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DOCTORAL THESIS

Condition-based maintenance policies and some properties of geometric-like processes

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Declaration of Authorship

I, Jiaqi Yin, declare that this thesis titled, "Condition-based maintenance policies and some properties of geometric-like processes" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University,
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- Where I have consulted the published work of others, this is always clearly attributed,
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work,
- I have acknowledged all main sources of help.
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Signed: Jiaqi Yin

Date: 15 March 2024

"Thanks to my solid academic training, today I can write hundreds of words on virtually any topic without possessing a shred of information, which is how I got a good job in journalism."

Dave Barry

Abstract

by Jiaqi YIN

This thesis proposes maintenance polices from different conditions in real life and investigates some properties of geometric-like processes so that the economic impact due to the failures of a technical system can be reduced. Therefore, this thesis consists of two parts. Part A involves Chapter 3 and Chapter 4, which optimises maintenance policies for two scenarios. Scenario 1 aims at deriving maintenance policies for k-out-of-n technical systems, where the degradation process of each component is modelled by the gamma process and the uncertainties of models and their parameters, along with the expected maintenance cost of the maintenance policy, are regarded as model choice criteria in maintenance policy optimisation. Scenario 2 aims at optimising maintenance policies for multi-component systems and compares the maintenance policies under four cases, where the degradation process of each component is modelled by the Wiener process and the system fails when a linear combination of the degradation levels of the components exceeds a pre-specified value. Part B involves Chapter 5 and Chapter 6, which investigates the impact of the uncertainty of the parameter estimation of an extension of the geometric process (GP) on the expected maintenance cost, derives the least square estimation method for another extension of the GeP, and then compares the goodness-of-fit of seven GP-like models on twenty-five real datasets. As such, the main contributions of this thesis include the following

- A multi-criteria model-selecting method of k-out-of-n system for maintenance policy optimisation is proposed, where the degradation process of each component in the system is modelled by the gamma process.
- It proposes a maintenance policy for a multi-component system in which the degradation level of each component is modelled with a Wiener process and the system is said "failed" once a weighted linear combination of the degradation level of the components exceeds a pre-specified value. The proposed maintenance policy is compared with four other maintenance policies.
- Sensitivity analysis due to the uncertainty of the parameter estimation of an extension of the geometric process is carried out. The impact of the uncertainty on the expected cost ratio and the optimal PM (preventive maintenance) intervals is analysed.

- The least square method for estimating the parameters in an extension of the geometric process is derived and the performance of the goodness-of-fit of the geometric process and six of its extensions on twenty-five real-world datasets are compared.

Keywords: Reliability, Maintenance policies optimisation, Model selection, Degradation process, Cost process

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List of Abbreviations

AIC **A**kaike **I**nformation **C**riterion
AICc **C**orrected AIC
AIGeP **A**lternating **Geometric Process**
ANN **A**rtificial **N**eural **N**etworks
ARD **A**rithmetic **R**eduction of **D**eterioration order
α-SP α **S**eries **P**rocess
CbM **C**ondition-**b**ased **M**aintenance
CM **C**orrective **M**aintenance
CV **C**oefficient of **V**ariation
DGeP **D**oubly **G**eometric **P**rocess
EGeP **E**xtended **G**eometric **P**rocess
GaP **G**amma **P**rocess
GeP **G**eometric **P**rocess
i.i.d **I**ndependent and **I**dentically **D**istributed
MLE **M**aximum **L**ikelihood **E**stimation
NHPP **N**on-**h**omogeneous **P**oisson **P**rocess
PM **P**reventive **M**aintenance
ReCM **Re**liability **C**entred **M**aintenance
RiCM **Ri**sk **C**entred **M**aintenance
RUL **R**emaining **U**seful **L**ife
SVR **S**upport **V**ector **R**egression
TbM **T**ime-**b**ased **M**aintenance
TGeP **T**hreshold **G**eometric **P**rocess
TPM **T**otal **P**roductive **M**aintenance
VDM **V**alue-**D**riven **M**aintenance
WP **W**iener **P**rocess
IG **I**nverse **G**aussian
ArGeP **Ar**ithmetic **G**eometric **P**rocess
EPP **E**xtended **P**oisson **P**rocess
BGeP **B**inary **G**eometric **P**rocess
SGeP **S**emi-**G**eometric **P**rocess
GePAP **Ge**ometric **P**olya-**A**eppli **P**rocess
DRGeP **D**ouble **R**atio **G**eometric **P**rocess

List of Symbols

Chapter 3

t	time
i	the i-th component
$X_i(t)$	the degradation level of the i-th component at time t
c_r	the replacement cost
j	the j-th maintenance
α	the shape parameter of the geometric process
β	the scale parameter of the geometric process after j-th maintenance
a	the ratio of the geometric process
$F_j(x;\alpha(t),\beta_j)$	the cdf of $X_{j,i}(t)$ after the $(j-1)$-th maintenance
$Y_j(t)$	the degradation level of the whole system after the $(j-1)$-th maintenance
$g_j(x;\alpha(t),\beta_j)$	the pdf of $Y_j(t)$
$U_j(t)$	the expected repair cost of the whole system
$C(N\|L)$	the expected cost ratio
L_{GeP}	the likelihood function under the geometric process
L_{EPP}	the likelihood function under the extended Poisson process
L_α	the likelihood function under the α-series process
ℓ_{GeP}	the log-likelihood function under the geometric process
ℓ_{EPP}	the log-likelihood function under the extended Poisson process
ℓ_α	the log-likelihood function under the α-series process
$H(N)$	the objective function for minimizing
w_1	the weighted value of $C(N\|L)$
w_2	the weighted value of AIC
w_3	the weighted value of CV

Chapter 4

$X_k(t)$	the degradation level of the k-th component
$f_{GaP}(x;\alpha_k(t),\beta_k)$	the pdf of $X_k(t)$ under the gamma process
Y_{gap}	the overall degradation level of the system under the gamma process
Y_{wp}	the overall degradation level of the system under the Wiener process
$f_{Y(t)_{GaP}}$	the pdf of Y_{GaP}
$X_{k_{wp}}(t)$	the degradation level of the k-th component under the Wiener process
$\mu_{k_{wp}}$	the mean under the Wiener process
$\sigma_{k_{wp}}$	the standard deviation of the Wiener process

$f_{gap}(x, \alpha_k(t), \beta_k)$	the pdf under the gamma process
$U_{gap}(t)$	the repair cost process under the gamma process at time t
$U_{wp}(t)$	the repair cost process under the Wiener process at time t
c_m	the maintenance cost
c_r	the repair cost
T_b	time interval of replacement under the block replacement policy

Chapter 5

X_k	the time interval between the $(k-1)$-th failure and the k-th failure
α	the parameter under the extended Poisson process
β	the parameter under the extended Poisson process
a	the parameter under the extended Poisson process
b	the parameter under the extended Poisson process
k^*	the turning point under the extended Poisson process
$g(k^*)$	the pdf of the turning point
$g'(k^*)$	the derivation of $g(k^*)$
γ	the shape parameter under the Weibull distribution
ρ	the scale parameter under the Weibull distribution
$\hat{h}(\cdot)$	the estimator of lower bound of k^*
$\hat{g}(\cdot)$	the estimator of upper bound of k^*

Chapter 6

Y_k	the sequence of i.i.d random variables under the α-series process
k	the k-th failure
a	the parameter under the α-series process
Q_α	the sum squares of errors under the α-series process

Publication

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Chapter 1

Introduction

Reliability and maintenance play an important role in many sectors such as the railway transport industry and the nuclear industry. Normally, a technical system deteriorates and ages over time since its reliability may be affected by aging, usage intensity, and environmental factors. For many industries, technical systems such as petroleum equipment in the oil and gas industry and medical equipment in the healthcare industry are expensive. Keeping them properly maintained can ensure a certain level of availability and may prevent them from causing financial losses, health damage, life loss, and environmental detriment (Smith, 1993).

Abundant existing research has reported how maintenance is closely related to cost efficiency. Below show examples from three industries.

Production industry Mobley (2002) indicated that from 15 percent to 40 percent of production cost is closely associated with several maintenance activities. Over 15 percent of the total workforce in the process industry deal with maintenance operations. Jonge, Teunter, and Tinga (2017) reviewed that about 15-70 percent of the cost for a product has an increasing trend on the maintenance cost.

Chemical industry In the chemical industry, maintenance accounts for up to 30 percent of the total cost (Waeyenbergh and Pintelon, 2002). Another source shows that maintenance can generate additional production benefits in the range of 5 percent on offshore oil and gas wells (Telford, Mazhar, and Howard, 2011).

