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Confluent Darboux Transformations and Wronskians for Algebraic Solutions of the Painlevé III (D_7) Equation

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Abstract

Darboux transformations are relations between the eigenfunctions and coefficients of a pair of linear differential operators, while Painlevé equations are nonlinear ordinary differential equations whose solutions arise in diverse areas of applied mathematics and mathematical physics. Here, we describe the use of confluent Darboux transformations for Schrödinger operators, and how they give rise to explicit Wronskian formulae for certain algebraic solutions of Painlevé equations. As a preliminary illustration, we briefly describe how the Yablonskii–Vorob'ev polynomials arise in this way, thus providing well-known expressions for the tau functions of the rational solutions of the Painlevé II equation. We then proceed to apply the method to obtain the main result, namely, a new Wronskian representation for the Ohyama polynomials, which correspond to the algebraic solutions of the Painlevé III equation of type D_7.

Keywords: Darboux transformation; Painleve equation; Wronskian

MSC: 34M55; 35J10; 37K10

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Academic Editors: Henrik Aratyn and Andrei D. Polyanin

Received: 17 March 2025
Revised: 6 June 2025
Accepted: 19 June 2025
Published: 10 July 2025

Citation: Harrow, J.W.E.; Hone, A.N.W. Confluent Darboux Transformations and Wronskians for Algebraic Solutions of the Painlevé III (D_7) Equation. *Mathematics* **2025**, *13*, 2236. https://doi.org/10.3390/math13142236

1. Introduction

In its original form, as discovered by Darboux [1], the Darboux transformation is a relation between the solutions and coefficients of a pair of linear ordinary differential equations of the second order. Hence, in the case of a 1D Schrödinger equation, it gives a technique for obtaining a new eigenfunction and potential, starting from some initially known eigenfunction and potential. As such, this technique has led to a powerful algebraic approach for constructing families of exactly solvable potentials, within the framework of supersymmetric quantum mechanics [2]. Furthermore, the generalization of the Darboux transformation to the case of higher-order linear operators, or compatible matrix linear systems (zero-curvature equations), has resulted in an extremely effective tool for deriving explicit solutions of integrable nonlinear partial differential equations (PDEs), including soliton solutions [3].

The focus of this article is on explicit solutions of nonlinear ordinary differential equations (ODEs), rather than PDEs. Specifically, here, we are concerned with solutions of Painlevé equations, which are second-order ODEs of the general form

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = F\left(z, q, \frac{\mathrm{d}q}{\mathrm{d}z}\right), \tag{1}$$

where the function F on the right-hand side is rational in the first derivative of q, algebraic in q, and analytic in z, having the property that all solutions are meromorphic away from a finite number of fixed critical points (which are determined by the equation itself). Up to certain coordinate transformations, the Painlevé equations are classified into six canonical forms, referred to as Painlevé I-VI, given by particular functions F depending on certain parameters, which are at most four in number. (See Chapter XIV in [4] for details of this classification.)

It is known that the general solution of each Painlevé equation is a higher transcendental function, which cannot be expressed in terms of simpler functions, e.g., elliptic functions or classical special functions given by solutions of linear ODEs. For this reason, the solutions of Painlevé equations should be regarded as quintessentially nonlinear special functions, which provide the solutions of many fundamental problems appearing in diverse areas of application, including probability theory, random matrices, quantum gravity, orthogonal polynomials, and asymptotics of PDEs (see [5] and the references therein). An example of fundamental significance can be seen in the work of Tracy and Widom [6], who derived some particular solutions of Painlevé II that yielded probability distributions for certain eigenvalue statistics in random matrix ensembles, and were subsequently found to provide exact analytic solutions of the famous Kardar–Parisi–Zhang equation which were in very precise agreement with experimental measurements of interface growth in a nematic liquid crystal [7]. However, aside from higher transcendental solutions of this type, there are certain parameter values for which Painlevé equations admit special solutions that are expressed in terms of simpler functions. For instance, the Painlevé II equation, which is usually written in the form

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + \alpha, \tag{2}$$

has a set of particular solutions $q_n(z)$ for integer parameter values $\alpha = n \in \mathbb{Z}$, given by the sequence of rational functions with $q_{-n}(z) = -q_n(z)$, as shown in Table 1, and, for half-integer values of α, it has a one-parameter family of special solutions that can be written in terms of Airy functions and their derivatives.

Table 1. Rational solutions of Painlevé II.

α	0	1	2	3	4
q	0	$-\frac{1}{z}$	$\frac{1}{z} - \frac{3z^2}{z^3+4}$	$\frac{3z^2}{z^3+4} - \frac{6z^2(z^3+10)}{z^6+20z^3-80}$	$\frac{6z^2(z^3+10)}{z^6+20z^3-80} - \frac{10(z^9+42z^6+1120)}{z(z^9+60z^6+11200)}$

The standard way to obtain sequences of solutions such as these is via the application of Bäcklund transformations (BTs), which are discrete symmetries of Painlevé equations that map solutions to solutions while changing the parameters. In the case of Painlevé II, there are two independent symmetries of this kind, given by

$$\begin{array}{lll} S: & q \mapsto -q, & \alpha \mapsto -\alpha \\ T: & q \mapsto \tilde{q} = q + \left(\alpha + \frac{1}{2}\right) / \left(\frac{\mathrm{d}q}{\mathrm{d}z} + q^2 + \frac{z}{2}\right), & \alpha \mapsto -\alpha - 1. \end{array} \tag{3}$$

The composition $S \circ T$ sends $q \mapsto -\tilde{q}$, maps the parameter $\alpha \mapsto \alpha + 1$, and generates the sequence in Table 1, starting from the seed solution $y = 0$ when $\alpha = 0$. It has been known since the work of Okamoto that the BTs of each Painlevé equation are associated with the birational actions of an (extended) affine Weyl group on the space of initial conditions, with the parameters corresponding to root variables (see [8], for instance). It was subsequently found by Sakai that the space of initial conditions for both continuous and discrete Painlevé equations can be identified with a smooth rational surface $\mathcal{X}$, whose anti-canonical class determines a pair of orthogonal affine root subsystems inside the $E_8^{(1)}$ root lattice. One of these root subsystems corresponds to the exceptional divisors obtained via a blowing up procedure, and determines the surface type, while the other is associated with the affine Weyl group that determines the discrete symmetries of the equation (and hence its BTs). For Painlevé II equation (2), the surface $\mathcal{X}$ is of type $E_7^{(1)}$, while the symmetry type is $A_1^{(1)}$. Therefore, in particular, there is only one parameter α, corresponding to the fact that the A_1 root system has rank 1.

Another essential aspect of Okamoto's work was the representation of Painlevé equations as non-autonomous Hamiltonian systems, and the tau function associated with this representation. In the case of Painlevé II, we have a variable p, which is the canonically conjugate momentum associated with q and the pair of Hamilton's equations

$$\begin{array}{rclcl} q' & = & -q^2 - p - \frac{z}{2} & = & \frac{\partial h}{\partial p} \\ p' & = & 2pq - \ell & = & -\frac{\partial h}{\partial q}, \end{array} \tag{4}$$

where the prime denotes the time evolution (z derivative), while the Hamiltonian h and auxiliary parameter ℓ are defined by

$$h = -\tfrac{1}{2}p^2 - pq^2 - \tfrac{z}{2}p + \ell q, \qquad \ell = \alpha + \tfrac{1}{2}. \tag{5}$$

Regarding the Hamiltonian $h = h(q, p, z)$ for Painlevé II, evaluated on a solution $q(z), p(z)$ of the system (4), as a function of the independent variable z, Okamoto's tau function $\tau(z)$ is defined (up to an overall constant multiplier) by

$$h(z) = -\frac{\mathrm{d}}{\mathrm{d}z}\log\tau(z). \tag{6}$$

The Painlevé property for Equation (2) is equivalent to the statement that the tau function $\tau(z)$ is holomorphic. Indeed, the birational action of $S \circ T$ on the extended phase space coordinates (q, p, z, ℓ) is

$$S \circ T: \quad \begin{cases} q & \mapsto -q + \frac{\ell}{p} \\ p & \mapsto -p - 2(q - \frac{\ell}{p})^2 - z \\ z & \mapsto z \\ \ell & \mapsto \ell + 1, \end{cases} \tag{7}$$

which preserves a contact 2-form Ω, that is,

$$(S \circ T)^*\Omega = \Omega, \qquad \Omega = \mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}h \wedge \mathrm{d}z,$$

and, if the variables, Hamiltonians, and tau functions are indexed by the parameter ℓ, then

$$h_\ell = (S \circ T)^*(h_{\ell-1}) = h_{\ell-1} + q_\ell.$$

Therefore, the solution of Painlevé II is given by the logarithmic derivative of a ratio of tau functions as

$$q_\ell(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\left(\frac{\tau_{\ell-1}(z)}{\tau_\ell(z)}\right), \tag{8}$$

while comparing (6) with (5), and, using the fact that

$$h'_\ell(z) = \frac{\partial}{\partial z}h_\ell(q, p, z),$$

the corresponding conjugate momentum variable is given by

$$p_\ell(z) = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\big(\log \tau_\ell(z)\big). \tag{9}$$

Hence, the solution of (2) has simple poles at the places where one of the adjacent tau functions $\tau_{\ell-1}, \tau_\ell$ has a zero, while p_ℓ has double poles where τ_ℓ has a zero.

For the particular sequence of rational solutions in Table 1, a determinantal representation was found by Kajiwara and Ohta, by making a scaling reduction from the rational solutions of the KP hierarchy. The following is a slightly adapted version of the statement in [9].

Theorem 1. *For the values $\ell = n + 1/2$ of the parameter in the Hamiltonian (5), the Painlevé II equation (2) has rational solutions given by*

$$q_{n+1/2} = \frac{\mathrm{d}}{\mathrm{d}z}\log\left(\frac{\tau_{n-1/2}(z)}{\tau_{n+1/2}(z)}\right),$$

where the tau function associated with $h_{n+1/2}$ is given by

$$\tau_{n+1/2}(z) = \exp\left(-\frac{z^3}{24}\right)\begin{vmatrix} \mathrm{p}_1(z) & \mathrm{p}_3(z) & \cdots & \mathrm{p}_{2n-1}(z) \\ \mathrm{p}_0(z) & \mathrm{p}_2(z) & \cdots & \mathrm{p}_{2n-2}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{p}_{-n+2}(z) & \mathrm{p}_{-n+4}(z) & \cdots & \mathrm{p}_n(z) \end{vmatrix} \tag{10}$$

for integer $n \geq 1$, with $y_{n+1/2} = -y_{-n-1/2}$ for $n \leq 0$, and polynomials $p_k(z)$ defined by the generating function

$$\exp\left(z\lambda - \frac{4}{3}\lambda^3\right) = \sum_{k=0}^{\infty} \mathrm{p}_k(z)\lambda^k, \qquad \textit{with} \quad \mathrm{p}_k(z) = 0 \quad \textit{for} \quad k < 0. \tag{11}$$

Observe that, from differentiating the above generating function with respect to z, it follows that the derivatives of the polynomial entries appearing in (10) satisfy

$$\mathrm{p}_k'(z) = \mathrm{p}_{k-1},$$

so that each row is the derivative of the one above it, and hence each determinant is a Wronskian. The sequence of tau functions begins with

$$e^{-z^3/24}, e^{-z^3/24}z, \frac{e^{-z^3/24}(z^3+4)}{3}, \frac{e^{-z^3/24}(z^6+20z^3-80)}{45}, \frac{e^{-z^3/24}(z^{10}+60z^7+11200z)}{4725},$$

for $n = 0, 1, 2, 3, 4$. The exponential prefactor $e^{-z^3/24}$ cancels out from the ratio of tau functions in (8), and the numerical prefactors make no difference to the logarithmic derivative; therefore, the rational solutions are determined by a sequence of monic polynomials whose degrees are triangular numbers, known as the Yablonskii–Vorob'ev polynomials (see [10] and its references). In [9], these polynomials are constructed by making a reduction from the well-known polynomial tau functions for the KP hierarchy. However, in Section 2, we will show how the rational solutions of Painlevé II arise naturally from a sequence of Darboux transformations applied to Schrödinger operators, starting from a zero potential, in a construction due to Adler and Moser [11]. However, it turns out that these Darboux transformations are of a degenerate type, known as confluent, where the same eigenvalue is repeated in successive steps. In the confluent case, the original formulation of Crum's theorem [12] about iterated Darboux transformations does not apply.

Confluent Darboux transformations, and their associated Wronskian representation under iteration, were studied systematically by Contreras-Astorga and Schulze-Halberg [13,14]. In the next section, we briefly review the theory of Darboux transformations, and the confluent case in particular, before the following section, where we discuss the relation with the work of Adler and Moser on the vanishing rational solutions of the KdV equation, thereby explaining how this leads to determinantal formulae for the Yablonskii–Vorob'ev polynomials via similarity reduction. In Section 4, we describe the Ohyama polynomials, which correspond to a sequence of tau functions for the algebraic solutions of the Painlevé equation

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2} = \frac{1}{P}\left(\frac{\mathrm{d}P}{\mathrm{d}z}\right)^2 - \frac{1}{z}\left(\frac{\mathrm{d}P}{\mathrm{d}z}\right) + \frac{1}{z}(2P^2 - \beta) - \frac{1}{P}. \tag{12}$$

The latter is referred to as the Painlevé III (D_7) equation because it has surface type $D_7^{(1)}$, and it has symmetry type $(A_1^{(1)})_{|\alpha|^2=4}$, with the additional suffix denoting a simple root having double the usual squared length (see Table 4 in [15]). The rest of the paper is devoted to describing the main result, namely, an explicit Wronskian representation for the algebraic solutions of (12), which is constructed by means of a sequence of confluent Darboux transformations applied to a Schrödinger operator that was obtained in [16] by reduction from the Lax pair for the Camassa–Holm equation (see also [17]). The corresponding algebraic solutions of the Painlevé III (D_7) equation are rational functions in $z^{\frac{1}{3}}$, while the associated sequence of Ohyama polynomials are polynomials in $z^{\frac{2}{3}}$. The Riemann–Hilbert problem for these solutions was recently analyzed and used to determine their asymptotic behavior in [18]. Here, our primary concern is to present, for the first time, a determinantal formula for these solutions, which was identified as an open problem in [19]. The proof of our main result is given in Section 5, with an associated generating function and its connection with Lax pairs being presented in Section 6. We end with some brief conclusions and an outlook on future work, while some further technical details about intertwining operators and the associated algebraic structure for confluent Darboux transformations have been relegated to Appendix A.

