{i\in C\backslash F_\varepsilon} |x(i)| < \varepsilon$. We therefore have

$$\begin{aligned}|[\varphi(h^{I_\alpha,\varepsilon_\alpha,z_\alpha}) - \varphi(h^{I,\varepsilon,z})](x)| < \Bigg| &\sum_{i\in I_\alpha\cap F_\varepsilon} -\varepsilon_\alpha(i)x(i) + \sum_{j\in(J\backslash I_\alpha)\cap F_\varepsilon} x(j)\tanh(z_\alpha(j)) \\ &- \sum_{i\in I\cap F_\varepsilon} -\varepsilon(i)x(i) - \sum_{j\in(J\backslash I)\cap F_\varepsilon} x(j)\tanh(z(j))\Bigg| + 2\varepsilon.\end{aligned}$$

As F_ε is finite, by the observations in the preceding paragraphs there must exist a β, so that for all $\alpha \geq \beta$ we have $I_\alpha \cap F_\varepsilon = I \cap F_\varepsilon$ and $(J\backslash I_\alpha) \cap F_\varepsilon = (J\backslash I) \cap F_\varepsilon$. Therefore, because the above inequality is true for all α, we must have that

$$\begin{aligned}\lim_\alpha |[\varphi(h^{I_\alpha,\varepsilon_\alpha,z_\alpha}) - \varphi(h^{I,\varepsilon,z})](x)| &\leq \lim_\alpha \Bigg| \sum_{i\in I\cap F_\varepsilon} -\varepsilon_\alpha(i)x(i) + \sum_{j\in(J\setminus I)\cap F_\varepsilon} x(j)\tanh(z_\alpha(j)) \\ &\quad - \sum_{i\in I\cap F_\varepsilon} -\varepsilon(i)x(i) - \sum_{j\in(J\setminus I)\cap F_\varepsilon} x(j)\tanh(z(j))\Bigg| + 2\varepsilon \\ &= 2\varepsilon.\end{aligned}$$

As ε and x were arbitrary, it follows that indeed $\varphi(h^{I_\alpha,\varepsilon_\alpha,z_\alpha}) \to \varphi(h^{I,\varepsilon,z})$ in the weak* topology on $B_{\ell^\infty(J)}$. Thus, φ is a continuous bijection between compact Hausdorff spaces, so it is a homeomorphism.

Recalling the definition of the face F_ε^I in Proposition 7.3.2, the definition of φ immediately means that $\varphi(h^{I,\varepsilon,z}) \in F_\varepsilon^I$ for all $z \in \mathbb{R}^{J\setminus I}$. Proposition 7.3.4 therefore shows that every part of the Busemann boundary is entirely contained within a weak* closed face of $B_{\ell^\infty(J)}$. $\square$

Proposition 7.3.4 shows that two horofunctions $h^{I,\varepsilon,z}$ and $h^{I,\varepsilon,w}$ are in the same part whenever $z - w \in \ell^1(J\setminus I)$, so if $J\setminus I$ is infinite, the proof of Theorem 7.3.5 means that uncountably many parts of the boundary are injected into the face F_ε^I. This is strikingly different from the situation in smooth $\ell^p(J)$ spaces. The author suspects that this discrepancy is fundamentally linked to the fact that $\partial\overline{\ell^1(J)}^h$ contains only Busemann points, but that investigation will be left to future work.

Chapter 8

Infinite Dimensional Spin Factors

Let $V = \mathbb{R} \oplus \mathcal{H}$ be an infinite dimensional spin factor as in Section 2.9.5, with unit $e = (1, 0)$, and equipped with the JB-algebra norm $\|\cdot\|$, which we recall is just the order unit norm $\|\cdot\|_e$. We denote by V_+ its cone of squares, which is equal to the set $\{(\lambda, x) \in V : \lambda \geq \|x\|_{\mathcal{H}}\}$. The open cone $V_+^\circ = \{(\lambda, x) \in V : \lambda > \|x\|_{\mathcal{H}}\}$ is an infinite dimensional symmetric cone, and we can equip it with a Finsler structure by defining $F \colon TV_+^\circ \to \mathbb{R}$ by $F(v, u) = \|U_{v^{-1/2}} u\|_e$. This gives rise to a Finsler metric ρ on V_+° by defining

$$\rho(u, v) = \inf_\gamma \left\{ \int_0^1 F(\gamma(t), \gamma'(t)) dt : \gamma(0) = u, \gamma(1) = v,\ \gamma \text{ is piecewise } C^1 \right\}.$$

It turns out that ρ is precisely the Thompson metric d_T on V_+° [55, Theorem 1.1]. Recall that the Thompson metric d_T is defined by, for $u, v \in V_+^\circ$

$$d_T(u, v) = \max\{\log M(u/v), \log M(v/u)\},$$

where $M \colon V \times V_+^\circ \to \mathbb{R}$ is the gauge function (2.7.1). We should note that the above setting is applicable to any JB-algebra A, but in this chapter we restrict our attention to the infinite dimensional spin-factor V. We investigate the horofunction compactification of $(V, \|\cdot\|)$ with 0 as the basepoint, which we denote by $\overline{V}^h$, as well as the horofunction compactification of (V_+°, d_T) with e as the basepoint, which we denote by $\overline{V_+^\circ}^h$. By the above remark we are thus investigating the horofunction compactification of a Finsler metric space along with the horofunction compactification of the tangent space of the base point equipped with the Finsler norm. We explicitly calculate the horofunctions in $\partial\overline{V}^h$ and $\partial\overline{V_+^\circ}^h$, and in the spirit of [42], we prove the following:

Theorem 8.0.1. *The exponential map* $\exp\colon V \to V_+^\circ$ *can be extended to a homeomorphism between* $\overline{V}^h$ *and* $\overline{V_+^\circ}^h$*, and this extension maps parts of* $\partial_B\overline{V}^h$ *onto parts of* $\partial_B\overline{V_+^\circ}^h$*.*

It is also interesting to projectivise the open cone by identifying rays, and so define the space $PV_+^\circ = V_+^\circ/\sim$, where $x \sim y$ if $y = \lambda x$ for some $\lambda > 0$. We choose to identify PV_+° with the hyperboloid

$$\mathbf{H} = \{v \in V_+^\circ : \det(v) = 1\}.$$

The spin factor is equipped with a natural quadratic form $Q\colon V \to \mathbb{R}$ defined by

$$Q((\gamma, x)) = \gamma^2 - \|x\|_{\mathcal{H}}^2,$$

and in [13], Claassens defines the hyperboloid to be the set

$$\hat{\mathbf{H}} = \{v \in V_+^\circ : Q(v) = 1\}.$$

These definitions are equivalent, because if $v \in \mathbf{H}\backslash\{e\}$ has a spectral decomposition $\lambda p + \mu(e-p)$, then the fact that $\det(v) = 1$ means that $\lambda\mu = 1$, so each $v \in \mathbf{H}\backslash\{e\}$ has a unique spectral decomposition $\lambda p + \lambda^{-1}(e-p)$, where $\lambda > 1$ and $p = (\frac{1}{2}, x)$ for $x \in \frac{1}{2}B_{\mathcal{H}}$, which means that

$$v = \left(\frac{\lambda + \lambda^{-1}}{2}, (\lambda - \lambda^{-1})x\right). \tag{8.0.1}$$

From this we can calculate, for each $v \in \mathbf{H}\backslash\{e\}$,

$$Q(v) = \frac{1}{4}\left[(\lambda + \lambda^{-1})^2 - (\lambda - \lambda^{-1})^2\right] = \frac{1}{4}\left[(\lambda^2 + 2 + \lambda^{-2}) - (\lambda^2 - 2 + \lambda^{-2})\right] = 1.$$

Conversely, if $v = (\gamma, x) \in \hat{\mathbf{H}}$, Lemma 2.9.13 shows that the two eigenvalues of v are $\gamma + \|x\|_{\mathcal{H}}$ and $\gamma - \|x\|_{\mathcal{H}}$, so $\det(v) = \gamma^2 - \|x\|_{\mathcal{H}}^2 = 1$, so indeed $\mathbf{H} = \hat{\mathbf{H}}$. Equation (8.0.1) is equivalent to saying that, for each $v \in \mathbf{H} \backslash \{e\}$, there exists a unique $\tau > 0$ and $x \in \frac{1}{2} B_{\mathcal{H}}$ such that

$$v = \left(\frac{f(\tau) + f(\tau)^{-1}}{2}, \tau x \right), \text{ where } f(\tau) = \sqrt{\frac{\tau^2}{4} + 1} + \frac{\tau}{2}.$$

Thus, the map $\varphi \colon \mathbf{H} \to V$ defined by

$$\varphi(v) = \begin{cases} 0 & \text{if } v = e \\ (0, \tau x) & \text{if } v = \left(\frac{f(\tau) + f(\tau)^{-1}}{2}, \tau x \right). \end{cases}$$

is a bijection, so $\mathbf{H}$ is a codimension-1 submanifold of V, and $\mathbf{H}$ is a C^∞ manifold modelled on $\mathcal{H}$. We now want to characterise the tangent space of $\mathbf{H}$ at the unit e, which we denote by T_e. To that end, let us consider an arbitrary $u \in V$ with $\operatorname{tr}(u) = 0$. Thus u has spectral decomposition $u = \lambda p - \lambda(e - p)$ for some $\lambda \in \mathbb{R}$ and primitive idempotent p. Let us define the path $\gamma \colon (-1, 1) \to V$ by $\gamma(t) = \exp tu$. Therefore, for any $t \in (-1, 1)$,

$$\det(\gamma(t)) = e^{t\lambda} e^{-t\lambda} = 1,$$

meaning that γ is a path in $\mathbf{H}$ with $\gamma(0) = e$. Furthermore, for any $t \in (-1, 1)$ we can calculate

$$\left. \frac{d}{dt} \right|_{t=0} \gamma(t) = \left. \frac{d}{dt} \right|_{t=0} [e^{\lambda t} p + e^{-\lambda t}(e - p)] = \lambda p - \lambda(e - p) = u.$$

This implies that $\{v \in V : \operatorname{tr}(v) = 0\} \subseteq T_e$. However, $\{v \in V : \operatorname{tr}(v) = 0\}$ is a closed subspace of codimension 1, indeed Lemma 2.9.13 means that

$$\{v \in V : \operatorname{tr}(v) = 0\} = \{0\} \times \mathcal{H},$$

so this inclusions means that $T_e = \{v \in V : \operatorname{tr}(v) = 0\}$.

As in the finite dimensional case dealt with by Lemmens in [42], it is possible to equip

$\mathbf{H}$ with a Finsler structure. We define the map $G\colon T\mathbf{H} \to \mathbb{R}$ by first defining the norm $|\cdot|_e$ on T_e by

$$|v|_e = \frac{1}{2}(M(v/e) - m(v/e)) = \frac{1}{2}(\max \sigma(v) - \min \sigma(v)) = \frac{1}{2}\operatorname{diam} \sigma(v).$$

This is the variation norm on T_e (with a factor of (1/2)). We can then define, for any $w \in \mathbf{H}$ and $v \in T_w$,

$$G(w, v) = |U_{w^{-1/2}} v|_e.$$

As U_a is a linear automorphism for any $a \in \mathbf{H}$, it follows that the pair $(\mathbf{H}, G)$ is a homogeneous Finsler manifold. The factor space PV_+° inherits this manifold structure in the natural way. The proof of Proposition 6.1.2 does not rely on finite dimensionality, and so shows that the Finsler distance on PV_+° generated by G coincides with Birkhoff's version of the Hilbert metric, d_H, where

$$d_H(u, v) = \frac{1}{2}\log(M(u/v)M(v/u)).$$

In [13], Claassens provides an explicit representation of the horofunctions of (PV_+°, d_H). He shows that the elements of $\overline{PV_+^\circ}^h$ are precisely the functions of the form h_r^x [13, Theorem 2], for $x \in \mathcal{B}_{\mathcal{H}}$ and $r \in [0, 1]$ with $\|x\|_{\mathcal{H}} \leq r$, where for any $(\gamma, y) \in PV_+^\circ$

$$h_x^r(\gamma, y) = \log\left(\frac{\gamma - \langle x, y\rangle + \sqrt{(\gamma - \langle x, y\rangle)^2 - (1 - r^2)(\gamma^2 - \|y\|_{\mathcal{H}}^2)}}{(1 + r)\sqrt{\gamma^2 - \|y\|_{\mathcal{H}}^2}}\right). \tag{8.0.2}$$

He also shows that if $\|x\|_{\mathcal{H}} = r < 1$, then $h^r_{\|x\|_{\mathcal{H}}} = h_{(1,x)}$. Claassens does not analyse the parts of $\overline{PV_+^\circ}^h$ in [13], so before proceeding we prove the following proposition:

Proposition 8.0.2. *The boundary of $\overline{PV_+^\circ}^h$ consists solely of singleton parts.*

Proof. In the proof of [13, Theorem 2], Claassens shows that if h_x^1 is a horofunction with $\|x\|_{\mathcal{H}} = 1$, then, if we define $u_n = (1, (1 - \frac{1}{n}x))$, $h_{u_n} \to h_x^1$. Recalling the definition of the gauge function M in Section 2.7, for any $n > m$,

$$M(u_n/u_m) = \inf\{\beta \in \mathbb{R} : u_n \leq \beta u_m\}.$$

Thus,

$$M(u_n/u_m) = \inf\left\{\beta \geq 1 : \beta - 1 \geq |\beta(1-\frac{1}{m}) - (1-\frac{1}{n})|\right\}.$$

For any $\beta \geq 1$, we can calculate that

$$\beta(1-\frac{1}{m}) - (1-\frac{1}{n}) \geq 0 \text{ if and only if } \beta \geq \frac{m}{n}\frac{n-1}{m-1}.$$

Thus, if $\beta \leq \frac{m}{n}\frac{n-1}{m-1}$, we can calculate that

$$\beta - 1 \geq (1-\frac{1}{n}) - \beta(1-\frac{1}{m}) \text{ if and only if } \beta \geq \frac{m}{n}\frac{2n-1}{2m-1}.$$

Thus, because $\frac{2n-1}{2m-1} \leq \frac{n-1}{m-1}$, we can conclude that

$$M(u_n/u_m) = \frac{m}{n}\frac{2n-1}{2m-1}.$$

A similar calculation shows that

$$M(u_m/u_n) = \frac{n}{m}, M(u_n/e) = 2 - \frac{1}{n}, \text{ and } M(e/u_n) = n.$$

Therefore

$$d_H(u_n, u_m) = \log\sqrt{\frac{2n-1}{2m-1}}, \text{ and } d_H(u_n, e) = \log\sqrt{2n-1},$$

from which we see that (u_n) is an almost-geodesic sequence. In fact, it lies on a geodesic. Claassens also proves in [13, Theorem 3] that the Busemann points of $\overline{PV_+^\circ}^h$ are precisely those horofunctions of the form h_x^1, where $\|x\|_{\mathcal{H}} = 1$. Now suppose that h_x^1 and h_y^1 are two Busemann points in $\overline{PV_+^\circ}^h$, and (u_n) is the geodesic defined above converging to h_x^1. Using the above in combination with Lemma 3.2.5, we can calculate, using (8.0.2), the detour cost $H(h_x^1, h_y^1)$:

$$\begin{aligned} H(h_x^1, h_y^1) &= \lim_{n\to\infty} \log\sqrt{2n-1} + h_y^r(u_n) \\ &= \lim_{n\to\infty} \log\sqrt{2n-1} + \log\left(\frac{1-\left(1-\frac{1}{n}\right)\langle x,y\rangle}{\sqrt{1-(1-\frac{1}{n})^2}}\right) \\ &= \lim_{n\to\infty} \log(n-(n-1)\langle x,y\rangle). \end{aligned}$$

This allows us to deduce that $H(h_x^1, h_y^1) < \infty$ if and only if $\langle x, y\rangle = 1$, which, by the Cauchy-Schwarz inequality, is true if and only if $x = y$. □

We are thus in the position to state the next main theorem of this chapter, which we prove in Section 8.4.

Theorem 8.0.3. *The exponential map* $\exp\colon T_e \to PV_+^\circ$ *can be extended to a homeomorphism between* $\overline{T}_e^h$ *and* $\overline{PV_+^\circ}^h$*, and this extension maps parts of* $\partial_B\overline{T}_e^h$ *onto parts of* $\partial_B\overline{PV}^h$*.*

8.1 Horoboundary of Spin factors

Let $V = \mathbb{R}\oplus\mathcal{H}$ be an infinite dimensional spin factor with unit e as above, equipped with the standard inner product and Jordan multiplication. We want to examine the horofunction boundary of the JB-algebra $(V, \|\cdot\|)$, with $\|\cdot\|$ the JB-algebra norm. Recall that $\|\cdot\| = \|\cdot\|_e = \|\cdot\|_\sigma$ where $\|\cdot\|_e$ is the order unit norm with e the distinguished order unit [4, Theorem 1.11] and $\|\cdot\|_\sigma$ is the spectral radius norm [4, Corollary 1.22]. In section 2.9.5 we showed that for any $(\gamma, x) \in V$

$$\|(\gamma, x)\| = |\gamma| + \|x\|_{\mathcal{H}}.$$

Thanks to this and Lemmas 3.4.2 and 3.4.3, which show that $\overline{V}^h = \overline{\mathbb{R}}^h \oplus \overline{\mathcal{H}}^h$, this investigation is made easier by the fact that we know the precise horofunction boundary of both $\mathcal{H}$ (see [27]) and $\mathbb{R}$. We recall that there are two classes of metric functionals in $\overline{\mathbb{R}}^h$, the internal metric functionals h_λ, and functionals of the form h^ε for $\varepsilon \in \{-1, 1\}$, where $h^\varepsilon(x) = -\varepsilon x$. In [27], Gutierrez showed that, if $\mathcal{H}$ is infinite dimensional then $\partial\overline{\mathcal{H}}^h = \overline{\mathcal{H}}^{h,e} \cup \overline{\mathcal{H}}^{h,\infty}$, where

$$\begin{aligned} \overline{\mathcal{H}}^{h,e} &= \{h^{x,c} : x \in \mathcal{H},\ c \geq \|x\|_{\mathcal{H}}\} \cup \{h^0\} \\ \overline{\mathcal{H}}^{h,\infty} &= \{h^x : 0 < \|x\|_{\mathcal{H}} \leq 1\}, \end{aligned} \tag{8.1.1}$$

where for all $z \in \mathcal{H}$,

$$h^{x,c}(z) = \sqrt{\|z\|_{\mathcal{H}}^2 - 2\langle x, z\rangle + c^2} - c, \text{ and, } h^x(z) = -\langle x, z\rangle. \tag{8.1.2}$$

With this, and introducing the notation, for $h_1 \in \overline{\mathbb{R}}^h$ and $h_2 \in \overline{\mathcal{H}}^h$,

$$h_1 \oplus h_2((\gamma, x)) = h_1(\lambda) + h_2(x)$$

we can prove:

Theorem 8.1.1. *If $\mathcal{H}$ is infinite dimensional, the horofunction boundary of V is given by*

$$\begin{aligned} \partial\overline{V}^{h,e} &= F = \{h_\lambda \oplus h^{x,c} : \lambda \in \mathbb{R},\ x \in \mathcal{H},\ c > \|x\|_{\mathcal{H}}\} \cup \{h_\lambda \oplus h^0 : \lambda \in \mathbb{R}\} \\ \partial\overline{V}^{h,\infty} &= I = \{h_\lambda \oplus h^x : \lambda \in \mathbb{R},\ 0 \leq \|x\|_{\mathcal{H}} \leq 1\} \cup \{h^\varepsilon \oplus h^x : \varepsilon \in \{1, -1\},\ 0 \leq \|x\|_{\mathcal{H}} \leq 1\} \\ &\quad \cup \{h^\varepsilon \oplus h^{x,c} : \varepsilon \in \{1, -1\},\ x \in \mathcal{H},\ c > \|x\|_{\mathcal{H}}\} \cup \{h^\varepsilon \oplus h_x : \varepsilon \in \{1, -1\},\ x \in \mathcal{H}\}. \end{aligned}$$

Proof. Lemma 3.4.2 in conjunction with (8.1.1) immediately gives that $\overline{V}^{h,\infty} \subseteq I$ and $\overline{V}^{h,e} \subseteq F$, because $I \cup F$ contains all possible functions $h_1 \oplus h_2$, where $h_1 \in \overline{\mathbb{R}}^h$, $h_2 \in \overline{\mathcal{H}}^h$, and at least one of h_1 or h_2 is not an internal metric functional. Conversely we now want to show that all functions in F and I satisfy the conditions of Lemma 3.4.3. We first note that for any $x \in \mathcal{H}$ and c with $c \geq \|x\|_{\mathcal{H}}$ Gutierrez showed in [27, Theorem 4.4] that if we choose the sequence $(y_n) \subseteq \mathcal{H}$ defined by, if $(e_n) \subseteq \mathcal{H}$ is an orthonormal sequence converging weakly to 0,

$$y_n = \sqrt{c^2 - \|x\|_{\mathcal{H}}^2} e_n + x$$

then $h_{y_n} \to h^{x,c}$. By picking the constant sequence, there also exists a sequence in $\mathbb{R}$ resulting in h_λ, so Lemma 3.4.3 means that $h_\lambda \oplus h^{x,c} \in \overline{V}^{h,e}$. For any $x \in B_{\mathcal{H}}$ Gutierrez also showed in [27, Theorem 4.4] that if we choose the sequence $(y_n) \subseteq \mathcal{H}$ defined by,

$$y_n = n\sqrt{1 - \|x\|_{\mathcal{H}}^2}e_n + nx$$

then $h_{y_n} \to h^{x,c}$. Choosing $x = 0$ we thus see $h_\lambda \oplus 0 \in \overline{V}^{h,e}$, too. Therefore $F \subseteq \overline{V}^{h,e}$. Using the same sequences for non-zero $x \in \mathcal{H}$ we immediately see that $h_\lambda \oplus h^x \in \overline{V}^{h,\infty}$. It is a simple calculation to show that $h_{\varepsilon n} \to h^\varepsilon \in \partial\overline{\mathbb{R}}^h$, so again Lemma 3.4.3 means that $h^\varepsilon \oplus h^x \in \overline{V}^{h,\infty}$ for any $x \in \mathcal{H}$ with $\|x\|_{\mathcal{H}} \leq 1$ and $\varepsilon \in \{-1, 1\}$, and also that $h^\varepsilon \oplus h^{x,c} \in \overline{V}^{h,\infty}$ for any $\varepsilon \in \{-1, 1\}$ and $x \in \mathcal{H}$ and $c \geq \|x\|_{\mathcal{H}}$. Finally, it also means that any $h^\varepsilon \oplus h_x \in \overline{V}^{h,\infty}$. Thus $I \subseteq \overline{V}^{h,\infty}$. □

If $\mathcal{H}$ is finite dimensional, then V is a proper geodesic metric space, so $\overline{V}^{h,e} = \emptyset$.

Corollary 8.1.2. *If $\mathcal{H}$ is finite dimensional then the metric compactification of V is given by*

$$\begin{aligned}\partial\overline{V}^h = &\{h_\lambda \oplus h^x : \lambda \in \mathbb{R},\ \|x\|_{\mathcal{H}} = 1\} \cup \{h^\varepsilon \oplus h^x : \varepsilon \in \{1, -1\},\ \|x\|_{\mathcal{H}} = 1\} \\ &\cup \{h^\varepsilon \oplus h_x : \varepsilon \in \{1, -1\},\ x \in \mathcal{H}\}.\end{aligned}$$

By combining Lemma 3.4.4, Proposition 7.1.3 and Proposition 7.1.4 we see that:

Proposition 8.1.3. *The Busemann points of V are precisely:*

$$\begin{aligned}\partial_B\overline{V}^h = &\{h_\lambda \oplus h^x : \lambda \in \mathbb{R},\ \|x\|_{\mathcal{H}} = 1\} \cup \{h^\varepsilon \oplus h^x : \varepsilon \in \{1, -1\},\ \|x\|_{\mathcal{H}} = 1\} \\ &\cup \{h^\varepsilon \oplus h_x : \varepsilon \in \{1, -1\},\ x \in \mathcal{H}\}.\end{aligned}$$

It is interesting to note that a spin-factor is finite dimensional if and only if its horofunction boundary consists entirely of Busemann points.

Remark 8.1.4. Note that the unions defining both $\partial\overline{V}^h$ in Theorem and $8.1.1\partial_B\overline{V}^h$ in Proposition 8.1.3 are disjoint, which gives us a useful way of categorising these elements. We shall say two horofunctions in $\partial\overline{V}^h$ are of the same *type* if they belong to the same constituent set in the disjoint union defined in Theorem 8.1.1. For example $h_{\lambda_1} \oplus h^{x_1}$ is of the same type as $h_{\lambda_2} \oplus h^{x_2}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x_1, x_2 \in B_{\mathcal{H}}$, but $h_\lambda \oplus h^{x_1}$ is *not* of the same type as $h_\lambda \oplus h^{x_1,c}$ for $c, \lambda \in \mathbb{R}$ and $x \in cB^\circ_{\mathcal{H}}$. We shall also call the superscript data of each Busemann point in Proposition 8.1.3 its *data at infinity*. For example, for $h_\lambda \oplus h^x \in \partial_B\overline{V}^h$ its data at infinity is the point $x \in S_{\mathcal{H}}$, whereas for the point $h^\varepsilon \oplus h^x \in \partial_B\overline{V}^h$ its data at infinity is the pair $(\varepsilon, x) \in \{-1, 1\} \times S_{\mathcal{H}}$.

Although using the fact that the spin-factor can be seen as an ℓ^1 sum provides an easy proof for the characterising of its horofunction compactification, it does not make clear the relation between a spin factor's horofunctions and its JB-algebra structure. We would expect to be able to express the Busemann points in terms of a spectral decomposition similar to Theorem 5.2.1. The next theorem shows that we can, indeed, do this. Recall that for a subalgebra $A \subseteq V$, we define, for any $v \in A$, $\Lambda_A(v)$ to be the maximum eigenvalue of v considered as an element of the algebra A, and we define $V(p)$ to be the Peirce 1-space of an idempotent p. Also recall that for a collection of orthogonal idempotents $\{p_1, \ldots, p_k\}$ and any $I \subseteq \{1, \ldots, k\}$ we define $p_I = \sum_{i \in I} p_i$, where as usual the sum over the empty set is defined to be 0.

Theorem 8.1.5. *For any Jordan frame $p_1 = p, p_2 = e - p$, $I, J \subseteq \{1, 2\}$ with $I \cap J = \emptyset$, and $I \cup J \neq \emptyset$, and $\alpha \in \mathbb{R}^{I \cup J}$ where $\min\{\alpha_i\} = 0$, we define the function $h^{I,J}_{p,\alpha} \colon V \to \mathbb{R}$ by, for any $v \in V$,*

$$h^{I,J}_{p,\alpha}(v) = \max\left\{\Lambda_{V(p_I)}\left(-U_{p_I}(v) - \sum_{i \in I} \alpha_i p_i\right), \Lambda_{V(p_J)}\left(U_{p_J}(v) - \sum_{j \in J} \alpha_j p_j\right)\right\}, \qquad (8.1.3)$$

where if I or J is empty, the corresponding term in the max is omitted. Each $h^{I,J}_{p,\alpha}$ is a Busemann point, and furthermore every $h \in \partial_B\overline{V}^h$ is of the form $h^{I,J}_{p,\alpha}$.