Healthcare industry For many hospitals, the budget for maintenance of medical equipment nowadays also faces a huge maintenance budget problem. Patients rely on their medical professionals to provide the best treatment and accurate diagnosis, and properly functioning medical equipment is an important way to do so. Thus, not only for hospitals but also other healthcare industries, maintenance should be a high priority to ensure the safety and reliability of equipment to avoid downtime and to reduce related costs.

A survey by Rapoza (2016) indicated the average hourly cost of downtime across all businesses is about 260,000 US dollars in the United States, and it seems to be increasing in the future. Therefore, it is necessary to predict the possible failures of and design different maintenance policies for different systems to make the company reduce the maintenance cost and increase the profit.

1.1 Aim and objectives

This thesis aims to improve system availability so that the economic impact due to the failures of a technical system can be reduced. The objectives are

(1) to perform a literature review in the area of the degradation modelling of multicomponent systems and the extensions of the geometric process;

(2) to develop a multi-criteria model-selecting method for the optimisation of maintenance policies for multi-component system;

(3) to compare the performance of some maintenance policies for multi-component systems;

(4) to investigate the sensitivity to the values of parameters on the expected cost due to turning points;

(5) to compare the performance of the geometric-process-like models.

1.2 Structure of the thesis

The rest of the thesis is structured as follows.

Chapter 2 describes the background and literature review of reliability and maintenance. A framework will be introduced. Knowledge gaps are then identified

Chapter 3 proposes a multi-criteria model-selecting method for maintenance policy optimisation, which considers three indexes: the expected cost ratio, the uncertainty of the model that fits times between failures, and the fitness of the model.

Chapter 4 proposes a Wiener process-based method for the case where the weighted sum of the degradation levels is larger than a pre-specified value, then the system needs to be repaired or replaced. Four maintenance policies are proposed and compared.

Chapter 5 investigates the impact of the uncertainty of estimated parameters on the expected cost per unit of time of a maintenance policy of the turning point under the extended Poisson process.

Chapter 6 compares the performance of the goodness-of-fit of the geometric process and its extensions on twenty-five real-world datasets.

Chapter 7 concludes the contributions and limitations of this research. Future work will be proposed.

In this thesis, Chapter 3 and Chapter 4 constitute the Part A, which optimises maintenance policies for two scenarios. Chapter 5 and Chpater 6 constitute the Part B, which investigates the impact of the uncertainty of the parameter estimation of an extension of the geometric process (GP) on the expected maintenance cost, derives the least square estimation method for another extension of the GeP, and then compares the goodness-of-fit of seven GP-like models on twenty-five real datasets.

Chapter 2

Literature review

2.1 Introduction

Reliability is the ability of an item to perform a required function under given conditions for a given time interval (British Standard, 2017). It is the central concern of many engineering systems in different sectors to evaluate the state of a system, which may be either simple systems (single-component systems) such as pavement and oil pipelines or complex systems (multi-component systems) such as electric power systems, nuclear systems, and aerospace systems. Asset owners usually hope to increase profits by developing cost-effective maintenance policies to reduce operating or maintenance costs. To achieve this goal, it is necessary to evaluate the existing system state, the effect of maintenance, and the system state after maintenance. However, it is worth noting that all models for describing the change of system state or the effect of maintenance have their assumptions, which means they are approximations of the situation in the real world. In this thesis, such problems will be discussed from the perspective of single-component systems and multi-component systems.

2.2 Degradation process modelling

For maintenance, different methodologies can be flexibly used in different stages. In fact, many researchers attempt to combine these methods to better solve real-world problems. Several models can be used to model the failure processes of systems. One common taxonomy of maintenance models (Zhang, Li, and Yu, 2006; Galar et al., 2013; Kumar, Shankar, and Thakur, 2018), which is followed in this thesis, is to classify models into three classes: model-based approach, data-driven approach, and hybrid approach, as shown in Table 2.1. Other common methods such as the geometric process, regression, ANN and SVR can be seen in these examples: Dong et al. (2014), Liu, Jiang, and Zhang (2017), and Lo et al. (2019). Table 2.1 shows a summary of methodologies for modelling degradation processes.

In this section, commonly used models such as the gamma process (GaP), the Wiener Process (WP), the inverse Gaussian process (IGeP), and the Non-Homogeneous

TABLE 2.1: Summary of approaches

Approach	Technicals	Relevant studies
Model-based approach	Gamma process	Cinlar, Bazant, and Osman (1977), Lawless and Crowder (2004), Wu and Castro (2020), Cholette et al. (2019), Nicolai, Dekker, and Van Noortwijk (2007), Qi and Huang (2023), and Cai, Teunter, and Jonge (2023)
	Wiener process	Wang, Balakrishnan, and Guo (2014), Tsui et al. (2015), Liu et al. (2017), Zhang et al. (2018), Gao et al. (2023), and Cheng and Zhao (2023)
	Inverse Gaussian	Lawless and Crowder (2004), Ye et al. (2014), Chen and Tsui (2013), Chen et al. (2015), Hao, Yang, and Berenguer (2019), Yang et al. (2023), and Ruoran, Li, and Yi (2023)
	Geometric process	Houten and Kimura (2000), Liu, Jiang, and Zhang (2017), and Lone, Alam, and Rahman (2023)
	Hidden Markov models (HMM)	Kamlu and Laxmi (2019) and Gámiz, Limnios, and Carmen Segovia-García (2023)
Data-based approach	Bayesian Network	Jun and Kim (2017), Li et al. (2018), Xu et al. (2019), Sharma and Rai (2020), Davila-Frias et al. (2023), and Lee and Kwon (2023)
	Artifical neural network	Farsi and Hosseini (2019) and Davila-Frias et al. (2023)
	Support vector regression	Benkedjouh et al. (2013) and Jiang et al. (2023)
	Proportional hazard model (PHM)	Liu et al. (2020)
	Cluster analysis	Skormin et al. (1999)
Hybrid approach		Galar et al. (2013), Feng et al. (2017), Yang et al. (2018a), Chang et al. (2019), and Liu et al. (2023)

Poisson Process (NHPP) will be discussed in detail with their properties and existing applications.

2.2.1 Basic degradation processes

The degradation processes of some technical systems can be observed when systems are continuously working. For example, cracks and pavements on the road gradually widen or become large over daily use. For such systems, models such as the GaP, the WP, and the IGeP are widely used for estimating the state of a system.

Gamma process was initially proposed by Abdel-Hameed (1975) as a specific model characterised by the gamma distribution with identical scale parameters. One obvious feature of the gamma process is that it has a monotonic increase trend. Therefore, this stochastic cumulative process with a simple mathematical structure is widely used for describing degradation processes such as the creep of concrete (Cinlar, Bazant, and Osman, 1977), fatigue crack growth (Lawless and Crowder, 2004; Wu and Castro, 2020) and bursting of a tube (Cholette et al., 2019) in different systems, equipment, and components. Kong and Park (1997) used a stationary gamma process with the Weibull distribution to estimate the level of failures with some random factor, which aims to optimise the long-run expected cost rate. Research by Nicolai, Dekker, and Van Noortwijk (2007) indicated that the gamma process with the non-linear shape function describe the deterioration of the organic coating layer very well. Cholette et al. (2019) developed a new degradation model with gamma process and physic-based erosion model for boiler heat exchange erosion. Their model gave a support to forecast the overall maintenance for exchanger renewals in an Australian sugar factory.

Let $X(t)$ be the degradation level of a degradation process at time t. Suppose it follows the following assumptions:

1) $X(0) = 0$;
2) Increments $\Delta X(t) = X(t + \Delta t) - X(t)$ are independent of t;
3) $\Delta X(t)$ also follows a gamma distribution $\Gamma(\alpha(t + \Delta t) - \alpha(t), \beta)$ whose shape parameter is $\alpha(t + \Delta t) - \alpha(t)$ and scale parameter is β.

$X(t)$ follows probability distribution $\Gamma(\alpha(t), \beta)$ with mean $\beta\alpha(t)$, variance $\beta^2\alpha(t)$, and its probability density function being given by

$$f(x; \alpha(t), \beta) = \frac{\beta^{-\alpha(t)}}{\Gamma(\alpha(t))} x^{\alpha(t)-1} e^{-x/\beta} 1_{\{x>0\}}, \tag{2.1}$$

where $\Gamma(.)$ is the gamma function and x should be positive value.

Then $\{X(t), x > 0\}$ is a gamma process.

The gamma process has the characteristics that it can only be used in optimisation of maintenance policies for single-component systems, but not for multi-component systems or systems with multi-failure modes (Noortwijk, 2009). Thus, the gamma process is normally combined with other models to be more general for different complex systems.