2. Confluent Darboux Transformations

The concept of the Darboux transformation was first proposed in [1] as a covariance property of a general linear equation of the second order, transforming any such equation into a new one with related coefficients. In the simplest case of an equation of the Schrödinger type, written as

$$\varphi'' + (V_0 + \lambda)\varphi = 0, \tag{13}$$

with eigenvalue λ and potential $V_0(z)$ (strictly speaking, this is minus the potential in the context of quantum mechanics), the new equation obtained under the action of the Darboux transformation is

$$\tilde{\varphi}'' + \left(-\psi \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{\psi}\right) + \lambda - \mu\right) = 0, \tag{14}$$

where the primes denote derivatives with respect to the independent variable z, and the new eigenfunction is

$$\tilde{\varphi} = \varphi' - \frac{\psi'}{\psi}\varphi, \tag{15}$$

for some particular solution ψ to the original Schrödinger equation (13) with an arbitrary fixed choice of eigenvalue, μ say. Notably, (14) can also be written as

$$\tilde{\varphi}'' + (V_1 + \lambda)\tilde{\varphi} = 0 \tag{16}$$

so that the transformation has produced an eigenfunction for an equation of the same type but with a new potential $V_1(z)$ given by

$$V_1 = V_0 + 2(\log \psi)''. \tag{17}$$

One of the most direct ways to understand the algebraic structure of the Darboux transformation is through factorization of the Schrödinger operator. Upon introducing the first order operators

$$L = \frac{\mathrm{d}}{\mathrm{d}z} - y, \qquad L^\dagger = -\frac{\mathrm{d}}{\mathrm{d}z} - y, \qquad \text{with} \quad y = \frac{\mathrm{d}}{\mathrm{d}z}(\log \psi), \tag{18}$$

which, from (15), are such that

$$L\psi = 0, \qquad L\varphi = \tilde{\varphi}, \tag{19}$$

one has the factorizations

$$-L^\dagger L = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_0 + \mu, \qquad -LL^\dagger = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_1 + \mu,$$

with $V_1 = V_0 + 2y'$ by (17), and then, by (13) and (19), it follows that

$$L^\dagger \tilde{\varphi} = (\lambda - \mu)\varphi,$$

from which the new Schrödinger equation (16) is obtained by applying L to both sides. Also, from (18), it follows that

$$L^\dagger \psi^{-1} = 0 \implies -LL^\dagger \psi^{-1} = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_1 + \mu\right)\psi^{-1} = 0. \tag{20}$$

An elegant extension of the above expressions, describing the repeated action of successive Darboux transformations, was given by Crum [12], who provided a neat formulation of the overall transformation required to take the initial potential and solution from a given base system to one that is obtained by n applications of the transformation defined by (15) and (17), essentially reducing n steps to a single step. The key is to take n independent eigenfunctions ψ_i for the original equation, with associated eigenvalues μ_i, that is,

$$\psi_i'' + (V_0 + \mu_i)\psi_i = 0, \qquad i = 1, \ldots, n, \tag{21}$$

and consider their Wronskian

$$Wr(\psi_1, \psi_2, ..., \psi_n) = \det\left(\frac{\mathrm{d}^{i-1}\psi_j}{\mathrm{d}z^{i-1}}\right)_{i,j=1,\ldots,n}.$$

Then, the solution φ_n of the Schrödinger equation

$$\varphi_n'' + (V_n + \lambda)\varphi_n = 0 \tag{22}$$

obtained by iterating the Darboux transformation n times, starting from an initial solution $\varphi = \varphi_0$, is given by a ratio of Wronskians, namely,

$$\varphi_n = \frac{Wr(\psi_1, \psi_2, ..., \psi_n, \varphi)}{Wr(\psi_1, \psi_2, ..., \psi_n)}, \tag{23}$$

while the potential in (22) is given in terms of the original one by

$$V_n = V_0 + 2\big(\log Wr(\psi_1, ..., \psi_n)\big)''. \tag{24}$$

Note that, in order to have distinct potentials V_i and non-vanishing Wronskians $Wr(\psi_1, \psi_2, ..., \psi_i)$ at each stage where $0 \leq i \leq n$, Crum's description requires that the eigenfunctions ψ_i chosen should have distinct eigenvalues μ_i. However, it turns out that an analogous Wronskian description can be found in the case of so-called confluent Darboux transformations, where the new eigenfunction introduced at each stage has the same eigenvalue. This confluent case is the one that is relevant to the repeated application of the BTs (3) for Painlevé II, and also applicable to the BTs for the Painlevé III (D_7) Equation (12), which are the main object of our study here, ultimately giving rise to a Wronskian representation for the Ohyama polynomials. Before proceeding with the latter, we will first summarize some of the results on confluent Darboux transformations from [13,14].

In the confluent case, the entries of the Wronskian that produces the nth iteration of the Darboux transformation are no longer simple eigenfunctions of the original potential V_0, but, instead, are replaced by a sequence of generalized eigenfunctions ψ_i satisfying a so-called Jordan chain, which means that, in particular, for each i, the condition

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_0 + \mu\right)^i \psi_i = 0, \qquad i = 1, 2, \ldots \tag{25}$$

must hold, where μ is the common eigenvalue shared by all the solutions used in the application of the Darboux transformation at each step i. The condition (25) implies that

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_0 + \mu\right)\psi_i \in \ker\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_0 + \mu\right)^{i-1},$$

but, in order to iterate the Darboux transformation, the Jordan chain should satisfy the stronger condition

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_0 + \mu\right)\psi_i = \psi_{i-1}, \tag{26}$$

up to an overall non-zero constant multiplier, corresponding to a choice of normalization for the eigenfunctions (and possible addition to the right-hand side of linear combinations of ψ_k for $k < i - 1$ has been suppressed without loss of generality).

It is convenient to set $\psi_0 = 0$ so that the latter relation is valid for all $i \geq 1$. It can then be shown by induction (see the proof of Theorem 1 in [14]) that, for each integer $n \geq 0$, after n steps, the function

$$\phi_n = \frac{Wr(\psi_1, \ldots, \psi_{n+1})}{Wr(\psi_1, \ldots, \psi_n)}, \tag{27}$$

is a solution to

$$\phi_n'' + (V_n + \mu)\phi_n = 0, \qquad \text{where} \quad V_n = V_0 + 2\big(\log Wr(\psi_1, \ldots, \psi_n)\big)''. \tag{28}$$

Furthermore, for all $n \geq 1$, ϕ_{n-1}^{-1} is another independent eigenfunction of the operator with potential V_n:

$$(\phi_{n-1}^{-1})'' + (V_n + \mu)\phi_{n-1}^{-1} = 0, \qquad \text{with} \quad Wr(\phi_{n-1}^{-1}, \phi_n) = 1. \tag{29}$$

It is convenient to use θ_n to denote the Wronskian of the first n generalized eigenfunctions, and then it follows directly from (27) that

$$\theta_n = Wr(\psi_1, \ldots, \psi_n) = \prod_{j=0}^{n-1} \phi_j, \qquad \text{for} \quad n \geq 1, \tag{30}$$

with $\theta_0 = 1$ corresponding to the "empty" Wronskian.

For completeness, some further details about Jordan chains and associated Wronskian formulae have been included in the appendix.

3. Yablonskii–Vorob'ev Polynomials via Adler and Moser

In this section, we describe a simple example of a Jordan chain of generalized eigenfunctions that produces a sequence of Schrödinger operators, which appeared in the work of Adler and Moser [11], who used iterated Darboux transformations to construct the rational solutions of the Korteweg–deVries (KdV) equation, which is the PDE

$$V_t = V_{xxx} + 6VV_x. \tag{31}$$

By a direct argument, they were able to construct a Wronskian representation for these solutions from scratch, by matching them to the polynomial solutions of the recurrence relation

$$\theta'_{k+1}\theta_{k-1} - \theta_{k+1}\theta'_{k-1} = \theta_k^2 \qquad \text{for } k \geq 1, \tag{32}$$

starting with $\theta_0 = 1$, $\theta_1 = x$, where (above and in most of the rest of this section) the $'$ denotes differentiation by x. (Note that, compared with [11], we have removed a factor $(2k+1)$ from the right-hand side above.) In fact, these polynomials and the recurrence (32) were first discovered by Burchnall and Chaundy in the 1920s; see [20] and references for further details. However, here, we show how the Wronskian expressions for the polynomials θ_k are obtained immediately by applying the theory of confluent Darboux transformations to a specific Jordan chain. In addition, we explain how the rational solutions of Painlevé II, as in Table 1, arise as a special case of this construction, by taking a scaling similarity reduction of the KdV equation.

The solutions of the recurrence (32) admit the freedom to replace $\theta_{k+1} \to \theta_{k+1} + c\theta_{k-1}$, for an arbitrary integration constant c, since each θ_{k+1} can be found from the previous two terms in the sequence by integrating

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\theta_{k+1}}{\theta_{k-1}}\right) = \left(\frac{\theta_k}{\theta_{k-1}}\right)^2,$$

although it is by no means obvious that integration of the rational function on the right-hand side should automatically lead to a new polynomial θ_{k+1} at each stage. Adler and Moser showed directly that, for each $k \geq 0$, the pair of independent functions

$$\phi_k = \frac{\theta_{k+1}}{\theta_k}, \qquad (\phi_{k-1})^{-1} = \frac{\theta_{k-1}}{\theta_k}, \tag{33}$$

lies in the kernel of the Schrödinger operator

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_k, \qquad \text{with} \quad V_k = 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\log\theta_k), \tag{34}$$

where it is consistent to take $\theta_{-1} = 1$ so that (33) with $k = 0$ yields the two independent solutions $\phi_1 = x$, $\phi_0^{-1} = 1$ of the Schrödinger equation with the initial potential $V_0 = 0$. The Burchnall–Chaundy relation (32) is equivalent to the normalization of the Wronskian of the pair of solutions (33):

$$Wr(\phi_{k-1}^{-1}, \phi_k) = 1.$$

To apply a sequence of confluent Darboux transformations with a repeated eigenvalue $\mu = 0$, starting from the potential $V_0 = 0$, one could start from any solution $\phi_0 = ax + b$. However, by using the freedom to rescale and translate the independent variable, we take $\phi_0 = x$. Then, the Jordan chain associated with this initial potential $V_0 = 0$ and zero eigenvalue $\mu = 0$ is particularly simple: the condition (25) becomes

$$\left(\frac{d^2}{dx^2} + V_0 + \mu\right)^k \psi_k = \frac{\mathrm{d}^{2k}\psi_k}{\mathrm{d}x^{2k}} = 0, \tag{35}$$

which implies that ψ_k is a polynomial of degree at most $2k-1$. For consistency with (27), we can fix $\psi_1 = x$, and then, from (26), all the other generalized eigenfunctions ψ_k are found recursively by integrating the relation

$$\psi_k'' = \psi_{k-1}, \tag{36}$$

which yields

$$\psi_k = \frac{x^{2k-1}}{(2k-1)!} + \sum_{i=0}^{k-2} c_{k-i}\frac{x^{2i}}{(2i)!}, \tag{37}$$

for a set of arbitrary constants c_i. Observe that, despite performing two integrations at each step, the formula (37) only contains half as many constants as one would expect, and only even powers of x are added to the initial term of odd degree, because we have exploited the freedom to subtract from ψ_k any multiples of ψ_j with $j < k$, which makes no difference to the sequence of Wronskians θ_k. Upon assigning a weight $2i-1$ to each constant c_i, we see that each generalized eigenfunction ψ_k is a weighted homogeneous polynomial of degree $2k-1$, and, hence, θ_k has weight $\sum_{i=1}^{k}(2i-1) = \frac{1}{2}k(k+1)$. Up to rescaling, the c_i are equivalent to the constants denoted by τ_i in [11], and they correspond to the times of the KdV hierarchy. Hence, we arrive at the following result:

Theorem 2. *The sequence of polynomials given by $\theta_{-1} = 1 = \theta_0$ and the Wronskians*

$$\theta_k(x, c_2, c_3, c_4, \ldots) = Wr(\psi_1, \ldots, \psi_k) \qquad \text{for} \quad k \geq 1 \tag{38}$$

with entries ψ_i defined by (37), satisfies the Burchnall–Chaundy relation (32). Moreover, if we identify $c_2 = 4t$, then each of the potentials

$$V_k = 2\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\log\theta_k)$$

is a rational solution of the KdV equation (31), and, similarly, for each c_i, an $i > 2$ is identified as the higher time of weight $2i-1$ in the KdV hierarchy.

The only part of the above result that we have not discussed so far is the dependence on time t, and the other times in the KdV hierarchy. This is best understood by considering the Lax pair for the KdV hierarchy, which corresponds to an isospectral evolution of the Schrödinger operator $\mathcal{L}$, defined as the compatibility condition

$$\partial_t \mathcal{L} = [\mathcal{M}, \mathcal{L}] \tag{39}$$

for the linear system

$$(\mathcal{L} + \lambda)\phi = 0, \tag{40}$$

$$\partial_t \phi = \mathcal{M}\phi, \tag{41}$$

where

$$\mathcal{L} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V, \qquad \mathcal{M} = 4\mathcal{L}_+^{3/2} = 4\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 6V\frac{\mathrm{d}}{\mathrm{d}x} + 3V.$$

(Despite the fact that we are now considering partial derivatives, we reserve the ordinary derivative symbol for the distinguished variable x.) It turns out that the Darboux transformation acts covariantly not only on the Schrödinger equation (40), which is the x part of the linear system, but also on the t evolution of the wave function ϕ, given by (41). This leads to the familiar result that the Darboux transformation for the Schrödinger operator induces a BT on the KdV equation, sending solutions to solutions. (For a detailed discussion of this property of the Darboux transformation, and its extension to other equations of the Lax or zero-curvature types, and associated integrable PDEs, see [3].) Similarly, for each i, up to scaling, we can identify c_i with a time variable t', and the corresponding member of the KdV hierarchy is given by the Lax flow

$$\partial_{t'} \mathcal{L} = [\mathcal{M}_i, \mathcal{L}]$$

where $\mathcal{M}_i = \mathcal{L}_+^{(2i-1)/2}$, which is the compatibility condition of the Schrödinger equation (40) with the time evolution

$$\partial_{t'} \phi = \mathcal{M}_i \phi.$$

Starting from the simplest (vacuum) solution $V_0 = 0$, a single Darboux transformation with eigenvalue zero produces the stationary rational solution, which is given by the second logarithmic derivative

$$V_1 = 2(\log \theta_1)'' = -\frac{2}{x^2}.$$

(In contrast, applying a Darboux transformation with a non-zero eigenvalue produces a soliton solution from the vacuum.) Under the action of the confluent Darboux transformation, the next solution is obtained from (34) for $k = 2$, that is,

$$\theta_2 = Wr(\psi_1, \psi_2) = \begin{vmatrix} x & \frac{x^3}{6} + c_2 \\ 1 & \frac{x^2}{2} \end{vmatrix} = \frac{x^3}{3} - c_2, \tag{42}$$

Therefore, with $c_2 = 4t$, the corresponding solution of (31) is

$$V_2 = 2\left(\log\left(\frac{x^3}{3} - c_2\right)\right)'' = -\frac{6x(x^3 + 24t)}{(x^3 - 12t)^2}. \tag{43}$$

To see how the rational solutions of Painlevé II can be deduced from this construction, it is sufficient to consider scaling similarity solutions of the KdV equation, which take the form

$$V(x,t) = (-3t)^{-2/3}\left(p(z) + \tfrac{z}{2}\right), \qquad \text{with} \quad z = x(-3t)^{-1/3}. \tag{44}$$

Upon substituting the above expression into (31), the third-order PDE reduces to the second-order ODE

$$\frac{d^2p}{dz^2} = \frac{1}{2p}\left(\frac{dp}{dz}\right)^2 - 2p^2 - zp - \frac{\ell^2}{2p}, \tag{45}$$

where the coefficient ℓ^2 arises as an integration constant. The latter ODE is called the Painlevé XXXIV equation [4], and it is precisely the equation satisfied by the conjugate momentum variable p when q is eliminated from the pair of Hamilton's equation (4). Thus, letting primes now denote derivatives with respect to z once again,

$$q = \frac{p' + \ell}{2p} \tag{46}$$

satisfies the Painlevé II equation (2) with parameter $\alpha = \ell - \frac{1}{2}$ whenever p is a solution of (45); and, conversely, whenever q is a solution of Painlevé II, it follows that

$$p = -q' - q^2 - \tfrac{z}{2} \tag{47}$$

satisfies Painlevé XXXIV with $\ell = \alpha + \frac{1}{2}$. The latter formula also arises by reduction of the Miura transformation

$$V = -y_x - y^2, \tag{48}$$

which maps solutions of the modified Korteweg–deVries (mKdV) equation

$$y_t = y_{xxx} - 6y^2 y_x \tag{49}$$

to solutions of the KdV equation (31). Indeed, if one takes the similarity reduction

$$y(x,t) = (-3t)^{-1/3} q(z) \qquad \text{with} \quad z = x(-3t)^{-1/3}, \tag{50}$$

then the mKdV equation reduces to Painlevé II, while, by substituting (50) into the Miura formula (48), it is clear that $V(x,t)$ is a similarity solution of the KdV equation, being of the form (44) with p given in terms of q by (47).