Proof. Fix some primitive idempotent $p = (\frac{1}{2}, x) \in V$, immediately implying that $e - p = (\frac{1}{2}, -x)$. A direct computation using the definition of the quadratic representation (2.9.5) shows that, for any $v = (\gamma, u) \in V$,

$$U_p(v) = (\gamma + 2\langle u, x\rangle)p, \text{ and } U_{e-p}(v) = (\gamma - 2\langle u, x\rangle)(e-p). \tag{8.1.4}$$

As $V(q) = \mathbb{R}q$ for any primitive idempotent q, this means that, for any $v = (\gamma, u) \in V$,

$$\begin{aligned} h_{p,\alpha}^{\{1\},\{2\}}(v) &= \max\{-\gamma - 2\langle u, x\rangle - \alpha_1, \gamma - 2\langle u, x\rangle - \alpha_2\} \\ &= \max\{-\gamma - \alpha_1, \gamma - \alpha_2\} - \langle u, 2x\rangle. \end{aligned}$$

Thus, for any $\lambda \geq 0$ and $x \in S_{\mathcal{H}}$, if we choose $\alpha = (0, 2\lambda)$ and $p = \frac{1}{2}(1, x)$ we see that $h_{p,\alpha}^{\{1\},\{2\}}$ is exactly equal to the Busemann point $h_\lambda \oplus h^x$ as in Proposition 8.1.3. Similarly, for any $\lambda < 0$, if we choose $\alpha = (-2\lambda, 0)$ we see that $h_{p,\alpha}^{\{1\},\{2\}} = h_\lambda \oplus h^x$. Likewise, using (8.1.4), we deduce that for any $x \in S_{\mathcal{H}}$, by choosing $p = \frac{1}{2}(1, x)$,

$$h^1 \oplus h^x = h_{p,0}^{\{1\},\emptyset}, \text{ and } h^{-1} \oplus h^x = h_{p,0}^{\emptyset,\{2\}}.$$

Finally, if $I = \{1, 2\}$, we see for any primitive idempotent $p = (\frac{1}{2}, w)$ and $\alpha \in \mathbb{R}^2$ with $\min\{\alpha_i\} = 0$ that, for any $v = (\gamma, u) \in V$,

$$\begin{aligned} h_{p,\alpha}^{I,\emptyset}(v) &= \Lambda_{V(e)}\left(-U_e(v) + (\alpha_2 - \alpha_1)p - \alpha_2 e\right) \\ &= \Lambda\left(-v + (\alpha_2 - \alpha_1)p\right) - \alpha_2 \\ &= \sup_{\phi \in S(V)} \phi(-v + (\alpha_2 - \alpha_1)p) - \alpha_2 \\ &= \sup_{\|y\|_{\mathcal{H}} \leq 1} -\gamma + \langle(\alpha_2 - \alpha_1)w - u, y\rangle - \frac{\alpha_1 + \alpha_2}{2} \end{aligned}$$

We see that the supremum is attained when $y = \frac{(\alpha_2 - \alpha_1)w - u}{\|(\alpha_2 - \alpha_1)w - u\|_{\mathcal{H}}}$, so for any $x \in \mathcal{H}$ we can find $\alpha_2 \in [0,\infty)$ and $w \in \frac{1}{2}S_{\mathcal{H}}$ such that $x = \alpha_2 w$, and choosing $\alpha_1 = 0$ we thus get

$$h_{p,\alpha}^{I,\emptyset}(v) = -\gamma + \|u-x\|_{\mathcal{H}} - \|x\|_{\mathcal{H}} = h^1 \oplus h_x(v).$$

Similarly, for any $x \in \mathcal{H}$, choosing $J = \{1,2\}$, we can find an appropriate $\alpha \in \mathbb{R}^2$ and idempotent p such that

$$h_{p,\alpha}^{\emptyset,J}(v) = h^{-1} \oplus h_x(v).$$

Thus every Busemann point in Proposition 8.1.3 is of the form (8.1.3), and similarly by just reversing the choices in the above arguments, we see that every function of the form (8.1.3) is in fact a Busemann point. □

8.1.1 Geometry of a Spin Factor's Busemann Points

It is well known that if $V = Y \oplus Z$ is a direct sum of Banach spaces with the ℓ^1 product norm, $\|(y,z)\| = \|y\|_Y + \|z\|_Z$, then the continuous dual V^* is the Banach space $V^* = Y^* \oplus Z^*$ equipped with the maximum norm, $\|(y^*,z^*)\|_* = \max\{\|y^*\|_{Y^*}, \|z^*\|_{Z^*}\}$ [50, Theorem 1.10.13]. Thus, the unit ball of the dual of the spin factor is simply given by

$$B_{V^*} = \{(\lambda,x) \in V : \max\{|\lambda|, \|x\|_{\mathcal{H}}\} \le 1\}.$$

If we recall the definition of type and data at infinity in Remark 8.1.4 above, then the following proposition is a direct consequence of Proposition 7.1.4 and Theorem 3.4.5:

Proposition 8.1.6. *Two Busemann points of V are in the same part of the boundary if and only if they are of the same type, and their data at infinity is equal.*

Theorem 3.4.5 also gives an easy way of calculating the detour distance on the Busemann boundary. Using Theorem 8.1.3 this allows us to prove an analogous result to Theorem

5.2.2. Recall that if p and q are orthogonal idempotents, we define $V_{(p,q)} = V_p \oplus V_q$, which is a sub-algebra of V. On $V_{(p,q)}$, we can define the variation semi-norm $\|\cdot\|_{\text{var}}$ on $V_{(p,q)}$, where

$$\|u\|_{\text{var}} = \Lambda_{V_{p,q}}(u) + \Lambda_{V_{p,q}}(-u).$$

Theorem 8.1.7. *Given two Busemann points $h^{I,J}_{p,\alpha}$ and $h^{I',J'}_{\beta,q}$ as in Theorem 8.1.5.*

(i) $h^{I,J}_{p,\alpha}$ and $h^{I',J'}_{\beta,q}$ are in the same part if and only if $p_I = q_{I'}$ and $p_J = q_{J'}$

(ii) If $h^{I,J}_{p,\alpha}$ and $h^{I',J'}_{\beta,q}$ are in the same part, then their detour distance is given by

$$\delta(h^{I,J}_{p,\alpha}, h^{I',J'}_{\beta,q}) = \|a-b\|_{\text{var}},$$

where $a = \sum_{i\in I}\alpha_i p_i + \sum_{j\in J}\alpha_j p_j$ and $b = \sum_{i\in I'}\beta_i q_i + \sum_{j\in J'}\beta_j q_j$.

Proof. Let us define, for all $x \in S_{\mathcal{H}}$, $\mathcal{P}^x = \{h_\lambda \oplus h^x : \lambda \in \mathbb{R}\}$, and for each $\varepsilon \in \{-1,1\}$, $P_\varepsilon = \{h^\varepsilon \oplus h_u : u \in \mathcal{H}\}$. Propositions 8.1.6 and 8.1.3 mean that the parts of $\partial_B \overline{V}^h$ are precisely all the sets $\mathcal{P}^x$ and P_ε, along with all the singletons $\{h^\varepsilon \oplus h^x\}$, where $\varepsilon \in \{-1,1\}$ and $x \in S_{\mathcal{H}}$.

Fix some $x \in \mathcal{H}$. For any $\lambda \geq 0$, the proof of Theorem 8.1.5 shows that if we choose $\alpha = (0, 2\lambda)$, $p = (\frac{1}{2}, x/2)$, then $h_\lambda \oplus h^x = h^{\{1\},\{2\}}_{p,\alpha}$, and similarly if $\lambda < 0$, then choosing $\alpha = (-2\lambda, 0)$ and $p = (\frac{1}{2}, x/2)$ gives $h_\lambda \oplus h^x = h^{\{1\},\{2\}}_{p,\alpha}$. By reversing the argument in the proof of Theorem 8.1.5, we also see that every horofunction of the form $h^{\{1\},\{2\}}_{p,\alpha}$ where $p = (\frac{1}{2}, x/2)$ is in $\mathcal{P}^x$. We should note that $h^{\{1\},\{2\}}_{p,\alpha} = h^{\{2\},\{1\}}_{e-p,\alpha'}$. Thus

$$\mathcal{P}^x = \left\{h^{\{1\},\{2\}}_{p,\alpha} : p = (\frac{1}{2}, x/2)\right\} = \left\{h^{\{2\},\{1\}}_{e-p,\alpha} : p = (\frac{1}{2}, x/2)\right\}.$$

Furthermore, we know, from Theorem 3.4.2, that for any $h_\lambda \oplus h^x, h_{\lambda'} \oplus h^x \in \mathcal{P}^x$,

$$\delta(h_\lambda \oplus h^x, h_{\lambda'} \oplus h^x) = 2|\lambda - \lambda'|.$$

Now, if $h_\lambda \oplus h^x = h_{p,\alpha}^{\{1\},\{2\}}$ and $h_{\lambda'} \oplus h^x = h_{p,\beta}^{\{1\},\{2\}}$, we set $a = \alpha_1 p + \alpha_2(e-p)$, and $b = \beta_1 p + \beta_2(e-p)$. We can calculate, due to the fact that $V_{p,e-p} = \mathbb{R}p \oplus \mathbb{R}(e-p)$, that

$$\|a-b\|_{\text{var}} = \max_{i=1,2}\{\alpha_i - \beta_i\} + \max_{i=1,2}\{\beta_i - \alpha_i\}.$$

As $\alpha = (-2\lambda, 0)$ or $\alpha = (0, 2\lambda)$ and $\beta = (-2\lambda', 0)$ or $\beta = (0, 2\lambda')$ depending on the sign of λ and λ' respectively, it is simple to verify that

$$\|a-b\|_{\text{var}} = 2|\lambda - \lambda'| = \delta(h_\lambda \oplus h^x, h_{\lambda'} \oplus h^x). \tag{8.1.5}$$

Now fix $\varepsilon \in \{-1, 1\}$. Once again, the proof of Theorem 8.1.5 shows that every $h \in \mathcal{P}_\varepsilon$ is of the form $h_{p,\alpha}^{\{1,2\},\emptyset}$ if $\varepsilon = -1$ or $h_{p,\alpha}^{\emptyset,\{1,2\}}$ if $\varepsilon = 1$. Conversely, it also shows that if $\alpha = (\alpha_1, \alpha_2)$, and $\alpha' = (\alpha_2, \alpha_1)$, then $h_{p,\alpha}^{\{1,2\},\emptyset} = h_{e-p,\alpha'}^{\{1,2\},\emptyset}$ and $h_{p,\alpha}^{\emptyset,\{1,2\}} = h_{e-p,\alpha'}^{\emptyset,\{1,2\}}$. It thus also allows us to conclude that every Busemann point of the form $h_{p,\alpha}^{\{1,2\},\emptyset}$ is in $\mathcal{P}_{-1}$ and every Busemann point of the form $h_{p,\alpha}^{\emptyset,\{1,2\}}$ is in $\mathcal{P}_1$. Thus:

$$\mathcal{P}_1 = \left\{h_{p,\alpha}^{\emptyset,\{1,2\}} : p, \alpha \text{ as in Theorem. 8.1.5}\right\}, \text{and } \mathcal{P}_{-1} = \left\{h_{p,\alpha}^{\{1,2\},\emptyset} : p, \alpha \text{ as in Theorem. 8.1.5}\right\}.$$

Furthermore, we know, from Theorem 3.4.2, that for any $h^{-1} \oplus h_x, h^{-1} \oplus h_y \in \mathcal{P}_{-1}$,

$$\delta(h^\varepsilon \oplus h_x, h^\varepsilon \oplus h_y) = 2\|x-y\|_{\mathcal{H}}.$$

Let us now fix $h_{p,\alpha}^{\{1,2\},\emptyset}, h_{q,\beta}^{\{1,2\},\emptyset} \in \mathcal{P}_{-1}$ such that $h_{p,\alpha}^{\emptyset,\{1,2\}} = h^\varepsilon \oplus h_x$ and $h_{q,\beta}^{\emptyset,\{1,2\}} = h^\varepsilon \oplus h_y$. The proof of Theorem 8.1.5 shows that $p = (\frac{1}{2}, \frac{1}{2\|x\|_{\mathcal{H}}}x)$, $\alpha = (0, 2\|x\|_{\mathcal{H}})$, and $q = (\frac{1}{2}, \frac{1}{2\|y\|_{\mathcal{H}}}x)$, $\beta = (0, 2\|y\|_{\mathcal{H}})$. As above we define $a = \alpha_1 p + \alpha_2(e-p)$ and $b = \beta_1 q + \beta_2(e-q)$. As

$p_I = q_I = e$,

$$\begin{aligned}
\|a-b\|_{\text{var}} &= \Lambda(\alpha_1 p + \alpha_2(e-p) - \beta_1 q - \beta_2(e-q)) + \Lambda(\beta_1 q - \beta_2(e-q) - \alpha_1 p - \alpha_2(e-p)) \\
&= \sup_{\phi \in S(V)} \phi\left(\alpha_2(e-p) - \beta_2(e-q)\right) + \sup_{\varphi \in S(V)} \varphi\left(\beta_2(e-q) - \alpha_2(e-p)\right) \\
&= (\|x\|_{\mathcal{H}} - \|y\|_{\mathcal{H}}) + \sup_{\|w\|_{\mathcal{H}} \le 1} \langle y-x, w\rangle + (\|y\|_{\mathcal{H}} - \|x\|_{\mathcal{H}}) + \sup_{\|z\|_{\mathcal{H}} \le 1} \langle x-y, z\rangle \\
&= 2\|x-y\|_{\mathcal{H}} \\
&= \delta(h^\varepsilon \oplus h_x, h^\varepsilon \oplus h_y). \qquad (8.1.6)
\end{aligned}$$

Finally, let us consider a singleton Busemann point $h^\varepsilon \oplus h^x$. The proof of Theorem 8.1.5 shows that if we choose $p = (\frac{1}{2}, x/2)$, then $h^\varepsilon \oplus h^x = h^{\{1\},\emptyset}_{p,0}$ if $\varepsilon = 1$ and $h^\varepsilon \oplus h^x = h^{\emptyset,\{2\}}_{p,0}$ if $\varepsilon = -1$. It also shows that $h^{\{1\},\emptyset}_{p,0} = h^{\{2\},\emptyset}_{e-p,0}$ and $h^{\emptyset,\{2\}}_{p,0} = h^{\emptyset,\{1\}}_{e-p,0}$. We have thus shown that every part of the Busemann boundary consists solely of functions of the form given in Theorem 8.1.3 that satisfy the condition on their defining idempotents given in *(i)*. By reversing the above arguments it is simple to check by examination that if two Busemann points satisfy *(i)* then they must be in the same part. Equations (8.1.5) and (8.1.6) give part *(ii)*. □

8.2 Infinite Dimensional Spin Factor with the Thompson Metric on the Positive Cone

With V an infinite dimensional spin factor as before, we now consider the infinite dimensional Lorentz cone V_+°, equipped with the Thompson metric d_T. We use Claassens' representation of the gauge function M [13, Prop. 2] to calculate the horofunction compactification $\overline{V_+^\circ}^h$. For the convenience of the reader we recall those results: for any

$(\lambda, x), (\gamma, y) \in V_+^\circ$,

$$M\left({}^{(\lambda,x)}/_{(\gamma,y)}\right) = \frac{\gamma\lambda - \langle x, y\rangle + \sqrt{(\gamma\lambda - \langle x, y\rangle)^2 - (\gamma^2 - \|y\|_{\mathcal{H}}^2)(\lambda^2 - \|x\|_{\mathcal{H}}^2)}}{\gamma^2 - \|y\|_{\mathcal{H}}^2}. \tag{8.2.1}$$

In case one of the elements is the identity, we can simplify this to

$$M\left({}^{(\lambda,x)}/_{(1,0)}\right) = \lambda + \|x\|_{\mathcal{H}}, \quad \text{and} \quad M\left({}^{(1,0)}/_{(\lambda,x)}\right) = \frac{1}{\lambda - \|x\|_{\mathcal{H}}}. \tag{8.2.2}$$

In the finite dimensional case, this analytical expression of M is not needed, as shown by Lemmens in [42], because for any JB-algebra A, the gauge function $M\colon A \times A_+^\circ \to \mathbb{R}$ is continuous with respect to the product topology where each factor is equipped with the JB-algebra norm (see Lemma 2.7.8), and in finite dimensions the compactness of the unit ball means we only need to consider norm convergent sequences. In the infinite dimensional case, we have to consider nets with no norm convergent subnets, and the M function is not in general continuous when A and V_+° are equipped with the weak topology. For example, in our case with V an infinite dimensional spin factor, let $(e_n) \subseteq S_{\mathcal{H}}$ be an orthonormal sequence converging weakly to 0. If we define $p_n = \frac{1}{2}(1, e_n)$, $M(p_n/e) = 1$ for all $n \in \mathbb{N}$, but (p_n) converges weakly to $\frac{1}{2}e$, and $M(\frac{1}{2}e/e) = \frac{1}{2}$. That being said, we shall see that all Busemann points arise as the limit of sequences, which, when normalised, have norm convergent subsequences, and as such we need a lemma from [42].

Lemma 8.2.1. *For any JB-algebra $(A, \|\cdot\|)$, the exponential map $\exp\colon (A, \|\cdot\|) \to (A_+^\circ, d_T)$ is an isometry when restricted to $\operatorname{span}\{p_1, \ldots, p_k\}$, where $\{p_1, \ldots, p_k\}$ is a collection of orthogonal primitive idempotents. Furthermore, for any $u, v \in \operatorname{span}\{p_1, \ldots, p_k\}$, with $\|v\| = 1$, if $\gamma\colon A \to \mathbb{R}$ is defined by $\gamma(t) = tv + u$, then $\varphi\colon t \mapsto \exp(\gamma(t))$ is a geodesic.*

Proof. The proof is exactly that of [42, Lemma 3.1], as it does not rely on finite dimensionality in any way. □

We start with a lemma that is not needed in the finite dimensional case.

Lemma 8.2.2. *If (u_α) is an eventually bounded net in V_+, there exists a subnet (u_β), a $\lambda > 0$, an $x \in V_+$, and a $c \in \mathbb{R}$ with $\lambda > c \geq \|x\|$ such that $h_{u_\beta} \to h^{\lambda,x,c}$, where, for any $(\gamma, y) \in V_+$,*

$$h^{\lambda,x,c}((\gamma,y)) = \max\left\{ \log\left(\frac{\gamma\lambda - \langle x,y\rangle + \sqrt{(\gamma\lambda - \langle x,y\rangle)^2 - (\gamma^2 - \|y\|_{\mathcal{H}}^2)(\lambda^2 - c^2)}}{\lambda^2 - c^2}\right), \right.$$
$$\left. \log\left(\frac{\gamma\lambda - \langle x,y\rangle + \sqrt{(\gamma\lambda - \langle x,y\rangle)^2 - (\gamma^2 - \|y\|_{\mathcal{H}}^2)(\lambda^2 - c^2)}}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right) \right\}$$
$$- \max\left\{\log(\lambda + c), \log\left(\frac{1}{\lambda - c}\right)\right\}. \tag{8.2.3}$$

Furthermore, if $\|x\|_{\mathcal{H}} = c$ then $h^{\lambda,x,c} = h_{(\lambda,x)}$, and if $\|x\|_{\mathcal{H}} < c$, then $h^{\lambda,x,c}$ is not an internal metric functional.

Proof. As (u_α) is eventually bounded, there exists a subnet of (u_α), which we again label (u_α), which is bounded. As $d_T(e, u_\alpha) \geq \log(\lambda_\alpha + \|x_\alpha\|_{\mathcal{H}})$ by (8.2.2) this means (λ_α) and $(\|x_\alpha\|_{\mathcal{H}})$ must be bounded too. There must thus exist a subnet so that $\lambda_\beta \to \lambda \in \mathbb{R}_{>0}$, $\|x_\beta\|_{\mathcal{H}} \to c \in \mathbb{R}$, and by Banach-Alaoglu there exists some $x \in \mathcal{H}$ which x_β converges to weakly, with $\|x\|_{\mathcal{H}} \leq c$. As we are assuming $d_T(e, u_\beta)$ is bounded, there exists some $K > 0$ such that $\lambda_\beta - \|x_\beta\|_{\mathcal{H}} > K$ for all β, because by (8.2.2)

$$d_T(e, u_\beta) \geq \log\left(\frac{1}{\lambda_\beta - \|x_\beta\|_{\mathcal{H}}}\right).$$

Thus, $\lambda > c$. Therefore, for any β and any $(\gamma, y) \in V_+^\circ$ we have, using equations (8.2.1) and (8.2.2), that

$$\lim_\beta h_{u_\beta}((\gamma,y)) = \max\left\{ \log\left(\frac{\gamma\lambda - \langle x,y\rangle + \sqrt{(\gamma\lambda - \langle x,y\rangle)^2 - (\gamma^2 - \|y\|_{\mathcal{H}}^2)(\lambda^2 - c^2)}}{\lambda^2 - c^2}\right), \right.$$
$$\left. \log\left(\frac{\gamma\lambda - \langle x,y\rangle + \sqrt{(\gamma\lambda - \langle x,y\rangle)^2 - (\gamma^2 - \|y\|_{\mathcal{H}}^2)(\lambda^2 - c^2)}}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right) \right\}$$
$$- \max\left\{\log(\lambda + c), \log\left(\frac{1}{\lambda - c}\right)\right\}.$$

It is clear from the above equation that if $c = \|x\|_{\mathcal{H}}$, then $h_{u_\beta} \to h_{(\lambda,x)}$. Conversely, assume by way of contradiction that $\|x\|_{\mathcal{H}} < c$, but $h^{\lambda,x,c} = h_{(\eta,z)}$ for some $(\eta, z) \in V_+^\circ$. Let us first consider the case where $x = 0$. Thus, for any $(\gamma, y) \in V_+^\circ$,

$$\begin{aligned} h^{\lambda,0,c}(\gamma, y) = \max\Bigg\{ & \log\left(\frac{\gamma\lambda + \sqrt{(\gamma\lambda)^2 - (\gamma^2 - \|y\|_{\mathcal{H}}^2)(\lambda^2 - c^2)}}{\lambda^2 - c^2}\right), \\ & \log\left(\frac{\gamma\lambda + \sqrt{(\gamma\lambda)^2 - (\gamma^2 - \|y\|_{\mathcal{H}}^2)(\lambda^2 - c^2)}}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right)\Bigg\} \\ & - \max\left\{\log(\lambda + c), \log\left(\frac{1}{\lambda - c}\right)\right\}. \end{aligned}$$

It is therefore clear that $h^{\lambda,0,c}((\gamma, y)) = h^{\lambda,0,c}((\gamma, y'))$ whenever $\|y\|_{\mathcal{H}} = \|y'\|_{\mathcal{H}}$. However, if $z \neq 0$, then

$$\begin{aligned} h_{(\eta,z)}((\gamma, y)) = \max\Bigg\{ & \log\left(\frac{\gamma\eta - \langle z, y\rangle + \sqrt{(\gamma\eta - \langle z, y\rangle)^2 - (\gamma^2 - \|y\|_{\mathcal{H}}^2)(\eta^2 - \|z\|_{\mathcal{H}}^2)}}{\eta^2 - \|z\|_{\mathcal{H}}^2}\right), \\ & \log\left(\frac{\gamma\eta - \langle z, y\rangle + \sqrt{(\gamma\eta - \langle z, y\rangle)^2 - (\gamma^2 - \|y\|_{\mathcal{H}}^2)(\eta^2 - \|z\|_{\mathcal{H}}^2)}}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right)\Bigg\} \\ & - \max\left\{\log(\eta + \|z\|_{\mathcal{H}}), \log\left(\frac{1}{\eta - \|z\|_{\mathcal{H}}}\right)\right\}, \end{aligned}$$

from which we see that $h_{(\eta,x)}((\eta, -z)) > h_{(\eta,x)}((\eta, z))$. Thus we must have $z = 0$. We can therefore write, for any $(\gamma, 0) \in V_+$,

$$h_{(\eta,0)}((\gamma, 0)) = \max\left\{ \log\left(\frac{\gamma}{\eta}\right), \log\left(\frac{\eta}{\gamma}\right) \right\} - \max\left\{\log(\eta), \log\left(\frac{1}{\eta}\right)\right\},$$

whereas

$$h^{\lambda,0,c}((\gamma, 0)) = \max\left\{ \log\left(\frac{\gamma}{\lambda - c}\right), \log\left(\frac{\lambda + c}{\gamma}\right) \right\} - \max\left\{\log(\lambda + c), \log\left(\frac{1}{\lambda - c}\right)\right\}.$$

Now

$$\min_{\gamma} h^{\lambda,0,c}((\gamma,0)) = h^{\lambda,0,c}((\sqrt{\lambda^2-c^2},0)) = \min\left\{\log(\sqrt{\lambda^2-c^2}), \log\left(\frac{1}{\sqrt{\lambda^2-c^2}}\right)\right\},$$

and

$$\min_{\gamma} h_{(\eta,0)}((\gamma,0)) = -\max\left\{\log(\eta), \log\left(\frac{1}{\eta}\right)\right\}.$$

By assumption these two minima must be equal, so $\eta = \sqrt{\lambda^2-c^2}$. Thus for any $(\gamma,y) \in V_+^\circ$ we have

$$\begin{aligned} h_{(\sqrt{\lambda^2-c^2},0)}((\gamma,y)) = &\max\left\{ \log\left(\frac{\sqrt{\lambda^2-c^2}(\gamma+\|y\|_{\mathcal{H}})}{\lambda^2-c^2}\right), \log\left(\frac{\sqrt{\lambda^2-c^2}(\gamma+\|y\|_{\mathcal{H}})}{\gamma^2-\|y\|_{\mathcal{H}}^2}\right) \right\} \\ &- \max\left\{\log(\sqrt{\lambda^2-c^2}), \log\left(\frac{1}{\sqrt{\lambda^2-c^2}}\right)\right\}, \end{aligned}$$

whereas

$$\begin{aligned} h^{\lambda,0,c}((\gamma,y)) = &\max\left\{ \log\left(\frac{\lambda\gamma+\sqrt{\|y\|_{\mathcal{H}}^2(\lambda^2-c^2)+\gamma^2c^2}}{\lambda^2-c^2}\right), \log\left(\frac{\lambda\gamma+\sqrt{\|y\|_{\mathcal{H}}^2(\lambda^2-c^2)+\gamma^2c^2}}{\gamma^2-\|y\|_{\mathcal{H}}^2}\right) \right\} \\ &- \max\left\{\log(\lambda+c), \log\left(\frac{1}{\lambda-c}\right)\right\}. \end{aligned}$$

However, these two cannot be equal for all $(\gamma,y) \in V_+^\circ$. Indeed, fix (γ,y) where $\gamma^2 - \|y\|^2 > 10 + \lambda^2 - c^2$, and define $f\colon (1,\log 10) \to \mathbb{R}$ by

$$f(\varepsilon) = h^{\lambda,0,c}((\gamma,(1+e^{\varepsilon})y)) - h^{\lambda,0,c}((\gamma,y)) = \log\left(\frac{\lambda\gamma+\sqrt{(1+e^{\varepsilon})\|y\|_{\mathcal{H}}^2(\lambda^2-c^2)+\gamma^2c^2}}{\lambda\gamma+\sqrt{\|y\|_{\mathcal{H}}^2(\lambda^2-c^2)+\gamma^2c^2}}\right),$$

and $g\colon (1,\log 10) \to \mathbb{R}$ by

$$g(\varepsilon) = h_{(\sqrt{\lambda^2-c^2},0)}((\gamma,(1+e^{\varepsilon})y)) - h_{(\sqrt{\lambda^2-c^2},0)}((\gamma,y)) = \varepsilon + \log\left(\frac{\|y\|_{\mathcal{H}}}{\gamma+\|y\|_{\mathcal{H}}}\right).$$

We see that g is linear in ε and f isn't, but f and g must be equal if $h^{\lambda,0,c} = h_{(\sqrt{\lambda^2-c^2},0)}$, causing a contradiction. We now consider the case when $x \neq 0$. Examining $\min_\gamma h^{\lambda,x,c}((\gamma,0)) =$

$\min_\gamma h^{(\eta,z)}((\gamma,0))$ similarly to the case when $x=0$, we calculate that $\eta^2-\|z\|^2=\lambda^2-c^2$. Furthermore, we know that $\min_y h_{(\eta,z)}((\eta,y))=h_{(\eta,z)}((\eta,z))$, meaning that

$$\begin{aligned}\min_y h^{\lambda,x,c}((\eta,y)) = \max\Bigg\{ &\log\left(\frac{\eta\lambda-\langle x,z\rangle+\sqrt{(\eta\lambda-\langle x,z\rangle)^2-(\eta^2-\|z\|_{\mathcal{H}}^2)(\lambda^2-c^2)}}{\lambda^2-c^2}\right),\\ &\log\left(\frac{\eta\lambda-\langle x,z\rangle+\sqrt{(\eta\lambda-\langle x,z\rangle)^2-(\eta^2-\|z\|_{\mathcal{H}}^2)(\lambda^2-c^2)}}{\eta^2-\|z\|_{\mathcal{H}}^2}\right)\Bigg\}\\ -\max&\left\{\log(\lambda+c),\log\left(\frac{1}{\lambda-c}\right)\right\}\\ &= h^{\lambda,x,c}((\eta,z)).\end{aligned}$$

The fact that equality in the Cauchy-Schwarz inequality is achieved if and only if the two elements are linearly dependent means that by applying the Cauchy-Schwarz inequality to the above equation means $h^{\lambda,x,c}((\eta,z))>h^{\lambda,x,c}((\eta,\frac{\|z\|_{\mathcal{H}}}{\|x\|_{\mathcal{H}}}x))$ unless $x=\xi z$ for some $\xi>0$, in which case $h^{\lambda,x,c}((\eta,z))=h^{\lambda,x,c}((\eta,\frac{\|z\|_{\mathcal{H}}}{\|x\|_{\mathcal{H}}}x))$. Thus, to avoid a contradiction, $x=\xi z$ for $\xi>0$. Now, let us define the functions $\Phi_z,\Psi_z\colon(\|z\|_{\mathcal{H}},\infty)\times(0,\infty)\to\mathbb{R}$ by

$$\Phi_z(\gamma)=h_{(\eta,z)}(\gamma,-z)-h_{(\eta,z)}(\gamma,z),\ \text{and}\ \Psi_z(\gamma)=h^{\lambda.x,c}(\gamma,-z)-h_{\lambda,x,c}(\gamma,z). \tag{8.2.4}$$