Wiener process is another commonly used degradation process, which is applicable for non-monotonous degradation under a maintenance policy. In fact, the Wiener process has received a lot of attention as it can be flexibly used to model the deterioration of many common physical structures such as fatigue crack growth (Ebrahimi, 2005), wear of structure (Mishra and Vanli, 2016) and bearings or rotating machinery (Wen, Gao, and Zhang, 2018). Another typical application of the Wiener process is for lithium-ion batteries due to its non-monotonic characteristic.

In addition, several authors have described applications and the development of the Wiener process. Wang, Balakrishnan, and Guo (2014) indicated that the Wiener process with a time-scale transformation is an important advantage and extended it to a generalized Wiener process model based on residual time estimation for the target product. Tsui et al. (2015) provided a review for the Wiener process as independent increment process-based models. Zhang et al. (2018) firstly introduced a new classification of Wiener processes based on the five indicators: linearity, non-linearity, variability, covariates and multivariate. The authors also presented some challenges and problems of the Wiener process including multiple failure and degradation modes, effect of random-shocks, multiple phases for some special systems or components. Liu et al. (2017) proposed a Wiener process with linear drift to consider the situation that the operating cost increases with age increase while the system is functioning. It indicated that the optimal decision is a monotone control limit policy under this model. However, they also mentioned that the Wiener process with linear drift is only available to a limited number of systems and cannot describe multi-components or multi-failure modes systems.

Here we introduce the definition of the Wiener process:

Let $X(t)$ be the degradation level at time t, as a Wiener process, it normally follows these assumptions:

1) $X(0) = 0$, which also means that $W(0) = 0$;
2) $W(t)$ has independent increments which follows normal distribution. For ever $0 < s < t$, $W(t-s)$-$W(s)$ follows $N(0, (t-s))$.
3) $W(t)$ is continuous in t.

Then $\{X(t), t > 0\}$ is a Wiener process and is often expressed as:

$$X(t) = \mu t + \sigma W(t), \tag{2.2}$$

where μ is its drift coefficient and σ is its diffusion coefficient.

The probability density function is given by:

$$f_W(t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}. \tag{2.3}$$

The Wiener process has excellent mathematical properties, it is widely used for describing a system with its degradation level fluctuating around a constant with an increase or decrease trend.

Inverse Gaussian process is also a widely used model with similarities to the gamma process. The reason has been described by the bulk of existing research due to its elegant mathematical properties (a monotonous degradation) and physical meaning (Ye et al., 2014; Wang et al., 2018). This process was introduced by Wang and Xu (2010) to the reliability and investigated by Ye et al. (2014). Therefore, it obtains significant attention recently.

An inverse Gaussian process $\{Y(t), t > 0\}$ normally has the following properties:

1) $Y(t)$ has independent increments, such as $Y(t_2) - Y(t_1)$ and $Y(t_4) - Y(t_3)$ are independent;
2) each increment has an inverse Gaussian distribution;
3) $Y(0)$=0 with probability one.

The pdf of an inverse Gaussian random variable IG(μ, θ) with mean μ and variance μ^3/θ is:

$$f(x;\mu,\theta) = \sqrt{\frac{\theta}{2\pi}} x^{-3/2} \exp(-\frac{\theta(x-\mu)^2}{2\mu^2 x}) \tag{2.4}$$

Different from the gamma process and the Wiener process, a significant advantage of the inverse Gaussian process is that it is flexible to incorporate random effects and covariates to account for various types of heterogeneities (Ye et al., 2014; Chen et al., 2015). Either the gamma process or the Wiener process normally assumes a stable environment such as a fixed failure rate as a theory basis. Nevertheless, this assumption is increasingly challenged by many factors. For example, degradation patterns are usually different between two identical systems due to environmental differences, which have been reported in many studies (Lawless and Crowder, 2004; Chen and Tsui, 2013; Chen et al., 2015). Common approaches to decreasing this effect include using random effect models and using imposing distributions to estimate parameters. Therefore, the inverse Gaussian model is more competitive in recent years and is often combined with other models to solve the problem of heterogeneities. Chen et al. (2015) developed an optimal policy based on IG for systems with different mechanism failures. A development of this IG model was that it considers that the inspection costs are related to optimal inspection intervals due to different maintenance schedules. Li et al. (2017) built a nonlinear inverse Gaussian degradation model with imperfect maintenance to maximize the availability of the constraints of cost. Hao, Yang, and Berenguer (2019) extended the IG process by incorporating skew-normal random effects which is applicable for a gas laser degradation process and fatigue crack growth.

NHPP was introduced by Cox (1955) as a counting process, whose average rate of arrivals is allowed to vary with time and is different from a homogeneous Poisson process. The Homogeneous Poisson Process (HPP) is a basic model when the interarrival times between failures are independent and identically distributed according to the exponential distribution. Let $N(t)$ denote the number of random points in the interval $(0, t]$ in the NHPP, so that $N(t)$ is a counting process, then it has following conditions

1) $N(0) = 0$;
2) $N(t)$ has independent increment;
3) λ is the arrival rate of $N(t)$, which is fixed and positive;
4) the expected number of failures by time t is λt;
5) no events happen simultaneously.

The NHPP is similar to the HPP but relaxes the assumption of stationary increments of the HPP. Thus, the arrival rate can vary with time. Therefore, it is widely used for modelling the occurrences of defects and the effectiveness of maintenance. Srivastava and Mondal (2016) used the NHPP to analyse failure data and to model the failure occurrence of a repairable system. Chien et al. (2021) used the NHPP to develop a failure process following the Non-Homogeneous Pure Bith Process (NHBPP), which is a generalized version the the NHPP, and developed a two-stage maintenance policy to minimize the long-run expected cost rate. Lebied and Bacha (2022) used the NHPP to estimate the effectiveness of the minimal repair. It considered the intensity reduction due to the frequency of maintenance of an air compressor. Bacha, Bellaouar, and Dron (2023) proposed a proportional intensity model based on the NHPP to assess the effectiveness of maintenance of an oil pump. The probability of a given number of failures for the NHPP is calculated by

$$P(N(t) = k) = \frac{\Lambda(t)^k e^{-\Lambda(t)}}{k!}, \tag{2.5}$$

where $\Lambda(t)$ is the cumulative failure intensity and $\Lambda(t) = \int_0^t \lambda(u)\mathrm{d}u$.

CbM has been one of the most advanced maintenance techniques in the last two decades (Ahmad and Kamaruddin, 2012). Existing research demonstrates the benefits of CbM. One reason for the competitiveness of the CbM is its concern for the degradation process from normal states to abnormalities. This enables asset owners to save costs, with the lifetime being estimated, and the maintenance policy being planned (Shin and Jun, 2015). At the same time, the ability of CbM to assess the condition status of equipment with real-time information is constantly improving based on the fast development of computer science and sensing techniques.

The application of CbM in the relevant research and industry heavily depends on the development of degradation process modelling. For those system conditions that can be directly observed, Si et al. (2011) and Ye and Xie (2015) indicated that the type of stochastic deterioration models can be classified into discrete or continuous. For those whose environment is more complex, the deterioration is possibly caused by multiple factors. In addition, the proportional hazard model (PHM), which is a type of statistical model to describe relationship between the time that passes and covariates that may be relevant to such quantity of time. Therefore, these publications classify CbM models into discrete and continuous (Alaswad and Xiang, 2017).

Other literature for classifying CbM is based on the type of methodology or maintenance techniques to modelling the degradation process: model-based approach, data-driven approach, and hybrid approach (Zhang, Li, and Yu, 2006; Galar et al., 2013; Kumar, Shankar, and Thakur, 2018). Caesarendra (2010) and Jardine, Lin, and Banjevic (2006) indicated that the data-driven approach can transform high-dimensional data into lower-dimensional information, which is also referred to as the *machine learning* approach. This means that this approach can be used to analyse historical data and learn system behaviour. This is helpful in improving the accuracy of the model and modelling the real deterioration of an item better. In contrast, the model-based approach focuses on the physical understanding of the target item. Such mythologies generally rely on the use of mathematical methods.

It is worth mentioning that many of the existing studies are now concentrated on the hybrid approach to dealing with the increasingly complex systems (Galar et al., 2013; Feng et al., 2017; Chang et al., 2019). In the existing literature, there have been many CbM algorithms and models in various industrial applications of the CbM processes.

The maintenance cost generally increases with the increase of operating cost due to deterioration and ageing though the equipment is in a functioning state. Liu et al. (2017) developed a CbM strategy with age and state-dependent operating cost to solve this problem. The authors claimed that their model can be applied in a variety of systems. However, the versatility makes the model difficult to handle a more complicated degradation process. Besides, Liu et al. (2017)'s model firstly involved side effects of degradation.

For the degradation process, many empirical studies tend to explain this process of equipment within various opinions. Wu and Castro (2020) suggested a CbM model in which maintenance is conducted once a linear combination of several degradation processes exceeds a pre-specified threshold.