Under the similarity reduction (44), the trivial potential $V_0 = 0$ corresponds to the solution $p = -\frac{z}{2}$ of (45) with $\ell = 1/2$, while V_1 corresponds to

$$p = -\frac{2}{z^2} - -\frac{z}{2}, \qquad \ell = 3/2,$$

and, from V_2, as in (43), the scaling reduction gives

$$p = -\frac{6z(z^3 - 8)}{(z^3 + 4)^2} - \frac{z}{2}, \qquad \ell = 5/2.$$

These solutions all have the correct weighted homogeneity under the scaling $x \to \gamma x$, $t \to \gamma^3 t$ of the KdV independent variables, but the other KdV solutions V_k for $k > 2$ do not have the right scaling behavior unless we fix the higher time parameters c_i to be zero for $i > 2$. After making this adjustment, the fact that every θ_k has homogeneous degree $\frac{1}{2}k(k+1)$ means that they can be rescaled to give polynomials in the similarity variable z.

Lemma 1. *After setting $c_2 = 4t$ and $c_i = 0$ for $i > 2$, the polynomial solutions of the Burchnall–Chaundy relation satisfy the scaling property*

$$\theta_k(x, c_2, 0, 0, \ldots) = (-3t)^{k(k+1)/6}\, \theta_k\big(z, -\tfrac{4}{3}, 0, 0, \ldots\big). \tag{51}$$

If we now compare the Jordan chain for these scaling similarity solutions to the iterated action of the BT (7) for Painlevé II, we find that they match up exactly. To see this, note that for any fixed parameter ℓ, since (45) is invariant under $\ell \to -\ell$, the formula (46) relates two solutions of Painlevé II to the same solution of Painlevé XXXIV by writing

$$q_\ell = \frac{p'_\ell + \ell}{2p_\ell}, \qquad q_{-\ell} = \frac{p'_\ell - \ell}{2p_\ell}. \tag{52}$$

Upon adding these two equations, substituting for $q_{\pm\ell}$ in terms of the Okamoto tau function $\tau_\ell = \tau_{-\ell}$ via (8) and integrating the logarithmic derivative that appears on both sides, we find that p_ℓ is given by the ratio

$$p_\ell = C_\ell \frac{\tau_{\ell-1}\tau_{\ell+1}}{\tau_\ell^2} \tag{53}$$

for some normalization constant C_ℓ. On the one hand, comparing the above expression with (9) yields an equation for the sequence of tau functions,

$$C_\ell \tau_{\ell-1}\tau_{\ell+1} = 2\big(\tau_\ell\tau''_\ell - (\tau'_\ell)^2\big), \tag{54}$$

which is a bilinear form of the Toda lattice. On the other hand, subtracting one of the two equations (52) from the other, then substituting $y_{\pm\ell}$ using (8) and p_ℓ using (53), produces the Burchnall–Chaundy relation in the modified form

$$\tau'_{\ell+1}\tau_{\ell-1} - \tau_{\ell+1}\tau'_{\ell-1} = -\ell\, C_\ell\, \tau_\ell^2. \tag{55}$$

Furthermore, when we restrict to the sequence of rational solutions with parameters $\ell = n + 1/2$, and compare (44) with (51), for each integer n, we see that the ratios $\tau_{n+3/2}/\tau_{n+1/2}$, $\tau_{n-1/2}/\tau_{n+1/2}$ provide a pair of independent eigenfunctions for the potential

$$p_{n+1/2} + \frac{z}{2} = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big(\log\theta_n(z, -\tfrac{4}{3}, 0, 0, \dots)\Big) = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big(\log\tau_{n+1/2}(z)\Big) + \frac{z}{2}.$$

Then we can fix the normalization $-(n+1/2)C_{n+1/2} = 1$ to match (55) with (32), and note that this relation is invariant under the rescaling of all the tau functions by the same factor $e^{-z^3/24}$. Hence, from the above lemma, we have the following:

Corollary 1. *Up to normalizing constants, the Yablonskii–Vorob'ev polynomials are obtained from the Burchnall–Chaundy polynomials (38) by replacing $x \to z$, $c_2 \to -\frac{4}{3}$, and setting $c_i = 0$ for all $i > 2$. Moreover, they are related to the Okamoto tau functions by*

$$\theta_n(z, -\tfrac{4}{3}, 0, 0, \dots) = \exp\left(\frac{z^3}{24}\right)\tau_{n+1/2}(z).$$

An alternative route to the Yablonskii–Vorob'ev polynomials, and the one taken in [9], is to start from the rational solutions of the KP hierarchy, then reduce these to the rational solutions of the KdV equation, and, finally, make the similarity reduction to the corresponding solutions of Painlevé XXXIV/Painlevé II. The polynomial tau functions $\tau = \tau_Y(\underline{t})$ of the KP hierarchy are the Schur functions

$$\tau_Y(\underline{t}) = Wr(\mathrm{p}_{j_n}, \mathrm{p}_{j_{n-1}+1}, \dots, \mathrm{p}_{j_1+n-1}),$$

associated with a Young diagram Y defined by integers $j_1 \geq j_2 \geq \cdots \geq j_n$, where $\underline{t} = (t_1, t_2, t_3, \ldots)$ is the sequence of KP times, and p_j are the elementary Schur polynomials, with generating function

$$\exp\left(\sum_{i=1}^{\infty} t_i \lambda^i\right) = \sum_{j=0}^{\infty} \mathrm{p}_j(\underline{t})\lambda^j, \tag{56}$$

which satisfy

$$\partial_{t_i} \mathrm{p}_j = \mathrm{p}_{j-i} \tag{57}$$

and the infinite hierarchy of symmetries of the heat equation, that is,

$$\partial_{t_k} \mathrm{p}_j = \partial_x^k \mathrm{p}_j$$

(with $t_1 = x$). (For more details on the KP hierarchy and its solutions, see [21].) The reduction from the KP equation to the KdV equation requires that all the even times should be discarded so that only dependence on the odd times $t_1, t_3, t_5, \ldots$ remains, and the tau functions which survive are those that satisfy

$$\partial_{t_{2i}} \tau = 0,$$

which, in the case of polynomial solutions, requires that only the Schur functions with triangular Young diagrams should remain, namely,

$$\tau_{Y,KdV}(\underline{t}) = Wr(\mathrm{p}_1, \mathrm{p}_3, \ldots, \mathrm{p}_{2n-1}). \tag{58}$$

By comparison with (38), we see that these are precisely the Burchnall–Chaundy polynomials when we identify $t_1 = x$, $t_3 = c_2 = 4t$, and $t_{2i-1} = c_i$ for $i > 2$, and $\mathrm{p}_{2i-1} = \psi_i$, with the Jordan chain condition (36) being a particular consequence of the general derivative property (57). Then, under the scaling similarity reduction, the entries of the determinant (10) and the generating function (11) arise from (56) by replacing $t_1 \to z$, $t_3 \to -\frac{4}{3}$, and all other $t_i \to 0$. In a similar manner, for the PII hierarchy, which arises by taking scaling similarity reductions of the higher flows of the mKdV hierarchy (see, e.g., [22]), one can obtain the rational solutions by replacing $x = t_1 \to z$, fixing a non-zero value of the appropriate time t_{2i-1}, and setting all the other times to 0.

4. Ohyama Polynomials and BTs for Painlevé III (D_7)

The Ohyama polynomials $\rho_n(s)$ are a sequence of polynomials defined recursively by the relation

$$(s+n)\rho_n^2 - 2s\rho_n\ddot{\rho}_n + 2s(\dot{\rho}_n)^2 - 2\rho_n\dot{\rho}_n = \begin{cases} \rho_{n+1}\rho_{n-1} & \text{for } n \text{ odd,} \\ s\rho_{n+1}\rho_{n-1} & \text{for } n \text{ even,} \end{cases} \tag{59}$$

for $\rho_0 = \rho_{\pm 1} = 1$, where the dots denote differentiation with respect to the variable s, and $n \in \mathbb{Z}$. Despite it not being obvious from the form of this relation, it has been proven [23] that each ρ_n is a monic polynomial in s, with integer coefficients, and $\rho_n(0) \neq 0$. As we shall see, these polynomials are in direct correspondence with the algebraic solutions of a special case of the Painlevé III equation, given by (12), arising as particular solutions when the parameter β therein is an even integer. As such, they play an analogous role to that of the Yablonskii–Vorob'ev polynomials for Painlevé II, as in the previous section, and to that of other families of polynomials like the Umemura polynomials [24] and the Okamoto polynomials [8,19], which are associated with another family of rational solutions of Painlevé III and rational solutions of Painlevé IV, respectively. However, unlike these other polynomial families, until now, no Wronskian or other determinantal representation was known for the Ohyama polynomials. In this section, we outline how such a representation arises from Darboux transformations.

The Painlevé III (D_7) equation, as in (12), can be derived from the system of Hamilton's equations

$$\begin{array}{rclr} zQ' & = & Q(2PQ-\kappa)-z = & \frac{\partial \mathrm{h}}{\partial P}, \\ zP' & = & -P(2PQ-\kappa)+z = & -\frac{\partial \mathrm{h}}{\partial Q}, \end{array} \tag{60}$$

where the prime denotes the z derivative, and

$$\mathrm{h} = Q^2P^2 - \kappa\, QP - z(Q+P), \qquad \kappa = \beta + 1. \tag{61}$$

By eliminating Q from the system, P is found to satisfy Equation (12), that is,

$$P'' = \frac{1}{P}(P')^2 - \frac{1}{z}(P') + \frac{1}{z}(2P^2 - \beta) - \frac{1}{P}$$

If P is eliminated instead, then $Q = P_+$ is found to satisfy the same ODE but with $\beta \to \beta + 2$, and, by reversing the roles of P and Q, one can shift β down by 2, leading to a BT for (12) together with its inverse, namely, the pair of transformations

$$P \mapsto P_\pm = \frac{z(\mp P' + 1)}{2P^2} + \frac{(\pm 1 + \beta)}{2P}, \qquad \beta \mapsto \beta \pm 2. \tag{62}$$

The forward shift can be written as a birational transformation T_+ acting on the extended phase space with coordinates (Q, P, z, κ), given by

$$T_+ : \qquad \begin{cases} Q & \mapsto -P + \frac{\kappa+1}{Q} + \frac{z}{Q^2} \\ P & \mapsto Q \\ z & \mapsto z \\ \kappa & \mapsto \kappa + 2, \end{cases} \tag{63}$$

preserving a contact 2-form Ω, namely,

$$T_+^*\Omega = \Omega, \qquad \Omega = \mathrm{d}P \wedge \mathrm{d}Q - \frac{1}{z}\mathrm{dh} \wedge \mathrm{d}z,$$

and there is a similar set of expressions defining the inverse $T_- = T_+^{-1}$. Moreover, if the sequence of Hamiltonians obtained under the iterated action of T_+ is indexed by the parameter κ, then

$$\mathrm{h}_\kappa = T_+^*(\mathrm{h}_{\kappa-2}) = \mathrm{h}_{\kappa-2} - \frac{z}{P} - \kappa + 1. \tag{64}$$

(For the full set of affine Weyl group symmetries of the Painlevé III (D_7) equation, see [23].)

A deeper insight into the structure of the BT T_+, and an understanding of its connection with confluent Darboux transformations, was achieved due to the investigations in [16], which were further clarified in [17], where it was shown that the Painlevé III (D_7) Equation (12) arises as a similarity reduction of the Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = uu_{xxx} + 2u_xu_{xx}. \tag{65}$$

Although the full details are somewhat involved (cf. Theorem 3.2 in [17]), the similarity solutions of the PDE (65) can be specified in parametric form by the hodograph transformation

$$u(x,t) = \frac{1}{2t}\left(\frac{z}{P(z)} + \beta\right), \quad x - \frac{\beta}{2}\log t = \log\left(\frac{\phi_-(z)}{\phi_+(z)}\right) + \text{const}, \tag{66}$$

where $P(z)$ is a solution of (12) and $\phi_\pm$ are two solutions of an associated Schrödinger equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V\right)\phi_\pm = 0, \tag{67}$$

with the potential $V = V(z)$ defined in terms of $P(z)$ by

$$V = -\frac{1}{4P^2}\left((P')^2 - 1\right) + \frac{1}{2zP}\left(P' - 2P^2 + \beta\right), \tag{68}$$

and these two solutions of (67) are constrained by the requirement that their product is P and their Wronskian is 1:

$$P = \phi_+\phi_-, \qquad Wr(\phi_+,\phi_-) = 1. \tag{69}$$

As was explained in [16], the fact that the Camassa–Holm equation is related to a negative KdV flow means that the BT T_+ for (12) can be obtained from a (confluent) Darboux transformation acting on the Schrödinger operator in (67), where the potential V is viewed as coming from a similarity solution of a member of the KdV hierarchy. (However, note that the variables x, t in (65) are not directly related to x, t in the previous section.) Indeed, observe that, for any solution $P(z)$ of the Painlevé III (D_7) equation, we can introduce a tau function $\sigma(z)$, which is defined by considering the quantity

$$\eta(z) = -P(z) - \frac{z}{2}V(z), \tag{70}$$

and then a direct calculation using (12) shows that

$$\eta'(z) = \frac{1}{2}V(z). \tag{71}$$

Thus, if σ is related to η by

$$\eta(z) = \frac{\mathrm{d}}{\mathrm{d}z}\big(\log\sigma(z)\big), \tag{72}$$

then it transpires that the corresponding KdV potential V in (68) is given in terms of the same tau function by the standard relation

$$V(z) = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\big(\log\sigma(z)\big). \tag{73}$$

Then, in turn, by rearranging (70) and substituting for η and V in terms of σ, it follows that P is specified by the tau function according to the formula

$$P(z) = -\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\big(\log\sigma(z)\big)\right). \tag{74}$$

The connection between (62) and Darboux transformations is explained by the next result.