A direct calculation shows that

$$\Phi_z(\gamma)=\log\left(\frac{\eta\gamma+\|z\|_{\mathcal{H}}^2+\sqrt{(\eta\gamma+\|z\|_{\mathcal{H}}^2)^2-(\gamma^2-\|z\|_{\mathcal{H}}^2)(\lambda^2-c^2)}}{\eta\gamma-\|z\|_{\mathcal{H}}^2+\sqrt{(\eta\gamma-\|z\|_{\mathcal{H}}^2)^2-(\gamma^2-\|z\|_{\mathcal{H}}^2)(\lambda^2-c^2)}}\right),$$

and

$$\Psi_z(\gamma)=\log\left(\frac{\lambda\gamma+\xi\|z\|_{\mathcal{H}}^2+\sqrt{(\lambda\gamma+\xi\|z\|_{\mathcal{H}}^2)^2-(\gamma^2-\|z\|_{\mathcal{H}}^2)(\lambda^2-c^2)}}{\lambda\gamma-\xi\|z\|_{\mathcal{H}}^2+\sqrt{(\lambda\gamma-\xi\|z\|_{\mathcal{H}}^2)^2-(\gamma^2-\|z\|_{\mathcal{H}}^2)(\lambda^2-c^2)}}\right).$$

Our assumption means that $\Phi_z(\gamma) = \Psi_z(\gamma)$ for all $\gamma \in (\|z\|_{\mathcal{H}}, \infty)$, which implies that

$$\frac{\eta\gamma + \|z\|_{\mathcal{H}}^2 + \sqrt{(\eta\gamma + \|z\|_{\mathcal{H}}^2)^2 - (\gamma^2 - \|z\|_{\mathcal{H}}^2)(\lambda^2 - c^2)}}{\lambda\gamma + \xi\|z\|_{\mathcal{H}}^2 + \sqrt{(\lambda\gamma + \xi\|z\|_{\mathcal{H}}^2)^2 - (\gamma^2 - \|z\|_{\mathcal{H}}^2)(\lambda^2 - c^2)}}$$
$$= \frac{\eta\gamma - \|z\|_{\mathcal{H}}^2 + \sqrt{(\eta\gamma - \|z\|_{\mathcal{H}}^2)^2 - (\gamma^2 - \|z\|_{\mathcal{H}}^2)(\lambda^2 - c^2)}}{\lambda\gamma - \xi\|z\|_{\mathcal{H}}^2 + \sqrt{(\lambda\gamma - \xi\|z\|_{\mathcal{H}}^2)^2 - (\gamma^2 - \|z\|_{\mathcal{H}}^2)(\lambda^2 - c^2)}}$$

for all $\gamma \in (\|z\|_{\mathcal{H}}, \infty)$. In particular this above equation must hold in the limit as $\gamma \to \|z\|_{\mathcal{H}}$, meaning that

$$\frac{\eta + \|z\|_{\mathcal{H}}}{\lambda + \xi\|z\|_{\mathcal{H}}} = \frac{\eta - \|z\|_{\mathcal{H}}}{\lambda - \xi\|z\|_{\mathcal{H}}}.$$

Thus, $\xi = \frac{\lambda}{\eta}$. By definition, we also know that $\xi = \|x\|_{\mathcal{H}}/\|z\|_{\mathcal{H}}$, so

$$\frac{\|x\|_{\mathcal{H}}}{\|z\|_{\mathcal{H}}} = \frac{\lambda}{\eta}.$$

Combining this with the fact that $\eta = \sqrt{\lambda^2 - c^2 + \|z\|_{\mathcal{H}}^2}$ leads to the equation

$$\|z\|_{\mathcal{H}} = \|x\|_{\mathcal{H}} \sqrt{\frac{\lambda^2 - c^2}{\lambda^2 - \|x\|_{\mathcal{H}}^2}},$$

and because we know that $c > \|x\|_{\mathcal{H}}$ this means that $\|z\|_{\mathcal{H}} < \|x\|_{\mathcal{H}}$. Thus $\lambda > \eta$. Now let us define functions $\Gamma_z \colon [0, \eta/\|z\|_{\mathcal{H}}) \to \mathbb{R}$ and $\Lambda_z \colon [0, \eta/\|z\|_{\mathcal{H}}) \to \mathbb{R}$ by

$$\Gamma_z(t) = h^{\lambda,x,c}((\eta, tz)), \text{ and } \Lambda_z(t) = h_{(\eta,z)}((\eta, tz)).$$

If we set

$$r_1 = \max\left\{\log(\lambda + c), \log\left(\frac{1}{\lambda - c}\right)\right\}, \text{and } r_2 = \max\left\{\log(\eta + \|z\|_{\mathcal{H}}), \log\left(\frac{1}{\eta - \|z\|_{\mathcal{H}}}\right)\right\},$$

we can calculate that, for any $t \leq 1$

$$\Gamma_z(t) = \log\left(\frac{\xi(\eta^2 - t\|z\|_{\mathcal{H}}^2) + \sqrt{\xi^2(\eta^2 - t\|z\|_{\mathcal{H}}^2)^2 - (\eta^2 - t^2\|z\|_{\mathcal{H}}^2)(\eta - \|z\|_{\mathcal{H}}^2)}}{\eta^2 - \|z\|_{\mathcal{H}}^2}\right) - r_1,$$

and

$$\Lambda_z(t) = \log\left(\frac{\eta^2 - t\|z\|_{\mathcal{H}}^2 + \sqrt{(\eta^2 - t\|z\|_{\mathcal{H}}^2)^2 - (\eta^2 - t^2\|z\|_{\mathcal{H}}^2)(\eta - \|z\|_{\mathcal{H}}^2)}}{\eta^2 - \|z\|_{\mathcal{H}}^2}\right) - r_2.$$

Employing the chain rule multiple times, we can calculate the left hand derivatives of Γ_z and Λ_z at $t = 0^+$ and find that

$$\left.\frac{d}{dt}\right|_{t=0^+} \Gamma_z(t) = \frac{\|z\|_{\mathcal{H}}}{\eta^2 - \|z\|_{\mathcal{H}}^2} \frac{\sqrt{\xi^2 - 1} - \xi}{\sqrt{\xi^2 - 1} + \xi},$$

whereas

$$\left.\frac{d}{dt}\right|_{t=0^+} \Lambda_z(t) = -\frac{\|z\|_{\mathcal{H}}}{\eta^2 - \|z\|_{\mathcal{H}}^2}.$$

As $\xi > 1$, these derivatives are different, which means that if $\Gamma_z(1) = \Lambda_z(1)$, there must exist some $\delta > 0$ such that $\Gamma_z(1-s) \neq \Lambda_z(1-s)$ for all $0 < s < \delta$. This contradicts the fact that $h^{\lambda,x,c} = h_{(\eta,z)}$. □

We now investigate the behaviour of unbounded nets, and borrow the ideas of [42, Theorem 3.2],

Lemma 8.2.3. *If $(u_\alpha) = ((\lambda_\alpha, x_\alpha))$ is an unbounded net in (V_+°, d_T), then there must exist a subnet (u_β) such that h_{u_β} converges to h, where h is of one of the following forms, where* *we use the convention that* $\log 0 = -\infty$*:*

$$h = h^{u,v},\ \text{where } u, v \in V_+,\ \max\{\|u\|_e, \|v\|_e\} = 1,\ v \bullet u = 0,\ \text{and, for all } w \in V_+^\circ,$$

$$h^{u,v}(w) = \max\{\log M(u/w), \log M(v/w^{-1})\}. \tag{8.2.5}$$

$$h = h_x^{\hat\lambda,\hat\mu,\bar\lambda,\bar\mu}\ \text{where } \hat\lambda, \bar\lambda, \hat\mu, \bar\mu \in [0,1],\ \max\{\hat\lambda, \bar\lambda, \hat\mu, \bar\mu\} = 1,\ \hat\lambda\bar\lambda = \hat\mu\bar\mu = 0,$$

$$\hat\lambda\hat\mu < 1, \bar\lambda\bar\mu < 1,\ \|x\|_{\mathcal{H}} < 1/2,\ \text{and, for all } w = (\gamma, y) \in V_+^\circ,\ \text{with } w^{-1} = (\gamma', y')$$

$$\begin{aligned} h_x^{\hat\lambda,\hat\mu,\bar\lambda,\bar\mu}(w) = \max\Bigg\{ & \log\Bigg(\frac{\gamma\frac{\hat\lambda+\hat\mu}{2} - (\hat\lambda - \hat\mu)\langle x, y\rangle}{\gamma^2 - \|y\|_{\mathcal{H}}^2} \\ & + \frac{\sqrt{(\gamma\frac{\hat\lambda+\hat\mu}{2} - (\hat\lambda - \hat\mu)\langle x, y\rangle)^2 - \hat\lambda\hat\mu(\gamma^2 - \|y\|_{\mathcal{H}}^2)}}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\Bigg), \\ & \log\Bigg(\frac{\gamma'\frac{\bar\lambda+\bar\mu}{2} - (\bar\lambda - \bar\mu)\langle x, y'\rangle}{\gamma'^2 - \|y'\|^2} \\ & + \frac{\sqrt{(\gamma'\frac{\bar\lambda+\bar\mu}{2} - (\bar\lambda - \bar\mu)\langle x, y'\rangle)^2 - \bar\lambda\bar\mu(\gamma'^2 - \|y'\|_{\mathcal{H}}^2)}}{\gamma'^2 - \|y'\|_{\mathcal{H}}^2}\Bigg)\Bigg\}. \end{aligned} \tag{8.2.6}$$

Proof. By passing to a subnet, we assume that $d_T(e, u_\alpha) \to \infty$. We make use of the approach of Lemmens in his proof of Theorem 3.2 in [42]. To that end let us define, for each α, $r_\alpha = e^{d_T(e,u_\alpha)}$, $v_\alpha = u_\alpha^{-1}$, and $\hat u_\alpha = \frac{u_\alpha}{r_\alpha}$, $\hat v_\alpha = \frac{v_\alpha}{r_\alpha}$. As V has rank 2 and admits a spectral decomposition (Theorem 2.9.12), for each α there exists a primitive idempotent $p_\alpha \in V$, and eigenvalues $\lambda_\alpha, \mu_\alpha \in \mathbb{R}$ such that $u_\alpha = \lambda_\alpha p_\alpha + \mu_\alpha(e - p_\alpha)$. As $u_\alpha \in V_+^\circ$, we know that $\lambda_\alpha, \mu_\alpha > 0$. Thus we know, for all α, that $v_\alpha = \lambda_\alpha^{-1} p_\alpha + \mu_\alpha^{-1}(e - p_\alpha)$. The spectral calculus means that

$$d_T(e, u_\alpha) = \|\log u_\alpha\|_e = \|\log v_\alpha\|_e,$$

so we know $\hat u_\alpha \le e$ and $\hat v_\alpha \le e$. As $\|u_\alpha\|_e = \max\{\lambda_\alpha, \mu_\alpha\}$, $\|v_\alpha\|_e = \max\{\lambda_\alpha^{-1}, \mu_\alpha^{-1}\}$, we can thus write:

$$\hat{u}_\alpha = \hat{\lambda}_\alpha p_\alpha + \hat{\mu}_\alpha(e - p_\alpha), \text{ and } \hat{v}_\alpha = \bar{\lambda}_\alpha p_\alpha + \bar{\mu}_\alpha(e - p_\alpha),$$
$$\text{where } \hat{\lambda}_\alpha, \hat{\mu}_\alpha, \bar{\lambda}_\alpha, \bar{\mu}_\alpha \in (0, 1] \text{ and } \max\{\hat{\lambda}_\alpha, \hat{\mu}_\alpha \bar{\lambda}_\alpha, \bar{\mu}_\alpha\} = 1.$$

Now, for any $w \in V_+^\circ$, we can utilise the properties of the gauge function M (Lemma 2.7.8) to calculate

$$\begin{aligned} h_{u_\alpha}(w) &= \max\{\log M(u_\alpha/w), \log M(w/u_\alpha)\} - \log r_\alpha \\ &= \max\{\log M(u_\alpha/w), \log M(u_\alpha^{-1}/w^{-1})\} - \log r_\alpha \\ &= \max\{\log\left(r_\alpha^{-1} M(u_\alpha/w)\right), \log\left(r_\alpha^{-1} M(u_\alpha^{-1}/w^{-1})\right)\} \\ &= \max\{\log M(\hat{u}_\alpha/w), \log M(\hat{v}_\alpha/w^{-1})\}. \end{aligned} \tag{8.2.7}$$

We now need to consider two cases: either $(\hat{u}_\alpha)$ has subnet $(\hat{u}_\beta)$ such that (p_β) converges in the JB-algebra norm to some $p \leq e$, or there is no such subnet. Suppose the former, that there exists some subnet $(\hat{u}_\beta)$ and a $p \leq e$ with $\|p_\beta - p\|_e \to 0$. We can choose the subnet so that 1 is obtained by the same eigenvalue among $\hat{\lambda}_\beta, \hat{\mu}_\beta, \bar{\lambda}_\beta, \bar{\mu}_\beta$ for all β. As $p_\beta = (\frac{1}{2}, x_\beta)$, where $\|x_\beta\|_{\mathcal{H}} = \frac{1}{2}$ for all β, it is clear that p must also be a primitive idempotent. We can, by choosing further subnets if necessary and relabelling (see Lemma 2.3.6), find $\hat{\lambda}, \hat{\mu}, \bar{\lambda}, \bar{\mu} \in [0, 1]$, with $\max\{\hat{\lambda}, \hat{\mu}, \bar{\lambda}, \bar{\mu}\} = 1$, such that

$$\hat{\lambda}_\beta \to \hat{\lambda},\ \hat{\mu}_\beta \to \hat{\mu},\ \bar{\lambda}_\beta \to \bar{\lambda}, \text{ and } \bar{\mu}_\beta \to \bar{\mu}. \tag{8.2.8}$$

It is simple to see that if a limiting eigenvalue equals 1, its hat or bar twin must equal 0 (i.e. if $\hat{\lambda} = 1$ then $\bar{\lambda} = 0$), so we also know $\min\{\hat{\lambda}, \hat{\mu}, \bar{\lambda}, \bar{\mu}\} = 0$. The triangle inequality thus means that $\hat{u}_\beta \to u = \hat{\lambda} p + \hat{\mu}(e-p)$ and $\hat{v}_\beta \to v = \bar{\lambda} p + \bar{\mu}(e-p)$ in the JB-algebra norm, and $\max\{\|v\|_e, \|u\|_e\} = 1$. As the Jordan product is continuous with respect to the JB-algebra norm, and $\hat{u}_\beta \bullet \hat{v}_\beta = r_\beta^{-2} e \to 0$, we must then have $u \bullet v = 0$. As $u \bullet v = \hat{\lambda}\bar{\lambda} p + \hat{\mu}\bar{\mu}(e - p)$, this means that $\hat{\lambda}\bar{\lambda} = -\hat{\mu}\bar{\mu}$. However, we know that one of the four eigenvalues is 0, from which we deduce that $\min\{\hat{\lambda}, \bar{\lambda}\} = 0 = \min\{\hat{\mu}, \bar{\mu}\}$. As the map $M\colon V \times V_+^\circ \to \mathbb{R}$ is norm continuous [44, Lemma 2.2], (8.2.7) in conjunction with the above implies that, for any $w \in V_+^\circ$,

$$\lim_\beta h_{u_\beta}(w) = \max\{\log M(u/w), \log M(v/w^{-1})\} = h^{u,v}(w).$$

Let us now consider the case where (p_α) has no norm convergent subnet. We know that, for each β, $p_\beta = (\frac{1}{2}, x_\beta)$, where $\|x_\alpha\|_{\mathcal{H}} = \frac{1}{2}$. The Banach-Alaoglu Theorem means that there must exist a $x \in \mathcal{H}$ with $\|x\| \leq 1/2$, and a subnet (x_β), such that x_β converges weakly to x in $\mathcal{H}$. Now, if $\|x\|_{\mathcal{H}} = \frac{1}{2}$, then $\|x - x_\beta\|^2_{\mathcal{H}} = 1/4 - 2\langle x, x_\beta\rangle - 1/4 \to 0$, which contradicts our assumption on (p_α) having no norm convergent subnets. As above we can choose further subnets and relabel such that (8.2.8) holds. Using the same notation as above we can thus write, for all β,

$$\hat{u}_\beta = \left(\frac{\hat{\lambda}_\beta + \hat{\mu}_\beta}{2}, (\hat{\lambda}_\beta - \hat{\mu}_\beta)x_\beta\right), \text{ and } \hat{v}_\beta = \left(\frac{\bar{\lambda}_\beta + \bar{\mu}_\beta}{2}, (\bar{\lambda}_\beta - \bar{\mu}_\beta)x_\beta\right).$$

Furthermore, for any $w \in V_+^\circ$, we have the spectral decomposition $w = (\frac{\eta+\xi}{2}, (\eta - \xi)z)$ for some $\eta, \xi > 0$ and $z \in \mathcal{H}$ with $\|z\|_{\mathcal{H}} = 1/2$, from which we deduce that $w^{-1} = (\frac{\eta^{-1}+\xi^{-1}}{2}, (\eta^{-1} - \xi^{-1})z)$. For convenience we will write $w = (\gamma, y)$ and $w^{-1} = (\gamma', y')$

Combining this with (8.2.2) and (8.2.7), we thus see for any $w = (\gamma, y) \in V_+^\circ$ and any β

$$
\begin{aligned}
h_{u_\beta}(w) &= \max\{\log M(\hat{u}_\alpha/w), \log M(\hat{v}_\alpha/w^{-1})\} \\
&= \max\left\{ \log M\left(\left(\frac{\hat{\lambda}_\beta+\hat{\mu}_\beta}{2}, (\hat{\lambda}_\beta-\hat{\mu}_\beta)x_\beta\right)/(\gamma,y)\right),\right. \\
&\qquad \left. \log M\left(\left(\frac{\bar{\lambda}_\beta+\bar{\mu}_\beta}{2}, (\bar{\lambda}_\beta-\bar{\mu}_\beta)x_\beta\right)/(\gamma',y')\right)\right\} \\
&= \max\left\{ \log\left(\frac{\gamma\frac{\hat{\lambda}_\beta+\hat{\mu}_\beta}{2} - (\hat{\lambda}_\beta-\hat{\mu}_\beta)\langle x_\beta, y\rangle}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right.\right. \\
&\qquad \left. + \frac{\sqrt{(\gamma\frac{\hat{\lambda}_\beta+\hat{\mu}_\beta}{2} - (\hat{\lambda}_\beta-\hat{\mu}_\beta)\langle x_\beta, y\rangle)^2 - \hat{\lambda}_\beta\hat{\mu}_\beta(\gamma^2 - \|y\|_{\mathcal{H}}^2)}}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right), \\
&\qquad \log\left(\frac{\gamma'\frac{\bar{\lambda}_\beta+\bar{\mu}_\beta}{2} - (\bar{\lambda}_\beta-\bar{\mu}_\beta)\langle x_\beta, y'\rangle}{\gamma'^2 - \|y'\|^2}\right. \\
&\qquad \left.\left. + \frac{\sqrt{(\gamma'\frac{\bar{\lambda}_\beta+\bar{\mu}_\beta}{2} - (\bar{\lambda}_\beta-\bar{\mu}_\beta)\langle x_\beta, y'\rangle)^2 - \bar{\lambda}_\beta\bar{\mu}_\beta(\gamma'^2 - \|y'\|_{\mathcal{H}}^2)}}{\gamma'^2 - \|y'\|_{\mathcal{H}}^2}\right)\right\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lim_\beta h_{u_\beta}(w) &= \max\left\{ \log\left(\frac{\gamma\frac{\hat{\lambda}+\hat{\mu}}{2} - (\hat{\lambda}-\hat{\mu})\langle x, y\rangle}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right.\right. \\
&\qquad \left. + \frac{\sqrt{(\gamma\frac{\hat{\lambda}+\hat{\mu}}{2} - (\hat{\lambda}-\hat{\mu})\langle x, y\rangle)^2 - \hat{\lambda}\hat{\mu}(\gamma^2 - \|y\|_{\mathcal{H}}^2)}}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right), \\
&\qquad \log\left(\frac{\gamma'\frac{\bar{\lambda}+\bar{\mu}}{2} - (\bar{\lambda}-\bar{\mu})\langle x, y'\rangle}{\gamma'^2 - \|y'\|^2}\right. \\
&\qquad \left.\left. + \frac{\sqrt{(\gamma'\frac{\bar{\lambda}+\bar{\mu}}{2} - (\bar{\lambda}-\bar{\mu})\langle x, y'\rangle)^2 - \bar{\lambda}\bar{\mu}(\gamma'^2 - \|y'\|_{\mathcal{H}}^2)}}{\gamma'^2 - \|y'\|_{\mathcal{H}}^2}\right)\right\} \\
&= h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}(w), \qquad (8.2.9)
\end{aligned}
$$

where $\hat{\lambda}, \hat{\mu}, \bar{\lambda}, \bar{\mu} \in [0,1]$, and $\max\{\hat{\lambda}, \hat{\mu}, \bar{\lambda}, \bar{\mu}\} = 1$. To conclude, note that $\|\hat{u}_\beta \bullet \hat{v}_\beta\|_e = r_\beta^{-2} \to 0$, but also that, for every β, $\|\hat{u}_\beta \bullet \hat{v}_\beta\|_e = \max\{\hat{\lambda}_\beta\bar{\lambda}_\beta, \hat{\mu}_\beta\bar{\mu}_\beta\}$, so indeed $\min\{\hat{\lambda}, \bar{\lambda}\} =$

$\min\{\hat{\mu}, \bar{\mu}\} = 0$. It is a simple exercise to check, using (8.2.1), that $h_x^{1,1,0,0} = h^{e,0}$ and $h_x^{0,0,1,1} = h^{0,e}$ for all $x \in \mathcal{H}$, which is why we also exclude these possibilities in our definition of (8.2.6) □

Theorem 8.2.4. *A function $h \in \overline{V_+^\circ}^h$ is a horofunction of (V_+°, d_T) if and only if h is of the form* (8.2.3)*,* (8.2.5)*, or* (8.2.6)*.*

Proof. Let $h \in \partial\overline{V_+^\circ}^h$. There thus exists a net $(x_\alpha) \subseteq V_+^\circ$ with $h_{x_\alpha} \to h$ in the topology of pointwise convergence. Now, $(d(e, x_\alpha))$ is a net contained in the compact space $[0, \infty]$, so there must exist a subnet (x_β) and a $k \in [0, \infty]$ such that $d(e, x_\beta) \to k$. If $k \in \mathbb{R}$, Lemma 8.2.2 means that $h = h^{\lambda,x,c}$ for $\|x\| < c < \lambda$. If $k = \infty$, then Lemma 8.2.3 means that either $h = h^{u,v}$, or $h = h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}$.

Now consider an arbitrary $u, v \in V_+$ such that $\max\{\|u\|_e, \|v\|_e\} = 1$ and $u \bullet v = 0$. Let us assume that both u and v are non-zero. Lemma 2.9.14 means that there exists a primitive idempotent p, and eigenvalues $\lambda, \mu > 0$ with $\max\{\lambda, \mu\} = 1$, such that $u = \lambda p$ and $v = \mu(e-p)$. Now let us define $x = \log(\lambda)p - \log(\mu)(e-p)$ and $y = p-(e-p)$. Define $\gamma(t) = ty+x$, and $\varphi\colon \mathbb{R}_+ \to V$ by $\varphi(t) = \exp(\gamma(t))$. Now, $\|ty+x\|_e = \max\{|t+\log(\lambda)|, |t+\log(\mu)|\}$, but $\max\{\log(\lambda), \log(\mu)\} = 0$, so for all $t \geq -\min\{\log(\lambda), \log(\mu)\}$,

$$d_T(e, \varphi(t)) = \|\log(\varphi(t))\|_e = \|ty + x\|_e = t.$$

Thus, just as we did for (8.2.7), we can calculate, for any $w \in V_+^\circ$ and $t \geq -\min\{\log(\lambda), \log(\mu)\}$,

$$h_{\varphi(t)}(w) = \max\left\{\log M(e^{-t}\varphi(t)/w), \log M(e^{-t}\varphi(t)^{-1}/w^{-1})\right\}.$$

We know, thanks to the spectral calculus, that $\varphi(t)^{-1} = \exp(-ty - x)$ for all $t > 0$, so

$$\lim_{t\to\infty} e^{-t}\varphi(t) = \lim_{t\to\infty} e^{-t}\left(\lambda e^t p + \mu^{-1}e^{-t}(e-p)\right) = \lambda p = u,$$

and

$$\lim_{t\to\infty} e^{-t}\varphi(t)^{-1} = \lim_{t\to\infty} e^{-t}\left(\lambda^{-1}e^{-t}p + \mu e^{t}(e-p)\right) = \mu(e-p) = v.$$

Thus, due to the order-norm continuity of M,

$$\lim_{t\to\infty} h_{\varphi(t)}(w) = \max\left\{\log M(u/w), \log M(v/w^{-1})\right\} = h^{u,v}(w).$$

As φ is an unbounded geodesic (Lemma 8.2.1), the above means that $h^{u,v}$ is a Busemann point and horofunction (Corollary 3.2.4). If one of u or v is zero, we can slightly modify the same argument. Suppose without loss of generality that $v = 0$. We thus know that $u = p + \lambda(e-p)$ for some $\lambda \in [0,1]$. Define $\varphi\colon \mathbb{R}_+ \to V_+^\circ$ by $\varphi(t) = \exp(tp + t\lambda(e-p))$. The same argument as above thus show that φ is a geodesic and $h_{\varphi(t)} \to h^{u,0}$. Thus, every $h^{u,v}$ with $u, v \in V_+$, $\max\{\|u\|_e, \|v\|_e\} = 1$ and $u \bullet v = 0$ is a Busemann point.

Now let us assume that we are given some $h_x^{\hat\lambda,\hat\mu,\bar\lambda,\bar\mu}$ as in (8.2.6). As $\mathcal{H}$ is an infinite dimensional Hilbert space, we know there exists an orthonormal sequence $(e_n) \subseteq \mathcal{H}$, such that $e_n \xrightarrow{w} 0$. We now define, for all $n \in \mathbb{N}$,

$$c_n = \frac{-\langle x, e_n\rangle + \sqrt{\langle x, e_n\rangle^2 - 4\|x\|_{\mathcal{H}}^2 + 1}}{2}, \quad \text{and} \quad x_n = x + c_n e_n.$$

As $c_n \to \sqrt{\frac{1}{4} - \|x\|_{\mathcal{H}}^2}$, we see that x_n converges weakly to x, and we can also calculate that $\|x_n\|_{\mathcal{H}} = \frac{1}{2}$ for all $n \in \mathbb{N}$. Thus, each $p_n = (\frac{1}{2}, x_n)$ is a primitive idempotent in V_+. We now need to consider cases: first let us assume that $\hat\lambda, \hat\mu > 0$, meaning that $\bar\lambda = \bar\mu = 0$. Define, for each $n \in \mathbb{N}$, $u_n = n\hat\lambda p_n + n\hat\mu(e-p_n)$, meaning $u_n^{-1} = (n\hat\lambda)^{-1}p_n + (n\hat\mu)^{-1}(e-p_n)$. Let us define $r_n = e^{d_T(e,u_n)}$, so, because $\max\{\hat\lambda, \hat\mu\} = 1$, $r_n = n$ for n large enough. Thus, following a similar approach as in the proof of Lemma 8.2.3, we see, for any $w = (\gamma, y) \in V_+^\circ$, and large n,

$$h_{u_n}(w) = \max\{\log M(n^{-1}u_n/w), \log M(n^{-1}u_n^{-1}/w^{-1})\}.$$

Now if we note that $n^{-1}u_n = (\frac{\hat{\lambda}+\hat{\mu}}{2}, (\hat{\lambda}-\hat{\mu})x_n)$, and $n^{-1}u_n^{-1} \to 0$ in the order unit norm, we can use the above to show that $\lim_{n\to\infty} h_{u_n}(w) = h_x^{\hat{\lambda},\hat{\mu},0,0}(w)$ using Claassens' representation (8.2.1), following the same approach as in the latter part of the proof of Lemma 8.2.3. It is simple to adjust this argument if one of $\hat{\lambda}$ and $\hat{\mu}$ is 0. If instead $\hat{\lambda}, \hat{\mu} > 0$, by choosing $v_n = (n\bar{)}^{-1}\lambda p_n + (n\bar{\mu})^{-1}(e-p_n)$ we can replicate the same argument to show that $\lim_{n\to\infty} h_{v_n}(w) = h_x^{0,0,\bar{\lambda},\bar{\mu}}(w)$ for all $w \in V_+^\circ$. Now let us assume that $\max\{\hat{\lambda}, \hat{\mu}\} > 0$ and $\max\{\bar{\lambda}, \bar{\mu}\} > 0$. As we know hat and bar "conjugates" cannot both be non-zero, so we can assume without loss of generality that $\hat{\lambda} > 0$ and $\bar{\mu} > 0$. Let us now define, for each $n \in \mathbb{N}$, $u_n = n\hat{\lambda}p_n + (n\bar{\mu})^{-1}(e-p_n)$, meaning $u_n^{-1} = (n\hat{\lambda})^{-1}p_n + n\bar{\mu}(e-p_n)$ and because $\max\{\hat{\lambda}, \bar{\mu}\} = 1$ we have $e^{d_T(e,u_n)} = n$ for large n. Thus, once again following a similar approach as in the proof of Lemma 8.2.3, we see, for any $w = (\gamma, y) \in V_+^\circ$, and large n,

$$h_{u_n}(w) = \max\{\log M(n^{-1}u_n/w), \log M(n^{-1}u_n^{-1}/w^{-1})\},$$

but also,

$$n^{-1}u_n = \left(\frac{\hat{\lambda}+n^{-2}\bar{\mu}^{-1}}{2}, (\hat{\lambda}-n^{-2}\bar{\mu}^{-1})x_n\right), \text{ and } n^{-1}u_n^{-1} = \left(\frac{n^{-2}\hat{\lambda}^{-1}+\bar{\mu}}{2}, (n^{-2}\hat{\lambda}^{-1}-\bar{\mu})x_n\right).$$

Using this we can once again use (8.2.1) and follow the same argument as in the latter part of the proof of Lemma 8.2.3 to show that $\lim_{n\to\infty} h_{u_n}(w) = h_x^{\hat{\lambda},0,0,\bar{\mu}}(w)$ for all $w \in V_+^\circ$. The case where $\bar{\lambda} > 0$ and $\hat{\mu} > 0$ follows in the same way. We can thus conclude that every function $h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}$ as in (8.2.6) is an element of $\overline{V_+^\circ}^h$.