The review paper by Shin and Jun (2015) also indicated that the frequency of collecting monitoring data is a challenge of CbM. In addition, most CbM approaches considered that the monitored item is independent but not a part of an integrated system. This means that the influence among each parameter from other subsystems may be neglected.

For multi-component systems, the inverse Gaussian process can be applied to estimate the reliability of such systems due to its flexibility. Sun, Ye, and Chen (2017) provided an optimal maintenance policy into a Markov decision framework with the inverse Gaussian process, which can be used to investigate the optimal control limits and improve the efficiency of estimating the reliability.

Although there is a large amount of existing research showing that several models have good performance, their disadvantages are necessary to be investigated due to the changing real-world environment and some uncertain factors such as unexpected failure and natural disasters. Therefore, many current studies tend to combine different models and methodologies to avoid the limitations of one model and make the development closer to the real-world condition.

Below we summarize various combined degradation processes according to several existing research.

Caballé et al. (2015) proposed a condition-based maintenance policy by combining the NHPP and the GeP to modelling a multiple degradation process within a dependent degradation-threshold-shock model. This was a typical example of multi-failure modes. It carried out two incremental processes in two different methodologies, so as to achieve the situation where the decline mode of a single system changes. They also pointed out that the dependence analysis between the causes of failure was a potential development. Wu and Castro (2020) developed a weighted linear combination of degradation processes that combine gamma processes and the geometric process to optimise the time interval of maintenance for a pavement network. Xu et al. (2018) proposed an extended PHM model for a multi-component system with dependence analysis based on a state discretization technique. A two-component system-based gamma process is proposed by Don and Khan (2019).

Table 2.2 lists some existing publications about combination models.

2.2.2 Single-component systems and multi-component systems

In reality, a single-component system and a multi-component system refer to different types of systems based on the number of components they consist of. A single-component system is a system that consists of just one main component or element. For this type of system, multiple failure modes are usually considered. For example, a pavement can be considered as a single-component system with several failure modes, such as cracking and potholes. Whether the failure models will affect each

TABLE 2.2: Research on combined deterioration models

Reference	Used models/methodologies	Application/Solution
Wang (2012)	Bayesian theory, Monte Carlo simulation, Weibull distribution	Multi-observation, Complex system
Si et al. (2013)	Bayesian network, expectation maximisation algorithm	RUL estimation
Peng et al. (2014)	Inverse Gaussian process, Bayesian framework	scarce observation, random effects
Feng et al. (2015)	HPP, Wiener process	multi-failure mode, multi-component, failure dependence
Yang et al. (2018a)	NHPP, age-based maintenance and condition-based maintenance	Failure behaviours, random environment
Wu and Castro (2020)	Gamma process, geometric process and NHPP	Optimise the time between PM and the number of the PM
Zhang et al. (2020)	K-out-of-N, Levy process, Markov renewal, Monte-carlo simulations	Multi-component, failure dependence
Shi et al. (2020a)	Bayesian updating, k-out-of-n	Dynamic system, multi-component, decision-making policy
Hao, Yang, and Berenguer (2020)	Gamma process, NHPP	two-stage inspections
Ma et al. (2020)	Wiener process, exponential distribution, cumulative exposure model	Multi-component, two-stage degradation, failure renewal, multi-failure modes

other is also a question that needs to be considered. A multi-component system consists of multiple main components. For this type of system, the dependence between several components is a core problem.

Single-component systems

For one-component systems, stochastic processes such as the gamma process (Lawless and Crowder, 2004; Wu and Castro, 2020), the inverse Gaussian process (Li et al., 2017; Chen et al., 2015), and the Wiener process (Ebrahimi, 2005; Wen, Gao, and Zhang, 2018) are widely used for different applications due to their mathematical properties. With the development of these models, people gradually introduced other different models to improve and optimise maintenance policies in order to address their shortcomings. Models that can be used to express more complex situations are gradually developed and used to simulate singe-component systems with complex failure modes. Below are some examples of recent developments in single-component systems.

- Wu (2018) extended the geometric process to the doubly geometric process (DGeP). The DGeP can model stochastically increasing or decreasing inter-arrival times of recurrent event processes. Therefore, this DGeP model can be applied to warranty claim analysis and its performance may obtain a better fit to real-world data than the GeP, with respect to the Akaike information criterion (AIC).

- Mercier and Castro (2019) used the likelihood ratio ordering and convex ordering to show the result and difference between two imperfect repair models within two stochastic processes. For one model based on Arithmetic Reduction of Deterioration order (ARD), they assumed that the accumulated degradation caused by the last maintenance activity can be reduced by an imperfect repair.

- Syamsundar, Naikan, and Wu (2020) combined two scales as index for modelling the reliability of a repairable. They presented that the failure process is normally measured by time scale in the literature. Nevertheless, usage or throughput also can be used to measure the condition of system such as the accumulated number of miles for a vehicle but not the age. This development gives users more options to choose the alternative scale in terms of usage or time, either individually or in combination.

- Li and Nilkitsaranont (2009) proposed a combined regression technique for CbM to assess the remaining useful life of gas turbine engines, which improved engine reliability and availability and reduces life cycle costs.

- Coraddu et al. (2016) used some approaches to effectively predicting potential future failures of naval propulsion plants.

- Research such as Zhu, Fouladirad, and Bérenguer (2015) presented a deterioration model that included two system deterioration: wear and shock and gives an optimal maintenance policy for the minimal cost criterion. This study pointed a direction for future research: how to build a better model that is more suitable for a complex system with multiple components or failure types.

Therefore, developing a degradation process model, which is suitable for multi-component systems with multiple failure modes, and its corresponding maintenance policies are inevitable problems for increasingly complex systems, which can help decision makers to develop a reasonable long-term maintenance plan to reduce maintenance costs and increase the availability of a system.

Multi-component systems

Existing research is now concentrated on the combination approach to dealing with the increasingly complex system (Galar et al., 2013; Feng et al., 2017; Chang et al., 2019).

Caballé et al. (2015) proposed a condition-based maintenance policy by combining the non-homogeneous Poisson process (NHPP) and the gamma process (GaP). They modeled a multiple degradation processes within a dependent degradation-threshold-shock model. This study was a typical example of multi-failure modes. It carried out two incremental processes in two different methodologies, so as to achieve the situation where the decline mode of a single system changes. They also pointed out that the dependence analysis between the causes of failure was a potential development and the variability of the threshold should be considered in the future.

Zhu, Fouladirad, and Bérenguer (2015) simulated the wear damage by a non-stationary gamma process and the random shock damage with a generalized Pareto distribution satisfying Poisson arrivals. They derived the mathematical expression of the stationary behaviour of the system and calculated the long-term average total cost by using the semi-regenerative properties. It is worthwhile to notice that this study does not consider the impact of shocks or inspection costs which may influence the result of a long-term optimised maintenance policy.

Liu et al. (2017) proposed a new CbM model based on three-state degradation and the influence of external environmental shocks. The degradation process of the system was modelled by a two-state Wiener process with a Doubly Stochastic Poisson Process (DSPP). It considered two different thresholds, namely normal threshold and defective threshold which depend on the system state. Moreover, it rarely takes into account that downtime may lead to extra cost which is based on degradation-based failure or shock-based failure respectively.

Other common methods such as the geometric process, regression, ANN and SVR can be seen in these examples: Dong et al. (2014), Liu, Jiang, and Zhang (2017), and Lo et al. (2019).

Zhang et al. (2018) reviewed some developments and applications of the Wiener process. It also summarized some challenges and problems which mainly include: the Wiener process with multiple time-scales, the Wiener process integrating various types of data, the Wiener process with state recoveries and the Wiener process with non-Markovian feature. Change points on degradation modelling and prognostics largely occur randomly.

Yang et al. (2019) proposed a two-phase preventive maintenance policy for a single-component system. The first stage is the imperfect maintenance phase which aims to keep the system working. The second stage is the postponed replacement phase which considers a preventive replacement. This means that this maintenance policy will be sufficient and flexible for resource allocation due to its phase variability.

Wu and Castro (2020) developed a weighted linear combination of degradation processes that combine gamma processes and the geometric process to optimise the time interval of maintenance for a pavement network. However, it also points that the degradation may follow different degradation processes in one system. Besides, different failure modes can be correspond to different thresholds which is a potential development as well.

Zhao et al. (2021) proposed a multi-criteria mission abort policy which consider the normal and defective stages based on the time threshold. It also indicated that the performance of the optimal policy is compared in detail against several heuristic policies. Besides, the dynamic risk for controlling policy was also a possible extension for phased mission systems.

Liu et al. (2021a) proposed a condition-based maintenance model in a finite-time horizon which considers a system with two heterogeneous dependent components with economic dependence. Moreover, this research pointed that the two-unit system could be extended to multi-unit systems by generalizing the degradation process and Bellman equation, and the maintenance level could be extended to imperfect repair in future.