Lemma 2. *Given a solution $P(z)$ of the Painlevé III (D_7) equation, let*

$$y_\pm = \frac{P' \mp 1}{2P}. \tag{75}$$

Then the BT T_+ and its inverse $T_- = T_+^{-1}$, as in (62), can be expressed as

$$P_\pm = P - \frac{\mathrm{d}}{\mathrm{d}z}(zy_\pm). \tag{76}$$

Moreover, the associated potential, defined by (68), can be written as

$$V = -y_+' - y_+^2 = -y_-' - y_-^2, \tag{77}$$

and the corresponding action of $T_\pm$ on V is equivalent to a Darboux transformation, being given by

$$V_\pm = T_\pm^*(V) = V + 2y_\pm'. \tag{78}$$

Proof. A direct calculation, using the Painlevé III (D_7) Equation (12), shows that

$$\begin{aligned} y_+' = \tfrac{\mathrm{d}}{\mathrm{d}z}\left(\tfrac{P'-1}{2P}\right) &= \tfrac{P''}{2P} - \tfrac{(P')^2}{2P^2} + \tfrac{P'}{2P^2} \\ &= -\tfrac{1}{z}\left(\tfrac{P'+\beta}{2P} - P\right) + \tfrac{P'-1}{2P^2} \\ &= -\tfrac{1}{z}\left(y_+ + \tfrac{\beta+1}{2P} - P\right) + \tfrac{P'-1}{2P^2}, \end{aligned}$$

and hence, from the formula for T_+ in (62), $y_+' = z^{-1}(y_+ + P - P_+)$, which yields the $+$ case of (76), and a similar calculation yields the $-$ case. Furthermore, for both choices of sign, we find

$$-y_\pm' - y_\pm^2 = -\frac{P''}{2P} + \frac{(P')^2 - 1}{4P^2},$$

and, upon using the Painlevé III (D_7) equation once again to remove the P'' term, we arrive at the formula (68) for V. Thus, since (77) holds, we have two different factorizations of the Schrödinger operator with this potential:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V = \left(\frac{\mathrm{d}}{\mathrm{d}z} + y_+\right)\left(\frac{\mathrm{d}}{\mathrm{d}z} - y_+\right) = \left(\frac{\mathrm{d}}{\mathrm{d}z} + y_-\right)\left(\frac{\mathrm{d}}{\mathrm{d}z} - y_-\right).$$

Now, let us introduce the corresponding pair of eigenfunctions with eigenvalue zero:

$$y_\pm = \frac{\mathrm{d}}{\mathrm{d}z}\log\phi_\pm \implies \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V\right)\phi_\pm = 0.$$

Then, from the definitions of $y_\pm$ in terms of P, by adding, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\log(\phi_+\phi_-) = y_+ + y_- = \frac{\mathrm{d}}{\mathrm{d}z}\log P \implies P = \phi_+\phi_-,$$

where we have fixed an overall normalizing constant, while, by subtracting, we find

$$\frac{\mathrm{d}}{\mathrm{d}z}\log\left(\frac{\phi_-}{\phi_+}\right) = y_- - y_+ = \frac{1}{P} = \frac{1}{\phi_+\phi_-} \implies Wr(\phi_+, \phi_-) = 1.$$

Therefore, we have verified the assertions in (69). To relate the BT T_+ to a Darboux transformation, it is helpful to determine its action on the quantity η defined by (70). After a slightly tedious computation, using the second equation of the system (60) to substitute $P' = 1 - z^{-1}P(2PQ - \kappa)$ in the expression (68), we find that η is very closely linked with the Hamiltonian, being given by

$$\eta = \frac{1}{2z}\left(\mathrm{h} + PQ + \tfrac{\kappa(\kappa-2)}{4}\right). \tag{79}$$

(In fact, η is the same as the Hamiltonian denoted H in [23], where a different Hamiltonian structure is used.) Then, by applying the BT to shift all the variables in (64), we have

$$T_+^*(\mathrm{h}) = \mathrm{h} - \frac{z}{Q} - \kappa - 1,$$

Therefore, from (63), we obtain

$$\begin{aligned} T_+^*(\eta) &= \tfrac{1}{2z}\Big(\mathrm{h} - \tfrac{z}{Q} - \kappa - 1 + Q\Big(-P + \tfrac{\kappa+1}{Q} + \tfrac{z}{Q^2}\Big) + \tfrac{(\kappa+2)\kappa)}{4}\Big) \\ &= \tfrac{1}{2z}\Big(\mathrm{h} - QP + \tfrac{\kappa(\kappa+2)}{4}\Big) \\ &= \eta - \tfrac{1}{2z}\Big(2PQ - \kappa\Big). \end{aligned}$$

By using the formula for P' in (60), once again we see that $y_+ = -\frac{1}{2z}(2PQ - \kappa)$, and so

$$T_+^*(\eta) = \eta + y_+. \tag{80}$$

Hence, by differentiating and applying the relation (71), we see that

$$T_+^*(V) = V + 2y_+' = V + 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big(\log\phi_+\Big),$$

which shows that T_+ corresponds to a Darboux transformation acting on the potential V. An analogous chain of reasoning produces the $-$ case of (78), which shows that $T_- = T_+^{-1}$ also corresponds to a Darboux transformation. □

Remark 1. *We can introduce additional tau functions that appear under the action of the BT and its inverse so that*

$$T_\pm(\eta) = \frac{\mathrm{d}}{\mathrm{d}z}\log\sigma_\pm,$$

and then, from (80) and the analogous formula for T_-, it follows that

$$y_\pm = \frac{\mathrm{d}}{\mathrm{d}z}\Big(\log\Big(\frac{\sigma_\pm}{\sigma}\Big)\Big). \tag{81}$$

Also, the formula (79) implies that

$$\eta = \frac{1}{2z}\Big(\mathrm{h} + \tfrac{\kappa^2}{4}\Big) - \frac{1}{2}y_+,$$

which means that the Hamiltonian can be expressed in terms of tau functions as

$$\mathrm{h} = z\frac{\mathrm{d}}{\mathrm{d}z}\Big(\log(\sigma\sigma_+)\Big) - \frac{\kappa^2}{4}.$$

We now introduce some recursion relations for the tau functions of the Painlevé III (D_7) equation, which are key to understanding how the Ohyama polynomials, as well as the relation (59), appear in this context.

Proposition 1. *Suppose that $\sigma_-, \sigma, \sigma_+$ are three adjacent tau functions for Equation (12), connected via the action of the BTs T_- and T_+. Then, these three tau functions are connected by a bilinear equation of the Toda lattice type, namely,*

$$\sigma_+\sigma_- = C\Big(z\big((\sigma')^2 - \sigma\sigma''\big) - \sigma\sigma'\Big), \tag{82}$$

and also by a Burchnall–Chaundy relation, that is,

$$\sigma'_- \sigma_+ - \sigma_- \sigma'_+ = C\,\sigma^2 \tag{83}$$

where C is a non-zero constant.

Proof. Either by adding the definitions (75) of $y_\pm$ in Lemma 2, and integrating the logarithmic derivative that appears on both sides after substituting with the tau function expressions (81), or by observing that the eigenfunctions in (69) must be given by

$$\phi_+ = K_+ \frac{\sigma_+}{\sigma}, \qquad \phi_- = K_- \frac{\sigma_-}{\sigma}$$

for some non-zero constants $K_\pm$, we see that the solution P of the Painlevé III (D_7) equation is given in terms of these three tau functions by the ratio

$$P = C^{-1}\,\frac{\sigma_+\sigma_-}{\sigma^2} \tag{84}$$

for some constant $C = 1/(K_+K_-) \neq 0$. Upon equating this with the logarithmic derivative in (74), and clearing σ^2 from the denominator, the bilinear equation (82) is obtained. If, instead, the two definitions (75) are subtracted one from the other, then we have

$$y_- - y_+ = \frac{1}{P}$$

so that, from substituting the expressions (81) on the left-hand side and the ratio (84) on the right-hand side above, after clearing the denominator $\sigma_+\sigma_-$, this produces (83). □

Remark 2. *Both of the bilinear relations for the tau functions can be written somewhat more concisely by using the Hirota derivative D_z: the Toda-type relation is*

$$\sigma_+\sigma_- + C\left(\tfrac{z}{2}\,D_z^2\,\sigma\cdot\sigma + \sigma\,\sigma'\right) = 0,$$

while the Burchnall–Chaundy relation is

$$D_z\,\sigma_-\cdot\sigma_+ = C\,\sigma^2.$$

The algebraic solutions of the Painlevé III (D_7) equation arise for even integer values of the parameter β, and are related to a family of ramp-like similarity solutions of the Camassa–Holm equation (see [17]). The PDE (65) has the elementary solution

$$u(x,t) = \frac{x}{3t},$$

which is the same as the ramp solution for the inviscid Burgers' (Hopf) equation $u_t + 3uu_x = 0$, and is a particular similarity solution of the form (66) with $\beta = 0$, corresponding to the solution

$$P = \left(\frac{z}{2}\right)^{1/3}$$

of (12) for this value of β. The action of one of the BTs (62) either raises or lowers the value of the parameter by 2 with each application, so, by taking $P_0 = \zeta$ as the seed solution, with $\zeta = (z/2)^{1/3}$, a family of algebraic solutions is obtained for all even integer values $\beta = 2n$, which we denote by P_n for $n \in \mathbb{Z}$, and these are all rational functions of ζ. A few of these solutions are presented in the first row of Table 2. Similarly, since the action of each BT or Darboux transformation increases or decreases n by 1 at each step, it is convenient to index all relevant quantities with this integer, while, for the independent variables, it is necessary to switch between ζ and the variables

$$z = 2\zeta^3, \qquad s = 3\zeta^2.$$

The majority of the subsequent formulae are written most simply in terms of ζ, but, in all of the Wronskians, the derivatives are taken with respect to the variable z, while the Ohyama polynomials defined by (59) are polynomials in s.

Table 2. Algebraic solutions of (12), and associated potentials, tau functions, Wronskians, and eigenfunctions in terms of $\zeta = \left(\frac{z}{2}\right)^{\frac{1}{3}}$, with Ohyama polynomials in $s = 3\zeta^2$ for $n = 0, 1, 2, 3$.

$n = \beta/2$	**0**	**1**	**2**	**3**
P_n	ζ	$\frac{3\zeta^2+1}{3\zeta}$	$\frac{\zeta(9\zeta^4+12\zeta^2+5)}{(3\zeta^2+1)^2}$	$\frac{(3\zeta^2+1)(81\zeta^8+270\zeta^6+360\zeta^4+210\zeta^2+35)}{3\zeta(9\zeta^4+12\zeta^2+5)^2}$
V_n	$-\frac{36\zeta^4-5}{144\zeta^6}$	$-\frac{36\zeta^4-24\zeta^2+7}{144\zeta^6}$	$-\frac{324\zeta^8-216\zeta^6+135\zeta^4-30\zeta^2-5}{144\zeta^6(3\zeta^2+1)^2}$	$-\frac{2916\zeta^{12}+1944\zeta^{10}+1215\zeta^8+1080\zeta^6-630\zeta^4+175}{144\zeta^6(9\zeta^4+12\zeta^2+5)^2}$
σ_n	$\zeta^{-\frac{5}{24}}e^{-\frac{9}{8}\zeta^4}$	$3^{1/4}\zeta^{\frac{7}{24}}e^{-\frac{9}{8}\zeta^4-\frac{3}{2}\zeta^2}$	$(3\zeta^2+1)\zeta^{-\frac{5}{24}}e^{-\frac{9}{8}\zeta^4-3\zeta^2}$	$3^{1/4}(9\zeta^4+12\zeta^2+5)\zeta^{\frac{7}{24}}e^{-\frac{9}{8}\zeta^4-\frac{9}{2}\zeta^2}$
θ_n	1	$3^{1/4}\zeta^{\frac{1}{2}}e^{-\frac{3}{2}\zeta^2}$	$(3\zeta^2+1)e^{-3\zeta^2}$	$3^{1/4}(9\zeta^4+12\zeta^2+5)\zeta^{\frac{1}{2}}e^{-\frac{9}{2}\zeta^2}$
ϕ_n	$3^{1/4}\zeta^{\frac{1}{2}}e^{-\frac{3}{2}\zeta^2}$	$3^{-1/4}(3\zeta^2+1)\zeta^{-\frac{1}{2}}e^{-\frac{3}{2}\zeta^2}$	$3^{1/4}\frac{(9\zeta^4+12\zeta^2+5)}{(3\zeta^2+1)}\zeta^{\frac{1}{2}}e^{-\frac{3}{2}\zeta^2}$	$3^{-1/4}\frac{(81\zeta^8+270\zeta^6+360\zeta^4+210\zeta^2+35)}{(9\zeta^4+12\zeta^2+5)}\zeta^{-\frac{1}{2}}e^{-\frac{3}{2}\zeta^2}$
ρ_n	1	1	$s+1$	s^2+4s+5

The precise connection between the Ohyama polynomials and the algebraic solutions of the Painlevé III (D_7) equation is that, up to multiplying by certain n-dependent gauge factors and a change of independent variable, the polynomials $\rho_n(s)$ are equivalent to the tau functions $\sigma_n(z)$. Now, to fix our notation, we consider the sequence of potentials $V_n = V_n(z)$ associated with the algebraic solutions $P_n(z)$. In terms of tau functions $\sigma_n(z)$, we have

$$V_n = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\log \sigma_n), \qquad P_n = -\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}(\log \sigma_n)\right),$$

while the action of the corresponding (confluent) Darboux transformations on the potentials can be written in terms of eigenfunctions $\phi_n, \tilde{\phi}_n$ so that

$$V_{n+1} = V_n + 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\log \phi_n), \qquad V_{n-1} = V_n + 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\log \tilde{\phi}_n).$$

Comparing this with the notation used above in Lemma 2 and Proposition 1, we see that, for $\kappa = 2n+1$, we have a solution $P = P_n$ of (12), with two adjacent solutions $T^*_\pm(P) = P_{n\pm1}$ obtained by the action of the BT that shifts the parameter one step up/down. The associated potential $V = V_n$ is sent to a new potential $V_+ = V_{n+1}$, via the Darboux transformation generated by the eigenfunction $\phi_+ = \phi_n$ (with eigenvalue zero) satisfying the Schrödinger equation (67), or is sent to the potential $V_- = V_{n-1}$, by the Darboux transformation generated by the eigenfunction $\phi_- = \tilde{\phi}_n$. Finally, our goal is to construct the Jordan chain for the associated sequence of confluent Darboux transformations applied to the initial potential

$$V_0 = \frac{5}{36z^2} - \frac{1}{4(z/2)^{\frac{2}{3}}} = \frac{(5-36\zeta^4)}{144\zeta^6}, \tag{85}$$

which will allow us to write each potential in the form

$$V_n = V_0 + 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\log \theta_n), \tag{86}$$

where θ_n is a Wronskian built from generalized eigenfunctions ψ_i.