To prove that each $h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}} \in \partial\overline{V_+^\circ}^h$ we thus just need to show that it isn't an internal metric functional. We need to consider two cases. Either $\hat{\lambda}\hat{\mu} = 0 = \bar{\lambda}\bar{\mu}$, or $\max\{\hat{\lambda}\hat{\mu}, \bar{\lambda}\bar{\mu}\} > 0$. Let us first consider some fixed $h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}$ as in (8.2.6), where $\hat{\lambda}\hat{\mu} = 0 = \bar{\lambda}\bar{\mu}$. For any $w = (\gamma, y) \in V_+^\circ$ with $w' = (\gamma', y')$, we have

$$h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}(w) = \max\left\{ \log\left(\frac{\gamma(\hat{\lambda}+\hat{\mu}) - 2(\hat{\lambda}-\hat{\mu})\langle x, y\rangle}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right), \log\left(\frac{\gamma'(\bar{\lambda}+\bar{\mu}) - 2(\bar{\lambda}-\bar{\mu})\langle x, y'\rangle}{\gamma'^2 - \|y'\|^2}\right)\right\}.$$

Combining this with Lemma 2.9.15 leads to

$$h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}(w) = \max\left\{ \log\left(\frac{\gamma(\hat{\lambda}+\hat{\mu}) - 2(\hat{\lambda}-\hat{\mu})\langle x, y\rangle}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right), \log\left(\gamma(\bar{\lambda}+\bar{\mu}) + 2(\bar{\lambda}-\bar{\mu})\langle x, y\rangle\right)\right\}. \tag{8.2.10}$$

Due to symmetry we may as well assume without loss of generality that $\hat{\lambda} = 1$. By way of contradiction, let us assume that $h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}} = h_{\eta,z}$ for $(\eta, z) \in V_+^\circ$. However, we know from (8.2.1) that

$$h_{\eta,z}(\eta, z) = -\max\left\{\log\left(\frac{1}{\eta - \|z\|_{\mathcal{H}}}\right), \log(\eta + \|z\|_{\mathcal{H}})\right\},$$

but from (8.2.10),

$$h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}(\eta, z) = \max\left\{\left(\log\frac{\eta - 2\langle x, z\rangle}{\eta^2 - \|z\|_{\mathcal{H}}^2}\right), \log\left(\eta\bar{\mu} - 2\bar{\mu}\langle x, z\rangle\right)\right\}.$$

As $\|x\|_{\mathcal{H}} < \frac{1}{2}$, the Cauchy-Schwarz inequality thus gives

$$h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}(\eta, z) > \max\left\{\log\left(\frac{1}{\eta + \|z\|_{\mathcal{H}}}\right), \log(\eta - \|z\|_{\mathcal{H}}) + \log\bar{\mu}\right\},$$

meaning that, to avoid an immediate contradiction, we require $\bar{\mu} < 1$, and $\frac{1}{\eta - \|z\|_{\mathcal{H}}} \geq \eta + \|z\|_{\mathcal{H}}$. For any, $(\gamma, 0) \in V_+^\circ$, we can also calculate, using (8.2.1) and (8.2.10), that

$$h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}(\gamma, 0) = \max\left\{\log\left(\frac{1}{\gamma}\right), \log(\bar{\mu}\gamma)\right\}, \text{ and } h_{\eta,z}(\gamma, 0) = \max\left\{\log\left(\frac{1}{\gamma}\right), \log(\gamma)\right\}.$$

As we have already shown that $\bar{\mu} < 1$ the above means that for $\gamma > \sqrt{\eta^2 - \|z\|_{\mathcal{H}}^2}$, we have $h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}(\gamma, 0) \neq h_{\eta,z}(\gamma, 0)$, a contradiction. We can thus conclude that, whenever $\hat{\lambda}\hat{\mu} = 0 = \bar{\lambda}\bar{\mu}$, the function $h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}$ as in (8.2.6) is a horofunction. Now let us consider the case when $\max\{\hat{\lambda}\bar{\mu}, \bar{\lambda}\hat{\mu}\} > 0$. Once again, by symmetry, we may as well assume that $\hat{\lambda}\hat{\mu} > 0$, which means that $\bar{\lambda} = \bar{\mu} = 0$. Thus, for any $(\gamma, 0) \in V_+^\circ$, we have,

$$h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}(\gamma, 0) = \log\left(\frac{\frac{\hat{\lambda}+\hat{\mu}}{2} + \sqrt{\frac{1}{4}(\hat{\lambda}^2 - 2\hat{\lambda}\hat{\mu} + \hat{\mu}^2)}}{\gamma}\right) = \log\left(\frac{\hat{\lambda}}{\gamma}\right),$$

meaning that $h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}$ is not bounded below, meaning it must be a horofunction.

□

Unlike the finite dimensional case, we have non-Busemann points in $\partial\overline{V_+^\circ}^h$:

Theorem 8.2.5. *A function $h \in \partial\overline{V_+^\circ}$ is a Busemann point if and only if $h = h^{u,v}$ as in* (8.2.5).

Proof. We prove in the first part of the proof of Theorem 8.2.4 that each $h^{u,v}$ is a Busemann point. What is left is to show that these are the only Busemann points. We first show that no $h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}$ as in (8.2.6) is a Busemann point. First let us consider some fixed $h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}$, where $\hat{\lambda}\hat{\mu} = 0 = \bar{\lambda}\bar{\mu}$, and $\max\{\hat{\lambda}, \hat{\mu}\}, \max\{\bar{\lambda}, \bar{\mu}\} > 0$. Without loss of generality we may as well assume that $\hat{\lambda} = 1$. Combining (8.2.10) with the Cauchy-Schwarz inequality, and the fact that $\|x\|_{\mathcal{H}} < \frac{1}{2}$ means that there exists a $\delta \in [0, 1)$, so that, for any $(\lambda, y) \in V_+^\circ$

$$\begin{aligned} h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}(\gamma, y) &> \max\left\{\log\left(\frac{1}{\gamma + \|y\|_{\mathcal{H}}}\right), \log(\bar{\mu}) + \log(\gamma - \delta\|y\|_{\mathcal{H}})\right\} \\ &> \max\left\{\log\left(\frac{1}{2\gamma}\right), \log(\bar{\mu}) + \log(\gamma(1-\delta))\right\} \\ &\geq \max\left\{\log\left(\frac{1}{2\sqrt{2\bar{\mu}(1-\delta)}}\right), \log(\bar{\mu}) - \frac{1}{2}\log(2\bar{\mu}) + \log(\sqrt{1-\delta})\right\}, \end{aligned}$$

where the last inequality comes from noting that $\log(\frac{1}{2\gamma})$ is monotone decreasing in γ whereas $\log(\bar{\mu}) + \log(\gamma(1-\delta))$ is monotone increasing in γ, and they are equal when $\gamma = (2\bar{\mu}(1-\delta))^{-1/2}$. Therefore $h_x^{\hat{\lambda},\hat{\mu},\bar{\lambda},\bar{\mu}}$ is bounded below, so cannot be a Busemann point (Corollary 3.2.4). Let us now consider the case $\bar{\lambda} = 0 = \bar{\mu}$. Assume, by way of contradiction, that $h_x^{\hat{\lambda},\hat{\mu},0,0}$ is a Busemann point. There thus exists an almost geodesic $(u_\alpha) \subseteq V_+^\circ$, where $u_\alpha = (\gamma_\beta, y_\beta)$, with associated internal metric functionals converging to $h_x^{\hat{\lambda},\hat{\mu},0,0}$. Following the proof of Lemma 8.2.3, we know that we can use the spectral decomposition to write $u_\alpha = \lambda_\alpha p_\alpha + \mu_\alpha(e - p_\alpha)$ for primitive idempotents p_α. Following that proof, if we define $r_\alpha = e^{d_T(e,u_\alpha)}$, $\hat{\lambda}_\alpha = r_\alpha^{-1}\lambda_\alpha$, $\hat{\mu}_\alpha = r_\alpha^{-1}\mu_\alpha$, $\bar{\lambda}_\alpha = r_\alpha^{-1}\lambda_\alpha^{-1}$, and $\bar{\mu}_\alpha = r_\alpha^{-1}\mu_\alpha^{-1}$, there must exist a subnet, say $u_{\alpha'}$, such that $p_{\alpha'}$ converges weakly to some $z \in \mathcal{H}$ with $\|z\|_{\mathcal{H}} \leq \frac{1}{2}$, and there must exist $\hat{\lambda}', \hat{\mu}', \bar{\lambda}', \bar{\mu}' \in [0,1]$ such that

$$\hat{\lambda}_{\alpha'} \to \hat{\lambda}',\ \hat{\mu}_{\alpha'} \to \hat{\mu}',\ \bar{\lambda}_{\alpha'} \to \bar{\lambda}',\ \text{and } \bar{\mu}_{\alpha'} \to \bar{\mu}'.$$

Furthermore, we must have $h_{u_{\alpha'}} \to h_z^{\hat{\lambda}',\hat{\mu}',\bar{\lambda}',\bar{\mu}'}$. However, $\overline{V_+^\circ}^h$ is Hausdorff, so $h_z^{\hat{\lambda}',\hat{\mu}',\bar{\lambda}',\bar{\mu}'} = h_x^{\hat{\lambda},\hat{\mu},0,0}$, meaning $\hat{\lambda} = \hat{\lambda}'$ and $\hat{\mu} = \hat{\mu}'$. As the subnet of an almost geodesic net is an almost-geodesic net, we might as well relabel $(u_{\alpha'})$ and consider it as (u_α). By Lemma 3.2.6, for any $\varepsilon > 0$ there exists a η, so that for all $\alpha \geq \eta$

$$h_x^{\hat{\lambda},\hat{\mu},0,0}(u_\alpha) + d_T(u_\alpha, e) \leq \varepsilon. \tag{8.2.11}$$

Now, for any α,

$$h_x^{\hat{\lambda},\hat{\mu},0,0}(u_\alpha) = \log\left(\frac{\gamma_\alpha\frac{\hat{\lambda}+\hat{\mu}}{2} - (\hat{\lambda}-\hat{\mu})\langle x, y_\alpha\rangle}{\gamma_\alpha^2 - \|y_\alpha\|_{\mathcal{H}}^2} + \frac{\sqrt{(\gamma_\alpha\frac{\hat{\lambda}+\hat{\mu}}{2} - (\hat{\lambda}-\hat{\mu})\langle x, y_\alpha\rangle)^2 - \hat{\lambda}\hat{\mu}(\gamma_\alpha^2 - \|y_\alpha\|_{\mathcal{H}}^2)}}{\gamma_\alpha^2 - \|y_\alpha\|_{\mathcal{H}}^2}\right),$$

so, setting $\delta = \|x\|_{\mathcal{H}} < 1/2$, and $\Delta = \min\{\hat{\lambda}, \hat{\mu}\} < 1$, we can use the Cauchy-Schwarz inequality to calculate

$$\begin{aligned} h_x^{\hat{\lambda},\hat{\mu},0,0}(u_\alpha) \geq \log\Bigg(& \frac{\gamma_\alpha \frac{1+\Delta}{2} - \frac{\delta(1-\Delta)}{2}\|y_\alpha\|_{\mathcal{H}}}{\gamma_\alpha^2 - \|y_\alpha\|_{\mathcal{H}}^2} + \\ & \frac{\sqrt{(\gamma_\alpha \frac{1+\Delta}{2} - \frac{1-\Delta}{2}\|y_\alpha\|)^2 - \Delta(\gamma_\alpha^2 - \|y_\alpha\|_{\mathcal{H}}^2)}}{\gamma_\alpha^2 - \|y_\alpha\|_{\mathcal{H}}^2} \Bigg), \\ = \log & \left(\frac{\gamma_\alpha \frac{1+\Delta}{2} - \frac{\delta(1-\Delta)}{2}\|y_\alpha\|_{\mathcal{H}} + \left|\gamma_\alpha \frac{1-\Delta}{2} - \frac{1+\Delta}{2}\|y_\alpha\|_{\mathcal{H}}\right|}{\gamma_\alpha^2 - \|y_\alpha\|_{\mathcal{H}}^2} \right). \end{aligned}$$

Thus, for all α,

$$\begin{aligned} h_x^{\hat{\lambda},\hat{\mu},0,0}(u_\alpha) + d_T(e,u_\alpha) &\geq \log\left(\frac{\gamma_\alpha \frac{1+\Delta}{2} - \frac{\delta(1-\Delta)}{2}\|y_\alpha\|_{\mathcal{H}} + \left|\gamma_\alpha \frac{1-\Delta}{2} - \frac{1+\Delta}{2}\|y_\alpha\|_{\mathcal{H}}\right|}{\gamma_\alpha - \|y_\alpha\|_{\mathcal{H}}} \right) \\ &= \log\left(\frac{r_\alpha^{-1}}{r_\alpha^{-1}} \frac{\left[\gamma_\alpha \frac{1+\Delta}{2} - \frac{\delta(1-\Delta)}{2}\|y_\alpha\|_{\mathcal{H}} + \left|\gamma_\alpha \frac{1-\Delta}{2} - \frac{1+\Delta}{2}\|y_\alpha\|_{\mathcal{H}}\right|\right]}{\gamma_\alpha - \|y_\alpha\|_{\mathcal{H}}} \right). \end{aligned} \tag{8.2.12}$$

However, we know, for all α, that $r_\alpha^{-1}\gamma_\alpha = \frac{\hat{\lambda}_\alpha + \hat{\mu}_\alpha}{2} \to \frac{1+\Delta}{2}$ and $r_\alpha^{-1}\|y_\alpha\|_{\mathcal{H}} = \frac{|\hat{\lambda}_\alpha - \hat{\mu}_\alpha|}{2} \to \frac{1-\Delta}{2}$. As inequality (8.2.12) is true for all α, it thus follows that, if $\Delta > 0$, then for every $\varepsilon > 0$, there exists a η, so that for all $\alpha \geq \eta$,

$$h_x^{\hat{\lambda},\hat{\mu},0,0}(u_\alpha) + d_T(e,u_\alpha) \geq \frac{(1-\delta)(1+\Delta^2) + 2\Delta(1+\delta)}{2\Delta} - \varepsilon,$$

but this contradicts (8.2.11), as for all $\delta \in [0,1/2)$ and $\Delta \in (0,1)$,

$$\frac{(1-\delta)(1+\Delta^2) + 2\Delta(1+\delta)}{2\Delta} > 1.$$

If $\Delta = 0$, taking the limit in (8.2.12) shows that $h_x^{\hat{\lambda},\hat{\mu},0,0}(u_\alpha) + d_T(e,u_\alpha) \to \infty$, again a contradiction. □

Proposition 8.2.6. *Let $h = h^{u,v}$ and $h' = h^{u',v'}$ be two Busemann points as in* (8.2.5)*, with* *data having corresponding spectral decompositions* $u = \sum_{i\in I} e^{-\lambda_i} p_i$ *and* $v = \sum_{j\in J} e^{-\lambda_j} p_j$. *Denote* $p_I = \sum_{i\in I} p_i$ *and* $p_J = \sum_{j\in J} p_J$ *Then,* h *and* h' *are in the same part of the boundary if and only if* $u \sim u'$ *and* $v \sim v'$. *Furthermore, if* h *and* h' *are in the same part, then*

$$\delta(h, h') = d_H((u, v), (u', v')),$$

where δ_H *is the Hilbert metric on the product cone* $U_{p_I}(V_+) \oplus U_{q_J}(V_+)$. *If either* u *or* v *is* 0, *then we drop the corresponding term in the direct sum, and the Hilbert metric just reduces to the Hilbert metric on the face of the cone containing the remaining element.*

Proof. We closely follow the proof of Lemmens in [42, Theorem 3.4]. Let us first assume that $u, v \neq 0$. As in the proof of Theorem 8.2.4 we know there exists a primitive idempotent p, and eigenvalues $\lambda, \mu > 0$ with $\max\{\lambda, \mu\} = 1$, such that $u = \lambda p$ and $v = \mu(e - p)$. Similarly we define $x = \log(\lambda)p - \log(\mu)(e - p)$ and $y = p - (e - p)$. Define $\gamma(t) = ty + x$, and $\varphi\colon \mathbb{R}_+ \to V$ by $\varphi(t) = \exp(\gamma(t))$. Now, $\|ty + x\|_e = \max\{|t + \log(\lambda)|, |t + \log(\mu)|\}$, but $\max\{\log(\lambda), \log(\mu)\} = 0$, so for all $t \geq -\min\{\log(\lambda), \log(\mu)\}$, $d_T(e, \varphi(t)) = t$. In Theorem 8.2.4 we prove that φ is an almost geodesic such that $\lim_{t\to\infty} h_{\varphi(t)} = h^{u,v}$. We thus know from [69] that

$$\begin{aligned} H(h, h') &= \lim_{t\to\infty} d_T(e, \varphi(t)) + h'(\varphi(t)) \\ &= \lim_{t\to\infty} t + \max\{\log M(u'/\varphi(t), \log M(v'/\varphi(t)^{-1})\} \\ &= \lim_{t\to\infty} \max\{\log M(u'/e^{-t}\varphi(t)), \log M(v'/e^{-t}\varphi(t)^{-1})\} \end{aligned}$$

Now, we know from the proof of Theorem 8.2.4 that $e^{-t}\varphi(t) \to u$ and $e^{-t}\varphi(t)^{-1} \to v$ in the JB algebra norm. Furthermore, for any $t \geq s$,

$$e^{-t}\varphi(t) = \lambda p + \mu^{-1}e^{-2t}(e - p) \leq \lambda p + \mu^{-1}e^{-2s}(e - p) = e^{-s}\varphi(s),$$

and similarly $e^{-t}\varphi(t)^{-1} \leq e^{-s}\varphi(s)^{-1}$, thus Lemma 2.7.9 means that $H(h, h') < \infty$ if and only if u dominates u' and v dominates v', in which case $H(h,h') = \max\{\log M(u'/u), \log M(v'/v)\}$. A symmetrical argument shows that h and h' are in the same part if and only if $u \sim u'$ and $v \sim v'$, in which case, by Corollary 2.8.2

$$\begin{aligned}\delta(h,h') &= \max\{\log M(u'/u), \log M(v'/v)\} + \max\{\log M(u/u'), \log M(v/v')\}\\ &= d_H((u,v),(u',v')).\end{aligned}$$

Now let us assume, without loss of generality, that $v = 0$. Once again as in the proof of Theorem 8.2.4 we then know that $u = p + \lambda(e-p)$ for some primitive idempotent p and $\lambda \in [0,1]$. If we define $\varphi(t) = \exp(tp + t\log(\lambda)(e-p))$, following the same proof, we know that φ is an almost-geodesic such that $\lim_{t\to\infty} h_{\varphi(t)} = h^{u,0}$. Thus,

$$\begin{aligned}H(h,h') &= \lim_{t\to\infty} d_T(e,\varphi(t)) + h'(\varphi(t))\\ &= \lim_{t\to\infty} t + \max\{\log M(u'/\varphi(t), \log M(v'/\varphi(t)^{-1})\}\\ &= \lim_{t\to\infty} \max\{\log M(u'/e^{-t}\varphi(t)), \log M(v'/e^{-t}\varphi(t)^{-1})\},\end{aligned}$$

but we know that $e^{-t}\varphi(t)^{-1} \to 0$, $e^{-t}\varphi(t) \to u$ and for all $t \geq s$,

$$e^{-t}\varphi(t) = p + \lambda(e-p) = e^{-s}\varphi(s),$$

and

$$e^{-t}\varphi(t)^{-1} = e^{-2t}p + e^{-2t}\lambda(e-p) \leq e^{-s}\varphi(s)^{-1}.$$

Lemma 2.7.9 thus means that $H(h,h')$ can only be finite if u dominates u' and $v' = 0$, and symmetry means that $H(h',h)$ can only be finite if u' dominates u and $v = 0$. Thus, $\delta(h,h')$ is finite if and only if $u \sim u'$ and $v = v' = 0$. As above, we thus have

$$\delta(h,h') = \log M(u'/u) + \log M(u/u').$$

□

To make the proofs in the next section more readable, it makes sense at this point to introduce the following corollary:

Corollary 8.2.7. *The non-Busemann elements of $\partial\overline{V_+^\circ}^h$ can be divided further, and consist entirely of functions of the following two forms:*

$$
\begin{aligned}
&h = h^{\mu}_{x,\varepsilon}, \text{ where } \mu \in [0,1),\ \varepsilon \in \{-1,1\},\ \|x\|_{\mathcal{H}} < 1/2,\\
&\text{and, for } w = (\gamma, y) \in V_+^\circ, \text{ with } w^{-1} = (\gamma', y')\\
&\quad h^{\mu}_{x,1}(w) = \log\Bigg(\frac{\gamma\frac{1+\mu}{2} - (1-\mu)\langle x, y\rangle}{\gamma^2 - \|y\|^2_{\mathcal{H}}}\\
&\qquad\qquad + \frac{\sqrt{(\gamma\frac{1+\mu}{2} - (1-\mu)\langle x, y\rangle)^2 - \mu(\gamma^2 - \|y\|^2_{\mathcal{H}})}}{\gamma^2 - \|y\|^2_{\mathcal{H}}}\Bigg), \text{ and}\\
&\quad h^{\mu}_{x,-1}(w) = \log\Bigg(\frac{\gamma'\frac{1+\mu}{2} + (1-\mu)\langle x, y'\rangle}{\gamma'^2 - \|y'\|^2_{\mathcal{H}}}\\
&\qquad\qquad + \frac{\sqrt{(\gamma'\frac{1+\mu}{2} + (1-\mu)\langle x, y'\rangle)^2 - \mu(\gamma'^2 - \|y'\|^2_{\mathcal{H}})}}{\gamma'^2 - \|y'\|^2_{\mathcal{H}}}\Bigg).
\end{aligned}
\tag{8.2.13}
$$

$$
\begin{aligned}
&h = h^{\lambda,\mu}_{x}, \text{ where } \lambda, \mu \in (0,1],\ \max\{\lambda,\mu\} = 1,\ \|x\|_{\mathcal{H}} < 1/2,\\
&\text{and, for } w = (\gamma, y) \in V_+^\circ, \text{ with } w^{-1} = (\gamma', y')\\
&\quad h^{\lambda,\mu}_{x}(w) = \max\left\{\log\left(\frac{\gamma\lambda - 2\lambda\langle x, y\rangle}{\gamma^2 - \|y\|^2_{\mathcal{H}}}\right),\ \log\left(\frac{\gamma'\mu + 2\mu\langle x, y'\rangle}{\gamma'^2 - \|y'\|^2_{\mathcal{H}}}\right)\right\}.
\end{aligned}
\tag{8.2.14}
$$

Proof. It is clear from their definition that all functions of the form (8.2.13) and (8.2.14) are also functions of the form (8.2.6), so are therefore non-Busemann horofunctions by Theorems 8.2.4 and 8.2.5. Conversely, by examining (8.2.6), we see that, for any $\hat{\mu} \in [0,1)$ and $x \in \frac{1}{2}B_{\mathcal{H}}$, $h^{1,\hat{\mu},0,0}_{x} = h^{\hat{\mu},1,0,0}_{-x}$, and similarly $h^{0,0,1,\bar{\mu}}_{x} = h^{0,0,\bar{\mu},1}_{-x}$ for all $\bar{\mu} \in [0,1)$ and $x \in \frac{1}{2}B^\circ_{\mathcal{H}}$.

Thus all horofunctions in the class (8.2.6) of the form $h_x^{\hat{\lambda},\hat{\mu},0,0}$ and $h_x^{0,0,\bar{\lambda},\bar{\mu}}$ are of the form $h_{x,\varepsilon}^{\mu}$ as in (8.2.13) (remembering that in these cases $\max\{\hat{\lambda},\hat{\mu}\} = 1$ and $\max\{\bar{\lambda},\bar{\mu}\} = 1$). Similarly we see, for all $\hat{\lambda},\bar{\mu} \in (0,1]$ with $\max\{\hat{\lambda},\bar{\mu}\} = 1$, and all $x \in \frac{1}{2}B_{\mathcal{H}}^{\circ}$, that $h_x^{\hat{\lambda},0,0,\bar{\mu}} = h_{-x}^{0,\hat{\lambda},\bar{\mu},0}$, which allows us to conclude that all horofunctions in the class (8.2.6) of the form $h_x^{\hat{\lambda},0,0,\bar{\mu}}$ and $h_x^{0,\bar{\lambda},\hat{\mu},0}$ are actually of the form $h_x^{\lambda,\mu}$ as in (8.2.14). This shows that all non-Busemann horofunctions of the form (8.2.6) are actually of the form (8.2.13) or (8.2.14), proving the corollary. □

Finally, it is useful to know that our characterisation of non-Busemann horofunctions uniquely partitions the set $\partial\overline{V_+^{\circ}}^h \backslash \partial_B \overline{V_+^{\circ}}^h$:

Proposition 8.2.8. *If $h, h' \in \partial\overline{V_+^{\circ}}^h$ are two non-Busemann points with $h = h'$, then either both h and h' are of the form* (8.2.3)*, or both are of the form* (8.2.13)*, or both are of the form* (8.2.14)*.*

Proof. First assume that $h = h^{\lambda,x,c}$ as in (8.2.3). It is clear that h' cannot be of the form (8.2.13), as $h_{x,\varepsilon}^{\mu}$ is unbounded below, whereas $h^{\lambda,x,c}$ is bounded below. So, assume by way of contradiction that $h' = h_{x'}^{\lambda',\mu}$ as in (8.2.14). Using Lemma 2.9.15 we can write, for any $w = (\gamma, y) \in V_+^{\circ}$,

$$h_{x'}^{\lambda',\mu}(w) = \max\left\{ \log\left(\frac{\gamma\lambda' - 2\lambda'\langle x', y\rangle}{\gamma^2 - \|y\|_{\mathcal{H}}^2} \right),\ \log\left(\gamma\mu + 2\mu\langle x', y\rangle \right) \right\}. \tag{8.2.15}$$

By assumption, $h^{\lambda,x,c}((\gamma,0)) = h_{x'}^{\lambda',\mu}((\gamma,0))$ for any $\gamma > 0$, so, if we define

$$r = \max\left\{ \log(\lambda + c), \log\left(\frac{1}{\lambda - c}\right) \right\},$$

evaluating (8.2.3) and (8.2.15) at $(\gamma, 0)$ means that

$$\max\left\{ \log\left(\frac{e^{-r}\gamma}{\lambda - c}\right), \log\left(\frac{e^{-r}(\lambda + c)}{\gamma}\right) \right\} = \max\left\{ \log\left(\frac{\lambda'}{\gamma}\right), \log(\mu\gamma) \right\}.$$

As this is true for all γ, it means that

$$\frac{e^{-r}}{\lambda - c} = \mu, \text{ and } e^{-r}(\lambda + c) = \lambda'$$

Let us suppose without loss of generality that $\lambda' = 1$. Then $r = \log(\lambda + c)$, and $\mu = \frac{1}{\lambda^2 - c^2}$. As $\mu \leq 1$, $\lambda^2 - c^2 \geq 1$. As $\mathcal{H}$ is infinite dimensional we can choose $y = \alpha z$, where $z \neq 0$ is perpendicular to both x and x' in $\mathcal{H}$, and $\alpha > 0$. Then, for $w = (\gamma, \alpha z)$,

$$h_{x'}^{\lambda',\mu}(w) = \max\left\{ \log\left(\frac{\gamma}{\gamma^2 - \|\alpha z\|_{\mathcal{H}}^2} \right), \log\left(\frac{\gamma}{\lambda^2 - c^2} \right) \right\},$$

whereas

$$h^{\lambda,x,c}(w) = \max\left\{ \log\left(\frac{\lambda\gamma + \sqrt{\|\alpha z\|_{\mathcal{H}}^2(\lambda^2 - c^2) + c^2\gamma^2}}{(\lambda + c)(\gamma^2 - \|\alpha z\|_{\mathcal{H}}^2)} \right), \log\left(\frac{\lambda\gamma + \sqrt{\|\alpha z\|_{\mathcal{H}}^2(\lambda^2 - c^2) + c^2\gamma^2}}{(\lambda + c)(\lambda^2 - c^2)} \right) \right\}.$$

Thus, if we choose γ and α such that $\gamma^2 - \|\alpha z\|^2 \geq 2(\lambda^2 - c^2)$, and fix γ, it is clear that $h^{\lambda,x,c}(\gamma, \alpha z)$ varies with small perturbations of α, whereas $h_{x'}^{\lambda',\mu}(\gamma, \alpha z)$ does not, meaning that $h^{\lambda,x,c} \neq h_{x'}^{\lambda',\mu}$.