There is one more classification of the effectiveness of maintenance: worse-than-minimal maintenance (Wang and Pham, 2006; De Carlo and Arleo, 2017). This type of maintenance action brings the state of a system to a worse situation. The vitural age or the degradation level of an item increases after maintenance activates. This situation is normally caused by the incorrect operation of operators. Besides, an opposite situation to the previous one is called as better-than-perfect maintenance. This kind of maintenance action will make the state of a system return to a better state than it was, Nevertheless, these two types of maintenance effectiveness will not be described in this thesis as there is not much research on better-than-perfect repair and the scenario of worse-than-minimal repair rarely happens.

2.3 The geometric process and its extensions

In addition to the technical systems that can be directly observed for their state, there are also some systems whose degradation process is difficult to be observed. For example, it is hard to observe how a light bulb fails during its working time. Usually, we only can find the time point when it fails and stops working. Alternatively, the time of failures and the number of failures can be accurately recorded. Fig 2.1 shows the failures and repair times of such systems.

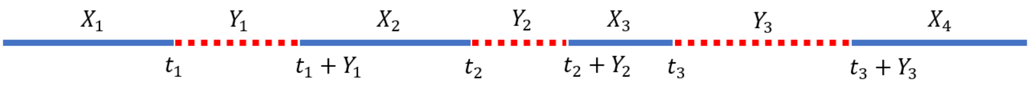

FIGURE 2.1: Non-observable degradation process

1) X_k: the kth working time;
2) Y_k: time duration of the kth repair;
3) t_k: when the kth failure occurred;
4) $t_k + Y_k$: when the kth failure is repaired.

The geometric process (GeP) will be introduced in this section with its properties. Besides, with the development of the GeP, several extensions have been developed and will be discussed in this section.

2.3.1 Geometric process

GeP was first introduced by Lam (1988b), which is also a counting process. A common phenomenon is that the working times between failures (WTBF) of a system after repairs may become shorter and shorter, and the repair times may become longer and longer. This phenomenon can be described by the GeP with its properties of stochastically increasing and decreasing. Since then, the GeP has been widely applied in different fields such as software reliability analysis (Pham and Wang, 2001; Das, Dewanji, and Kundu, 2020), reliability analysis (Kamal et al., 2022), maintenance policy optimisation (Yevkin and Krivtsov, 2020; Yang and Gao, 2022), warranty analysis (Arnold et al., 2020) and electricity pricing, etc. (Zhang, 1999; Lam, 1988a; Wang and Yam, 2017; Niu et al., 2022). According to Ross (1996), assume that X and Y are two random variables. For every real number u, then

$$P(X \geqslant u) \geqslant P(Y \geqslant u),$$

and X is stochastically greater than or equal to Y, or Y is stochastically less than or equal to X. Thus, the monotonicity of a stochastic process can be defined. Then, if $X_k \leq_{\text{st}} X_{k+1}$ or $X_k \geq_{\text{st}} X_{k+1}$ for $k = 1, 2...$, $\{X_k, k = 1, 2, ...\}$ is stochastically increasing or decreasing. Lam (1988b) proposed the definition of the geometric process as shown below.

Definition 1. *Lam (1988b) Given a sequence of non-negative random variables* $\{X_k, k = 1, 2, \ldots\}$*, if they are independent and the cdf of* X_k *is given by* $F(a^{k-1}x)$ *for* $k = 1, 2, \ldots$*, where* a *is a positive constant, then* $\{X_k, k = 1, 2, \cdots\}$ *is called a geometric process (GeP).*

We refer to the random variable X_k as the kth inter-arrival time in following remark.

Remark 1. *From Definitions 1 and 2, the following results have been obtained.*

- *If* $a > 1$*, then* $\{X_k, k = 1, 2, \cdots\}$ *is stochastically decreasing.*
- *If* $a < 1$*, then* $\{X_k, k = 1, 2, \cdots\}$ *is stochastically increasing.*
- *If* $a = 1$*, then* $\{X_k, k = 1, 2, \cdots\}$ *is a renewal process (RP).*

Since the GeP was introduced by Lam (1988b) to model the failure process of a repairable system and used in optimisation of maintenance policies, it has been extended to several models to overcome its various drawbacks, which include: (1) it cannot describe the failure process with nonmonotonous inter-arrival times of a system, (2) the probability distributions of the inter-arrival times have the fixed shape parameter during different inter-arrival times if the probability distribution of the time to first failure is the Weibull distribution, and (3) it assumes that the times between failures are independent. These limitations may prevent it from wide applications in the real world. In this chapter we review the extensions of the GeP, including the arithmetic geometric process (AGeP)(Francis, 2001), the α-series process (α-SP) (Braun, Li, and Zhao, 2005), the threshold geometric process (TGeP) (Chan et al., 2006), the extended Poisson process (EPP) (Wu and Clements-Croome, 2006), the exponent extended geometric process (EEGeP) (Bordes and Mercier, 2013), the extended geometric process (EGeP) (Zhang and Wang, 2016), the doubly geometric process (DGeP) (Wu, 2018), the semi-geometric process (SGeP) (Wu and Wang, 2018), the alternating geometric process (alternating GeP) (Arnold et al., 2020), and the double ratio geometric process (DRGeP) (Wu, 2022). For such extensions, we call them **GeP-like models** in this thesis. Next, we will introduce the definitions, properties, and the applications of several GeP-like models which have been mentioned earlier.

2.3.2 Arithmetic geometric process (ArGeP)

The AGeP is proposed by Francis (2001) with the following definition.

Definition 2. *(Francis, 2001) Given a sequence of random variable* $\{X_k, k = 1, 2, ...\}$*, if for real positive number* a *and real number* d*, and the cdf of* X_k *is given by* $F(a^{k-1}x + a^{k-1}(k-1)d)$*, where* d *and* a *are called the common difference and common ratio of the AGeP, respectively, then* $\{X_k, k = 1, 2, ...\}$ *is called an arithmetic geometric process (ArGeP).*

The expectation of X_k is given by:

$$\mathrm{E}[X_k] = \frac{\mathrm{E}[X_1]}{a^{k-1}} - (k-1)d, \tag{2.6}$$

and the variance is:

$$\mathrm{Var}[X_k] = \frac{\mathrm{Var}[X_1]}{a^{2(k-1)}}. \tag{2.7}$$

The AGeP has the following properties:

- if $a > 1$ and $d \in (0, \frac{\mu_{X_1}}{(n-1)a^{k-1}})$, where μ_{X_1} is the mean of the first random variable X_1, then the AGeP decreases stochastically.
- if $0 < a < 1$ and $d < 0$, then the AGeP increases stochastically.
- if $d = 0$ and $a = 1$, then it reduces to a RP.

2.3.3 α-series process (α-SP)

The α-series process (short for: α-SP) is proposed by Braun, Li, and Zhao (2005). It considers that the expected number of counts at an arbitrary time does not exist for the decreasing geometric process. The α-series process has a finite expected number of counts within certain conditions, and is defined as follow.

Definition 3. *(Braun, Li, and Zhao, 2005) Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is given by $F(k^\alpha x)$ for $k = 1, 2...$, where α is a positive constant, then $\{X_k, k = 1, 2, ...\}$ is called an α-series process (α-SP).*

The expectation is presented by:

$$\mathrm{E}[X_k] = \frac{\mathrm{E}[X_1]}{k^\alpha}, \tag{2.8}$$

and the variance is:

$$\mathrm{Var}[X_k] = \frac{\mathrm{Var}[X_1]}{k^{2\alpha}}. \tag{2.9}$$

It has following properties:

- if $\alpha > 0$, then the α-SP is stochastically decreasing.
- if $\alpha < 0$, then the α-SP is stochastically increasing.
- if $\alpha = 0$, then the α-SP reduces to a RP.

Demirci Biçer (2019) discussed the statistical inference for the α-SP with the generalized Rayleigh distribution. This chapter also compares the model performance based on maximum likelihood estimation, maximum spacing estimation, least-squares estimators, modified methods and L-moments estimators. Zuo and Xiao (2022) considered a multi-state one-component system and design an order-replacement policy based on the long-term cost function according to the Markov renewal reward in the case of the α-series process.

2.3.4 Threshold geometric process (TGeP)

The threshold geometric process (TGeP) is proposed by Chan et al. (2006). The main characterise of TGeP is that it can describe non-monotonous trends in a failure process. According to Chan et al. (2006), it has separate GePs and the kth GeP is denoted by

$$\mathrm{GeP}_k = \{X_i, T_n \leqslant i < T_{n+1}\}, k = 1, ..., K,$$

to each trend with turning point T_n. Then each GeP has its parameter a. The threshold geometric process is defined as follows.