Any consecutive triple of tau functions $\sigma_{n-1}, \sigma_n, \sigma_{n+1}$ can be identified with the triple $\sigma_-, \sigma, \sigma_+$ in Proposition 1, and, with a suitable choice of normalization, we can write the Wronskians and eigenfunctions in terms of these tau functions, which leads to

$$\theta_n = \frac{\sigma_n}{\sigma_0}, \tag{87}$$

meaning that each of the Wronskians θ_n can be seen to be a kind of renormalized tau function, as well as the relations

$$\phi_n = \frac{\sigma_{n+1}}{\sigma_n} = \frac{\theta_{n+1}}{\theta_n}, \quad \text{and} \quad \tilde{\phi}_n = \frac{\sigma_{n-1}}{\sigma_n} = \frac{\theta_{n-1}}{\theta_n} = (\phi_{n-1})^{-1}. \tag{88}$$

Note that, for each n, the corresponding quantity $\eta = \eta_n$ is determined by substituting $P = P_n$ and $V = V_n$ into (70), which means that, by (72), the corresponding tau function is given by

$$\sigma_n = \exp\left(\int \eta_n \, \mathrm{d}z \right),$$

and hence is fixed up to an overall constant multiplier (coming from the implicit integration constant above). It is instructive to list here the first few tau functions for $-3 \le n \le 3$:

$$\begin{aligned} &\sigma_0 = \zeta^{-\frac{5}{24}} e^{-\frac{9}{8}\zeta^4}, && \sigma_{\pm 1} = 3^{1/4} \zeta^{\frac{7}{24}} e^{-\frac{9}{8}\zeta^4 \mp \frac{3}{2}\zeta^2}, \\ &\sigma_{\pm 2} = (3\zeta^2 \pm 1)\zeta^{-\frac{5}{24}} e^{-\frac{9}{8}\zeta^4 \mp 3\zeta^2}, && \sigma_{\pm 3} = 3^{1/4}(9\zeta^4 \pm 12\zeta^2 + 5)\zeta^{\frac{7}{24}} e^{-\frac{9}{8}\zeta^4 \mp \frac{9}{2}\zeta^2}. \end{aligned}$$

Observe that, compared with [17], we have switched $n \to -n$ in order to be consistent with the convention used to define the Ohyama polynomials by (59). Then, by the results of Proposition 1, this sequence of tau functions satisfies the bilinear Toda-type relation

$$\sigma_{n+1}\sigma_{n-1} + C_n \left(\tfrac{z}{2} D_z^2 \sigma_n \cdot \sigma_n + \sigma_n \sigma_n' \right) = 0, \tag{89}$$

where the constant C from (82) (which depends on the choice of scaling) is allowed to depend on n, while, from (83), we also have the Burchnall–Chaundy relation

$$\sigma_{n-1}' \sigma_{n+1} - \sigma_{n-1} \sigma_{n+1}' = C_n \, \sigma_n^2. \tag{90}$$

The scaling chosen in [17] is to take $C_n = 1$ for an even n, and $C_n = 3$ for an odd n, but, here, we find it convenient to choose the scaling for σ_n to be such that $C_n = \sqrt{3}$ for all n values. However, in several subsequent statements, we leave the choice of scaling arbitrary.

It follows that, for a given choice of (non-zero) initial tau functions σ_0, σ_1, if C_n has been fixed for all n, then all of the σ_n for $n \in \mathbb{Z}$ are completely determined from (89).

Lemma 3. *The tau functions $\sigma_n(z)$ that satisfy (89) with $C_n = \sqrt{3}$ for all n values, with initial conditions $\sigma_0 = \zeta^{-\frac{5}{24}} e^{-\frac{9}{8}\zeta^4}$, $\sigma_1 = 3^{1/4}\zeta^{\frac{7}{24}} e^{-\frac{9}{8}\zeta^4 - \frac{3}{2}\zeta^2}$, are given by*

$$\sigma_n = \begin{cases} 3^{1/4} \zeta^{\frac{7}{24}} e^{-\frac{9}{8}\zeta^4 - \frac{3}{2}n\zeta^2} \rho_n(3\zeta^2) & \text{for } n \text{ odd}, \\ \zeta^{-\frac{5}{24}} e^{-\frac{9}{8}\zeta^4 - \frac{3}{2}n\zeta^2} \rho_n(3\zeta^2) & \text{for } n \text{ even}, \end{cases} \tag{91}$$

where $z = 2\zeta^3$ and the Ohyama polynomials are evaluated at $s = 3\zeta^2$.

Proof. As in intermediate step, for each choice of parity of n, we can substitute the expressions (91) into the bilinear Toda-type equation (89) with $C_n = \sqrt{3}$, in order to write everything in terms of the variable $\zeta = (z/2)^{\frac{1}{3}}$, and thus obtain

$$(3\zeta^2+n)\rho_n^2 - \frac{1}{6\zeta}\left(\frac{\zeta}{2}D_\zeta^2\rho_n\cdot\rho_n + \rho_n\rho_{n,\zeta}\right) = \begin{cases}\rho_{n+1}\rho_{n-1}, & \text{for } n \text{ odd}\\ 3\zeta^2\rho_{n+1}\rho_{n-1}, & \text{for } n \text{ even,}\end{cases} \tag{92}$$

where D_ζ is the Hirota derivative with respect to ζ, and the additional subscript denotes an ordinary derivative

$$\rho_{n,\zeta} = \frac{\mathrm{d}}{\mathrm{d}\zeta}\,\rho_n(s).$$

Note that, for the initial conditions at $n = 0, 1$, the prescribed choice of σ_0 and σ_1 implies that $\rho_0 = 1 = \rho_1$. After making a further change of variables to the independent variable $s = 3\zeta^2$, the relation (92) is transformed into the recurrence (59) for Ohyama's polynomials (where there, and throughout, the dot is used to mean the s derivative). Since the two sequences σ_n and ρ_n are specified by an equivalent Toda-type relation, modulo the given n-dependent rescaling and the change of independent variables, and the two initial conditions match, this guarantees that σ_n is specified in terms of the Ohyama polynomials by (91) for all $n \in \mathbb{Z}$. □

As a consequence of the relation (90), we can reverse the correspondence (91) to obtain an analogous relation for $\rho_n(s)$.

Corollary 2. *The Ohyama polynomials also obey a relation of the Burchnall–Chaundy type, namely,*

$$\rho_{n+1}\rho_{n-1} + \rho_{n+1}\dot{\rho}_{n-1} - \dot{\rho}_{n+1}\rho_{n-1} = \begin{cases} s\rho_n^2 & \text{for } n \text{ odd,}\\ \rho_n^2 & \text{for } n \text{ even,}\end{cases} \tag{93}$$

with initial conditions $\rho_0 = \rho_1 = 1$, where the dot denotes the derivative with respect to s.

Using (59), it was proved in [23] that $\rho_n(s)$ are monic polynomials with integer coefficients, which remain invariant up to an overall sign after switching $n \to -n$ and $s \to -s$, with the precise relation being

$$\rho_{-n}(s) = \begin{cases}(-1)^{\frac{n^2}{4}}\ \rho_n(-s) & \text{for even } n,\\ (-1)^{\frac{n^2-1}{4}}\rho_n(-s) & \text{for odd } n.\end{cases} \tag{94}$$

As a result of the latter property, it would be sufficient to state all subsequent results for $n \geq 0$ only and just use this symmetry to obtain the corresponding statements for negative n; however, for completeness, we usually present everything with both choices of sign.

By using the formulae (91) for even/odd n, the algebraic solutions of Equation (12) are manifestly rational functions of the variable ζ, being given in terms of ρ_n by

$$P_n = \begin{cases}\frac{1}{3\zeta}\rho_{n+1}(3\zeta^2)\rho_{n-1}(3\zeta^2)/\rho_n(3\zeta^2)^2 & \text{for } n \text{ odd}\\ \zeta\ \rho_{n+1}(3\zeta^2)\rho_{n-1}(3\zeta^2)/\rho_n(3\zeta^2)^2 & \text{for } n \text{ even.}\end{cases} \tag{95}$$

Most of the rest of the paper is devoted to the proof of our main result, which can be stated as follows:

Theorem 3. *For $n \in \mathbb{Z}$, the Ohyama polynomials $\rho_n(s)$ with $s = 3(z/2)^{\frac{2}{3}}$ are given in terms of Wronskian determinants by*

$$\rho_n = \begin{cases}3^{-1/4}(s/3)^{\frac{|n|-1}{4}}\ Wr(W_{\pm1},\ldots,W_n), & \text{for } n \text{ odd}\\ (s/3)^{\frac{|n|}{4}}\ \ Wr(W_{\pm1},\ldots,W_n), & \text{for } n \text{ even}\end{cases} \tag{96}$$

where the $\pm$ sign is given by $\delta = \mathrm{sgn}(n)$ and each Wronskian is taken in the variable z. The entries W_n are given by the following polynomials:

$$W_n = \sum_{k=0}^{2|n|-2} A_{|n|-1,k} \left(\frac{\pm s}{3} \right)^k \tag{97}$$

where the coefficients $A_{m,k}$ can be non-zero only for $k, m \in \mathbb{Z}_{\geq 0}$ and are specified by the recursion

$$\frac{k+1}{3} A_{m,k+1} - A_{m,k} + \frac{3\sqrt{3}}{k} A_{m-1,k-2} = 0,$$

together with

$$A_{m,2m} = \left(\frac{3\sqrt{3}}{2} \right)^m \frac{3^{\frac{1}{4}}}{m!}, \quad \text{and} \quad A_{m,0} = 0 \quad \text{for} \quad m \geq 1, \quad A_{0,k} = 0 \quad \text{for} \quad k \geq 1.$$

The preceding result is based on finding the Wronskian determinants θ_n, which encode the action of confluent Darboux transformations applied to the potential V_0, given by (85), together with the sequence of eigenfunctions, as in (88). The details of the proof are provided in the next section, while, in Section 6, this is followed by a discussion of the generating function for the entries of the Wronskian and its connection with the Lax pair for (12). Some further technical details about the intertwining operators of the Jordan chain are included in the appendix.

Before proceeding with the complete proof, it is instructive to start by considering the first few generalized eigenfunctions ψ_i which appear as entries in the Wronskians θ_n, and how to go about constructing them. After the trivial case $\theta_0 = 1$, the first case to consider is θ_1, which is a Wronskian of just one function, namely, $\psi_1 = \phi_0$, an ordinary eigenfunction (with eigenvalue zero) for the initial Schrödinger operator with potential V_0. Hence, we have simply

$$\theta_1 = \psi_1 = 3^{\frac{1}{4}} \left(\frac{z}{2} \right)^{\frac{1}{6}} e^{-\frac{3}{2}\left(\frac{z}{2}\right)^{\frac{2}{3}}} = 3^{\frac{1}{4}} \zeta^{\frac{1}{2}} e^{-\frac{3}{2}\zeta^2}, \tag{98}$$

and a solution of the Schrödinger equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_0 \right) \phi_0 = 0, \qquad \phi_0 = 3^{\frac{1}{4}} \zeta^{\frac{1}{2}} e^{-\frac{3}{2}\zeta^2},$$

with V_0 given by (85). Another independent solution of the same Schrödinger equation is provided by

$$\tilde{\phi}_0 = (\phi_{-1})^{-1} = 3^{\frac{1}{4}} \zeta^{\frac{1}{2}} e^{\frac{3}{2}\zeta^2}, \qquad Wr\big(\phi_0, (\phi_{-1})^{-1}\big) = \sqrt{3}.$$

The new potential obtained by applying a Darboux transformation with the eigenfunction ϕ_0 is

$$V_1 = V_0 + 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\log \phi_0) = \frac{-7 + 24\zeta^2 - 36\zeta^4}{144\zeta^6},$$

which is the same as the result of applying the BT T_+ to find P_1 and then using Formula (68). By construction, $(\phi_0)^{-1}$ is an eigenfunction of the Schrödinger operator with the latter potential, but we wish to construct another eigenfunction ϕ_1 such that

$$Wr\big(\phi_1, (\phi_0)^{-1}\big) = \sqrt{3}, \tag{99}$$

and write it as

$$\phi_1 = \frac{\theta_2}{\theta_1}$$

with $\theta_1 = \psi_1$ as before and θ_2 being a Wronskian, that is,

$$\theta_2 = \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{vmatrix},$$

where ψ_2 is to be determined. One way to go about this is to apply the BT T_+ once more to obtain the next potential V_2 (see Table 2), then integrate

$$V_2 = V_0 + 2(\log \theta_2)''$$

twice to find θ_2, and finally use the Wronskian formula for θ_2 to extract ψ_2 from another integral:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\psi_2}{\psi_1}\right) = \frac{\theta_2}{\psi_1^2} \implies \psi_2 \propto \left(\frac{z}{2}\right)^{\frac{1}{6}} e^{-\frac{3}{2}\left(\frac{z}{2}\right)^{\frac{2}{3}}} \left(\frac{9}{2}\left(\frac{z}{2}\right)^{\frac{4}{3}} + 3\left(\frac{z}{2}\right)^{\frac{2}{3}}\right) \tag{100}$$

(where we have ignored an overall choice of scale, and the addition of an arbitrary constant multiple of ψ_1).

However, a much more direct approach is to use the Jordan chain associated with the sequence of confluent Darboux transformations, upon which we will base the proof in the next section. Indeed, the condition (99) is equivalent to the Burchnall–Chaundy relation

$$-\theta_2' = \sqrt{3}\,\theta_1^2$$

(noting that $\theta_0 = 1$), and substituting $\theta_1 = \psi_1$ as before and θ_2 as a 2×2 Wronskian produces

$$-\begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{vmatrix}' = -\begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1'' & \psi_2'' \end{vmatrix} = \sqrt{3}\psi_1^2 \implies \psi_1\psi_2'' - \psi_2\psi_1'' = -\sqrt{3}\psi_1^2,$$

so that, from $\psi_1'' + V_0\psi_1 = 0$, we obtain the next step of the Jordan chain, namely, the generalized eigenfunction equation for ψ_2, that is,

$$\psi_2'' + V_0\psi_2 = -\sqrt{3}\psi_1.$$

Then, the latter inhomogeneous equation can be solved by variation of parameters, using two independent solutions of the homogeneous problem, namely, $\psi_1 = \phi_0$ and $\tilde{\psi}_1 = (\phi_{-1})^{-1}$, leading to

$$\psi_2 = \psi_1\left(\int \tilde{\psi}_1\psi_1\,\mathrm{d}z\right) - \tilde{\psi}_1\left(\int \psi_1^2\,\mathrm{d}z\right) = 3^{-\frac{1}{4}}\zeta^{\frac{1}{2}}e^{-\frac{3}{2}\zeta^2}\left(\tfrac{9}{2}\zeta^4 + 3\zeta^2 + 1 + c_1\right).$$

Note that the integration constant c_1 in the first integral above can be ignored, because adding constant multiples of ψ_1 to ψ_2 makes no difference to $Wr(\psi_1, \psi_2)$, but we can set $c_1 = -1$ so that it agrees with (100). Meanwhile, another integration constant in the second integral has been suppressed, because it corresponds to adding a multiple of $\tilde{\psi}_1$ with the wrong type of exponential factor $e^{\frac{3}{2}\zeta^2}$. We shall return to this point in due course. Some other examples of algebraic solutions, potentials, eigenfunctions, and Wronskians θ_n for a small n are listed in Table 2.

5. Proof of Main Theorem

In this section, we present the proof of Theorem 3, based on the properties of confluent Darboux transformations and the associated generalized eigenfunctions, forming a Jordan chain, in the context of the sequence of potentials V_n associated with the algebraic solutions of the Painlevé III (D_7) equation. Since we are considering the effect of the BTs (62) applied successively to the algebraic seed solution $P_0 = \zeta$ for $\beta = 0$, and the corresponding sequence of solutions P_n with parameter $\beta = 2n$ for $n \in \mathbb{Z}$, it is convenient to use ψ_n to denote each member of an associated set of generalized eigenfunctions, labeled by the same integer n. Thus, after $|n|$ applications of a confluent Darboux tranformation applied to the Schrödinger operator with potential V_0 given by (85), we find two different sequences of generalized eigenfunctions, namely, $\psi_\delta, \ldots, \psi_n$, where $\delta = \pm 1 = \mathrm{sgn}(n)$ (so one for positive values of n, the other for negative n). This allows us to write Wronskian formulae for the eigenfunctions (with eigenvalue zero) of each of the Schrödinger operators with potential V_n, which are valid for all $n \in \mathbb{Z}$. Hence, we have

$$\phi_n = \frac{\theta_{n+1}}{\theta_n}, \qquad \text{with} \quad \theta_n = Wr(\psi_\delta, \ldots, \psi_n),$$

according to the choice of sign δ, where

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_n\right)\phi_n = 0$$

holds for each integer n.