Now suppose that $h = h_x^{\lambda,\mu}$ as in (8.2.14). Cauchy-Schwarz and the fact that $\|x\|_{\mathcal{H}} < 1/2$ means that, by setting $\delta = 2\|x\|_{\mathcal{H}} < 1$, for any $(\gamma, y) \in V_+^\circ$,

$$\frac{\gamma\lambda - 2\lambda \langle x, y\rangle}{\gamma^2 - \|y\|_{\mathcal{H}}^2} > \frac{\lambda}{2\gamma}, \text{ and } \gamma\mu + 2\mu \langle x, y\rangle > \mu(1 - \delta)\gamma.$$

As $\lambda, \mu, 1 - \delta > 0$, it thus follows from applying these inequalities to (8.2.15) that

$$h_x^{\lambda,\mu} \geq \frac{\lambda}{2\mu(1 - \delta)}.$$

Thus we know that h' cannot be of the form (8.2.13), and we have already shown it cannot be of the form (8.2.3). □

Lemma 8.2.9. *The representation of non-Busemann horofunctions of the form* (8.2.3), (8.2.13), *and* (8.2.14) *depends uniquely on the data. That is* $h^{\lambda,x,c} = h^{\lambda',x',c'}$ *if and only if* $\lambda = \lambda'$, $x = x'$, *and* $c = c'$, *and similary for horofunctions of the form* $h_x^{\lambda,\mu}$ *and* $h_{x,\varepsilon}^{\mu}$

Proof. First suppose that $h^{\lambda,x,c} = h^{\lambda',x',c'}$. Utilising a similar strategy as in the proof of Proposition 8.2.8, we can calculate setting $r = \max\left\{\log(\lambda+c), \log\left(\frac{1}{\lambda-c}\right)\right\}$ and $r' = \max\left\{\log(\lambda'+c'), \log\left(\frac{1}{\lambda'-c'}\right)\right\}$, that for all $\gamma > 0$

$$\max\left\{\log\left(\frac{e^{-r}\gamma}{\lambda-c}\right), \log\left(\frac{e^{-r}(\lambda+c)}{\gamma}\right)\right\} = \max\left\{\log\left(\frac{e^{-r'}\gamma}{\lambda'-c'}\right), \log\left(\frac{e^{-r'}(\lambda'+c')}{\gamma}\right)\right\},$$

from which we deduce that

$$\frac{e^{-r}}{\lambda-c} = \frac{e^{-r'}}{\lambda'-c'}, \text{ and } e^{-r}(\lambda+c) = e^{-r'}(\lambda'+c'),$$

which immediately implies that $\lambda^2 - c^2 = \lambda'^2 - c'^2$. We can now utilise a similar strategy as the one used in the proof of Lemma 8.2.2. By choosing a unit vector y perpendicular to both x and x' in $\mathcal{H}$, we can examine $h^{\lambda,x,c}$ and $h^{\lambda',x',c'}$ acting on points $(\gamma, \alpha y)$ for γ and α such that $\gamma >> \alpha$. If $h^{\lambda,x,c} = h^{\lambda',x',c'}$ as we assume, this evaluation leads to the conclusion that the two expressions

$$\lambda\gamma + \sqrt{(\alpha(\lambda^2-c^2)+\gamma^2c^2}, \text{ and } \lambda'\gamma + \sqrt{(\alpha(\lambda^2-c^2)+\gamma^2c'^2}$$

must have the same partial derivatives in some neighbourhood of $(\gamma, \alpha) \in \mathbb{R}^2$. Calculating the partial derivatives with respect to α of the above expressions shows that $c = c'$, from which we immediately deduce that $\lambda = \lambda'$. If $x \neq x'$ it is clear that $x = \beta x'$ for some $\beta \in \mathbb{R}$, else we can choose a $0 \neq y \in \mathcal{H}$ such that y is perpendicular to x but not x', and see from (8.2.3) that $h^{\lambda,x,c}(\gamma, y) \neq h^{\lambda,x',c}(\gamma, y)$. It is similarly simple enough to see that $\beta = 1$

by evaluating both horofunctions at (γ, x) for any large enough γ. Thus, indeed, $\lambda = \lambda'$, $x = x'$, and $c = c'$.

A much simpler version of the same argument shows that if $h_x^{\lambda,\mu} = h_{x'}^{\lambda',\mu'}$ then $\lambda = \lambda'$, $\mu = \mu'$, and $x = x'$. Applying Lemma 2.9.15 to any $h_{x,-1}^{\mu}$ as in (8.2.13) shows that it cannot be equal to $h_{x',1}^{\mu'}$ for any allowed choice of μ' and x'. Thus if $h_{x,\varepsilon}^{\mu} = h_{x',\varepsilon'}^{\mu'}$, $\varepsilon = \varepsilon'$, and using a simple version of the argument above it is routine to show that $\mu = \mu'$ and $x = x'$. □

8.3 Extending the Exponential map to the Boundary

In [42], Lemmens showed that the exponential map extends as a homeomorphism from the horofunction compactification of a finite dimensional JB-algebra A onto the horofunction compactification of the interior of the positive cone of A equipped with the Thompson metric. In this section we show an analogous result for infinite dimensional spin Factors and their associated cone of squares, which we recall is the positive cone when considered as an order unit space with order unit e, the identity.

As above, let $(V, \|\cdot\|)$ be an infinite dimensional spin factor equipped with the JB-algebra norm, which is also the order unit norm $\|\cdot\|_e$. We use $\overline{V_+^\circ}^h$ to denote the horofunction compactification of V_+° equipped with the Thompson metric d_T. As in [42], let us define an extended exponential map $\mathrm{Exp}\colon \overline{V}^h \to \overline{V_+^\circ}^h$. For $u \in V$, we define $\mathrm{Exp}(u) = \exp(u)$ where exp is the standard exponential on V, which can be defined using the spectral calculus. We need to define $\mathrm{Exp}|_{\partial\overline{V}^h}$ piecewise. Recall that for a Busemann point $h \in \partial\overline{V}^h$, we know it is of the form (8.1.3) by Theorem 8.1.5. We can now define Exp piecewise, where all horofunctions used in the definition are as in Theorem 8.1.1, Corollary 8.2.7, (8.2.3), (8.1.3), or (8.2.5):

$$\mathrm{Exp}(u) = \exp(u) \text{ for } u \in V$$

$$\operatorname{Exp}(h^{I,J}_{p,\alpha}) = h^{u,v}, \text{ where } u = \sum_{i\in I} e^{-\alpha_i} p_i \text{ and } v = \sum_{j\in J} e^{-\alpha_j} p_j$$

$$\operatorname{Exp}(h) = \begin{cases} h^{\lambda',x',c'} & \text{if } h = h_\lambda \oplus h^{x,c}, \text{ where } \lambda' = \frac{e^{\lambda+c}+e^{\lambda-c}}{2}, c' = \frac{e^{\lambda+c}-e^{\lambda-c}}{2}, x' = \frac{e^{\lambda+c}-e^{\lambda-c}}{2c}x \\ h^{e^{-2c}}_{x/(2c),\varepsilon} & \text{if } h = h^\varepsilon \oplus h^{x,c}, \ c > \|x\|_{\mathcal{H}} \\ h^{0}_{x/2,\varepsilon} & \text{if } h = h^\varepsilon \oplus h^{x}, \ \|x\|_{\mathcal{H}} < 1 \\ h^{1,e^{-2\lambda}}_{x/2} & \text{if } h = h_\lambda \oplus h^{x}, \text{ and } \lambda \geq 0, \ \|x\|_{\mathcal{H}} < 1 \\ h^{e^{2\lambda},1}_{x/2} & \text{if } h = h_\lambda \oplus h^{x}, \text{ and } \lambda < 0, \ \|x\|_{\mathcal{H}} < 1. \end{cases} \tag{8.3.1}$$

If $u = (\gamma, x) \in V$, the spectral calculus in conjunction with Lemma 2.9.13 allow us to calculate

$$\exp(u) = \left(\frac{e^{\gamma+\|x\|_{\mathcal{H}}} + e^{\gamma-\|x\|_{\mathcal{H}}}}{2}, \frac{e^{\gamma+\|x\|_{\mathcal{H}}} - e^{\gamma-\|x\|_{\mathcal{H}}}}{2\|x\|_{\mathcal{H}}} x \right). \tag{8.3.2}$$

It is clear that Exp is well defined on V, as it is the usual exponential. It is also clear that it is well defined on the non-Busemann horofunctions of $\overline{V}^h$, because they are uniquely defined by their data in Theorem 8.1.1. To prove that Exp is well defined on $\partial_B \overline{V}^h$, we borrow [42, Lemma 4.2]:

Lemma 8.3.1. *Let $x, y \in V$ have spectral decompositions $x = \sum_{i\in I} \lambda_i p_i$ and $y = \sum_{j\in J} \mu_j q_j$. If $\sum_{i\in I} p_i = \sum_{j\in J} q_j$ and $x = y$, then $\sum_{i\in I} e^{-\lambda_i} p_i = \sum_{j\in J} e^{-\mu_j} q_j$.*

Proof. As above, we let $p_I = \sum_{i\in I} p_i$ and $q_J = \sum_{j\in J} q_j$. Now $x = 0(e-p_I) + \sum_{i\in I} \lambda_i p_i$, and $y = 0(e-q_J) + \sum_{j\in J} \mu_j q_j$. Note that the collections $\{p_i\}_{i\in I} \cup \{e-p_I\}$ and $\{q_j\}_{j\in J} \cup \{e-p_J\}$ are complete collections of orthogonal idempotents, so, using the assumption $x = y$,

$$\exp(x) = e - p_I + \sum_{i\in I} e^{-\lambda_i} p_i = e - q_J + \sum_{j\in J} e^{-\mu_j} q_j = \exp(y),$$

which gives the result, because by assumption $\exp(x) = \exp(y)$ and $p_I = q_J$. □

Now let us suppose we are given two representations of the same Busemann point h in $\partial_B \overline{V}^h$, say $h^{I,J}_{p,\alpha}$ and $h^{I',J'}_{q,\beta}$. Theorem 8.1.7 means that $p_I = q_{I'}$ and $p_J = q_{J'}$, and as

$\delta(h^{I,J}_{p,\alpha}, h^{I',J'}_{q,\beta}) = 0$ it also follows that

$$\sum_{i\in I} \alpha_i p_i + \sum_{j\in J} \alpha_j p_j = \sum_{i\in I'} \beta_i q_i + \sum_{j\in J'} \beta_j q_j.$$

As p_I is perpendicular to p_J, $q_{I'}$ is perpendicular to $q_{J'}$, and $p_I = q_{I'}$, it follows that

$$\sum_{i\in I} \alpha_i p_i = U_{p_I}\left(\sum_{i\in I} \alpha_i p_i + \sum_{j\in J} \alpha_j p_j\right) = U_{q_{I'}}\left(\sum_{i\in I'} \beta_i q_i + \sum_{j\in J'} \beta_j q_j\right) = \sum_{i\in I'} \beta_i q_i.$$

Similarly, $\sum_{j\in J} \alpha_j p_j = \sum_{j\in J'} \beta_j q_j$. Lemma 8.3.1 thus means that $\mathrm{Exp}(h^{I,J}_{p,\alpha}) = \mathrm{Exp}(h^{I',J'}_{q,\beta})$, so indeed Exp is well defined.

Lemma 8.3.2. *The map* $\mathrm{Exp}\colon \overline{V}^h \to \overline{V^\circ_+}^h$ *is a bijection mapping* V *onto* V°_+ *and* $\partial\overline{V}^h$ *onto* $\partial\overline{V^\circ_+}^h$*. It also maps* $\partial_B\overline{V}^h$ *bijectively onto* $\partial_B\overline{V^\circ_+}^h$*.*

Proof. By the spectral theorem, any $u \in V$ can be written as $u = \lambda p + \mu(e - p)$, for a primitive idempotent p, where λ and μ uniquely determine u. Thus $\mathrm{Exp}(u) = e^\lambda p + e^\mu(e-p)$, from which we immediately deduce that Exp maps V bijectively onto V°_+. For a Busemann point $h \in \partial_B\overline{V}^h$ with representation $h^{I,J}_{p,\alpha}$, $\mathrm{Exp}(h) = h^{u,v}$, where $u = \sum_{i\in I} e^{-\alpha_i} p_i$ and $v = \sum_{j\in J} e^{-\alpha_j} p_j$. As $\min \alpha_i = 0$, it follows that $\max\{\|u\|_e, \|v\|_e\} = 1$, and as p_I and q_I are orthogonal idempotents, u and v are also orthogonal. Theorem 8.2.5 thus implies Exp maps $\partial_B\overline{V}^h$ into $\partial_B\overline{V^\circ_+}^h$. Now suppose that $\mathrm{Exp}(h) = \mathrm{Exp}(h')$ for $h, h' \in \partial_B\overline{V}^h$, where $h = h^{I,J}_{p,\alpha}$ and $h' = h^{I',J'}_{q,\beta}$. We know that $\mathrm{Exp}(h) = h^{u,v}$ and $\mathrm{Exp}(h') = h^{u',v'}$, where

$$u = \sum_{i\in I} e^{-\alpha_i} p_i,\ v = \sum_{j\in J} e^{-\alpha_j} p_j,\ u' = \sum_{i\in I'} e^{-\beta_i} q_i \text{ and } v = \sum_{j\in J'} e^{-\beta_j} q_j.$$

As $\delta(h^{u,v}, h^{u',v'}) = 0$, Theorem 8.2.5 means that $u = u'$ and $v = v'$. Following a similar argument as in the proof of Lemma 8.3.1, $\sum_{i\in I} \alpha_i p_i = \sum_{i\in I'} \beta_i q_i$ and $\sum_{j\in J} \alpha_j p_j = \sum_{j\in J'} \beta_j q_j$ and also that $p_I = q_{I'}$ and $p_J = q_{J'}$. Thus $h = h'$, and Exp is an injective mapping from Busemann points to Busemann points. For any $h \in \partial_B\overline{V}^\circ_+$ there exists $u, v \in \partial V_+$ such that $h = h^{u,v}$ as per Theorem 8.2.5. We know that $u = \lambda_1 p + \mu_1(e - p)$ and $v = \lambda_2 q + \mu_2(e - q)$

for idempotents p, q and $\lambda_i \geq \mu_i \geq 0$. As $u \bullet v = 0$, Lemma 2.9.14 means that either $u = \lambda_1 p$ and $v = \lambda_2(e - p)$, or one of u and v is 0. Without loss of generality first assume that $v = 0$. If $\lambda_1, \mu_1 > 0$, then $h = \mathrm{Exp}(h_{p,\alpha}^{\{1,2\},\emptyset})$ where $\alpha = (-\log \lambda_1, -\log \mu_1)$. If one of λ_1, μ_1 is 0 then $h = \mathrm{Exp}(h_{0,p}^{\{1\},\emptyset})$ or $h = \mathrm{Exp}(h_{0,p}^{\{2\},\emptyset})$ depending on which one is 0. Now, if both u and v are non-zero, then Lemma 2.9.14 means that $q = e - p$, and $\mu_1 = 0 = \mu_2$. In this case, $h = \mathrm{Exp}(h_{p,\alpha}^{I,J})$, where $I = \{1\}, J = \{2\}$, and $\alpha = (-\log(\lambda_1), -\log(\lambda_2))$. We can thus conclude that indeed Exp maps $\partial_B \overline{V}^h$ bijectively onto $\partial_B \overline{V_+^\circ}^h$.

As the horofunction compactification of any metric space is the disjoint union of internal metric functionals, Busemann points, and non-Busemann horofunctions, what is left is to show that Exp maps the non-Busemann horofunctions in $\overline{V}^h$ bijectively onto the non-Busemann horofunctions in $\overline{V_+^\circ}^h$. Proposition 8.2.8 means that if $\mathrm{Exp}(h) = \mathrm{Exp}(h')$ for non-Busemann horofunctions in $\overline{V}^h$, then $\mathrm{Exp}(h)$ and $\mathrm{Exp}(h')$ have to be of the same form, by which we mean both $\mathrm{Exp}(h)$ and $\mathrm{Exp}(h')$ have to be of one of the three forms (8.2.3), (8.2.13), or (8.2.14). Lemma 8.2.9 shows that the defining data for $\mathrm{Exp}(h)$ and $\mathrm{Exp}(h')$ must be identical, but Exp is uniquely defined by the defining data, so $h = h'$. It is a simple matter of data matching to verify that Exp surjectively maps $\partial \overline{V}^h \backslash \partial_B \overline{V}^h$ onto $\partial \overline{V_+^\circ}^h \backslash \partial_B \overline{V_+^\circ}^h$. □

We now need to prove that Exp is continuous. We do this by proving that if (h_α) is a net in $\overline{V}^h$ converging to some $h \in \overline{V_+^\circ}^h$, then $\mathrm{Exp}(h_\alpha)$ converges to $\mathrm{Exp}(h)$. If (h_α) is a net of internal metric functionals, the continuity of exp immediately implies this. The other cases take more work to prove, so we break up the proof into a series of propositions and lemmas. The structure of the proof of each of Lemmas 8.3.4, 8.3.5, 8.3.6, 8.3.8, 8.3.9, 8.3.10, and 8.3.12 proved below is similar. We consider a net $(h_\alpha) \subseteq \overline{V}^h$, and show that every subnet of (h_α) has a further subnet, say (h_β) such that $\mathrm{Exp}(h_\beta)$ converges to $\mathrm{Exp}(h)$. This is equivalent to showing that every subnet of $\mathrm{Exp}(h_\alpha)$ has a further subnet converging to $\mathrm{Exp}(h)$, which shows that $\mathrm{Exp}(h_\alpha) \to \mathrm{Exp}(h)$.

Proposition 8.3.3. *If* $(h_{u_\alpha}) \subseteq \overline{V}^h$ *is a net of internal metric functionals converging to some* $h \in \partial\overline{V}^h$*, then* $\mathrm{Exp}(h_{u_\alpha})$ *converges to* $\mathrm{Exp}(h)$ *in* $\partial\overline{V_+^\circ}^h$*.*

Proof. The proof of Proposition 8.3.3 is given by combining Lemmas 8.3.4, 8.3.5, and 8.3.6 proved below. □

Lemma 8.3.4. *If* $(u_\alpha) \subseteq V$ *is an eventually bounded net such that* h_{u_α} *converges to some* $h \in \partial\overline{V}^h$*, then* $\mathrm{Exp}(h_{u_\alpha})$ *converges to* $\mathrm{Exp}(h)$ *in* $\partial\overline{V_+^\circ}^h$*.*

Proof. Assume $(u_\alpha) = ((\gamma_\alpha, y_\alpha))$ is bounded in the JB-algebra norm, so (γ_α) is bounded in $\mathbb{R}$ and (y_α) is bounded in $\mathcal{H}$. Lemma 4.1 in [27] thus means that $h = h_\lambda \oplus h^{x,c}$ for $\lambda \in \mathbb{R}$ and $\|x\| < c$. Let us assume without loss of generality that $y_\alpha \neq 0$ for large enough α. For each α, we have the spectral decomposition

$$u_\alpha = \left(\frac{\eta_\alpha + \mu_\alpha}{2}, \frac{\eta_\alpha - \mu_\alpha}{2\|y_\alpha\|_{\mathcal{H}}} y_\alpha\right),$$

where $\eta_\alpha \geq \mu_\alpha$ and both depend uniquely on u_α. As (u_α) is bounded in the JB-algebra norm, both (η_α) and (μ_α) must also be bounded. In combination with Banach Alaoglu there must then exist a subnet (u_β), $\eta, \mu \in \mathbb{R}$, $y \in \mathcal{H}$, and $c' \geq \|y\|$ such that $\eta_\beta \to \eta$, $\mu_\beta \to \mu$, $\frac{(\eta_\beta - \mu_\beta)}{2\|y_\beta\|_{\mathcal{H}}} y_\beta \xrightarrow{w} y$, and $\frac{\eta_\beta - \mu_\beta}{2} \to c'$. Furthermore we know that $y = \frac{\eta - \mu}{2} z$, where z is the weak limit of $\frac{y_\beta}{\|y_\beta\|_{\mathcal{H}}}$. The proof of Lemma 4.1 in [27] means that $h_{u_\beta} \to h_{(\eta+\mu)/2} \oplus h^{y,c'}$. Uniqueness of limits in a Hausdorff space in combination with Lemma 7.1.8 thus means that $\lambda = \frac{\eta+\mu}{2}$, $x = y$, and $c = c'$. Now,

$$\exp(u_\beta) = \left(\frac{e^{\eta_\beta} + e^{\mu_\beta}}{2}, \frac{e^{\eta_\beta} - e^{\mu_\beta}}{2\|y_\beta\|_{\mathcal{H}}} y_\beta\right),$$

which converges weakly to $\left(\frac{e^\eta + e^\mu}{2}, \frac{e^\eta - e^\mu}{2} z\right)$. We now note that, using the relationships we have established, we can calculate that

$$\eta = \lambda + c, \text{ and } \mu = \lambda - c,$$

Thus we can conclude that

$$\exp(u_\beta) \xrightarrow{w} \left(\frac{e^{\lambda+c}+e^{\lambda-c}}{2}, \frac{e^{\lambda+c}-e^{\lambda-c}}{2c}x\right).$$

The proof of Lemma 8.2.2 then shows that $\operatorname{Exp}(h_{u_\beta})$ converges to $h^{\lambda',x',c'} = \operatorname{Exp}(h)$, with λ', x', and c' as in definition (8.3.1). As every subnet of an eventually bounded net contains a bounded subnet, this argument actually shows that any subnet of (u_α) must have a further subnet (u_β), such that $\operatorname{Exp}(h_{u_\beta})$ converges to $h^{\lambda',x',c'} = \operatorname{Exp}(h)$ in $\partial\overline{V_+^\circ}^h$, so indeed $\operatorname{Exp}(h_\alpha) \to \operatorname{Exp}(h)$. □

Lemma 8.3.5. *If $(u_\alpha) \subseteq V$ is an unbounded net such that h_{u_α} converges to some non-Busemann $h \in \partial\overline{V}^h$, then $\operatorname{Exp}(h_{u_\alpha})$ converges to $\operatorname{Exp}(h)$ in $\partial\overline{V_+^\circ}^h$.*

Proof. As h is not a Busemann point, (u_α) is not an almost-geodesic. Once again, we spectrally decompose each u_α to write

$$u_\alpha = \left(\frac{\eta_\alpha+\mu_\alpha}{2}, \frac{\eta_\alpha-\mu_\alpha}{2\|y_\alpha\|_{\mathcal{H}}}y_\alpha\right).$$

We initially consider the case when (γ_α) is bounded in $\mathbb{R}$. (y_α) must then be unbounded in $\mathcal{H}$. The proof of Theorem 8.1.1 implies then that there exists a subnet (u_β) such that h_{u_β} converges to $h_\lambda \oplus h^x$, for $\|x\|_{\mathcal{H}} < 1$ and $\lambda \in \mathbb{R}$. Thus $h = h_\lambda \oplus h^x$. Similarly to the proof of the above lemma, there must exist a subnet (u_β) and a $\lambda' \in \mathbb{R}$ and $y \in \mathcal{H}$ such that $\frac{\eta_\beta+\mu_\beta}{2} \to \lambda'$ and $y_\beta/\|y_\beta\|_{\mathcal{H}}$ converges weakly to y. The proof of [27, Lemma 4.3] and uniqueness of limits means that $\lambda = \lambda'$ and $x = y$. Furthermore, we must have that η_β and μ_β are unbounded, and $\lim_\beta \frac{\eta_\beta}{\mu_\beta} = -1$ (else either (u_β) would be bounded or (λ_β) would be unbounded). As $\eta_\beta \geq \mu_\beta$ for all β this means that $\eta_\beta \to \infty$ and $\mu_\beta \to -\infty$. This argument shows that any subnet has a similarly convergent subnet, so we must have $y_\alpha/\|y_\alpha\|_{\mathcal{H}} \xrightarrow{w} x$, $\frac{\eta_\alpha+\mu_\alpha}{2} \to \lambda$, and $\lim_\alpha \frac{\eta_\alpha}{\mu_\alpha} = -1$. Using spectral decomposition, we can write

$$\operatorname{Exp} u_\alpha = e^{\eta_\alpha}p_\alpha + e^{\mu_\alpha}(e-p_\alpha), \text{ and}$$

$$(\operatorname{Exp} u_\alpha)^{-1} = e^{-\eta_\alpha} p_\alpha + e^{-\mu_\alpha}(e - p_\alpha)$$

where $p_\alpha = (\frac{1}{2}, \frac{1}{2\|y_\alpha\|_\mathcal{H}} y_\alpha)$. If $\lambda \geq 0$, then for large enough α, $\eta_\alpha \geq |\mu_\alpha|$, and if $\lambda < 0$ then $\eta_\alpha, < |\mu_\alpha|$. Keeping this in mind, if we then follow the proof of Lemma 8.2.3 in combination with the proof of Corollary 8.2.7, we see that $h_{\operatorname{Exp} u_\alpha}$ converges to $h_{x/2}^{1,e^{-2\lambda}}$ if $\lambda \geq 0$ and $h_x^{e^{-2\lambda},1}$ if $\lambda < 0$, so indeed $\lim_\alpha \operatorname{Exp}(h_\alpha) = \operatorname{Exp}(h)$.

If (γ_α) is unbounded but (y_α) is bounded, the proof of Theorem 8.1.1 then implies that $h = h^\varepsilon \oplus h^{x,c}$. For each α we again have the spectral decomposition

$$u_\alpha = \eta_\alpha p_\alpha + \mu_\alpha(e - p_\alpha),$$

where $p_\alpha = (\frac{1}{2}, \frac{1}{2\|y_\alpha\|_\mathcal{H}} y_\alpha)$, and $\eta_\alpha \geq u_\alpha$. We can combine the arguments in the two paragraphs above to see that $\frac{\eta_\alpha + \mu_\alpha}{2} \to \pm\infty$, where the sign is determined by ε, and $y_\alpha \xrightarrow{w} x$ and $\|y_\alpha\|_\mathcal{H} = \frac{\eta_\alpha - \mu_\alpha}{2} \to c$, which means that, for large enough α, η_α and μ_α must have the same sign, and $||\eta_\alpha| - |\mu_\alpha|| \leq K$ for some $K > 0$. Thus, the proof of Lemma 8.2.3 in combination with the proof of Corollary 8.2.7 shows that indeed $\operatorname{Exp}(h_{u_\alpha}) \to h_{x/(2c),\varepsilon}^{e^{-2c}} = \operatorname{Exp}(h)$.