Definition 4. *(Chan et al., 2006) Given a sequence of non-negative random variables* $\{X_k, k = 1, 2, ...\}$*, if they are independent and the cdf of* X_k *is given by* $F(a_k^{i-T_n} x)$ *for* $k = 1, 2...$*, where* a *is a positive constant, then* $\{X_k, k = 1, 2, ...\}$ *is called a threshold geometric process.*

The expectation of $X_k(t)$ is presented by:

$$\mathrm{E}[X_{T_{k+i}}] = \frac{\mu_k}{a_n^i},$$

and the variance is:

$$\mathrm{Var}[X_{T_{k+i}}] = \frac{\sigma_k^2}{a_n^{2i}}.$$

The TGeP can be narrowly considered as the piecewise GeP based on different ratio a_k, therefore, it has following properties:

- if $a_k > 1$, then the TGeP of kth phrase is stochastically decreasing.
- if $a_k < 1$, then the TGeP of kth phrase is stochastically increasing.
- if $a_k = 1$, then the TGeP of kth phrase reduces to the RP.

According to Arnold et al. (2020), the TGeP can be used for estimating SARS epidemic, electricity prices and the daily range stock market indices.

2.3.5 Extended Poisson Process (EPP)

The extended Poisson process (EPP) is proposed by Wu and Clements-Croome (2006). Similar to the TGeP, the EPP can model a failure process with non-monotonous trends.

Definition 5. *(Wu and Clements-Croome, 2006) Given a sequence of non-negative random variables* $\{X_k, k = 1, 2, ...\}$*, if they are independent and the cdf of* X_k *is given by* $F((\alpha a^{k-1} +$

$\beta b^{k-1})x)$ for $k = 1, 2...$, where $F(x)$ is an exponential cdf, $\alpha + \beta \neq 0$, $\alpha, \beta \geq 0$, $a \geq 0$ and $0 < b \leq 1$, then $\{X_k, k = 1, 2, ...\}$ is called as an extended Poisson process (EPP).

The expectation and the variance of the $X_k(t)$ is given by

$$\mathrm{E}[X_k] = \mathrm{E}[X_1]((\alpha a^{k-1} + \beta b^{k-1}))^{-1},$$

and

$$\mathrm{Var}[X_k] = \sigma^2((\alpha a^{k-1} + \beta b^{k-1}))^{-1}$$

where $\mathrm{E}(X_1)$ is the expectation of X_1 and σ^2 is the variance of X_1.

The EPP has the following properties:

- if $a = b = 1$, then the EPP reduces to the HPP with the cdf $F((\alpha + \beta)x)$;
- if $\alpha a^{k-1} \neq 0$ and $\beta b^{k-1} = 0$ (or $\alpha a^{k-1} = 0$ and $\beta b^{k-1} \neq 0$), then the EPP reduces to a GeP, in this case,
  - (1) $\alpha a > 1$ (or $\beta b^{k-1} > 1$), the EPP is stochastically decreasing;
  - (2) $0 < \alpha a < 1$ (or $0 < \beta b^{k-1} < 1$), the EPP is stochastically increasing;
  - (3) $\alpha a = 1$ (or $\beta b^{k-1} = 1$), the EPP reduces to the GeP.
- if $\alpha a^{k-1} \neq 0$ and $\beta b^{k-1} \neq 0$, in this case,
  - (a) if $a = 1$ and $b < 1$ (or $a > 1$ and $b = 1$), then the EPP can model a failure process with decreasing failure intensity functions with respect to k;
  - (b) if $a > 1$ and $b < 1$, then the EPP can model a failure process with a more complicated failure intensity functions with respect to k;

It can be seen that one can set $a = 1$ or $b = 1$ in applications when the EPP is used to model recurrent event data with non-monotonous trends.

If the EPP is used to describe a non-monotonic failure trend, it must satisfy the conditions: $\alpha a^{k-1} \neq 0$, a<1, 0<b<1, and $\beta b^{k-1} \neq 1$.

2.3.6 Binary geometric process (BGeP)

Chan and Leung (2010) proposed an indicator function

Definition 6. *(Chan and Leung, 2010) Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent, and there is a binary sequence $\{W_k, k = 1, 2, ...\}$ that is the indicator of whether X_k is greater than certain level b. If the cdf of $W_k = b$ is given by $1 - F(a^{k-1}x)$ for $k = 1, 2, \ldots$, where $F(a^{k-1}x)$ is the cdf of X_k and a is a positive constant, then $\{X_k, k = 1, 2, ...\}$ is called a binary geometric process (BGeP).*

Similarly, the BGeP has the following properties:

- If $a > 1$, then $\{X_k, k = 1, 2, \cdots\}$ is stochastically increasing.
- If $a < 1$, then $\{X_k, k = 1, 2, \cdots\}$ is stochastically decreasing.
- If $a = 1$, then $\{X_k, k = 1, 2, \cdots\}$ is a renewal process (RP).

2.3.7 Exponent extended geometric process (EEGeP)

Bordes and Mercier (2013) considered that one of the limitations of the GeP is the fast increase or decrease in successive periods. For the EEGeP, the multiplicative scaling factor is not necessarily a geometric progression, therefore, the assumptions of this model are more flexible.

The definition of the EEGeP is the following.

Definition 7. *(Bordes and Mercier, 2013) Given a sequence of non-negative random variables* $\{X_k, k = 1, 2, ...\}$*, if they are independent and non-negative, and the cdf of* X_k *is given by* $F(a^{b_k} Y_k)$ *for* $k = 1, 2...,$ *where a is a positive constant,* $(b_k)_{k \geq 1}$ *forms a non-decreasing sequence such that* $0 = b_1 \leq b_2, \ldots, \lim_{k \to \infty} b_k = \infty$ *and* Y_k *are the inter-arrival times of a RP, then* $\{X_k, k = 1, 2, ...\}$ *is called an exponent extended geometric process (EEGeP).*

As b_k is non-decreasing, the EEGeP has following properties:

- if $a > 1$, the the EEGeP is stochastically decreasing.
- if $0 < a < 1$, then the EEGeP is stochastically increasing.
- if $a = 1$, then the EEGeP reduces to the RP.

2.3.8 Doubly geometric process (DGeP)

The doubly geometric process (DGeP) is proposed by Wu (2018). Different from the GeP, the DGeP can model a situation that the shape parameters of the lifetime distributions of inter-arrival times X_k changes k. Besides, it can model not only monotonously increasing or decreasing stochastic processes, but also a failure process with different trends (stochastically increasing or decreasing), which is similar to TGeP. The definition of a DPG is given in the following definition.

Definition 8. *(Wu, 2018) Given a sequence of non-negative random variables* $\{X_k, k = 1, 2, ...\}$*, if they are independent and the cdf of* X_k *is given by* $F(a^{k-1} x^{h(k)})$ *for* $k = 1, 2...,$ *where a is a positive constant,* $h(k)$ *is a function of k and the likelihood of the parameters in* $h(k)$ *has a known closed form, and* $h(k) > 0$*, then* $\{X_k, k = 1, 2, ...\}$ *is called as a doubly geometric process (DGeP). The* a^{k-1} *is refereed as the scale impact factor and* $h(k)$ *as the shape impact factor.*

The series $\{a^{k-1}, k = 1, 2...\}$ and $\{h(k), k = 1, 2...\}$ can be two different geometric series. According to Definition 8, if $h(k)$ equals to 1, then the shape parameter becomes constant over k if X_1 follows a Weibull distribution. If $h(k)$ is not 1, Wu (2018) uses

$$h(k) = (1 + \log(k))^b$$

where log is the logarithm with base 10 and b is a parameter.

The expectation and the variance of the DGeP is given by

$$\mathrm{E}[X_k] = a^{(1-k)h^{-1}(k)} \mathrm{E}[X_1^{h^{-1}(k)}],$$

and

$$\text{Var}[X_k] = a^{(2-2k)h^{-1}(k)}\text{E}[X_1^{2h^{-1}(k)}] - \text{E}^2[X_1^{h^{-1}(k)}].$$

Based on Definition 8, a DGeP can model more flexible processes and it has the following properties

- if $0 < a < 1$ and $b < 0$, then $\{X_k, k = 1, 2, ...\}$ is stochastically increasing.
- if $a > 1$ and $b < 0$, then $\{X_k, k = 1, 2, ...\}$ is stochastically decreasing.
- if $0 < a < 1$ and $0 < b < 4.898226$, then $\{X_k, k = 1, 2, ...\}$ is stochastically increasing.
- if $a > 1$ and $0 < b < 4.898226$, then $\{X_k, k = 1, 2, ...\}$ is stochastically decreasing.