The main idea of the proof is to use the Frobenius method to obtain an explicit recursion for the generalized eigenfunctions ψ_n, which (up to an exponential factor and a prefactor of $\sqrt{\zeta}$) are given by polynomials in ζ^2. However, before we proceed with this, it is convenient to state a slightly technical result on the coefficients that will appear in the Frobenius expansion, which reflects the fact that the whole construction has a certain symmetry property under exchanging $n \leftrightarrow -n$. (This can also be seen from the property (94) of the Ohyama polynomials.)

Lemma 4. *Let coefficients $A_{m,k}$, labeled by integers m, k with $k \geq 0$, be generated recursively by the relations*

$$\begin{array}{rcl} A_{m,k} & = & (3a/k)A_{m-\delta,k-2} + \frac{k+1}{3}\delta A_{m,k+1} \quad \textit{for} \quad m \neq 0,\ k \geq 2, \\ A_{m,1} & = & \frac{2}{3}\delta A_{m,2} \end{array} \tag{101}$$

with boundary conditions

$$A_{m,0} = 0 \quad \textit{for} \quad m \neq 0, \qquad A_{m,2|m|} = \left(\frac{3a}{2}\right)^{|m|} \frac{c}{|m|!}, \quad \textit{for all} \quad m, \tag{102}$$

where $\delta = \mathrm{sgn}(n)$ and $a, c \in \mathbb{C}$ are arbitrary non-zero parameters. Then, the coefficients satisfy

$$A_{m,k} = (-1)^k A_{-m,k} \tag{103}$$

for all $m \in \mathbb{Z}$.

Proof. The result follows by a straightforward induction on $|m|$. First of all, the result is trivially true for $m = 0$, with $A_{0,0} = c$. Now we wish to show by induction that, for each $m \neq 0$, the recursion relations (101) and boundary values (102) completely determine the coefficients $A_{m,k}$ for all $k \geq 0$, and, moreover, that $A_{m,k} = 0$ when $k > 2|m|$. When $m > 0$, so $\delta = 1$, it is convenient to start by taking $k = 2m$ in (101), in which case the boundary values $A_{m,2m}$ and $A_{m-1,2m-2}$ give

$$\left(\frac{3a}{2}\right)^m \frac{c}{m!} = \left(\frac{3a}{2m}\right)\left(\frac{3a}{2}\right)^{m-1} \frac{c}{(m-1)!} + \frac{2m+1}{3} A_{m,2m+1}$$

which implies that $A_{m,2m+1} = 0$. Thus, by increasing the value of k and using the inductive hypothesis, we see that $A_{m,k} = 0$ for all $k \geq 2m + 1$, as required. Hence, by decreasing k at each step, we can use the recursion (101) to determine the other (non-zero) coefficients $A_{m,k}$ for $1 \leq k \leq 2m - 1$, while $A_{m,0} = 0$ is fixed by (102). The coefficients $A_{m,k}$ with $m < 0$ are determined similarly by the recursion with $\delta = -1$. Then, the statement is trivially true for $k > 2|m|$ and $k = 0$, and, for other values of k, it follows from the fact that (101) and the boundary conditions are left invariant under replacing $A_{m,k} \to (-1)^k A_{-m,k}$, $\delta \to -\delta$. $\square$

We are now ready to present generalized eigenfunctions ψ_n for the potential V_0 given by (85), associated with the algebraic seed solution of the Painlevé III (D_7) equation, which is given in terms of ζ by $P_0 = \zeta$.

Proposition 2. *For $n \in \mathbb{Z} \setminus \{0\}$, ψ_n, defined by*

$$\psi_n = \zeta^{\frac{1}{2}} e^{-\frac{3}{2}\delta\zeta^2} \sum_{k=0}^{2|n|-2} A_{|n|-1,k} (\delta\zeta^2)^k \tag{104}$$

where $\delta = \operatorname{sgn}(n) = \pm 1$, $a, c \in \mathbb{C}$ are arbitrary non-zero constants, and the coefficients $A_{m,k}$ are subject to the relations (101) and (102), is a generalized eigenfunction for the potential V_0 satisfying

$$\psi_n'' + V_0 \psi_n = -\delta a \psi_{n-\delta}, \tag{105}$$

where $'$ denotes the derivative with respect to z and $\psi_0 = 0$.

Proof. Firstly, considering the case $n > 0$, changing variables from z to ζ in (105), writing

$$\psi_n(z) = \hat{\psi}_n(\zeta)$$

and substituting in the Formula (85) for V_0 in terms of ζ gives

$$4\zeta^2 \hat{\psi}_n'' - 8\zeta \hat{\psi}_n' - 36\zeta^4 \hat{\psi}_n + 5\hat{\psi}_n = -144a\zeta^6 \hat{\psi}_{n-1}, \tag{106}$$

where the primes on $\hat{\psi}_n$ denote derivatives with respect to the argument ζ. We define $w_n = w_n(\zeta)$ for $n \geq 0$ by

$$\hat{\psi}_n(\zeta) = \zeta^{\frac{1}{2}} e^{-\frac{3}{2}\zeta^2} w_{n-1}(\zeta), \tag{107}$$

where shifting down the index to w_{n-1} makes certain powers of ζ easier to keep track of later, and we set $w_{-1} = 0$ to be consistent with the convention that $\psi_0 = 0$. This simplifies the modified form of the Jordan chain to be

$$\zeta^2 \frac{\mathrm{d}^2 w_n}{\mathrm{d}\zeta^2} - \left(6\zeta^3 + \zeta\right) \frac{\mathrm{d}w_n}{\mathrm{d}\zeta} + 36a\zeta^6 w_{n-1} = 0, \tag{108}$$

Taking the initial constant solution $w_0 = c \neq 0$ when $n = 0$ in (108), we may follow Frobenius' method by taking a series solution

$$w_n = \sum_{k=0}^{\infty} A_{n,k} \zeta^k, \tag{109}$$

which, when substituted into the preceding relation between w_n and w_{n-1}, produces

$$\sum_{k=0}^{\infty} k(k-2) A_{n,k} \zeta^{k-2} - 6 \sum_{k=0}^{\infty} k A_{n,k} \zeta^k + 36a \sum_{k=0}^{\infty} A_{n-1,k} \zeta^{k+4} = 0. \tag{110}$$

Shifting $k \to k-2$ and $k \to k-6$, and collecting terms with the same power of ζ in (110), we begin by explicitly calculating the coefficients for the first six values of k in the sum. As each coefficient must vanish, we see that $A_{n,0}$ and $A_{n,2}$ can be freely chosen, while $A_{n,1}$, $A_{n,3}$, and $A_{n,5}$ must all be zero, and, for $k=5$, we have that $A_{n,4} = \frac{3}{2}A_{n,2}$. The remaining infinite sum gives the relations defining $A_{n,k}$ for $k \geq 6$, which are obtained as

$$A_{n,k} = \frac{6}{k}A_{n,k-2} - \frac{36a}{k(k-2)}A_{n-1,k-6}.$$

From this and $A_{n,1} = A_{n,3} = A_{n,5} = 0$, we can see that, for all odd k values, $A_{n,k} = 0$. We also choose to take $A_{n,0} = 0$ for $n \neq 0$, since including a non-zero $A_{n,0}$ is equivalent to adding a multiple of the homogeneous solution ψ_1. (In the context of the Wronskian form of the solution with entries ψ_n, this amounts to adding one column onto another, having no effect on the determinant.)

From the examples of ψ_1 and ψ_2 calculated previously, we would expect there to be some choice of $A_{n,2}$ which truncates this infinite sum, which follows from (108) if we assume that the w_n are polynomials in ζ. If w_n and w_{n-1} are polynomials of degree m and p, respectively, then, in the relation between them, the highest powers appearing are $m+2$ and $p+6$; hence, in order for them to agree, we require $m = p+4$. Thus, if the w_n are a sequence of polynomials, each of degree 4 greater than the previous one, then the fact that $w_0 = c$ has degree 0 implies that w_n must be of degree $4n$, and so we must have $A_{n,k} = 0$ for $k > 4n$. All greater odd k terms are zero automatically, so the first that must be set to zero is

$$0 = A_{n,4n+2} = \frac{6}{4n+2}A_{n,4n} - \frac{36a}{4n(4n+2)}A_{n-1,4n-4} \implies A_{n,4n} = \frac{3a}{2n}A_{n-1,4n-4}.$$

By induction, setting $A_{n,4n+2} = 0$ also sets $A_{n,k} = 0$ for all higher k values as well, as they only rely linearly on it and on values of $A_{n-1,k}$ beyond the point at which all these coefficients are fixed to zero. Since $A_{0,0} = c = w_0$, iterating the above recursion for $A_{n,4n}$ in terms of $A_{n-1,4n-4}$ gives

$$A_{n,4n} = \left(\frac{3a}{2}\right)^n \frac{c}{n!}.$$

The relation for $A_{n,k}$ can also be rearranged to step down in k instead of up in the form

$$A_{n,k} = \frac{k+2}{6}A_{n,k+2} + \frac{6a}{k}A_{n-1,k-4},$$

and so, for a given n, starting from the above formula for $A_{n,4n}$, this relation is used to find all terms $A_{n,k}$ with a lower k all the way down to $k=2$, with the inductive assumption that the previous terms $A_{n-1,k}$ have been found already. This produces a sequence of polynomials $w_n = \sum_{k=0}^{n} A_{n,k}\zeta^{2k}$ for $n \geq 0$ (with no odd powers of ζ in the sum), whose coefficients are determined by (101) and (102), and, as a consequence of (103), we also have

$$A_{0,k} = 0 \qquad \text{for} \quad k \geq 1.$$

This completes the proof of the result on the generalized eigenfunctions ψ_n for $n > 0$, corresponding to the statement with $\delta = 1$.

For the case of negative n, we instead let

$$\psi_n(z) = \hat{\psi}_n(\zeta) = \zeta^{\frac{1}{2}} e^{\frac{3}{2}\zeta^2} u_{n+1}(\zeta), \tag{111}$$

and then substituting into the Jordan chain Equation (106) with the replacement $a \to -a$ (corresponding to $\delta = -1$) results in a relation for u_n, which is subtly different from (108), namely,

$$\zeta^2 \frac{\mathrm{d}^2 u_n}{\mathrm{d}\zeta^2} + \left(6\zeta^3 - \zeta\right) \frac{\mathrm{d}u_n}{\mathrm{d}\zeta} - 36a\zeta^6 u_{n+1} = 0,$$

where we set $u_1 = 0$ to ensure the validity of the above when $n = 0$. Mutatis mutandis, using the symmetry properties of the coefficients $A_{n,k}$ as in Lemma 4, the rest of the proof proceeds by induction in the same way as for the case of positive n. Hence, we obtain the expression (104) for the sequence of generalized eigenfunctions of the Schrödinger operator with potential V_0, valid for both choices of the sign $\delta = \mathrm{sgn}(n) = \pm 1$. □

The application of Frobenius' method in the above proof shows that the factors w_n in (107) and u_n in (111) are given by infinite series in general, as different choices of the coefficients $A_{n,0}$ and $A_{n,1}$ are allowed at each step, corresponding to the freedom to add arbitrary multiples of ψ_k with $|k| < |n|$ to each generalized eigenfunction ψ_n. While such choices lead to valid Darboux transformations of the potential V_0, they do not result in the correct forms for the Wronskians θ_n to produce the Ohyama polynomials, as required by Theorem 3, whose proof will be presented shortly.

Remark 3. *It is worth commenting on the choice of scaling at this stage, as reflected in the parameters a, c. The constant c determines the choice of scaling for ψ_1, the initial eigenfunction for the Schrödinger operator with potential V_0, while the parameter a corresponds to the Wronskian between pairs of eigenfunctions generated by subsequent iterations of the confluent Darboux transformation (see Appendix A for more details). In order to be consistent with [21], we usually fix $a = \sqrt{3}$ and $c = 3^{\frac{1}{4}}$, e.g., in the statement of Theorem 3. However, other choices make the form of the coefficients $A_{n,k}$ and the relations between them somewhat simpler. For instance, the choices $c = 1$ and $a = \frac{2}{3}$ can be taken, which gives*

$$\begin{aligned}
A_{n,2n} &= \frac{1}{n!}, \\
A_{n,2n-1} &= \frac{2}{9}(2n+1)\frac{1}{(n-1)!}, \\
A_{n,2n-2} &= \frac{2}{81}(4n^2+10n+9)\frac{1}{(n-2)!}, \\
A_{n,2n-3} &= \frac{4}{2187}(2n-1)(4n^3+18n^2+23n-15)\frac{1}{(n-2)!}, \\
A_{n,2n-4} &= \frac{2}{19683}(16n^5+128n^4+356n^3+292n^2+513n+1125)\frac{1}{(n-3)!},
\end{aligned}$$

and so on, which appear to be the most concise explicit forms of these coefficients. It is also possible to produce an exact generating function Ψ for the generalized eigenfunctions that appear as entries in the Wronskians. In the case of $\delta = +1$, upon multiplying (105) by λ^{n-1} and summing from $n = 1$ to ∞, we find

$$\Psi'' + V_0 \Psi = -a\lambda\Psi,$$

which is a Schrödinger equation with a non-zero eigenvalue term. After fixing the scale so that $a = -1$, an explicit formula for Ψ is given in the next section (see Proposition 4).

We now complete the proof of our main result on Ohyama polynomials.

Proof of Theorem 3. Recall from the results of Lemma 2, Proposition 1, and Lemma 3, that the Ohyama polynomials ρ_n are equivalent (up to some scale factors) to a sequence of tau functions σ_n for the Painlevé III (D_7) equation. In turn, via the formula

$$V_n = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\sigma_n, \qquad n \in \mathbb{Z},$$

these tau functions correspond to a sequence of Schrödinger potentials obtained from successive application of confluent Darboux transformations applied to the initial potential V_0, with one Jordan chain for $n \geq 0$ and another for $n \leq 0$. General results on confluent Darboux transformations (which are described in more detail in the appendix) imply that the tau functions and new eigenfunctions obtained via this process can be written in terms of quantities θ_n, as in (87) and (88), where

$$\theta_n = Wr(\psi_\delta, \ldots, \psi_n)$$

are Wronskians of generalized eigenfunctions ψ_j. Moreover, the Burchnall–Chaundy relation (83) fixes a choice of normalization for the new eigenfunction introduced at each stage.