Finally, suppose that both (γ_α) and (y_α) are unbounded. The proof of Theorem 8.1.1 implies then that there exists a subnet (u_β) such that h_{u_β} converges to $h^\varepsilon \oplus h^x$, for $\|x\|_\mathcal{H} < 1$ and $\varepsilon \in \{-1, 1\}$. Thus $h = h^\varepsilon \oplus h^x$. As above, for each α we can choose the spectral representation

$$u_\alpha = \eta_\alpha p_\alpha + \mu_\alpha(e - p_\alpha),$$

where $p_\alpha = (\frac{1}{2}, \frac{1}{2\|y_\alpha\|_\mathcal{H}} y_\alpha)$, and $\eta_\alpha \geq u_\alpha$. We can combine the arguments above to see that $\frac{\eta_\alpha + \mu_\alpha}{2} \to \pm\infty$, where the sign is determined by ε, and $x_\alpha / \|x_\alpha\|_\mathcal{H}$ converges weakly to x. As $\frac{\eta_\alpha - \mu_\alpha}{2} = \|y_\alpha\|_\mathcal{H} \to \infty$, and

$$\operatorname{Exp} u_\alpha = e^{\eta_\alpha} p_\alpha + e^{\mu_\alpha}(e - p_\alpha), \text{ and}$$

we must either have, for large enough α, that $\|\log(\mathrm{Exp}(u_\alpha))\|_e = \eta_\alpha$ and $-\eta_\alpha \pm \mu_\alpha \to -\infty$ if $\varepsilon = 1$, and $\|\log(\mathrm{Exp}(u_\alpha))\|_e = -\mu_\alpha$ and $\mu_\alpha \pm \eta_\alpha \to -\infty$ if $\varepsilon = -1$. Thus the proof of Lemma 8.2.3 in combination with the proof of Corollary 8.2.7 shows that indeed $\mathrm{Exp}(h_{u_\alpha}) \to h^0_{x/2,\varepsilon} = \mathrm{Exp}(h)$. □

Lemma 8.3.6. *If $(u_\alpha) \subseteq V$ is an unbounded net such that h_{u_α} converges to some Busemann $h \in \partial_B \overline{V}^h$, then $\mathrm{Exp}(h_{u_\alpha})$ converges to $\mathrm{Exp}(h)$ in $\partial\overline{V^\circ_+}^h$.*

Proof. As h is a Busemann point, it must be of the form $h^{I,J}_{p,\alpha}$ by Theorem 8.1.5. We can utilise the proof of [42, Lemma 4.4] with some tweaking. For each α, if $u_\alpha = (\lambda_\alpha, y_\alpha)$, we once again choose the spectral representation

$$u_\alpha = \eta_\alpha p_\alpha + \mu_\alpha(e - p_\alpha),$$

where $p_\alpha = (\frac{1}{2}, \frac{1}{2\|y_\alpha\|_{\mathcal{H}}} y_\alpha)$, and $\eta_\alpha \geq u_\alpha$. As h is not bounded below by Corollary 3.2.4, (u_α) must be unbounded. If (y_α) is bounded in $\mathcal{H}$, then there must exist a subnet (y_β) and a $c \geq 0$ and $x \in \mathcal{H}$ such that $\|y_\beta\|_{\mathcal{H}} \to c$ and $y_\beta \xrightarrow{w} x$. If $\|x\|_{\mathcal{H}} < c$, the proof of Theorem 8.1.1 shows that $h = h^\varepsilon \oplus h^{x,c}$, which is not a Busemann point by Theorem 8.1.3, a contradiction. Thus, if (y_α) is bounded and does not converge to 0 in norm, there must exist a subnet (u_β) and $y \in \mathcal{H}$ such that, if $y \neq 0$, the net $p_\beta = \frac{1}{2\|y_\beta\|_{\mathcal{H}}} y_\beta$ converges weakly to $\frac{1}{2\|y\|_{\mathcal{H}}} y$, which is a primitive idempotent, say p', and as in the proof of Lemma 8.2.3, this means that p_β must actually converge in norm to p'. If $y_\alpha \to 0$ we immediately see that $h_{u_\alpha} \to h^\varepsilon \oplus h_0$, and $\eta_\beta - \mu_\beta \to 0$. Now if (y_α) is unbounded, there must exist a subnet (u_β), and an $x \in \mathcal{H}$, such that $y_\beta / \|y_\beta\|_{\mathcal{H}}$ converges weakly to x. The proof of Theorem 8.1.1 thus means that $h = h^\varepsilon \oplus h^x$ or $h = h_\lambda \oplus h^x$. If $\|x\|_{\mathcal{H}} < 1$, Theorem 8.1.3 means that h cannot be a Busemann point. Therefore once again there must exist a subnet such that (p_β) converges in norm to a primitive idempotent $p' = (1/2, x)$. Thus, if (y_α) does not converge to 0, there exists a subnet (u_β) such that $p_\beta \to p' = (1/2, x)$ for a primitive idempotent p'. The proof of Theorem 8.1.1 thus shows that h_{u_β} must converge to one of either $h^\varepsilon \oplus h_{ax}$ for some $a \in \mathbb{R}$, or $h_\lambda \oplus h^{2x}$, or $h^\varepsilon \oplus h^{2x}$. The proof of

Theorem 8.1.5 thus shows that $p = p'$, and the statement of the same theorem along with its proof means that there must exist $I', J' \subseteq \{1,2\}$ satisfying $I' \cap J' = \emptyset$, $I' \cup U' \neq \emptyset$, and a $\alpha' \in \mathbb{R}^{I' \cup J'}$ with $\min \alpha' = 0$, so that $\lim_\beta h_{u_\beta} = h^{I'J'}_{p,\alpha'}$. Furthermore the proof of Theorem 8.1.5 combined with Lemma 2.9.13 shows that $\alpha'_i = \lim_\beta \|u_\beta\|_e - \mu_\beta$, if $i \in I'$, and $\alpha'_i = \lim_\beta \|u_\beta\|_e + \eta_\beta$ if $i \in J'$. Now, following the proof of Lemma 8.2.3, there must exist a further subnet (u_β), and $u', v' \in \partial V_+$, $u' = \hat{\lambda} p + \hat{\mu}(e - p)$ and $v' = \bar{\lambda} p + \bar{\mu}(e - p)$ with $\max\{\|u'\|_e, \|v'\|_e\} = 1$ and $u' \bullet v' = 0$, such that $\operatorname{Exp}(h_{u_\beta}) \to h^{u',v'}$. We also know from definition 8.3.1 that $\operatorname{Exp}(h) = h^{u,v}$, where $u = \sum_{i \in I} e^{-\alpha_i} p_i$ and $v = \sum_{j \in J} e^{-\alpha_j} p_j$. If $y_\alpha \to 0$, then $h_{u_\alpha} \to h^\varepsilon \oplus h_0$, so the proof of Lemma 8.2.3 shows that $(\operatorname{Exp}(h_u \alpha))$ converges to either $h^{e,0}$ or $h^{0,e}$ depending on the sign of ε, and the proof of Theorem 8.1.5 shows that $h^\varepsilon \oplus h_0 = h^{\{1,2\},\emptyset}_{p,0}$ or $h^\varepsilon \oplus h_0 = h^{\emptyset,\{1,2\}}_{p,0}$ depending on the sign of ε. We are now in the position where the proof of Lemma 4.4 in [42] applies directly, as we have avoided the need to rely on finite dimensionality with the above paragraph. □

Proposition 8.3.7. *If $(h_\alpha) \subseteq \partial \overline{V}^h$ is a net converging to some $h \in \partial \overline{V}^h$, then $\operatorname{Exp}(h_\alpha)$ converges to $\operatorname{Exp}(h)$ in $\partial \overline{V_+^\circ}^h$.*

Proof. Via Theorem 8.1.1, we know that we can write $h = h^{\mathbb{R}} \oplus h^{\mathcal{H}}$, where $h^{\mathbb{R}} \in \overline{\mathbb{R}}^h$, $h^{\mathcal{H}} \in \overline{\mathcal{H}}^h$, and at least one of $h^{\mathbb{R}}$ or $h^{\mathcal{H}}$ is a horofunction. Similarly we can decompose each $h_\alpha = h^{\mathbb{R}}_\alpha \oplus h^{\mathcal{H}}_\alpha$. Lemma 2.3.6 in conjunction with the uniqueness of limits means that $h^{\mathbb{R}}_\alpha \to h^{\mathbb{R}}$ and $h^{\mathcal{H}}_\alpha \to h^{\mathcal{H}}$. Using this fact we consider the various cases in Lemmas 8.3.8, 8.3.9, 8.3.10, and 8.3.12 below, which taken together prove the proposition. □

Lemma 8.3.8. *If $(h_\alpha) \subseteq \partial \overline{V}^h$ is a net converging to some $h_\lambda \oplus h^{x,c} \in \partial \overline{V}^h$, where $h^{x,c}$ is as in* (8.1.2), *then $\operatorname{Exp}(h_\alpha)$ converges to $\operatorname{Exp}(h)$ in $\partial \overline{V_+^\circ}^h$.*

Proof. As $h^\varepsilon(\varepsilon 2\lambda) = -2\lambda$ for $\varepsilon \in \{-1, 1\}$ we must have, for every tail, that $h^{\mathbb{R}}_\alpha = h_{\lambda_\alpha}$ for all α large enough, where $\lambda_\alpha \to \lambda$. We also claim that if $x \neq 0$, then there cannot exist any subnets $(h^{\mathcal{H}}_\beta)$ so that $h^{\mathcal{H}}_\beta = h^{x_\beta}$ for all β in a cofinal set. Indeed, suppose that such a subnet does exist. It is simple to calculate, if $x \neq 0$ that $h^{x,c}(2x) = 0$, and $h^{x,c}(-2x) > 0$. This means that $\lim_\beta -2\langle x, x_\beta \rangle = 0$ and $\lim_\beta 2\langle x, x_\beta \rangle > 0$, a clear impossibility. If $x = 0$ a similar sign argument evaluating at any $y, -y \in cB_{\mathcal{H}}$ shows the same impossibility. Thus, every subnet of (h_α) must have a further subnet such that $h_\beta = h^{x_\beta, c_\beta}$. We claim that (c_β) must be bounded. Indeed, suppose by way of contradiction that it is unbounded. For any $y \in \mathcal{H}$ the Cauchy-Schwarz inequality means that, for any β,

$$\sqrt{c_\beta^2 - 2\|y\|_{\mathcal{H}}\|x_\beta\|_{\mathcal{H}} + \|y\|_{\mathcal{H}}^2} - c_\beta \leq h^{x_\beta, c_\beta}(y) \leq \sqrt{c_\beta^2 + 2\|y\|_{\mathcal{H}}\|x_\beta\|_{\mathcal{H}} + \|y\|_{\mathcal{H}}^2} - c_\beta.$$

As $\|x_\beta\|_{\mathcal{H}} < c_\beta$, we can set $t_\beta = \|x_\beta\|_{\mathcal{H}}/c_\beta$, and choose a further subnet such that $t_\beta \to t \in [0, 1]$, and rewrite the above inequality to read

$$c_\beta \left(\sqrt{1 - \frac{2t_\beta\|y\|_{\mathcal{H}}}{c_\beta} + \frac{\|y\|_{\mathcal{H}}^2}{c_\beta^2}} - 1 \right) \leq h^{x_\beta, c_\beta}(y) \leq c_\beta \left(\sqrt{1 + \frac{2t_\beta\|y\|_{\mathcal{H}}}{c_\beta} + \frac{\|y\|_{\mathcal{H}}^2}{c_\beta^2}} - 1 \right).$$

Recall the Taylor expansion of $\sqrt{1+a}$ for small a, $\sqrt{1+a} = 1 + a/2 + O(a^2)$. It is clear that $\frac{2t_\beta\|y\|_{\mathcal{H}}}{c_\beta} + \frac{\|y\|_{\mathcal{H}}^2}{c_\beta^2} \to 0$, so we can use this expansion in the above inequality and calculate, for large β, that

$$c_\beta \left(-\frac{t_\beta\|y\|_{\mathcal{H}}}{c_\beta} + \frac{\|y\|_{\mathcal{H}}^2}{2c_\beta^2} + O\left(\frac{1}{c_\beta^2}\right) \right) \leq h^{x_\beta, c_\beta}(y) \leq c_\beta \left(\frac{t_\beta\|y\|_{\mathcal{H}}}{c_\beta} + \frac{\|y\|_{\mathcal{H}}^2}{2c_\beta^2} + O\left(\frac{1}{c_\beta^2}\right) \right).$$

Thus, for large β,

$$-t_\beta\|y\|_{\mathcal{H}} + \frac{\|y\|_{\mathcal{H}}^2}{2c_\beta} + O\left(\frac{1}{c_\beta}\right) \leq h^{x_\beta, c_\beta}(y) \leq t_\beta\|y\|_{\mathcal{H}} + \frac{\|y\|_{\mathcal{H}}^2}{2c_\beta} + O\left(\frac{1}{c_\beta}\right), \tag{8.3.3}$$

and because $h^{x,c} \neq 0$, this can only be true if $t > 0$. Thus $t^{-1} \in [1, \infty)$ is well defined. Furthermore, Banach-Alaoglu means we can pick a further subnet and a $z \in \mathcal{H}$ such that $z_\beta = x_\beta/\|x_\beta\|$ converges weakly to z. Now, we can use a similar approach as above to write, for all $y \in \mathcal{H}$ and large enough β,

$$h^{x_\beta,c_\beta}(y) = t_\beta^{-1}\|x_\beta\|_{\mathcal{H}}\left(\sqrt{1 - \frac{2\langle y, z_\beta\rangle}{t_\beta^{-2}\|x_\beta\|_{\mathcal{H}}} + \frac{\|y\|_{\mathcal{H}}^2}{t_\beta^{-2}\|x_\beta\|_{\mathcal{H}}^2}} - 1\right)$$
$$= -t_\beta\langle y, z_\beta\rangle + \frac{t_\beta\|y\|_{\mathcal{H}}^2}{2\|x_\beta\|_{\mathcal{H}}} + O\left(\frac{1}{\|x_\beta\|_{\mathcal{H}}}\right), \tag{8.3.4}$$

meaning that $\lim_\beta h^{x_\beta,c_\beta} = h^{tz}$, a contradiction. Thus, indeed, (c_β) is bounded, and so too is $\|x_\beta\|_{\mathcal{H}}$. There must thus exist a further subnet (h_β) such that $c_\beta \to d$ for some $d \in \mathbb{R}$, and $x_\beta \xrightarrow{w} z$ for some $z \in \mathcal{H}$. It is then clear to see, by definition, that $h_\beta^{\mathcal{H}} \to h^{z,d}$, so by Lemma 7.1.8 we have $x = z$ and $c = d$. Now, $\mathrm{Exp}(h_\beta) = h^{\lambda'_\beta,c'_\beta,x'_\beta}$, where $\lambda'_\beta = \frac{e^{\lambda_\beta+c_\beta}+e^{\lambda_\beta-c_\beta}}{2}$, $c'_\beta = \frac{e^{\lambda_\beta+c_\beta}-e^{\lambda_\beta-c_\beta}}{2}$, and $x'_\beta = \frac{e^{\lambda_\beta+c_\beta}-e^{\lambda_\beta-c_\beta}}{2c_\beta}x_\beta$. It is clear that $\lambda'_\beta \to \lambda'$, $c'_\beta \to c'$, and so $x'_\beta \xrightarrow{w} x'$. Thus, for any $(\gamma, y) \in V_+^\circ$ we see by (8.2.3) that $\lim_\beta \mathrm{Exp}(h_\beta) = h^{\lambda',x',c'} = \mathrm{Exp}(h)$. This argument shows that every subnet of (h_α) has a subnet (h_β) such that $\mathrm{Exp}(h_\beta) \to \mathrm{Exp}(h)$, proving the lemma. □

Lemma 8.3.9. *If $(h_\alpha) \subseteq \partial\overline{V}^h$ is a net converging to some $h = h^\varepsilon \oplus h^{x,c} \in \partial\overline{V}^h$, where $h^{x,c}$ is as in* (8.1.2)*, then* $\mathrm{Exp}(h_\alpha)$ *converges to* $\mathrm{Exp}(h)$ *in* $\partial\overline{V_+^\circ}^h$.

Proof. We have to consider two cases. Either there exists a subnet (h_β) such that $h_\beta^{\mathbb{R}} = h_\lambda$ for all β, meaning that $h_\beta^{\mathcal{H}} \in \partial\overline{\mathcal{H}}^h$, or there exists a subnet such that $h_\beta^{\mathbb{R}} = h^\varepsilon$, meaning that the $h_\beta^{\mathcal{H}}$ can either be internal functionals or horofunctions. Let us first assume we are in the former case. The same argument as above shows that every subnet of (h_β) has a further subnet (h_β) where $h_\beta^{\mathcal{H}} = h^{x_\beta,c_\beta}$, and $x_\beta \xrightarrow{w} x$ and $c_\beta \to c$. The only difference in the above argument is that now $\lambda_\beta \to \pm\infty$. To simplify calculations we define, for any $\delta > 0$, $z \in \mathcal{H}$, and $(\gamma, y) \in V_+^\circ$,

$$\zeta_\delta^z(\gamma, y) = \sqrt{\left(\frac{(1+e^{-2\delta})\gamma}{2} - \frac{1-e^{-2\delta}}{2\delta}\langle z, y\rangle\right)^2 - e^{-2\delta}(\gamma^2 - \|y\|_{\mathcal{H}}^2)}. \tag{8.3.5}$$

Factoring out e^{c_β} from the square root term in (8.2.3), we can thus write

$$\begin{aligned}\operatorname{Exp}(h_\beta)(\gamma, y) = \max\Bigg\{ &\log\left(\frac{(e^{c_\beta}+e^{-c_\beta})\gamma}{2} - \frac{e^{c_\beta}-e^{-c_\beta}}{2c_\beta}\langle x_\beta, y\rangle + e^{c_\beta}\zeta^{x_\beta}_{c_\beta}(\gamma,y)\right) - 2\lambda_\beta, \\ &\log\left(\frac{\frac{(e^{c_\beta}+e^{-c_\beta})\gamma}{2} - \frac{e^{c_\beta}-e^{-c_\beta}}{2c_\beta}\langle x_\beta, y\rangle + e^{c_\beta}\zeta^{x_\beta}_{c_\beta}(\gamma,y)}{\gamma^2 - \|y\|^2_{\mathcal{H}}}\right)\Bigg\} \\ &+ \lambda_\beta - \max\{\lambda_\beta + c_\beta, c_\beta - \lambda_\beta\}.\end{aligned}$$

If $\lambda_\beta \to -\infty$, it is clear from the above in combination with Lemma 2.9.15 that, for all $(y,\gamma) \in V_+^\circ$,

$$\begin{aligned}\lim_\beta \operatorname{Exp}(h_\beta)(\gamma, y) &= \lim_\beta \log\left(\frac{(e^{c_\beta}+e^{-c_\beta})\gamma}{2} - \frac{e^{c_\beta}-e^{-c_\beta}}{2c_\beta}\langle x_\beta, y\rangle + e^{c_\beta}\zeta^{x_\beta}_{c_\beta}(\gamma,y)\right) - c_\beta \\ &= \log\left(\frac{(1+e^{-2c})\gamma}{2} - (1-e^{-2c})\langle x/2c, y\rangle + \zeta^x_c(\gamma,y)\right) \\ &= h^{e^{-2c}}_{x/2c,-1}(\gamma,y) = \operatorname{Exp}(h^c_{x,-1})(\gamma,y).\end{aligned}$$

Similarly, if $\lambda_\beta \to \infty$, we see that $\lim_\beta \operatorname{Exp}(h_\beta) = \operatorname{Exp}(h^c_{x,1})$.

Let us now consider the case where there exists a subnet (h_β) such that $h^{\mathbb{R}}_\beta = h^\varepsilon$ and $h^{\mathcal{H}}_\beta = h^{x_\beta, c_\beta}$ for all β. As above, every subnet must have a further subnet such that $x_\beta \xrightarrow{w} x$ and $c_\beta \to c$. It is thus clear from the definition of (8.3.1) along with (8.2.13) that $\lim_\beta \operatorname{Exp}(h_\beta) = \operatorname{Exp}(h^c_{x,\varepsilon})$. Finally, let us consider the case where there exists a subnet (h_β) such that $h^{\mathbb{R}}_\beta = h^\varepsilon$ and $h^{\mathcal{H}}_\beta = h^{\mathcal{H}}_{x_\beta}$ for all β. As we proved above, it cannot be the case that (x_β) is unbounded in $\mathcal{H}$, so there must exist a further subnet such that $x_\beta \xrightarrow{w} x$ and $c_\beta \to c$. Now, if $\varepsilon = 1$, the proof of Theorem 8.1.5 shows that for all β we can write $h_\beta = h^{I,\emptyset}_{p_\beta,\alpha_\beta}$, where $I = \{1,2\}$, $p_\beta = (\frac{1}{2}, \frac{1}{2\|x_\beta\|_{\mathcal{H}}} x_\beta)$, and $\alpha = (0, 2\|x_\beta\|_{\mathcal{H}})$. Thus

$$\operatorname{Exp}(h_\beta) = h^{v_\beta, 0}, \text{ where } v_\beta = p_\beta + e^{-2\|x_\beta\|_{\mathcal{H}}}(e - p_\beta).$$

Using Claassens' representation of the gauge function (8.2.1) we can thus write, for any $(\gamma, y) \in V_+^\circ$ and any β

$$\mathrm{Exp}(h_\beta)(\gamma, y) = \log\left(\frac{\frac{(1+e^{-2\|x_\beta\|_{\mathcal{H}}})\gamma}{2} - (1-e^{-2\|x_\beta\|_{\mathcal{H}}})\,\langle x/2\|x_\beta\|_{\mathcal{H}}, y\rangle + \zeta^{x_\beta}_{\|x_\beta\|_{\mathcal{H}}}(\gamma, y)}{\gamma^2 - \|y\|^2_{\mathcal{H}}}\right),$$

meaning that

$$\lim_\beta \mathrm{Exp}(h_\beta)(\gamma, y) = h^{e^{-2c}}_{x/2c,1} = \mathrm{Exp}(h^1 \oplus h^{x,c}).$$

A similar argument shows that if $\varepsilon = -1$ then $\lim_\beta \mathrm{Exp}(h_\beta)(\gamma, y) = \mathrm{Exp}(h^{-1} \oplus h^{x,c})$. So, indeed, in all cases, $\lim_\alpha \mathrm{Exp}(h_\alpha) = \mathrm{Exp}(h)$. □

Lemma 8.3.10. *If* $(h_\alpha) \subseteq \partial\overline{V}^h$ *is a net converging to some* $h = h_\lambda \oplus h^x \in \partial\overline{V}^h$*, where* h^x *is as in* (8.1.2)*, then* $\mathrm{Exp}(h_\alpha)$ *converges to* $\mathrm{Exp}(h)$ *in* $\partial\overline{V_+^\circ}^h$*.*

Proof. As noted above, without loss of generality we can assume that $h^{\mathbb{R}}_\alpha = h_{\lambda_\alpha}$, where $\lambda_\alpha \to \lambda$. Thus, there must either exist a subnet (h_β) such that $h^{\mathcal{H}}_\beta = h^{x_\beta, c_\beta}$, or $h^{\mathcal{H}}_\beta = h^{x_\beta}$. Let us first assume the former. Note, for any β, that $\inf_{\mathcal{H}} h^{x_\beta} = -c_\beta$. Thus if (c_β) is a bounded net, $\inf_{\mathcal{H}} h^x \geq -\sup_\beta c_\beta > -\infty$, a contradiction unless $x = 0$. If $x = 0$, and (c_β) is bounded, these same remarks mean that $\inf c_\beta = 0$, meaning that $\inf \|x_\beta\|_{\mathcal{H}} = 0$, but then $\lim_\beta h^{x_\beta, c_\beta} = h_0$, a contradiction, so indeed (c_β) is unbounded. As in the above proofs we set $t_\beta = \|x_\beta\|_{\mathcal{H}}/c_\beta \in [0, 1]$. The argument employed around (8.3.4) shows that there exists a further subnet such that $h^{\mathcal{H}}_\beta$ converges to h^{tz}, where $t = \lim_\beta t_\beta$, and z is the weak limit of $z_\beta = x_\beta/\|x_\beta\|_{\mathcal{H}}$. The uniqueness of limits implies that $tz = x$. Now, $\mathrm{Exp}(h_\beta) = h^{\lambda'_\beta, c'_\beta, x'_\beta}$, where $\lambda'_\beta = \frac{e^{\lambda_\beta + c_\beta} + e^{\lambda_\beta - c_\beta}}{2}$, $c'_\beta = \frac{e^{\lambda_\beta + c_\beta} - e^{\lambda_\beta - c_\beta}}{2}$, and $x'_\beta = \frac{e^{\lambda_\beta + c_\beta} - e^{\lambda_\beta - c_\beta}}{2c_\beta} x_\beta$. Using (8.3.5), we can thus write

$$\begin{aligned}\operatorname{Exp}(h_\beta)(\gamma,y) = \max\Bigg\{ &\log\left(\frac{(1+e^{-2c_\beta})\gamma}{2} - \frac{1-e^{-2c_\beta}}{2c_\beta}\langle x_\beta, y\rangle + \zeta^{x_\beta}_{c_\beta}(\gamma,y)\right) - 2\lambda_\beta,\\ &\log\left(\frac{\frac{(1+e^{-2c_\beta})\gamma}{2} - \frac{1-e^{-2c_\beta}}{2c_\beta}\langle x_\beta, y\rangle + \zeta^{x_\beta}_{c_\beta}(\gamma,y)}{\gamma^2 - \|y\|^2_{\mathcal{H}}}\right)\Bigg\}\\ &+ \lambda_\beta + c_\beta - \max\{\lambda_\beta + c_\beta, c_\beta - \lambda_\beta\}.\end{aligned}$$

As we can write $x_\beta/c_\beta = t_\beta z_\beta$, it follows that, recalling Lemma 2.9.15, if $\lambda \geq 0$,

$$\begin{aligned}\lim_\beta \operatorname{Exp}(h_\beta)(\gamma,y) &= \max\left\{\log(\gamma - \langle tz, y\rangle) - 2\lambda, \log\left(\frac{\gamma - \langle tz,y\rangle}{\gamma^2 - \|y\|^2}\right)\right\}.\\ &= h^{1,e^{-2\lambda}}_{x/2}.\end{aligned}$$

Similarly, if $\lambda < 0$ it follows that $\lim_\beta \operatorname{Exp}(h_\beta) = h^{e^{-2\lambda},1}_{x/2}$. Thus, if $\|x\|_{\mathcal{H}} < 1$, $\lim_\beta \operatorname{Exp}(h_\beta) = \operatorname{Exp}(h)$. If $\|x\| = 1$, the proof of Theorem 8.1.5 shows that $h = h^{\{1\},\{2\}}_{p,\alpha}$, where $p = (\frac{1}{2}, \frac{1}{2}x)$ and $\alpha = (0, 2\lambda)$ if $\lambda \geq 0$ and $\alpha = (-2\lambda, 0)$ if $\lambda < 0$. Thus $\operatorname{Exp}(h) = h^{u,v}$, where $u = p$ and $v = e^{-2\lambda}(e-p)$ if $\lambda \geq 0$, and $u = e^{-2\lambda}p$ and $v = (e-p)$ if $\lambda < 0$. Claassens' representation (8.2.1) thus shows that $h^{u,v} = h^{1,e^{-2\lambda}}_{x/2}$ or $h^{e^{-2\lambda},1}_{x/2}$ depending on the sign of λ, where this horofunction is defined exactly as in (8.2.14), but we allow $\|x/2\|_{\mathcal{H}} = 1/2$. Thus, in all cases $\lim_\beta \operatorname{Exp}(h_\beta) = \operatorname{Exp}(h)$.

We now consider the case when there exists a subnet such that $h^{\mathcal{H}}_\beta = h^{x_\beta}$. By Banach Alaoglu and uniqueness of limits, there must exist a further subnet such that (x_β) converges weakly to x. Now, if $\lambda \geq 0$, for any $(\gamma, y) \in V^\circ_+$,

$$\lim_\beta \operatorname{Exp}(h_\beta) = \lim_\beta \max\left\{\log\left(\frac{\gamma - \langle x_\beta, y\rangle}{\gamma^2 - \|y\|^2_{\mathcal{H}}}\right),\ \log\left(\frac{\gamma' e^{-2\lambda_\beta} + e^{-2\lambda_\beta}\langle x_\beta, y'\rangle}{\gamma'^2 - \|y'\|^2_{\mathcal{H}}}\right)\right\},$$

so indeed $\lim_\beta \operatorname{Exp}(h_\beta) = \operatorname{Exp}(h)$. This result follows similarly if $\lambda < 0$. Thus we have shown that every subnet of (h_α) has a further subnet (h_β) such that $\lim_\beta \operatorname{Exp}(h_\beta) = \operatorname{Exp}(h)$, meaning that, indeed, $\lim_\alpha \mathrm{Exp}(h_\alpha) = \mathrm{Exp}(h)$. □

Remark 8.3.11. At the end of the proof above, we showed, using Claassens' characterisation of the M function (8.2.1), that if $p = \frac{1}{2}(1,x)$ for $x \in S_{\mathcal{H}}$ is a primitive idempotent, then for $\lambda \in (0,1]$, $h^{p,\lambda(e-p)} = h^{1,\lambda}_{x/2}$ and $h^{\lambda p,(e-p)} = h^{\lambda,1}_{x/2}$, where $h^{1,\lambda}_{x/2}$ and $h^{\lambda,1}_{x/2}$ are defined exactly as in (8.2.14). This observation can be extended, and directly applying (8.2.1) to an arbitrary Busemann point $h^{u,v} \in \partial_B \overline{V_+^\circ}^h$ we see that every Busemann point $h^{u,v} \in \partial_B \overline{V_+^\circ}^h$ is actually equal to a function of the form (8.2.13) or (8.2.14), but where $\|x\|_{\mathcal{H}} = \frac{1}{2}$, and every function of the form (8.2.13) or (8.2.14) where $\|x\|_{\mathcal{H}} = \frac{1}{2}$ is equal to some $h^{u,v} \in \partial_B \overline{V_+^\circ}^h$.