Pekalp, Eroğlu İnan, and Aydoğdu (2022) and Pekalp, Eroğlu İnan, and Aydoğdu (2020) discussed the statistical inference for the DGeP with Weibull inter-arrival times and exponential inter-renewal times, respectively. Jasim and Nauef (2021) applied the DGeP to the estimation of Covid-19 virus infection cases based on the chicken swarm optimisation algorithm.

2.3.9 Semi-geometric process (SGeP)

Wu and Wang (2018) considered that the independence assumption of the GeP is too restrictive due to a sequence of independent random variables $\{X_k, k = 1, 2, \ldots\}$. Working times between occurrences of failures may be statistically dependent in the real world. In this thesis, a relax assumption is that times between failures are independent.

The definition of the SGeP is following

Definition 9. *(Wu and Wang, 2018) Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if $P\{X_k < x \mid X_{k-1} = x_{k-1}, \ldots, X_1 = x_1\} = P\{X_k < x \mid X_{k-1} = x_{k-1}\}$ and the marginal distribution of X_k has cdf of $F(a^{k-1}x)$, where a is a positive constant, then $\{X_k, k = 1, 2, ...\}$ is called a semi-geometric process (SGeP).*

The SGeP has same expectation, variance and properties of the GeP.

2.3.10 Extended geometric process (EGeP)

Zhang and Wang (2016) considered that a system after repair is not always successive degenerative in practice. Some failures are slight and the impact of such failures after repair can be eliminated completely, therefore, the system is not degenerative. Moreover, when the number of system failures is increasing, the probability of the system degeneration will be increasing.

The definition of the EGeP is following.

Definition 10. *(Zhang and Wang, 2016) Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is $F_k = pF_{k-1}(x) + qF_{k-1}(ax)$, where a, p, q are all positive constants, and $p + q = 1$ for $k = 1, 2, \ldots$, then $\{X_k, k = 1, 2, ...\}$ is called an EGeP, and p_k is called the extended factor.*

The expectation and the variance of the EGeP is given by

$$E[X_k] = E[X_1](p + \frac{q}{a})^{k-1}, \tag{2.10}$$

and

$$\text{Var}[X_k] = \text{Var}[X_1](p + \frac{q}{a})^{2(k-1)}. \tag{2.11}$$

It has the following properties:

- if $p = 0$, then $F_k(x) = F_{k-1}(ax) = F_{k-2}(a^2 x) = \cdots = F(a^{k-1}x)$, and the EGeP reduces to the GeP. In this case,
  - (a) if $a > 1$, then it is stochastically decreasing.
  - (b) if $0 < a < 1$, then it is stochastically decreasing.
- if $p = 1$, then the EGeP reduces to a renewal process.

Zhang and Wang (2018) used the EGeP for a simple repairable system with imperfect delayed repair. Based on the failure number N of the system, an optimal replacement policy N^* is determined by minimizing the average cost and maximizing the average availability rate if the system after repair is not always successively degenerative. Junyuan, Jimin, and Pengfei (2019) proposed a new repairable system model if the preventive repair time and the failure correction time are described with the EGeP.

2.3.11 Geometric Pólya-Aeppli process (GePAP)

The geometric Pólya-Aeppli process (GePAP) is proposed by Chukova and Minkova (2020). It is a compound geometric process with exponential underlying distribution. The definition of the GePAP is as follows

Definition 11. *(Chukova and Minkova, 2020) Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is $F_k(x) = 1 - (1 - \rho)e^{-\mu a^{k-1} x}, k = 1, 2, ...$, where $\rho \in [0, 1)$, a is a positive constant and μ is the rate parameter.*

Chukova and Minkova (2020) proposed this model based on the advantages of the GeP and the Pólya-Appli process (Minkova, 2004) so that the GePAP is able to accommodate trends over time. However, they did not discuss the properties of stochastically increasing/decreasing due to the parameter a.

2.3.12 Double ratio geometric process (DRGeP)

The double ratio geometric process (DRGeP) is proposed by Wu (2022). Suppose that the hazard function of X_k is denoted by $r_k(x)$ and that $\{X_k, k = 1, 2, \dots\}$ follows the GeP, then

$$h_k(x) = ah_{k-1}(ax), \tag{2.12}$$

where the two a's play different roles in and have different implications in describing maintenance effectiveness: the first a describes the effectiveness on how the hazard function is affected and the second a (i.e., the one multiplying x in the parentheses) describes the effectiveness on how the age of the item under maintenance is affected. Therefore, the following definition of the DRGeP can be given

Definition 12. *(Wu, 2022) Given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is given by $F_k(x) = 1 - (1 - F_1(a_k x))^{b_k/a_k}$ for $k = 1, 2...$, where a_k and b_k are positive parameters and $a_1 = b_1 = 1$. Then $\{X_k, k = 1, 2, ...\}$ is a double-ratio geometric process.*

It has the following proprieties:

- if both a_k and b_k are increasing in k, then the DRGeP is stochastically decreasing,
- if both a_k and b_k are decreasing in k, then the DRGeP is stochastically increasing, and
- if a_k or b_k is increasing in k and b_k or a_k is decreasing in k, then the DRGeP may not be stochastically monotonic.

2.3.13 Alternating geometric process (AIGeP)

The alternating geometric process (AIGeP) is proposed by Arnold et al. (2022). It combines the ideas of the GeP and the alternating renewal process (Ross, 1996). The following definition of the alternating GeP can be given

Definition 13. *(Arnold et al., 2022) $\{X_k\}_1^\infty$ and $\{Y_k\}_1^\infty$ are independent sequences, if $\{X_k\}_1^\infty$ is a stochastically decreasing GeP with parameters $\{a, F_{X_1}(t)\}$, $a > 1$ and $\{Y_k\}_1^\infty$ is stochastically increasing GeP with parameters $\{b, F_{Y_1}(t)\}$, $0 < b < 1$. then the sequence of pairs of random variables $\{(X_1, Y_1), (X_2, Y_2), \dots, \}$ is called an alternating GeP with parameters $\{a, F_{X_1}(x); b, F_{Y_1}(x)\}$.*

It is worth noticing that, for the Alternating GeP, it denotes X_k as the kth operational time with cdf F_{X_k} and Y_k as the kth repair time with cdf F_{Y_k}. Therefore, denote by $Z_k = X_k + Y_k$, the length of the kth cycle, then the cdf of Z_k is denoted by $Q_k(z)$,

$$Q_k(z) = F_{X_k} \star F_{Y_k}, \tag{2.13}$$

and the time until the completion of the kth cycle T_k is defined as

$$T_k = \sum_{i=1}^{k} Z_i = \sum_{i=1}^{k}(X_i + Y_i), k = 1, 2, \ldots, \tag{2.14}$$

and denote the cdf of T_k is the convolution of of $G_1(z)$, $G_2(z)$, $\ldots$, $G_k(z)$.

For the alternating GeP, it has separate mean and variance for the number of completed cycles $N(z)$ and the number of failures $M(z)$, respectively. The expectation and variance of $N(z)$ are given by

$$\mathrm{E}[N_z] = \sum_{k=1}^{\infty} kP(N(z) = k),$$

and

$$\mathrm{Var}[N_z] = \sum_{k=1}^{\infty} k^2 P(N(z) = k) - \mathrm{E}^2[N_z].$$

The expectation and variance of $M(z)$ are given by

$$\mathrm{E}[M_z] = \sum_{k=1}^{\infty} Q_k'(z), z \geq 0,$$

and

$$\mathrm{Var}[M_z] = 2\sum_{k=1}^{\infty} kQ_k'(z) - \mathrm{E}[M_z](1 + \mathrm{E}[M_z]), z \geq 0.$$

2.3.14 Rate randomized geometric process(RRGeP)

The rate randomized geometric process is proposed by Asadi and Wu (2024).

Definition 14. *(Asadi and Wu, 2024) A sequence of independent non-negative random variables $\{X_k, k = 1, 2, ...\}$ is said to be a rate-randomized geometric process (RRGeP) if $Y_k \equiv (Z_{k-1})^{k-1} X_k$ for $\{k = 1, 2, ...\}$, where X_k are i.i.d with cdf H that has the domain (τ_1, τ_2), and the sequence $\{X_k, k \geq 1\}$ and $\{Z_k, k =\geq 1\}$ are independent.*

The cdf of Y_k then is given by

$$G_k(x) = \int_{\tau_1}^{\tau_2} F(\frac{x}{t^{k-1}})\mathrm{d}H(t),$$

where F and H are two cdf's on the positive real line with $\tau_1, \tau_2 \geq 0$.