The problem is then to specify the explicit choices of the generalized eigenfunctions that appear as entries in the Wronskians θ_n. The proof of Proposition 2 shows that, in general, these have the form

$$\psi_n = \zeta^{\frac{1}{2}}e^{-\frac{3}{2}\delta\zeta^2}W_n(\zeta),$$

(with $W_n = w_{n-1}$ for $n > 0$, $W_n = u_{n+1}$ for $n < 0$, respectively), where each W_n is given by an infinite series in ζ, unless suitable coefficients are fixed to be zero, in which case it is the polynomial

$$W_n = \sum_{k=0}^{2|n|-2} A_{|n|-1,k}(\delta\zeta^2)^k. \tag{112}$$

Using the Wronskian identity

$$Wr(g\,f_1, g\,f_2, \ldots, g\,f_n) = g^n\, Wr(f_1, f_2, \ldots, f_n),$$

we can extract all the prefactors $g = \zeta^{\frac{1}{2}}e^{-\frac{3}{2}\delta\zeta^2}$ to obtain the formula

$$\theta_n = Wr(\psi_\delta, \ldots, \psi_n) = \zeta^{\frac{|n|}{2}}e^{-\frac{3}{2}n\zeta^2}Wr(W_\delta, \ldots, W_n). \tag{113}$$

On the other hand, using (87) and (91) to express θ_n in terms of ρ_n, we find

$$\theta_n = \begin{cases} 3^{1/4}\zeta^{\frac{1}{2}}e^{-\frac{3}{2}n\zeta^2}\rho_n(3\zeta^2) & \text{for } n \text{ odd,} \\ e^{-\frac{3}{2}n\zeta^2}\rho_n(3\zeta^2) & \text{for } n \text{ even.} \end{cases} \tag{114}$$

By comparing (113) with (114), we obtain

$$\rho_n = \begin{cases} 3^{-\frac{1}{4}}\zeta^{\frac{|n|-1}{2}}Wr(W_\delta, \ldots, W_n) & \text{for } n \text{ odd,} \\ \zeta^{\frac{|n|}{2}}Wr(W_\delta, \ldots, W_n) & \text{for } n \text{ even,} \end{cases} \tag{115}$$

where the Wronskian entries are as in (112). Converting (115) to the variable $s = 3\zeta^2$ gives the statement of the theorem. □

Remark 4. *The formula (115) does not make all the properties of the Ohyama polynomials obvious. In particular, the presence of the derivatives* $\frac{d}{dz} = \frac{1}{6\zeta^2}\frac{d}{d\zeta}$ *in the Wronskian means that it can be inferred immediately that*

$$\rho_n = \zeta^{\ell_n} D_n(\zeta), \tag{116}$$

for some $\ell_n \in \mathbb{Z}$*, where* D_n *is a polynomial in* ζ *with a non-zero constant term, and it is also easy to see that* ρ_n *is an even function of* ζ*. However, it is not clear why* $\ell_n \geq 0$*, so the best conclusion that can be made from (116) is that* ρ_n *is a Laurent polynomial in* $s = 3\zeta^2$*. Furthermore, it is not immediately apparent why* $\rho_n(s)$ *should have integer coefficients, which is another feature of the Ohyama polynomials, with the normalization chosen as in [21].*

6. Lax Pair for Painlevé III (D_7) and Generating Functions

In this section, we consider a Lax pair for the Painlevé III (D_7) equation, and present a particular solution to this linear system. This solution is then used to derive a generating function for the generalized eigenfunctions ψ_i which appear as Wronskian entries in the tau functions for the algebraic solutions.

A scalar Lax pair for the Painlevé III (D_7) Equation (12) is given by

$$\begin{cases} \Psi_{zz} + V\Psi = \lambda\Psi, \\ \Psi_\lambda = \frac{1}{2}\left(\frac{z}{\lambda} - \frac{P}{\lambda^2}\right)\Psi_z - \frac{1}{4}\left(\frac{1}{\lambda} - \frac{P'}{\lambda^2}\right)\Psi, \end{cases} \tag{117}$$

where V is defined in terms of P and P' (its z derivative) by (68). With V specified in this way, the compatibility condition $\Psi_{zz\lambda} = \Psi_{\lambda zz}$ produces (12) directly. Alternatively, one can start with V, P as unspecified functions, and then the compatibility condition yields the system

$$\begin{aligned} P''' + 4VP' + 2V'P &= 0, \\ P' + \tfrac{z}{2}V' + V &= 0, \end{aligned}$$

from which both (12) and (68) can be derived, with the parameter β appearing as an integration constant. This Lax pair can be derived by applying the similarity reduction (66), as found in [16] and used in [17], to the Lax pair for the Camassa–Holm equation, which is related via a reciprocal (hodograph-type) transformation to the Lax pair for the first negative KdV flow [25], that is,

$$\begin{cases} \Phi_{XX} + \bar{V}\Phi = \xi\Phi, \\ \Phi_T = A\Phi_X - \frac{1}{2}A_X\Phi, \end{cases} \tag{118}$$

where ξ is the spectral parameter and

$$\bar{V} = -\frac{p_{XX}}{2p} + \frac{p_X^2}{4p^2} - \frac{1}{4p^2}, \qquad A = \frac{p}{2\xi}.$$

We can simultaneously perform a scaling similarity reduction on the coefficients and the wave function of this Lax pair, by setting

$$p(X,T) = T^{-1/2}P(z), \qquad z = XT^{1/2}, \qquad \Phi(X,T;\xi) = \lambda^{1/4}\Psi(z,\lambda), \qquad \lambda = \xi T^{-1},$$

which transforms the linear system (118) into the Lax pair (117) above.

Recently, Buckingham and Miller [18] presented a Riemann–Hilbert representation of the algebraic solutions of the Painlevé III (D_7) equation which involved solving the linear system coming from an alternative Lax pair of the Jimbo–Miwa type in terms of Airy functions. Since (117) is connected to the Jimbo–Miwa Lax pair by a gauge transformation, we can follow their lead somewhat and see that the wave function Ψ can be solved in terms of Airy functions when we take the seed solution $P_0 = (z/2)^{1/3}$.

Proposition 3. *The Lax pair for the algebraic seed solution of Painlevé III (D_7) with $\beta = 0$, given by the linear system (117), has the general solution*

$$\Psi = \lambda^{-\frac{1}{6}}\zeta^{\frac{1}{2}}\left[c_1\,\mathrm{Ai}\left(\left(\frac{9}{\lambda^2}\right)^{\frac{1}{3}}\left(\frac{1}{4}+\lambda\zeta^2\right)\right)+c_2\,\mathrm{Bi}\left(\left(\frac{9}{\lambda^2}\right)^{\frac{1}{3}}\left(\frac{1}{4}+\lambda\zeta^2\right)\right)\right], \tag{119}$$

for arbitrary constants c_1, c_2.

Proof. For the algebraic solution of Painlevé III (D_7), we substitute V_0 from (85) and $P_0 = (z/2)^{1/3} = \zeta$ into the Lax pair. Then, by rewriting the first equation in (117) in terms of ζ, we obtain

$$4\zeta^2\Psi_{\zeta\zeta} - 8\zeta\Psi_\zeta - (36\zeta^4 - 5)\Psi = 144\lambda\zeta^6\Psi,$$

and, making a further change of variables to $w = \frac{1}{4} + \lambda\zeta^2$, we obtain

$$16\left(w-\frac{1}{4}\right)^2\Psi_{ww} - 8\left(w-\frac{1}{4}\right)\Psi_w = \left(144(w-\tfrac{1}{4}) + 36\lambda^{-2}(w-\tfrac{1}{4})^2 - 5\right)\Psi.$$

To clean up the right-hand side of the preceding equation, we set

$$\Psi = \left(w-\frac{1}{4}\right)^{\frac{1}{4}}\vartheta(w) = \lambda^{\frac{1}{4}}\zeta^{\frac{1}{2}}\vartheta\left(\frac{1}{4}+\lambda\zeta^2\right),$$

which gives a scaled Airy equation for $\vartheta(w)$, that is,

$$\vartheta_{ww} = \left(\frac{9}{\lambda^2}w\right)\vartheta,$$

with general solution

$$\vartheta(w) = \tilde{A}\,\mathrm{Ai}\left(\left(\frac{9}{\lambda^2}\right)^{\frac{1}{3}}w\right) + \tilde{B}\,\mathrm{Bi}\left(\left(\frac{9}{\lambda^2}\right)^{\frac{1}{3}}w\right)$$

where $\mathrm{Ai}(w), \mathrm{Bi}(w)$ are the standard Airy functions and $\tilde{A}, \tilde{B}$ are independent of w but otherwise arbitrary. Hence, the solution of the first part of the Lax pair is given by

$$\Psi = \lambda^{\frac{1}{4}}\zeta^{\frac{1}{2}}\left[\tilde{A}(\lambda)\,\mathrm{Ai}\left(\left(\frac{9}{\lambda^2}\right)^{\frac{1}{3}}\left(\frac{1}{4}+\lambda\zeta^2\right)\right)+\tilde{B}(\lambda)\,\mathrm{Bi}\left(\left(\frac{9}{\lambda^2}\right)^{\frac{1}{3}}\left(\frac{1}{4}+\lambda\zeta^2\right)\right)\right], \tag{120}$$

where $A(\lambda), B(\lambda)$ are arbitrary functions of λ (that do not depend on ζ). We can fix these coefficients using terms via the second part of the Lax pair to obtain a general solution. Substituting the solution (120) into the second equation in (117) gives an equation in terms of Ai, Bi and their derivatives Ai$'$, Bi$'$. The derivative terms cancel identically, and hence place no restriction on the coefficients $A(\lambda), B(\lambda)$. The remaining part can be written concisely as

$$\lambda^{\frac{1}{4}}\zeta^{\frac{1}{2}}\left(A'(\lambda)\,\mathrm{Ai}\left(\left(\frac{9}{\lambda^2}\right)^{\frac{1}{3}}\left(\frac{1}{4}+\lambda\zeta^2\right)\right)+B'(\lambda)\,\mathrm{Bi}\left(\left(\frac{9}{\lambda^2}\right)^{\frac{1}{3}}\left(\frac{1}{4}+\lambda\zeta^2\right)\right)\right) = -\frac{5}{12}\lambda^{-1}\Psi,$$

which yields separate equations relating the coefficients in front of the independent functions Ai and Bi appearing on each side. For the coefficient of Ai, we obtain

$$A'(\lambda) = -\frac{5}{12\lambda}A(\lambda) \implies A(\lambda) = c_1\lambda^{-\frac{5}{12}},$$

and, likewise for the coefficient of Bi, we obtain the same linear ODE so that $B(\lambda) = c_2\lambda^{-\frac{5}{12}}$ for arbitrary constants c_1, c_2. Hence, overall, we have the solution (119), in terms of the standard Airy functions Ai, Bi, as required. □

Although the independent variable in (12) is z, it is more convenient for the algebraic solutions to use ζ instead, so we have written $\Psi(\zeta, \lambda)$ in terms of the latter variable in (119). Returning to the consideration of the generalized eigenfunctions ψ_i, which are entries of the Wronskians θ_n, we noted previously that a generating function Ψ for these entries would be a solution of the Schrödinger equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_0\right)\Psi = \lambda\Psi, \tag{121}$$

with V_0 given by (85), which is exactly the z part of the Lax pair for Painlevé III (D_7). Therefore, from the proof of the preceding proposition, we see that such a Ψ must be of the form (120), leading us to the following result:

Proposition 4. *For the generalized eigenfunctions ψ_i with $i > 0$ of the algebraic potential V_0 given by (85), the generating function*

$$\Psi = \sum_{n=1}^{\infty} \psi_n \lambda^{n-1} \tag{122}$$

is given by

$$\Psi = 3^{\frac{5}{12}}\lambda^{-\frac{1}{6}}e^{\frac{1}{4\lambda}}\sqrt{2\pi\zeta}\ \mathrm{Ai}\left(\left(\frac{9}{\lambda^2}\right)^{\frac{1}{3}}\left(\frac{1}{4} + \lambda\zeta^2\right)\right), \tag{123}$$

while the generating function

$$\tilde{\Psi} = \sum_{n=1}^{\infty} \psi_{-n}\lambda^{n-1}$$

for the generalized eigenfunctions ψ_i with $i < 0$ is given by

$$\tilde{\Psi} = 3^{\frac{5}{12}}\lambda^{-\frac{1}{6}}e^{-\frac{1}{4\lambda}}\sqrt{\frac{\pi\zeta}{2}}\ \mathrm{Bi}\left(\left(\frac{9}{\lambda^2}\right)^{\frac{1}{3}}\left(\frac{1}{4} + \lambda\zeta^2\right)\right). \tag{124}$$

Proof. Since (122) is a solution of the Schrödinger equation (121), we need to choose the coefficients $\tilde{A}(\lambda), \tilde{B}(\lambda)$ in (120) so that we have a solution with the correct asymptotic behavior written as $\lambda \to 0$, which means considering the asymptotics of the Airy functions as their argument goes to infinity. From the results in Section §9.7(ii) of [26], the relevant asymptotic expansions are

$$\mathrm{Ai}\left(\left(\frac{3\xi}{2}\right)^{\frac{2}{3}}\right) \sim \frac{e^{-\xi}}{2^{\frac{5}{6}}3^{\frac{1}{6}}\sqrt{\pi}\,\xi^{\frac{1}{6}}}\left(1 + \sum_{k=1}^{\infty}\frac{(-1)^k u_k}{\xi^k}\right), \qquad \mathrm{Bi}\left(\left(\frac{3\xi}{2}\right)^{\frac{2}{3}}\right) \sim \frac{2^{\frac{1}{6}}e^{\xi}}{3^{\frac{1}{6}}\sqrt{\pi}\,\xi^{\frac{1}{6}}}\left(1 + \sum_{k=1}^{\infty}\frac{u_k}{\xi^k}\right), \tag{125}$$

each valid as $\xi \to \infty$ in suitable sectors of the ξ plane (including the positive real axis), for a certain sequence of rational numbers u_k. In the case at hand, we have

$$\xi = \frac{1}{4\lambda}(1 + 4\lambda\zeta^2)^{\frac{3}{2}} = \frac{1}{4\lambda} + \frac{3}{2}\zeta^2 + O(\lambda) \quad \text{as } \lambda \to 0,$$

so, to leading order, we have

$$\mathrm{Ai}\left(\left(\frac{3\zeta}{2}\right)^{\frac{2}{3}}\right) \sim c\, e^{-\frac{1}{4\lambda}-\frac{3}{2}\zeta^2}\lambda^{\frac{1}{6}}(1+O(\lambda)), \quad \mathrm{Bi}\left(\left(\frac{3\zeta}{2}\right)^{\frac{2}{3}}\right) \sim \tilde{c}\, e^{\frac{1}{4\lambda}+\frac{3}{2}\zeta^2}\lambda^{\frac{1}{6}}(1+O(\lambda)),$$

for certain constants $c, \tilde{c}$. Hence, if we choose $\tilde{A}(\lambda)$ to cancel out the leading-order factors $e^{-\frac{1}{4\lambda}}\lambda^{\frac{1}{6}}$ in Ai, as well as fix the appropriate normalization, and set $\tilde{B}(\lambda) = 0$ in (120), then we find a solution Ψ of the Schrödinger equation given by (123), with leading-order behavior

$$\Psi = 3^{\frac{1}{4}}\zeta^{\frac{1}{2}}e^{-\frac{3}{2}\zeta^2} + o(1) \sim \psi_1 \quad \text{as } \lambda \to 0.$$

Moreover, because this Ψ satisfies (121), the coefficients of its expansion (122) in powers of λ generate precisely the positive Jordan chain (105) with $\delta = 1$ (up to the choice of normalization constant a, which can be adjusted by rescaling λ). Furthermore, from the dependence of Ψ on ζ and the given asymptotic expansion of Ai at infinity, it is clear that the coefficient ψ_n of each power of λ is a polynomial in ζ^2 multiplied by the prefactor $\zeta^{\frac{1}{2}}e^{-\frac{3}{2}\zeta^2}$, as required. Similarly, by choosing $\tilde{A}(\lambda) = 0$ and fixing $\tilde{B}(\lambda)$ appropriately, we obtain the solution (124) with

$$\tilde{\Psi} = 3^{\frac{1}{4}}\zeta^{\frac{1}{2}}e^{\frac{3}{2}\zeta^2} + o(1) \sim \psi_{-1} \quad \text{as } \lambda \to 0,$$

which generates the generalized eigenfunctions in the negative Jordan chain (105) with $\delta = -1$. □