Lemma 8.3.12. *If $(h_\alpha) \subseteq \partial \overline{V}^h$ is a net converging to some $h = h^\varepsilon \oplus h^x \in \partial \overline{V}^h$, where h^x is as in* (8.1.2)*, then* $\mathrm{Exp}(h_\alpha)$ *converges to* $\mathrm{Exp}(h)$ *in* $\partial \overline{V_+^\circ}^h$.

Proof. We first note, that if $\|x\|_{\mathcal{H}} = 1$, then by the proof of Theorem 8.1.5, setting $p = (\frac{1}{2}, \frac{1}{2\|x\|_{\mathcal{H}}} x)$, we have $h = h^{\{1\},\emptyset}_{p,0}$ if $\varepsilon = 1$, and $h = h^{\emptyset,\{2\}}_{p,0}$ if $\varepsilon = -1$. Thus, $\mathrm{Exp}(h) = h^{p,0}$ if $\varepsilon = 1$, and $\mathrm{Exp}(h) = h^{0,e-p}$ if $\varepsilon = -1$. Using Claassens' representation of the gauge function (8.2.1), we see that these correspond to $h^0_{x/2,\varepsilon}$ for $\varepsilon = -1$ and $\varepsilon = 1$ respectively, where $h^0_{x/2,\varepsilon}$ is exactly as in (8.2.13), except we now allow $\|x/2\|_{\mathcal{H}} = \frac{1}{2}$ in the definition. Let us first consider the case where there exists a subnet such that $h^{\mathcal{H}}_\beta = h_{x_\beta}$, meaning $h^{\mathbb{R}}_\beta = h^\varepsilon$ for sufficiently large β. It follows from section 4 of [27] and the uniqueness of limits that x_β is unbounded, and $x_\beta/\|x_\beta\|_{\mathcal{H}}$ converges weakly to x. As above, if $\varepsilon = 1$, the proof of Theorem 8.1.5 shows that for all β we can write $h_\beta = h^{I,\emptyset}_{p_\beta,\alpha_\beta}$, where $I = \{1,2\}$, $p_\beta = (\frac{1}{2}, \frac{1}{2\|x_\beta\|_{\mathcal{H}}} x_\beta)$, and $\alpha = (0, 2\|x_\beta\|_{\mathcal{H}})$. Thus

$$\mathrm{Exp}(h_\beta) = h^{v_\beta,0}, \text{ where } v_\beta = p_\beta + e^{-2\|x_\beta\|_{\mathcal{H}}}(e - p_\beta).$$

Using Claassens' representation of the gauge function (8.2.1) and (8.3.5) we can thus write, for any $(\gamma, y) \in V_+^\circ$ and any β

$$\mathrm{Exp}(h_\beta)(\gamma,y) = \log\left(\frac{\frac{(1+e^{-2\|x_\beta\|_{\mathcal{H}}})\gamma}{2} - (1 - e^{-2\|x_\beta\|_{\mathcal{H}}})\langle x_\beta/2\|x_\beta\|_{\mathcal{H}}, y\rangle + \zeta^{x_\beta}_{\|x_\beta\|_{\mathcal{H}}}(\gamma,y)}{\gamma^2 - \|y\|^2_{\mathcal{H}}}\right),$$

meaning that

$$\lim_{\beta} \operatorname{Exp}(h_\beta)(\gamma, y) = \log\left(\frac{\gamma - 2\langle x/2, y\rangle}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right) = h^0_{x/2,1}.$$

The same argument shows that, if $\varepsilon = -1$, $\lim_\beta \operatorname{Exp}(h_\beta) = h^0_{x/2,-1}$, so indeed $\lim_\beta \operatorname{Exp}(h_\beta) = \operatorname{Exp}(h)$.

If there exists a subnet such that $h^{\mathbb{R}}_\beta = h^\varepsilon$ and $h^{\mathcal{H}}_\beta = h^{x_\beta, c_\beta}$, the same argument as in the proof of Lemma 8.3.10 above shows that (c_β) cannot be bounded. We again set $t_\beta = \|x_\beta\|_{\mathcal{H}}/c_\beta \in [0,1]$, and $z_\beta = x_\beta/\|x_\beta\|_{\mathcal{H}}$. There must exist a further subnet, which we relabel by β, and a $t \in [0,1]$ and $z \in \mathcal{B}_{\mathcal{H}}$, such that $t = \lim_\beta t_\beta$, and $z_\beta \xrightarrow{w} z$. Equation (8.3.4) shows that $h^{\mathcal{H}}_\beta$ converges to h^{tz}. The uniqueness of limits implies that $tz = x$. Now, if $\varepsilon = 1$, for any β and $(\gamma, y) \in V^\circ_+$,

$$\operatorname{Exp}(h_\beta)(\gamma, y) = \log\left(\frac{\frac{(1+e^{-2c_\beta})\gamma}{2} - (1 - e^{-2c_\beta})\langle x_\beta/2c_\beta, y\rangle + \zeta^{x_\beta}_{c_\beta}(\gamma, y)}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right),$$

with $\zeta^{x_\beta}_{c_\beta}$ as in (8.3.5). Therefore, because $x_\beta/c_\beta = t_\beta z_\beta$, and

$$\lim_\beta \zeta^{x_\beta}_{c_\beta}(\gamma, y) = \left(\frac{\gamma}{2} - \frac{t}{2}\langle z, y\rangle\right),$$

we have

$$\lim_{\beta} \operatorname{Exp}(h_\beta)(\gamma, y) = \log\left(\frac{\gamma - 2\langle x/2, y\rangle}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right) = h^0_{x/2,1}.$$

Similarly, if $\varepsilon = -1$, $\lim_\beta \operatorname{Exp}(h_\beta) = h^0_{x/2,-1}$. If there exists a subnet such that $h^{\mathbb{R}}_\beta = h^\varepsilon$ and $h^{\mathcal{H}}_\beta = h^{x_\beta}$, just as in previous cases there must exist a further subnet such that x_β converges weakly to x. Thus, if $\varepsilon = 1$, for any $(\gamma, y) \in V^\circ_+$,

$$\begin{aligned}\lim_\beta \operatorname{Exp}(h_\beta) &= \lim_\beta \log\left(\frac{\gamma - 2\langle x_\beta/2, y\rangle}{\gamma^2 - \|y\|_{\mathcal{H}}^2}\right)\\ &= h^0_{x/2,1}.\end{aligned}$$

Similarly, if $\varepsilon = -1$, we can conclude that $\lim_\beta \operatorname{Exp}(h_\beta) = h^0_{x/2,-1}$.

We now switch to the cases where h is still of the form $h^\varepsilon \oplus h^x$, but there exists a subnet, say (h_β), such that $h^{\mathbb{R}}_\beta = h_{\lambda_\beta}$ for all β. In this case, there must exist a further subnet such that $h^{\mathcal{H}}_\beta = h^{x_\beta,c_\beta}$ for all β, or $h_\beta = h^{x_\beta}$. Let us first assume we are in the former case. As proven in Lemma 8.3.10 we know that (c_β) cannot be bounded. Recall that we set $t_\beta = \|x_\beta\|_{\mathcal{H}}/c_\beta \in [0,1]$. Yet again, the argument employed around (8.3.4) show that there exists a further subnet such that $h^{\mathcal{H}}_\beta$ converges to h^{tz}, where $t = \lim_\beta t_\beta$, and z is the weak limit of $z_\beta = x_\beta/\|x_\beta\|_{\mathcal{H}}$. The uniqueness of limits implies that $tz = x$. For any β,

$$\begin{aligned}
\operatorname{Exp}(h_\beta)(\gamma, y) = \max\Bigg\{ &\log\left(\frac{(1+e^{-2c_\beta})\gamma}{2} - \frac{1-e^{-2c_\beta}}{2c_\beta}\langle x_\beta, y\rangle + \zeta^{x_\beta}_{c_\beta}(\gamma, y)\right) - 2\lambda_\beta, \\
&\log\left(\frac{\frac{(1+e^{-2c_\beta})\gamma}{2} - \frac{1-e^{-2c_\beta}}{2c_\beta}\langle x_\beta, y\rangle + \zeta^{x_\beta}_{c_\beta}(\gamma, y)}{\gamma^2 - \|y\|^2_{\mathcal{H}}}\right)\Bigg\} \\
&+ \lambda_\beta + c_\beta - \max\{\lambda_\beta + c_\beta, c_\beta - \lambda_\beta\}.
\end{aligned}$$

Now, if $\varepsilon = 1$, $\lambda_\beta \to \infty$, so the above immediately implies that indeed $\lim_\beta \operatorname{Exp}(h_\beta) = h^0_{x/2,1}$. Similarly, if $\varepsilon = -1$, and if we recall Lemma (2.9.15), the above implies that $\lim_\beta \operatorname{Exp}(h_\beta) = h^0_{x/2,-1}$. To finish the proof, we just need to investigate the case where there exists a subnet such that $h^{\mathbb{R}}_\beta = h_{\lambda_\beta}$ and $h^{\mathcal{H}}_\beta = h^{x_\beta}$ for all β. As above, every further subsequence must have a subsequence such that $x_\beta \xrightarrow{w} x$. If $\varepsilon = 1$, $\lambda_\beta \to \infty$, so

$$\begin{aligned}
\lim_\beta \operatorname{Exp}(h_\beta) &= \lim_\beta \max\left\{\log\left(\frac{\gamma - 2\langle x_\beta/2, y\rangle}{\gamma^2 - \|y\|^2_{\mathcal{H}}}\right),\ \log\left(\frac{\gamma' e^{-2\lambda_\beta} + e^{-2\lambda_\beta}\langle x_\beta, y'\rangle}{\gamma'^2 - \|y'\|^2_{\mathcal{H}}}\right)\right\} \\
&= h^0_{x/2,1}.
\end{aligned}$$

Similarly, if $\varepsilon = -1$, $\lim_\beta \operatorname{Exp}(h_\beta) = h^0_{x/2,-1}$. As this argument applies to every subnet of (h_α), indeed $\operatorname{Exp}(h_\alpha) \to \operatorname{Exp}(h)$. □

Proposition 8.3.13. *If $(h_\alpha) \subseteq \partial\overline{V}^h$ is a net converging to some $h_{(\lambda,x)} \in \overline{V}^h$, then* $\operatorname{Exp}(h_\alpha)$

converges to $\operatorname{Exp}(h_{(\lambda,x)})$ *in* $\overline{V_+^\circ}^h$.

Proof. From the proof of Lemma 8.3.8, we know that we can assume without loss of generality that $h_\alpha^{\mathbb{R}} = h_{\lambda_\alpha}$, where $h_{\lambda_\alpha} \to h_\lambda$. Furthermore, if there exists a subnet such that $h_\beta^{\mathcal{H}} = h^{x_\beta}$, the Banach-Alaoglu theorem means there must exist a further subnet and $z \in B_{\mathcal{H}}$ such that $h^{x_\beta} \to h^z$, but h^z is either unbounded below, or is identically 0, so it cannot equal h_x. Thus we can assume without loss of generality that $h_\alpha^{\mathcal{H}} = h^{x_\alpha, c_\alpha}$. If c_α is unbounded we define $t_\alpha = \|x_\alpha\|_{\mathcal{H}}/c_\alpha \in [0,1]$, and through relabelling and passing to a subnet there exists a $t \in \mathbb{R}$ such that $t_\alpha \to t$. In this case, inequality (8.3.3) shows that $\lim_\alpha |h^{x_\alpha,c_\alpha}(y)| \leq t\|y\|_{\mathcal{H}}$ for all $y \in \mathcal{H}$, and because $|h_x(x)| = \|x\|_{\mathcal{H}}$, we must have $t = 1$, so $\|x_\alpha\|_{\mathcal{H}}$ is unbounded. However, if (x_α) is unbounded, there must exist a further subnet, (h_β), such that $x_\beta/\|x_\beta\|_{\mathcal{H}}$ converges weakly to some $z \in \mathcal{B}_H$, but then (8.3.4) means (h^{x_β,c_β}) converges to h^z, which is a contradiction. Thus (c_α) and (x_α) are bounded. By the Banach-Alaoglu theorem there exists a subnet (h^{x_β,c_β}) and a $z \in \mathcal{H}$ and $c \geq 0$ such that $c_\beta \to c$ and $x_\beta \xrightarrow{w} z$. Thus $h^{x_\beta,c_\beta} \to h^{z,c}$, and Corollary 7.1.9 in conjunction with uniqueness of limits then means that $z = x$ and $c = \|x\|_{\mathcal{H}}$. As this argument applies to every subnet, we must have that $c_\alpha \to c$ and $x_\alpha \xrightarrow{w} x$. Thus $\operatorname{Exp}(h_\alpha) = h^{\lambda'_\alpha, c'_\alpha, x'_\alpha}$, where $\lambda'_\alpha = \frac{e^{\lambda_\alpha + c_\alpha} + e^{\lambda_\alpha - c_\alpha}}{2}$, $c'_\alpha = \frac{e^{\lambda_\alpha + c_\alpha} - e^{\lambda_\alpha - c_\alpha}}{2}$, and $x'_\alpha = \frac{e^{\lambda_\alpha + c_\alpha} - e^{\lambda_\alpha - c_\alpha}}{2c_\alpha} x_\alpha$. As

$$\lambda' \to \lambda = \frac{e^{\lambda+c} + e^{\lambda-c}}{2}, \; c'_\alpha \to c' = \frac{e^{\lambda+c} - e^{\lambda-c}}{2}, \text{ and } x'_\alpha \xrightarrow{w} x' = \frac{e^{\lambda+c} - e^{\lambda-c}}{2c} x,$$

we see via (8.2.3) that $\operatorname{Exp}(h_\alpha) \to h^{\lambda',c',x'}$. However, as $\|x\|_{\mathcal{H}} = c$ the above shows that $\|x'\|_{\mathcal{H}} = c'$, so $h^{\lambda',c',x'} = h_{(\lambda',x')}$ by Lemma 8.2.2, but by (8.3.2) $h_{(\lambda',x')} = \operatorname{Exp}(h_{\lambda,x})$. □

Propositions 8.3.3, 8.3.7, and 8.3.13 show that Exp is a continuous bijection between compact Hausdorff spaces, so it is indeed a homeomorphism. Finally, we need to prove that Exp maps parts of $\partial_B \overline{V}^h$ onto parts of $\partial_B \overline{V_+^\circ}^h$.

Proposition 8.3.14. *Two Busemann points* $h, h' \in \partial_B \overline{V}^h$ *are in the same part of* $\partial_B \overline{V_+^\circ}^h$, *if and only if* $\operatorname{Exp}(h)$ *and* $\operatorname{Exp}(h')$ *lie in the same part of* $\partial_B \overline{V_+^\circ}^h$.

Proof. The proof of Theorem 1.1a) in [42] suffices, as it does not actually rely on finite dimensionality in any way. However, in our case we can present a more rudimental argument. Let us first consider elements $w=\lambda_1 p+\mu_1(e-p)\in V_+$ and $z=\lambda_2 p+\mu_2(e-p)\in V_+$ for $\lambda_i,\mu_i\in[0,\infty)$, where $\max\{\lambda_1,\mu_1\}>0$ and $\max\{\lambda_2,\mu_2\}>0$. Using Lemma 2.7.4, we can calculate that $w\sim z$ if and only if there exists $0<\alpha\leq\beta$ such that

$$\beta\lambda_2-\lambda_1+\beta\mu_2-\mu_1\geq 0,\ \text{ and } \beta(\lambda_2+\mu_2)-\lambda_1-\mu_1\geq|\beta(\lambda_2-\mu_2)+\mu_1-\lambda_1|,$$

and

$$\lambda_1-\alpha\lambda_2+\mu_1-\alpha\mu_2\geq 0,\ \text{ and } \lambda_1+\mu_1-\alpha(\lambda_2+\mu_2)\geq|\lambda_1-\mu_1-\alpha(\lambda_2-\mu_2)|.$$

This is true if and only if one of the following three conditions is satisfied:

(i) $\lambda_1,\lambda_2,\mu_1,\mu_2>0$.

(ii) $\lambda_1,\lambda_2>0$, and $\mu_1=\mu_2=0$.

(iii) $\mu_1,\mu_2>0$ and $\lambda_1=\lambda_2=0$.

Theorem 8.1.7 shows that $h=h^{I,J}_{p,\alpha}$ and $h^{I',J'}_{\beta,q}$ are in the same part if and only if $p_I=q_{I'}$ and $p_J=q_{J'}$. Thus if h and h' are in the same part, $\mathrm{Exp}(h)=h^{u,v}$ and $\mathrm{Exp}(h')=h^{u',v'}$ where either $u=\sum_{i\in I}e^{-\alpha_i}p_i$, $v=\sum_{j\in J}e^{-\alpha_j}p_j$, and $u'=\sum_{i\in I'}e^{-\beta_i}q_i$, $v'=\sum_{j\in J'}e^{-\beta_j}q_j$, where $q\in\{p,e-p\}$, or precisely one of $\{u,u'\}$ or $\{v,v'\}$ lies in V_+° while the other contains only 0. Lemma 2.7.4 immediately implies that in the latter case $\mathrm{Exp}(h)$ and $\mathrm{Exp}(h')$ lie in the same part of $\partial_B\overline{V_+^\circ}^h$, and in the former case, points $(i)-(iii)$ above imply the same. Thus $\mathrm{Exp}(h)$ and $\mathrm{Exp}(h')$ are in the same part if h and h' are in the same part. Conversely, if $h^{u,v}$ and $h^{u',v'}$ lie in the same part of $\partial_B\overline{V_+^\circ}$, Proposition 8.2.6 in conjunction with Lemma 2.9.14 and $(i)-(iii)$ above means that either precisely one of $\{u,u'\}$ or $\{v,v'\}$ lies in V_+° while the other contains only 0, or $u=\lambda p$, $u'=\lambda' p$, $v=\mu(e-p)$ and $v'=\mu'(e-p)$ for some primitive idempotent p, where $\lambda,\lambda',\mu,\mu'\in[0,\infty)$, and $\lambda>0$ if and only if $\lambda'>0$ and similarly $\mu > 0$ if and only if $\mu' > 0$. If we are in the former case, then $\mathrm{Exp}^{-1}(h^{u,v}) = h^{I,J}_{p,\alpha}$ and $\mathrm{Exp}^{-1}(h^{u',v'}) = h^{I',J'}_{\beta,q}$ where either $I = I' = \{1,2\}$ or $J = J' = \{1,2\}$, so either $p_I = q_{I'} = e$ or $p_J = q_{J'} = e$, meaning that $h^{I,J}_{p,\alpha}$ and $h^{I',J'}_{\beta,q}$ are in the same part of $\partial_B \overline{V}^h$ by Theorem 8.1.7. If we are in the latter case $\mathrm{Exp}^{-1}(h^{u,v}) = h^{I,J}_{p,\alpha}$ and $\mathrm{Exp}^{-1}(h^{u',v'}) = h^{I',J'}_{\beta,p}$, where $I = I'$ and $J = J'$, meaning that again $h^{I,J}_{p,\alpha}$ and $h^{I',J'}_{\beta,p}$ are in the same part of $\partial_B \overline{V}^h$ by Theorem 8.1.7. □

Theorem 8.0.1 is thus proved by Propositions 8.3.2 8.3.3, 8.3.7, 8.3.13 and 8.3.14. Methods used in the above proofs also allow us to prove something similar in spirit to the previous chapters. Namely, that there is a continuous bijection from $V \cup \partial_B \overline{V}^h$ to B_{V^*} equipped with the weak* topology, and this bijection is a homeomorphism between $\partial_B \overline{V}^h$ and S_{V^*}, and maps each part of $\partial_B \overline{V}^h$ onto the relative interior of a single boundary face of S_{V^*}. First let us recall that V^* is the Banach space $(\mathbb{R} \oplus \mathcal{H}, \|\cdot\|_*)$, where $\|(\lambda, x)\|_* = \max\{|\lambda|, \|x\|_{\mathcal{H}}\}$ [50, Theorem 1.10.13]. Therefore, the boundary faces of B_{V^*} are precisely the sets F^ε for $\varepsilon \in \{-1, 1\}$, F_z for $z \in S_X$, and F^ε_z where

$$F^\varepsilon = \{(\varepsilon, x) : x \in B_X\}, \text{ and } F_z = \{(\lambda, z) : \lambda \in B_{\mathbb{R}}\} \text{ ,and } F^\varepsilon_z = \{(\varepsilon, z)\}.$$

This suggests a candidate bijection, and we define a map $\varphi \colon V \cup \partial_B \overline{V}^h \to B_{V^*}$ by:

$$\varphi((\lambda, x)) = \begin{cases} \left(\tanh(\lambda), \frac{\tanh(\|x\|_{\mathcal{H}})}{\|x\|_{\mathcal{H}}} x\right) & \text{if } x \neq 0 \\ (\tanh(\lambda), 0) & \text{if } x = 0 \end{cases}, \tag{8.3.6}$$

if $(\lambda, x) \in V$, and for Busemann points of the form given in Proposition 8.1.3,

$$\varphi(h_\lambda \oplus h^x) = (\tanh(\lambda), x),\ \varphi(h^\varepsilon \oplus h^x) = (\varepsilon, x),\ \varphi(h^\varepsilon \oplus h_x) = \left(\varepsilon, \frac{\tanh(\|x\|_{\mathcal{H}})}{\|x\|_{\mathcal{H}}} x\right), \tag{8.3.7}$$

where in the last equation we have implicitly assumed that $x \neq 0$, and we define $\varphi(h^\varepsilon \oplus h_0) = (\varepsilon, 0)$. We prove the following:

Theorem 8.3.15. *The map* $\varphi\colon V \cup \partial\overline{V}^B \to B_{V^*}$ *is a continuous bijection when* B_{V^*} *is equipped with the weak* topology, and when restricted to the Busemann boundary it is a homeomorphism onto* S_{V^*} *equipped with the weak* topology. Furthermore* φ *maps parts of* $\partial_B\overline{V}^h$ *bijectively onto the relative interiors of boundary faces of* S_{V^*}*.*

Theorem 8.3.15 means that we can think of $V \cup \partial\overline{V}^B$ as the cylinder $[-1,1] \times B_{\mathcal{H}}$ illustrated in Figure 8.1 below, where all internal points live in the interior of the cylinder. The part of the Busemann boundary consisting of Busemann points of the type $h^1 \oplus h_x$ is associated to the top face of the cylinder, the part of the Busemann boundary consisting of Busemann points of the type $h^{-1} \oplus h_x$ is associated to the interior of the bottom face of the cylinder. The singleton Busemann points of the type $h^\varepsilon \oplus h^x$ are associated to points in the top and bottom boundary disks of the cylinder, and the parts of the Busemann boundary consisting of Busemann points of the type $h_\lambda \oplus h^x$ are associated to the interior of vertical lines on the surface of the cylinder.

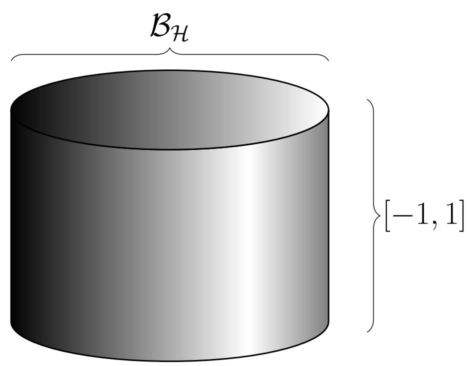

Figure 8.1: The closed unit ball of V^*.

We prove Theorem 8.3.15 via a sequence of lemmas.

Lemma 8.3.16. *The map* φ *is a bijection onto* $V \cup \partial\overline{V}^B \to B_{V^*}$*.*

Proof. To prove injectivity, assume that $\varphi(h_1) = \varphi(h_2)$, for $h_1, h_2 \in V \cup \partial\overline{V}^B$. By definition of φ we see this is only possible if both h_1 and h_2 lie in V or both lie in $\partial\overline{V}^B$. We also know that $\tanh\colon \mathbb{R} \to (-1,1)$ is bijective, so if

$$\frac{\tanh(\|x\|_{\mathcal{H}})}{\|x\|_{\mathcal{H}}}x = \frac{\tanh(\|y\|_{\mathcal{H}})}{\|y\|_{\mathcal{H}}}y,$$

we can take the norm of both sides to get $\|x\|_{\mathcal{H}} = \|y\|_{\mathcal{H}}$, meaning $x = y$. The bijectivity of tanh also makes it routine to verify that $h_1 = h_2$ if their images under φ are equal on the boundary. Surjectivity can also be checked directly case by case, where all that needs to be kept in mind is that

$$B^{\circ}_{\mathcal{H}}\backslash\{0\} \ni y \mapsto \frac{\tanh^{-1}(\|y\|_{\mathcal{H}})}{\|y\|_{\mathcal{H}}}y \in \mathcal{H}\backslash\{0\}$$

is the inverse of $x \mapsto \frac{\tanh(\|x\|_{\mathcal{H}})}{\|x\|_{\mathcal{H}}}x$, and $\tanh^{-1}\colon (-1,1) \to \mathbb{R}$ is a bijection. □

Lemma 8.3.17. *The maps φ and φ^{-1} are continuous when B_{V^*} is considered with the weak* topology.*

Proof. Let us first suppose that $(h_{(\lambda_\alpha,x_\alpha)}) = (h_{\lambda_\alpha} \oplus h_{x_\alpha})$ is a net converging to some $h_{(\lambda,x)} = h_\lambda \oplus h_x \in \overline{V}^h$. For any $y \in \mathcal{H}$, $h_{(\lambda_\alpha,x_\alpha)}(0,y) = h_{x_\alpha}(y) \to h_{(\lambda,x)}(0,y) = h_x(y)$, so by 7.1.7 we know that $\|x_\alpha - x\|_{\mathcal{H}} \to 0$. Similarly we see that $\lambda_\alpha \to \lambda$. Thus, by the continuity of tanh, and the fact that if $x_\alpha \to 0$ then $\frac{\tanh(\|x_\alpha\|_{\mathcal{H}})}{\|x_\alpha\|_{\mathcal{H}}}x_\alpha \to 0$, we see that $\lim_\alpha \varphi(h_{(\lambda_\alpha,x_\alpha)}) = \varphi(h_{(\lambda,x)})$.

We now consider the case when $(h_{(\lambda_\alpha,x_\alpha)})$ is a net converging to some $h \in \partial\overline{V}^B$. The same reasoning as in the proof of Proposition 8.3.3 combined with the continuity of tanh allow us to put restrictions on the behaviour of the nets (λ_α) and (x_α) depending on the data at infinity of h, and because a net $(h_{x_\alpha}) \subseteq \overline{\mathcal{H}}^h$ converges to some Busemann point h^x in $\partial_B\overline{\mathcal{H}}^h$ if and only if the net $(x_\alpha/\|x_\alpha\|_{\mathcal{H}})$ converges weakly to x in $\S_{\mathcal{H}}$, the same convergence arguments used to prove Proposition 8.3.3 show that $\varphi(h_\alpha) \to \varphi(h)$. To prove the continuity of φ on the boundary and the continuity of $\varphi^{-1}|_{S_{V^*}}$ we recall that V is reflexive and $\mathcal{H}$ and $\mathbb{R}$ are self-dual, so $((\lambda_\alpha, x_\alpha)) \subseteq S_{V*}$ converges to $(\lambda, x) \in S_{V^*}$ in the weak* topology if and only if $\lambda_\alpha \to \lambda$ and $\langle x_\alpha, z\rangle \to \langle x, z\rangle$ for every $z \in \mathcal{H}$. Thus, if $(h^\alpha) \subseteq \partial_B \overline{V}^h$ we can use the same reasoning as in the proof of Propositions 8.3.7 to determine the weak* limits of the data at infinity of the net to show that $\varphi(h_\alpha) \to \varphi(h)$. We can similarly use arguments akin to those in Proposition 8.3.7 to show that if $((\lambda_\alpha, x_\alpha)) \subseteq S_{V^*}$ converges to $(\lambda, x) \in S_{V^*}$ in the weak* topology then $\varphi^{-1}((\lambda_\alpha, x_\alpha)) \to \varphi^{-1}((\lambda, x))$. □

Lemma 8.3.18. *Parts of $\partial_B \overline{V}^h$ are mapped bijectively under φ onto the relative interiors of boundary faces of B_{V^*}*

Proof. Recall that the boundary faces of B_{V^*} are precisely the sets F^ε for $\varepsilon \in -1, 1$, F_z for $z \in S_{\mathcal{H}}$, and F_z^ε where

$$F^\varepsilon = \{(\varepsilon, x) : x \in B_{\mathcal{H}}\}, \text{ and } F_z = \{(\lambda, z) : \lambda \in B_{\mathbb{R}}\}\text{ ,and } F_z^\varepsilon = \{(\varepsilon, z)\}.$$

Proposition 8.1.6 in combined with the definition of φ then immediately proves the lemma. □

8.4 The Horofunction Compactification of (PV_+°, d_H)

In this section we give a specific infinite dimensional extension of Theorem 1.1 b) in [42], as well as situating some results about the horofunction compactification of infinite dimensional separable real hyperbolic space in [16] in a more general context. In [16], Duchesne restricts his attention to separable hyperbolic spaces and separable Hilbert spaces, and in this setting of separability shows that the horofunction compactification of (PV_+°, d_H) is homeomorphic to the horofunction compactification of $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$, by showing that both are homeomorphic to the truncated cone, what he calls the *frustum*:

$$\mathbf{F} = \{(\gamma, x) \in [0,1] \times \mathcal{B}_{\mathcal{H}} : \|x\|_{\mathcal{H}} \leq \gamma\},$$

which he equips with the weak topology. He makes extensive use of the stereographic projection $\sigma_{\mathcal{H}}\colon \mathcal{H} \to B_{\mathcal{H}}$ given by $x \mapsto \frac{1}{\sqrt{1+\|x\|_{\mathcal{H}}^2}}x$ with inverse $\sigma_{\mathcal{H}}^{-1}$ given by $x \mapsto \frac{1}{\sqrt{1-\|x\|_{\mathcal{H}}^2}}x$. He uses the results of Claassens [13], and claims that the map $g\colon \mathbf{F} \to \overline{PV_+^\circ}^h$ given by $(\gamma, x) \mapsto h_x^\gamma$ is a continuous bijection from $\mathbf{F}$ onto $\overline{PV_+^\circ}^h$. He then repeats the construction of Gutiérrez in [27, Section 4] (modulo some renorming) to compute the horofunction compactification of $\mathcal{H}$, and thus states that the map $f\colon \mathbf{F} \to \overline{\mathcal{H}}^h$ given by $(\gamma, x) \mapsto h^{\sigma_{\mathcal{H}}^{-1}(x), \sigma_{\mathbb{R}}^{-1}(\gamma)}$ for $\|x\|_{\mathcal{H}} \leq \gamma < 1$, and $f(\gamma, x) = h^x$ for $\gamma = 1$ is a continuous bijection. He does not deal with how Busemann points and the parts are affected by the the bijection $f^{-1} \circ g$. In keeping with the theme of this thesis, we believe that the relationship between $\overline{\mathcal{H}}^h$ and $\overline{PV_+^\circ}^h$ is best illuminated by viewing d_H as an infinite dimensional Finsler distance, and $\mathcal{H}$ as the tangent space at the unit.

Recall that $T_e = \{v \in V : \mathrm{tr}(v) = 0\}$. Thus,

$$T_e = \{\lambda p + (-\lambda)(e-p) : \lambda \in \mathbb{R},\ p \text{ is a primitive idempotent}\} = \{0\} \times \mathcal{H}.$$

Now, for any $y \in \mathcal{H}$, $(0, y)$ has spectral representation $\lambda p + (-\lambda)(e-p)$, where $p = (\frac{1}{2}, \frac{1}{2\|y\|_{\mathcal{H}}})$ and $\lambda = \|y\|_{\mathcal{H}}$, which gives us the form of the variation norm $|\cdot|_e$ on T_e arising from the Finsler metric:

$$|(0, y)|_e = |\lambda| = \|y\|_{\mathcal{H}}.$$

Thus, we immediately see that $\overline{T_e}^h = \overline{\mathcal{H}}^h$, where we make the natural identification $h(0, y) = h(y)$ for any $h \in \overline{\mathcal{H}}^h$. The standard exponential exp can be defined on T_e in the usual way, and by using spectral decomposition in combination with the above remarks we see that, for any $v = (0, y) \in T_e$, $e^{\|y\|_{\mathcal{H}}}$ and $e^{-\|y\|_{\mathcal{H}}}$ are the two eigenvalues of $\exp(v)$, meaning that

$$\exp(v) = \left(\frac{e^{\|y\|_{\mathcal{H}}} + e^{-\|y\|_{\mathcal{H}}}}{2}, \frac{e^{\|y\|_{\mathcal{H}}} - e^{-\|y\|_{\mathcal{H}}}}{2\|y\|_{\mathcal{H}}} y \right).$$

Therefore, $\det\exp(v) = 1$, so exp is indeed a map from T_e to PV_+°. We want to extend the exponential $\exp\colon T_e \to PV_+^\circ$ to a parts preserving homeomorphism between $\overline{T}_e^h$ and $\overline{PV_+^\circ}^h$. To that end, recall that the elements of $\overline{PV_+^\circ}^h$ are precisely those functions h_x^r as in (8.0.2) for $\|x\|_{\mathcal{H}} < r \leq 1$, and so we define $\mathrm{Exp}_H\colon \overline{T}_e^h \to \overline{PV_+^\circ}^h$ by

$$\mathrm{Exp}_H(y) = \exp(y) \text{ for } y \in T_e$$

$$\mathrm{Exp}_H(h) = \begin{cases} h^{\tanh(c)}_{\frac{\tanh(c)}{c}x} & \text{if } h = h^{x,c},\ c > \|x\|_{\mathcal{H}} \\ h_x^1 & \text{if } h = h^x, \end{cases} \tag{8.4.1}$$

where h^x and $h^{x.c}$ are elements of $\partial\overline{\mathcal{H}}^h$ as in (8.1.2).

Lemma 8.4.1. *The map* $\mathrm{Exp}_H\colon \overline{T}_e^h \to \overline{PV_+^\circ}^h$ *is a bijection mapping* T_e *onto* PV_+° *and* $\partial\overline{T}_e^h$ *onto* $\partial\overline{PV_+^\circ}^h$*.* Exp_H *also maps* $\partial_B\overline{T}_e^h$ *bijectively onto* $\partial_B\overline{PV_+^\circ}^h$*. Furthermore, two Busemann points* $h, h' \in \partial_B\overline{T}_e^h$ *are in the same part if and only if* $\mathrm{Exp}_H(h)$ *and* $\mathrm{Exp}_H(h')$ *lie in the same part of* $\partial_B\overline{PV_+^\circ}^h$*.*

Proof. By the Spectral Theorem, we know that any $y \in PV_+^\circ$ has the form $y = \lambda p + \lambda^{-1}(e - p)$ for a primitive idempotent p and $\lambda > 0$. As $\log(\lambda)p - \log(\lambda)(e-p) \in T_e$, Exp_H is bijective on T_e. A very similar argument to the one in the proof of Lemma 8.2.9 shows that $h_x^r = h_{x'}^{r'}$ if and only if $r = r'$ and $x = x'$, and because $\tanh\colon [0,\infty) \to [0,1)$ is bijective, we thus have that $\mathrm{Exp}_H|_{\partial\overline{T_e}^h}$ is a bijection. The only Busemann points in T_e are horofunctions h^x where $x \in S_{\mathcal{H}}$ by Proposition 7.1.3, and the only Busemann points in $\partial_B\overline{PV_+^\circ}^h$ are horofunctions h_x^1 where $x \in B_{\mathcal{H}}$ [13, Theorem 3], so it is clear that $\mathrm{Exp}_H|_{\partial_B\overline{T_e}^h}$ is a bijection. The final statement of the lemma thus follows trivially, because all parts of both $\partial_B\overline{T_e}^h$ and $\partial_B\overline{PV_+^\circ}^h$ are singletons by Propositions 7.1.4 and Proposition 8.0.2. □

We should note here that we are only choosing the set $\{v \in V_+^\circ : \det(v) = 1\}$ to represent PV_+°, but PV_+° is actually a quotient space consisting of equivalent classes of rays. Recall that Birkhoff's version of the Hilbert metric, d_H, is still a well defined function on the whole interior of the positive cone, but for any $v \in V_+^\circ$ and any $\alpha, \beta > 0$,

$$d_H(\alpha v, \beta v) = 0.$$

Thus if h_v is an internal metric functional on PV_+° for some $v \in \mathbf{H}$, then $h_{\alpha v}$ is well defined as a function on V_+° for any $\alpha > 0$, and $h_{\alpha v} = h_v$. Any $h \in \overline{PV_+^\circ}^h$ is also a function on V_+°, where, for any $v \in \mathbf{H}$, and $\alpha > 0$, $h(\alpha v) = h(v)$. This is useful, because it means to calculate horofunctions we can use whatever representation of a ray that is most convenient. Specifically, we note that for any $v = (0, x) \in T_e$,

$$\exp(v) = \left(\frac{e^{\|x\|_{\mathcal{H}}} + e^{-\|x\|_{\mathcal{H}}}}{2}, \frac{e^{\|x\|_{\mathcal{H}}} - e^{-\|x\|_{\mathcal{H}}}}{2\|x\|_{\mathcal{H}}} y \right),$$

so multiplying by the constant $2/(e^{\|x\|_{\mathcal{H}}} + e^{-\|x\|_{\mathcal{H}}})$, the above remarks mean that

$$h_{\exp(v)} = h_{(1, \frac{\tanh(\|x\|_{\mathcal{H}})}{\|x\|_{\mathcal{H}}} x)}.$$

From this, in combination with (8.2.2), we calculate that for any $(\gamma, y) \in V_+^\circ$

$$h_{\exp(v)}(\gamma, y)$$
$$= \log \left(\frac{\gamma - \frac{\tanh(\|x\|_{\mathcal{H}})}{\|x\|_{\mathcal{H}}} \langle x, y \rangle + \sqrt{(\gamma - \frac{\tanh(\|x\|_{\mathcal{H}})}{\|x\|_{\mathcal{H}}} \langle x, y \rangle)^2 - (1 - \tanh(\|x\|_{\mathcal{H}})^2)(\gamma^2 - \|y\|_{\mathcal{H}}^2)}}{(1 + \tanh(\|x\|_{\mathcal{H}}))\sqrt{\gamma^2 - \|y\|_{\mathcal{H}}^2}} \right) \tag{8.4.2}$$

The structure of the proof of the following lemma is very similar to those of Lemmas 8.3.3 and 8.3.7, so we may omit some routine details to spare the reader, but they can all be found in the proofs of the aforementioned lemmas.

Lemma 8.4.2. *If* $(h_\alpha) \subseteq \overline{T_e}^h$ *is a net converging to some* $h \in \overline{T_e}^h$, *then* $\mathrm{Exp}_H(h_\alpha)$ *converges to* $\mathrm{Exp}_H(h)$ *in* $\overline{PV_+^\circ}^h$.

Proof. First let us assume that $h = h_x \in T_e$. There is either a subnet (h_β) such that

$h_\beta = h_{x_\beta}$ for all β, or $h_\beta = h^{x_\beta, c_\beta}$. If we are in the former case, we know that $x_\beta \to x$ in $\mathcal{H}$ by Lemma 7.1.7, from which we immediately deduce that $\lim_\beta \operatorname{Exp}_H(h_\beta) = \operatorname{Exp}_H(h)$. If we are in the latter case we know that (c_β) must be bounded, so $h^{x_\beta, c_\beta} \to h_x$, meaning that x_β converges weakly to x, and c_β converges to $\|x\|_\mathcal{H}$. Thus, for any $(\gamma, y) \in PV_+^\circ$ and any β

$$\begin{aligned}
&\operatorname{Exp}(h_\beta)(\gamma, y)\\
&= \log\left(\frac{\gamma - \frac{\tanh(c_\beta)}{c_\beta}\langle x_\beta, y\rangle + \sqrt{(\gamma - \frac{\tanh(c_\beta)}{c_\beta}\langle x_\beta, y\rangle)^2 - (1-\tanh(c_\beta)^2)(\gamma^2 - \|y\|_\mathcal{H}^2)}}{(1+\tanh(c_\beta))\sqrt{\gamma^2 - \|y\|_\mathcal{H}^2}}\right)\\
&\to h^{\tanh(\|x\|_\mathcal{H})}_{\frac{\tanh(\|x\|_\mathcal{H})}{\|x\|_\mathcal{H}} x}(\gamma, y) = h_{(1, \frac{\tanh(\|x\|_\mathcal{H})}{\|x\|_\mathcal{H}} x)}(\gamma, y).
\end{aligned}$$

We can thus conclude that $\lim_\alpha \operatorname{Exp}(h_\alpha) = \operatorname{Exp}_H(h_x)$. Now let us assume that $h = h^{x,c}$. If there exists a subnet (h_β) such that $h_\beta = h^{c_\beta, x_\beta}$, the above argument can easily be adjusted to see that $\lim_\beta \operatorname{Exp}_H(h_\beta) = \operatorname{Exp}_H(h)$. Similarly, if there exists a subnet (h_β) such that $h_\beta = h_{x_\beta}$, we know from the proof of Lemma 8.3.3 that (x_β) is bounded, and $x_\beta \xrightarrow{w} x$, and $\|x_\beta\|_\mathcal{H} \to c$. Thus, using (8.4.2) we again see that $\lim_\beta \operatorname{Exp}_H(h_\beta) = h$, so we conclude that $\lim_\alpha \operatorname{Exp}(h_\alpha) = \operatorname{Exp}_H(h^{x,c})$. Finally suppose that $h = h^x$. If there exists a subnet $h_\beta = h^{x_\beta}$, we know that $x_\beta \xrightarrow{w} x$, from which we easily deduce that $\lim_\beta \operatorname{Exp}_H(h_\beta) = \operatorname{Exp}_H(h)$. If there exists a subnet such that $h_\beta = h^{x_\beta, c_\beta}$, the proof of Lemma 8.3.7 shows that (c_β) is unbounded, and there exists a $z \in \mathcal{B}_\mathcal{H}$ and $t \in [0,1]$ such that $x_\beta / \|x_\beta\|_\mathcal{H} \xrightarrow{w} z$, $t_\beta = \|x_\beta\|_\mathcal{H} / c_\beta \to t$, and $tz = x$. Taken together this shows that $x_\beta / c_\beta \xrightarrow{w} x$, and $\tanh(c_\beta) \to 1$, from which we can use the above expression for $\operatorname{Exp}(h_\beta)(\gamma, y)$ to calculate $\lim_\beta \operatorname{Exp}_H(h_\beta) = \operatorname{Exp}_H(h)$. If there exists a subnet such that $h_\beta = h_{x_\beta}$, we know that $x_\beta / \|x_\beta\|_\mathcal{H} \xrightarrow{w} x$, and $\tanh(\|x_\beta\|_\mathcal{H}) \to 1$. Equation (8.4.2) thus shows that $\lim_\beta \operatorname{Exp}_H(h_\beta) = \operatorname{Exp}_H(h)$, so indeed we can conclude that $\lim_\alpha \operatorname{Exp}(h_\alpha) = \operatorname{Exp}_H(h^x)$. □

Lemmas 8.4.1 and 8.4.2 prove Theorem 8.0.3.

Chapter 9

Concluding Remarks

In Chapters 4, 5, and 6, we provided further classes of homogeneous Finsler metric spaces whose horofunction compactifications are homeomorphic to their dual unit ball in the tangent space at the base point. This of course does not answer Question 1.0.1 fully, but does give further evidence. During the duration of this thesis, further positive evidence has also been provided in [12], where the authors show that this duality phenomenon also holds for Hermitian symmetric spaces and JB*-triples. A key observation is that in all cases dealt with in these chapters, as well as the cases in [12] and the other literature discussed in the introduction, every horofunction is a Busemann point, and so Question 1.0.1 can be reframed by asking whether parts of the boundary were mapped to the relative interiors of faces of the dual ball. In all proofs, this fact was utilised explicitly, and at present we are unable to see how these proofs could be replaced without using properties unique to a space having only Busemann points as horofunctions.

We currently are not aware of any evidence showing that the horofunction compactification of homogeneous Finsler metric spaces with non-Busemann points is homeomorphic to the dual ball in the tangent space at the base point, where the homeomorphism maps equivalence parts of the natural stratification of the boundary bijectively onto the relative interior of faces of the dual ball. In fact we are not aware of any evidence showing that this is true even for finite dimensional normed spaces possessing non-Busemann points. This suggests to us that a full answer to Question 1.0.1 is more likely to be obtained by restricting our attention to trying to prove that Question 1.0.1 has a positive answer for homogeneous Finsler metric spaces with horofunction boundaries consisting entirely of Busemann points, and separately searching for an example of a finite dimensional normed space possessing non-Busemann horofunctions that cannot be homeomorphic to the dual ball and maintain a bijection between the equivalence classes in the stratification and the relative interiors of faces of the dual ball. We stress that this problem is still wholly open.

The infinite dimensional case is fascinating for different reasons. In Chapter 7 we showed that all infinite dimensional ℓ^p spaces, for $1 < p < \infty$, possess a multitude of non-Busemann horofunctions, and in fact for any infinite dimensional Hilbert space $\mathcal{H}$ the non-Busemann horofunctions are dense in $\overline{\mathcal{H}}^h$. We have also shown that the Busemann points are homeomorphic to the dual unit sphere equipped with the weak* topology, and because all Busemann points are singletons and all faces of the dual ball are singletons, parts of the boundary are mapped bijectively to the relative interior of faces of the dual. We also showed that any such homeomorphism *cannot* be extended to a homeomorphism from the whole horofunction compactification to the closed unit ball in the dual.

The situation for infinite dimensional ℓ^1 spaces is strikingly different. We showed that every horofunction is a Busemann point, and that there does exist a homeomorphism from the whole horofunction compactification onto the dual unit ball, which maps Busemann points bijectively onto the dual unit sphere. However, this homeomorphism is not a bijection between the parts of the boundary and the relative interiors of faces of the dual ball. In fact this homeomorphism injects uncountably many parts of the boundary into faces of the dual ball. We were not able to adequately explain this phenomenon during the course of this thesis. However, we believe it may have something to do with the fact that the horofunction compactification of infinite dimensional uniformly smooth and strictly convex

Banach spaces is a topological compactification by the usual definition. The embedding is a homeomorphism onto its image. However, an infinite dimensional ℓ^1 space is *not* homeomorphically embedded in its horofunction compactification, as the inverse of the embedding fails to be continuous. There may be a link between this property, and the fact that all horofunctions of ℓ^1 are Busemann points, which naively seems to be why there are "too many" parts of the boundary.

Bibliography

[1] M. Abate. *Iteration theory of holomorphic maps on taut manifolds.* Research and Lecture Notes in Mathematics. Complex Analysis and Geometry. Mediterranean Press, Rende, 1989.

[2] M. Akian, S. Gaubert, and C. Walsh. The max-plus Martin boundary. *Doc. Math.*, 14:195–240, 2009.

[3] E. M. Alfsen and F. W. Shultz. *State spaces of operator algebras. Basic theory, orientations, and C^*-products.* Theory & Applications, Mathematics, 2001.

[4] E. M. Alfsen and F. W. Shultz. *Geometry of state spaces of operator algebras.* Theory & Applications, Birkhäuser Boston, Inc, Mathematics, 2003.

[5] L. Arosio, M. Fiacchi, S. Gontard, and L. Guerini. The horofunction boundary of a Gromov hyperbolic space. *Math. Ann.*, 388(2):1163–1204, 2024.

[6] Z. M. Balogh and M. Bonk. Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains. *Comment. Math. Helv.*, 75(3):504–533, 2000.

[7] A. F. Beardon. The dynamics of contractions. *Ergodic Theory Dynam. Systems*, 17(6):1257–1266, 1997.

[8] F. Bracci and H. Gaussier. Horosphere topology. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 20(1):239–289, 2020.

[9] F. Bracci, H. Gaussier, and A. Zimmer. Homeomorphic extension of quasi-isometries for convex domains in $\mathbb{C}^d$ and iteration theory. *Math. Ann.*, 379(1-2):691–718, 2021.

[10] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[11] C. Chu. Iteration of holomorphic maps on Lie balls. *Adv. Math.*, 264:114–154, 2014.

[12] C. Chu, M. Cueto-Avellaneda, and B. Lemmens. Horofunctions and metric compactification of noncompact Hermitian symmetric spaces. *Ann. Mat. Pura Appl. (4)*, 203(4):1703–1751, 2024.

[13] F. Claassens. The horofunction boundary of the infinite dimensional hyperbolic space. *Geom. Dedicata*, 207:255–263, 2020.

[14] J. A. Clarkson. Uniformly convex spaces. *Trans. Amer. Math. Soc.*, 40(3):396–414, 1936.

[15] J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[16] B. Duchesne. The Polish topology of the isometry group of the infinite-dimensional hyperbolic space. *Groups Geom. Dyn.*, 17(2):633–670, 2023.

[17] M. Duchin and N. Fisher. Stars at infinity in Teichmüller space. *Geom. Dedicata*, 213:531–545, 2021.

[18] C. M. Edwards and G. T. Rüttimann. On the facial structure of the unit balls in a gl-space and its dual. *Math. Proc. Cambridge Philos. Soc.*, 98(2):305–322, 1985.

[19] M. Fabian, P. Habala, P. Hájek, V. M. Santalucía, Jan Pelant, and V. Zizler. *Functional Analysis and Infinite-Dimensional Geometry.* Springer New York, New York, NY, 2001.

[20] J. Faraut and A. Korányi. *Analysis on symmetric cones.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1994. Oxford Science Publications.

[21] S. Friedland and P. Freitas. p-metrics on $GL(n, \mathbb{C})/U_n$ and their Busemann compactifications. *Linear Algebra Appl.*, 376:1–18, 2004.

[22] S. Gaubert and G. Vigeral. A maximin characterisation of the escape rate of non-expansive mappings in metrically convex spaces. *Math. Proc. Cambridge Philos. Soc.*, 152(2):341–363, 2012.

[23] S. Gouëzel and A. Karlsson. Subadditive and multiplicative ergodic theorems. *J. Eur. Math. Soc. (JEMS)*, 22(6):1893–1915, 2020.

[24] M. Greenfield and L. Ji. Metrics and compactifications of teichmüller spaces of flat tori. *Asian J. Math.*, 25(4):477–504, 2021.

[25] M. Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume No. 97 of *Ann. of Math. Stud.*, pages 183–213. Princeton Univ. Press, Princeton, NJ, 1981.

[26] A. Gutiérrez. The horofunction boundary of finite-dimensional ℓ_p spaces. *Colloq. Math.*, 155(1):51–65, 2019.

[27] A. Gutiérrez. On the metric compactification of infinite-dimensional ℓ_p spaces. *Canad. Math. Bull.*, 62(3):491–507, 2019.

[28] A. Gutiérrez. Characterizing the metric compactification of L_p spaces by random measures. *Ann. Funct. Anal.*, 11(2):227–243, 2020.

[29] T. Haettel, A. Schilling, C. Walsh, and A. Wienhard. Horofunction compactifications of symmetric spaces, 2018. *arXiv:1705.05026 [math.DG].*

[30] H. Hanche-Olsen and E. Stormer. *Jordan operator algebras.* Monographs and Studies in Mathematics, 21. Pitman, Boston, MA, 1984.

[31] U. Hirzebruch. Über jordan-algebren und kompakte riemannsche symmetrische räume vom rang 1. *Math. Z.*, 90:339–354, 1965.

[32] L. Ji and A. Schilling. Polyhedral horofunction compactification as polyhedral ball, 2016. *arXiv:1607.00564v2 [math.GT].*

[33] L. Ji and A. Schilling. Toric varieties vs. horofunction compactifications of polyhedral norms. *Enseign. Math.*, 63(3-4):375–401, 2017.

[34] M. Kapovich and B. Leeb. Finsler bordifications of symmetric and certain locally symmetric spaces. *Geom. Topol.*, 22(5):2533–2646, 2018.

[35] A. Karlsson. Non-expanding maps and Busemann functions. *Ergodic Theory Dynam. Systems*, 21(5):1447–1457, 2001.

[36] A. Karlsson. Two extensions of Thurston's spectral theorem for surface diffeomorphisms. *Bull. Lond. Math. Soc.*, 46(2):217–226, 2014.

[37] A. Karlsson. Hahn-Banach for metric functionals and horofunctions. *J. Funct. Anal.*, 281(2):Paper No. 109030, 17, 2021.

[38] A. Karlsson, V. Metz, and G. A. Noskov. Horoballs in simplices and minkowski spaces. *Int. J. Math. Math. Sci.*, Art. ID 23656, 2006. 20 pp.

[39] S. Kobayashi. *Hyperbolic complex spaces.* Grundlehren der Mathematischen Wissenschaften, 318. Springer-Verlag, Berlin, 1998.

[40] E. Kreyszig. *Introductory functional analysis with applications.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1989.

[41] B. Lemmens. A metric version of Poincaré's theorem concerning biholomorphic inequivalence of domains. *J. Geom. Anal.*, 32(5):Paper No. 160, 20, 2022.

[42] B. Lemmens. Horofunction compactifications of symmetric cones under Finsler distances. *Ann. Fenn. Math.*, 48(2):729–756, 2023.

[43] B. Lemmens, B. Lins, and R. Nussbaum. Detecting fixed points of nonexpansive maps by illuminating the unit ball. *Israel J. Math.*, 224(1):231–262, 2018.

[44] B. Lemmens, B. Lins, R. Nussbaum, and M. Wortel. Denjoy-Wolff theorems for Hilbert's and Thompson's metric spaces. *J. Anal. Math.*, 134(2):671–718, 2018.

[45] B. Lemmens, C. Milliken, and K. Power. The horofunction compactification of ℓ^1-products of metric spaces. *Involve*, Forthcoming.

[46] B. Lemmens and R. Nussbaum. *Nonlinear Perron-Frobenius theory*, volume 189 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2012.

[47] B. Lemmens and K. Power. Horofunction compactifications and duality. *J. Geom. Anal.*, 33(5):Paper No. 154, 57, 2023.

[48] B. Lemmens, M. Roelands, and M. Wortel. Hilbert and thompson isometries on cones in jb-algebras. *Math. Z.*, 292(3-4):1511–1547, 2019.

[49] B. Lemmens and C. Walsh. Isometries of polyhedral Hilbert geometries. *J. Topol. Anal.*, 3(2):213–241, 2011.

[50] R. Megginson. *An introduction to Banach space theory*, volume 183 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[51] P. Mellon. A general Wolff theorem for arbitrary Banach spaces. *Math. Proc. R. Ir. Acad.*, 104A(2):127–142, 2004.

[52] D. P. Milman. On some criteria for the regularity of spaces of the type (b). *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 20:243–246, 1938.

[53] H. Miyachi. Extremal length boundary of the Teichmüller space contains non-Busemann points. *Trans. Amer. Math. Soc.*, 366(10):5409–5430, 2014.

[54] J. R. Munkres. *Topology.* Prentice Hall, Inc., Upper Saddle River, NJ, second edition, 2000.

[55] R. D. Nussbaum. Finsler structures for the part metric and hilbert's projective metric and applications to ordinary differential equations. *Differential Integral Equations*, 7(5-6):1649–1707, 1994.

[56] A. Papadopoulos. *Metric spaces, convexity and nonpositive curvature*, volume 6 of *IRMA Lectures in Mathematics and Theoretical Physics.* European Mathematical Society (EMS), Zürich, 2005.

[57] A. Papadopoulos and M. Troyanov. Handbook of hilbert geometry. In *IRMA Lectures in Mathematics and Theoretical Physics*, volume 22. European Mathematical Society (EMS), Zürich, 2014.

[58] B. J. Pettis. A proof that every uniformly convex space is reflexive. *Duke Math. J.*, 5(2):249–253, 1939.

[59] P. Planche. *Géométrie de Finsler sur les Espaces Symétriques.* PhD thesis, Université de Genève, 1995.

[60] M. A. Rieffel. Group C^*-algebras as compact quantum metric spaces. *Doc. Math.*, 7:605–651, 2002.

[61] R. Rockafellar. *Convex analysis*, volume No. 28 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 1970.

[62] W. Rudin. *Functional analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[63] E. Schechter. *Handbook of analysis and its foundations.* Academic Press, Inc., San Diego, CA, 1997.

[64] A. Schilling. *The Horofunction Compactification of Finite-Dimensional Normed Vector Spaces and of Symmetric Spaces*. PhD thesis, Heidelberg Germany, 1996.

[65] V. Šmulian. Sur la dérivabilité de la norme dans l'espace de Banach. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 27:643–648, 1940.

[66] C. Walsh. The horofunction boundary of finite-dimensional normed spaces. *Math. Proc. Cambridge Philos. Soc.*, 142(3):497–507, 2007.

[67] C. Walsh. The horoboundary and isometry group of Thurston's Lipschitz metric. In *Handbook of Teichmüller theory. Vol. IV*, volume 19 of *IRMA Lect. Math. Theor. Phys.*, pages 327–353. Eur. Math. Soc., Zürich, 2014.

[68] C. Walsh. Gauge-reversing maps on cones, and Hilbert and Thompson isometries. *Geom. Topol.*, 22(1):55–104, 2018.

[69] C. Walsh. Hilbert and Thompson geometries isometric to infinite-dimensional Banach spaces. *Ann. Inst. Fourier (Grenoble)*, 68(5):1831–1877, 2018.

[70] A. Zimmer. Gromov hyperbolicity and the Kobayashi metric on convex domains of finite type. *Math. Ann.*, 365(3-4):1425–1498, 2016.