Observe that, compared with (119), the factors $e^{\pm\frac{1}{4\lambda}}$ appearing in the generating functions $\Psi, \tilde{\Psi}$ above mean that these cannot also satisfy the second (λ flow) part of the Lax pair for Painlevé III (D_7). Also, it is worth mentioning that the coefficients ψ_i obtained by expanding these generating functions do not all correspond precisely to the generalized eigenfunctions determined in Proposition 2, due to the freedom to add on multiples of lower terms in the Jordan chain at each step (which makes no difference to the sequence of their Wronskians). It is instructive to see how this works in the case of the negative part of Jordan chain (105) with $\delta = -1$. The expansion of Bi in (125) can be rewritten in a more precise form with gamma functions, taking the argument $t \to \infty$ with $|\arg(t)| < \frac{\pi}{3}$, as follows:

$$\mathrm{Bi}(t) \sim \frac{e^{\frac{2}{3}t^{\frac{3}{2}}}}{\sqrt{\pi}t^{\frac{1}{4}}}\sum_{n=0}^{\infty}\left[\frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{2\pi n! t^{\frac{3}{2}n}}\left(\frac{3}{4}\right)^n\right].$$

Using this expression allows the first few terms in the expansion of $\tilde{\Psi}$ as $\lambda \to 0$ to be found explicitly as

$$\tilde{\Psi} \sim 3^{\frac{1}{4}}\zeta^{\frac{1}{2}}e^{\frac{3}{2}\zeta^2}\left[1 + \left(\frac{3}{2}\zeta^4 - \zeta^2 + \frac{5}{18}\right)\lambda + \left(\frac{9}{8}\zeta^8 - \frac{5}{2}\zeta^6 + \frac{35}{12}\zeta^4 - \frac{35}{18}\zeta^2 + \frac{385}{648}\right)\lambda^2 + O(\lambda^3)\right]. \quad (126)$$

The leading-order term (coefficient of λ^0) has been fixed as the normalized eigenfunction $\psi_{-1} = 3^{\frac{1}{4}}\zeta^{\frac{1}{2}}e^{\frac{3}{2}\zeta^2}$, but, reading off the next term, the coefficient of λ gives the generalized eigenfunction

$$\psi_{-2} = 3^{\frac{1}{4}}\zeta^{\frac{1}{2}}e^{\frac{3}{2}\zeta^2}\left(\frac{3}{2}\zeta^4 - \zeta^2 + \frac{5}{18}\right),$$

which differs from the formula for ψ_{-2} in (104) in two ways: first, by virtue of the fact that, there, the lowest coefficient $A_{1,0} = 0$, whereas, above the final non-zero coefficient, $\frac{5}{18}$ is included, corresponding to the freedom to add a multiple of ψ_{-1} to ψ_{-2}; and, second, by an overall factor of $-\sqrt{3}$, since Theorem 3 has been stated with the convention that

$a = \sqrt{3}$ in (105), whereas the form of the Equation (121) for Ψ corresponds to the choice $a = -1$. Similar considerations apply to the form of ψ_{-3}, given by the coefficient of λ^2 in the above expansion of $\tilde{\Psi}$, which can be simplified by subtracting linear combinations of ψ_{-1} and ψ_{-2}.

Remark 5. *It is interesting to note that the Airy function also appears in two different ways in the theory of Painlevé II, as observed in [27]. Firstly, (2) has special classical solutions given in terms of Airy functions when the parameter ℓ is an integer (so α is a half-integer). Secondly, the rational solutions have a different determinantal representation from the one in Theorem 1, in terms of Hankel determinants, and, in that context, the Airy function* Ai *arises in the generating function of the entries of the Hankel matrix. We should also like to point out that yet another alternative representation for the tau functions of these rational solutions was found recently, in terms of Gram determinants [28]. Very recently, a broad class of Airy function solutions of the KP equation was constructed using Grammians [29].*

7. Conclusions

We have shown how the application of confluent Darboux transformations leads to Wronskian representations for algebraic solutions of both the Painlevé II equation and the Painlevé III (D_7) equation. It should be apparent from our presentation that there is nothing inherently special about the algebraic solutions, in the sense that every solution of these equations is connected with a Jordan chain of generalized eigenfunctions for a sequence of Schrödinger operators, linked to one another by the repeated action of a confluent Darboux transformation. Hence, every sequence of solutions of these Painlevé equations, related to one another by iterated BTs, admits a formal Wronskian representation. The reason we use the word "formal" here is that the generalized eigenfunctions, which appear as entries in the Wronksians, are typically higher transcendental functions, constructed from a Schrödinger operator whose potential is built out of Painlevé transcendents. So this is what distinguishes the algebraic solutions from the general case: for algebraic solutions, the entries of the Wronskians can be expressed in closed form, and/or as polynomials, and similar considerations apply to classical solutions of Painlevé equations such as the Airy solutions of Painlevé II, which are expressed in terms of Wronskians of Airy functions and their derivatives [8]. However, for the general transcendental solution, the seed eigenfunctions appear to be at least as complicated as the solution itself.

In recent work, which will appear shortly, we present two more (different) determinantal representations for the algebraic solutions of the Painlevé III (D_7) equation. One of these new representations is a Hankel determinant formula, generalizing known results on Hankel determinants for solutions of Painlevé II [30], which arises due to the connection of Painlevé III (D_7) with the Toda lattice, which we are able to exploit based on the bilinear Equation (89) using standard forms for Toda solutions (see, e.g., [31,32]). The other new determinantal representation for the Ohyama polynomials is an analog of a determinant of the Jacobi–Trudi type in terms of generalized Laguerre polynomials, which was presented to us in the form of long-standing conjecture by Kajiwara (private communication to the authors). A particular advantage of the latter representation is that the properties of the Ohyama polynomials in terms of the variable s are made manifest.

In conclusion, it is worth pointing out that Darboux transformations are applicable to matrix linear systems of a very general type [3], and, since all of the Painlevé equations admit 2×2 matrix Lax pairs, the approach of this paper is relevant in that more general context as well, for deriving other special solutions (see e.g., [33,34] for the case of Painlevé V). However, here, we were specifically interested in the case of Painlevé III (D_7), since, to the best of our knowledge, this was the only case of classical solutions for which no determinantal representation was previously known.

Author Contributions: Conceptualization, J.W.E.H. and A.N.W.H.; Formal analysis, J.W.E.H. and A.N.W.H.; Investigation, J.W.E.H.; Writing—original draft, J.W.E.H.; Writing—review & editing, A.N.W.H.; Supervision, A.N.W.H. All authors have read and agreed to the published version of the manuscript.

Funding: Royal Society: IEC\R3\193024.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Intertwining Relations and Wronskians

The purpose of this appendix is to collect together some algebraic facts about Jordan chains for confluent Darboux transformations. In particular, we focus on the specific role played by the Burchnall–Chaundy relation, and how it relates to the definition of the Jordan chain and the choice of normalization for the generalized eigenfunctions. This is especially pertinent to Adler and Moser's construction of the rational solutions of the KdV equation, as described in Section 3, and to our derivation of the Wronskians for the algebraic solutions of Painlevé III (D_7). While most of the facts presented here can probably be found somewhere in the extensive literature on Darboux transformations, it appears that the precise connection between the normalization of the Jordan chain and the Burchnall–Chaundy relation was not addressed in the recent works [13,14] on the confluent case.

Suppose that we have a sequence of Schrödinger operators, labeled by $n \in \mathbb{Z}_{\geq 0}$, related to one another by the repeated application of confluent Darboux transformations to some initial operator. In order to eliminate some inconvenient minus signs, we use H_n to denote the (formally self-adjoint) operator

$$H_n = -\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_n\right).$$

Hence, we can regard each H_n as the Hamiltonian operator for some 1D quantum mechanical system (with $-V_n$ being the potential energy in that context).

In order to construct the sequence of operators via the action of confluent Darboux transformations, we consider an eigenfunction ϕ_0 for the initial operator H_0, that is,

$$H_0\,\phi_0 = 0,$$

with eigenvalue zero. (Here, we consider only the case of repeated eigenvalue $\mu = 0$, because that is the only case of interest in the rest of the paper. However, the general case requires only minor modifications (cf. Section 2).) Then, we have the factorization

$$H_0 = L_0^\dagger L_0, \qquad L_0 = \frac{\mathrm{d}}{\mathrm{d}z} - \frac{\phi_0'}{\phi_0}, \quad L_0^\dagger = -\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\phi_0'}{\phi_0},$$

so that ϕ_0 spans the kernel of the first-order operator L_0, i.e.,

$$L_0\,\phi_0 = 0,$$

and the next operator is produced by reversing the order of factorization:

$$H_1 = L_0 L_0^\dagger.$$

So far, we just have a standard (single) Darboux transformation, and, by construction, the reciprocal of the original eigenfunction is an eigenfunction of the new operator, that is,

$$H_1 \, (\phi_0)^{-1} = 0.$$

Then the main question is how to iterate the construction and obtain a sequence of eigenfunctions ϕ_n for each operator with the same eigenvalue (in this case, zero) when the standard formula (15) for generating a new eigenfunction breaks down, and the usual Crum formula is no longer valid.

The answer is to build ϕ_n from a sequence of generalized eigenfunctions for H_0, denoted by ψ_n, which arise in the following way: Starting with the original eigenfunction of H_0, that is,

$$\phi_0 = \psi_1,$$

we define the new eigenfunction for H_1 as

$$\phi_1 = L_0 \, \psi_2,$$

So we find

$$H_1 \, \phi_1 = L_0 L_0^\dagger \, \phi_1 = L_0 L_0^\dagger L_0 \, \psi_2 = L_0 H_0 \, \psi_2 \implies H_0 \, \psi_2 \in \ker L_0.$$

Hence, $H_0 \psi_2$ is a multiple of $\phi_0 = \psi_1$, which gives the generalized eigenfunction equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V_0 \right)^2 \psi_2 = H_0^2 \, \psi_2 = 0.$$

However, we can say more: For a non-trivial result at the next stage of iteration, we require that ϕ_1 should be independent of ϕ_0^{-1}, the other eigenfunction of H_1 that we already know. Thus, their Wronskian is non-zero:

$$Wr\big(\phi_1, (\phi_0)^{-1}\big) = C_1 \neq 0. \tag{A1}$$

Furthermore, we can rewrite the latter condition in terms of operators as

$$(\phi_0)^{-1} \, L_0^\dagger \, \phi_1 = C_1 \implies L_0^\dagger \, \phi_1 = C_1 \, \phi_0,$$

and replacing ϕ_1 and ϕ_0 in terms of ψ_2 and ψ_1, respectively, this becomes

$$L_0^\dagger L_0 \, \psi_2 = C_1 \, \psi_1 \iff H_0 \, \psi_2 = C_1 \, \psi_1,$$

which is precisely the Jordan chain condition (26) but with an arbitrary choice of non-zero normalizing constant C_1, coming from the Wronskian (A1). The formulae in Sections 2 and 3 involved the particular choice $C_1 = -1$, whereas, in Section 4, in the context of the Ohyama polynomials, the choice $C_1 = \sqrt{3}$ was made.

For subsequent eigenfunctions, we proceed by induction. For each n, we have a Schrödinger operator factorized in two different ways, that is,

$$H_n = L_{n-1} L_{n-1}^\dagger = L_n^\dagger L_n, \tag{A2}$$

which gives the standard intertwining relation

$$L_{n-1}H_{n-1} = H_n L_{n-1}, \tag{A3}$$

and, at each iteration of the confluent Darboux transformation, the new eigenfunction is defined by

$$\phi_n = M_n\, \psi_{n+1}, \qquad \text{where} \quad M_n = L_{n-1}L_{n-2}\cdots L_0. \tag{A4}$$

By construction, the reciprocal of the eigenfunction from the previous step satisfies

$$L^\dagger_{n-1}\,(\phi_{n-1})^{-1} = 0 \implies H_n\,(\phi_{n-1})^{-1} = 0$$

by (A2), and we require that the new eigenfunction should be independent, with the Wronskian

$$Wr\big(\phi_n, (\phi_{n-1})^{-1}\big) = C_n, \tag{A5}$$

for arbitrary $C_n \neq 0$. Similarly to (A1), the latter condition can be rewritten as the operator equation

$$L^\dagger_{n-1}\,\phi_n = C_n\,\phi_{n-1}, \tag{A6}$$

and, after applying L_{n-1} to both sides, this implies

$$L_{n-1}L^\dagger_{n-1}\,\phi_n = L_{n-1}(C_n\,\phi_{n-1}) \implies H_n\,\phi_n = 0.$$

Furthermore, by (A4), we can rewrite (A6) as

$$L^\dagger_{n-1}M_n\,\psi_{n+1} = C_n\,M_{n-1}\,\psi_n,$$

and then note that, by repeated application of the intertwining relation (A3), we have

$$\begin{aligned}
L^\dagger_{n-1}M_n &= L^\dagger_{n-1}L_{n-1}L_{n-2}\cdots L_0 \\
&= H_{n-1}M_{n-1} \\
&= L_{n-2}H_{n-2}M_{n-2} \\
&= L_{n-2}L_{n-3}H_{n-3}M_{n-3} \\
&= \ldots \\
&= M_{n-1}H_0,
\end{aligned}$$

which means that another consequence of (A6) is the equation

$$M_{n-1}H_0\,\psi_{n+1} = C_n\,M_{n-1}\,\psi_n,$$

or, equivalently,

$$H_0\,\psi_{n+1} - C_n\,\psi_n \in \ker M_{n-1}. \tag{A7}$$

Also note that, at the next step of the Darboux transformation, we have

$$L_n\,\phi_n = 0 \implies L_n M_{n-1}\,\psi_{n+1} = M_n\,\psi_{n+1} = 0. \tag{A8}$$

The statement (A7), for each n, is the most general formulation of the Jordan chain condition, which, for $i = n+1$, includes (26) as a special case. To understand the general condition, observe that

$$M_n = \left(\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\phi'_{n-1}}{\phi_{n-1}}\right)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\phi'_0}{\phi_0}\right) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} + \cdots$$

is a differential operator of order n, and we claim that its action on any function χ is given as a ratio of Wronskians by the formula

$$M_n\,\chi = \frac{Wr(\psi_1,\ldots,\psi_n,\chi)}{Wr(\psi_1,\ldots,\psi_n)}. \tag{A9}$$

To see why the above formula is correct, observe that, from $M_n = L_{n-1}M_{n-1}$ and (A8), it is clear by induction that $\ker M_n$ is spanned by $\psi_1,\ldots,\psi_n$, and the Wronskian in the numerator vanishes whenever χ is a linear combination of these functions. Moreover, the linear operator defined by (A9) also has the same leading term as M_n, so they must be the same. It follows from this that the most general way to satisfy the condition (A7) is to take, at each step,

$$H_0\,\psi_{n+1} = C_n\,\psi_n + \sum_{j=1}^{n-1} c_{n,j}\,\psi_j, \tag{A10}$$

with additional arbitrary constants $c_{n,j}$. However, while the freedom to add these extra terms is always present, it can be absorbed into a redefinition of ψ_n on the right-hand side of (A10), which does not affect the values of the Wronskians.

Finally, by combining (A4) with (A9), it follows that the eigenfunctions of each Schrödinger operator H_n are given by the ratio of Wronskians (27), that is,

$$\phi_n = \frac{\theta_{n+1}}{\theta_n}, \qquad \text{with} \quad \theta_n = Wr(\psi_1,\ldots,\psi_n),$$

and the Wronskian condition (A5) for the pair of eigenfunctions obtained at each step is equivalent to the Burchnall–Chaundy relation in the form

$$\theta'_{n-1}\theta_{n+1} - \theta_{n-1}\theta'_{n+1} = C_n\,\theta_n^2,$$

where the normalization constant $C_n \neq 0$ can be freely chosen for each n.

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