The relevant expectation and variance are given by:

$$\mu_k = \mathrm{E}[Z^{k-1}X] = E(Z^{k-1}\mathrm{E}[X] = \mu\mathrm{E}[Z^{k-1}], k = 1, 2, \ldots,$$

where μ is the mean of F and mu_k is the mean of Y_k. And

$$\sigma_k^2 = \mathrm{Var}[Z^{k-1}X] = (\mathrm{E}[Z^{k-1}]^2\mathrm{Var}[X] + \mathrm{E}[X]\mathrm{Var}[Z^{k-1}]), k \geq 1.$$

It has the following basic proprieties:

- if the support of H is $(0, \tau_2)$, $\tau_2 \leq 1$(or $\tau_1 \leq 1$, the support is (τ_1, ∞)), then the RRGeP is decreasing (increasing) in the sense of usual stochastic order, that is, $Y_{k+1} \leq_{st} (\geq_{st}) Y_k$.
- if Z is a degenerate random variable concentrated on 1, then the RRGP reduces to the renewal process. Otherwise, the RRGP can be non-monotonic in k.

2.3.15 An overview of the extensions of the GeP

From the above analysis, we can summarise those GeP extensions in Table 2.3.

TABLE 2.3: Summary of the GeP and its extensions

Survival distribution	Model	Reference
$F(a^{k-1}x)$	GeP	Lam (1988b)
$F_k(g(k)x)$	GRP	Wu (1995)
$F(a^{k-1}x + a^{k-1}(k-1)d)$	ArGeP	Francis (2001)
$F(k^a x)$	α-SP	Braun, Li, and Zhao (2005)
$F(a^{k-M_i}x)$	TGeP	Chan et al. (2006)
$F((\alpha a^{k-1} + \beta b^{k-1})x)$	EPP	Wu and Clements-Croome (2006)
$1 - F(a^{k-1}x)$	BGeP	Chan and Leung (2010)
$F(a^{b_k}x)$	EGeP	Bordes and Mercier (2013)
X_k are depenedent and $X_k \sim F(a^{k-1}x)$	SGeP	Wu and Wang (2018)
$F_k = pF_{k-1}(x) + qF_{k-1}(ax)$	EGeP	Zhang and Wang (2016)
$F(a^{k-1}x^{g(k)})$	DGeP	Wu (2018)
$F_k(x) = 1 - (1-\rho)e^{\mu a^{k-1}}x$	GePAP	Chukova and Minkova (2020)
$1 - (1 - F(a_k x))^{b_k/a_k}$	DRGeP	Wu (2022)
$\{(X_1, Y_1), (X_2, Y_2), \ldots, \}$	AIGeP	Arnold et al. (2022)
$G_k(x) = \int_{\tau_1}^{\tau_2} F(\frac{x}{t^{k-1}})\mathrm{d}H(t)$	RRGeP	Asadi and Wu (2024)

According to the summary of the GeP and its extensions, it is obvious that $\{X_k, k = 1, 2, ...\}$ is monotonous over k (a is greater or smaller than 1), which means that the GeP cannot be used to describe the case with non-monotonous failure ratio/patterns. To overcome this problem, the parameter a is developed to a function over k, which can be generalized as $g(\delta_1, k)$, where δ_1 is a parameter vector of failure ratio. For example, a^{k-1} of the GeP and the DGeP, k^α of the α-SP process, a^{b_k} of the EGeP, a^{k-M_i} of the TGeP, and $(\alpha a^{k-1} + \beta b^{k-1})$ of the EPP. Wu, Peng, and Wu (2020) also summarized a similar conclusion and indicated that $g(\delta_1, k)$ is a positive function. They also indicated that the index of X_k can be considered as a positive scale function, which can be denoted by $h(\delta_2, k)$. However, this may be too stringent for many extensions of the GeP, such as the ArGeP, the EGeP, the GePAP, and the DRGeP. Therefore, in this thesis, we only consider $g(\delta_1, k)$ as a function over k for generalizing the GeP and its extensions. We will discuss several generalizing properties of the GeP-like models and special cases in Section 5.2.

2.3.16 Applications of the GeP and its extensions

The GeP-like models are normally used for modelling the working times after the repairs and the repair times for each maintenance activities. Commonly, the GeP-like models can be applied into the reliability and maintenance and warranty analysis for estimating the repair or replacement interval time, planning the corrective and preventative maintenance and predicting the number of warranty claims, etc. Several existing research publications have reviewed (Wu, Peng, and Wu, 2020; Arnold et al., 2020). In this thesis, we focus on other applications of the GeP-like models. Table 2.4 shows a summary of the application of the GeP-like models beyond reliability and maintenance.

TABLE 2.4: Summary of the applications of the GeP-like models

Application	GeP-like model	Problems
Coal mining disasters	GeP	Simulation (Lam et al., 2004)
SARS epidemic	TGeP	Forecasting epidemic (Chan et al., 2006)
Coal mining disasters	BGeP	Simulation (Chan and Leung, 2010)
Market stock	TGeP	High-low stock price (Chan et al., 2012)
Crime analysis	GeP	Number of arrests (Chan and Wan, 2014)
Electricity price	TGeP	Forecasting (Chan, Choy, and Lam, 2014)
Recruitment	GeP	Forecasting & Recruitment policy (Chan, Wan, and Yu, 2014)
Crime analysis	GeP	Cannabis offenses (Chan and Wan, 2016)
COVID-19 contagion	DGeP	Disease contagion(Jasim and Nauef, 2021)
Recruitment policy	GeP	Loss of recruitment (Samundeeswari, Kanchana, and Varuvel, 2022)

Chan et al. (2006) proposed the TGeP, which can be considered a collection of several GePs with different ratio parameters a_k based on the moving windows. The moving windows can separate a group of data into different subsets of fixed length starting from the first window with ratio a_1. In this research, the SARS data of Hong Kong, Singapore, Toronto and Taipei were separated into several subsets based on their moving windows. The model performance were be estimated by log-LSE and LSE method, respectively.

Chan, Choy, and Lam (2014) used the TGeP to modelling the changes of electricity market prices over time. One characteristic of the electricity market prices is that it is floating with high spikes at different time period for a whole day. The threshold of the TGeP corresponds to the high spikes of the electricity price. This is the reason why the TGeP had a good performance on this case.

Chan and Wan (2016) used the GeP to analyse the cannabis offences in New South Wales of the Australia. According to their data analysis, they found that there exists underdispersed and overdispersed data, which shows a non-monotonicity trend. The Markov chain Monte Carlo was used introduced to overcome such problems. Besides, they discussed the model performance under both underdispersed and overdispersed data and used the result of an equidispersion situation as comparison. Jasim and Nauef (2021) applied the DGeP to the estimation of Covid-19 virus infection cases based on the chicken swam optimisation algorithm.

Other research face to similar problems. Chan and Wan (2014) proposed the multivariate generalized Poisson log-t geometric process to analyse the number of arrests due to the use of two illicit drugs. Lam et al. (2004) and Chan and Leung (2010) used the data of coal mining disasters to compare the model performance among several times series models and their extended GeP-like models.

From the above-reviewed papers, the GeP-like models can be applied into other fields such as the business analysis. Besides, due to the properties of the GeP-like models, we consider that they can be used in the fields where data can be described by the counting process such as the queuing problem and the transportation.

2.4 Maintenance policy

Maintenance policy is sometimes also called maintenance strategy in reliability. Its purpose is to ensure that the system can achieve a certain level of reliability in a long-term period by setting and achieving decision-making objectives and implementing different maintenance activities so that the system can keep operation. For the users of a system, an optimised maintenance policy can effectively reduce the long-term operating costs of the system, while also improving the system availability. Engineers usually collect historical failure data and cost related data from the operation of a system and then build a failure process model on such data. All the factors such as maintenance performance (effectiveness of maintenance), frequency of maintenance, objectives of maintenance (minimizing cost or maximizing profit), etc, are the main components of a maintenance policy. A maintenance policy can be broke down into three main parts: how to perform maintenance, when to repair/replace, and how to optimise maintenance, which are shown in Figure 2.2. Each part will be described in the next three subsections.

2.4.1 Effectiveness of a repair

There are two common actions for restoring the state of a system: a) repair; and b) replacement. The former has two categories: minimal repair and imperfect repair, while the latter can be considered a special case of repair. In other words, a replacement can be also called a perfect repair. Since each repair has an impact on the existing status of a system, it is necessary to assess the effectiveness of the repair. Moreover, the effectiveness of repair is one of the core basic assumptions before a degradation process is built so that the system status after a repair can be effectively re-evaluated and then planned for a long-term maintenance policy. This concept is common for many degradation models and even for some dynamic maintenance policies(Liu et al., 2017; Mercier and Castro, 2019). The effectiveness of maintenance can be typically categorized into perfect repair, minimal repair, and imperfect repair (De Carlo and Arleo, 2017; Wu and Scarf, 2017). Commonly, the effectiveness of common maintenance actions is discussed as follows:

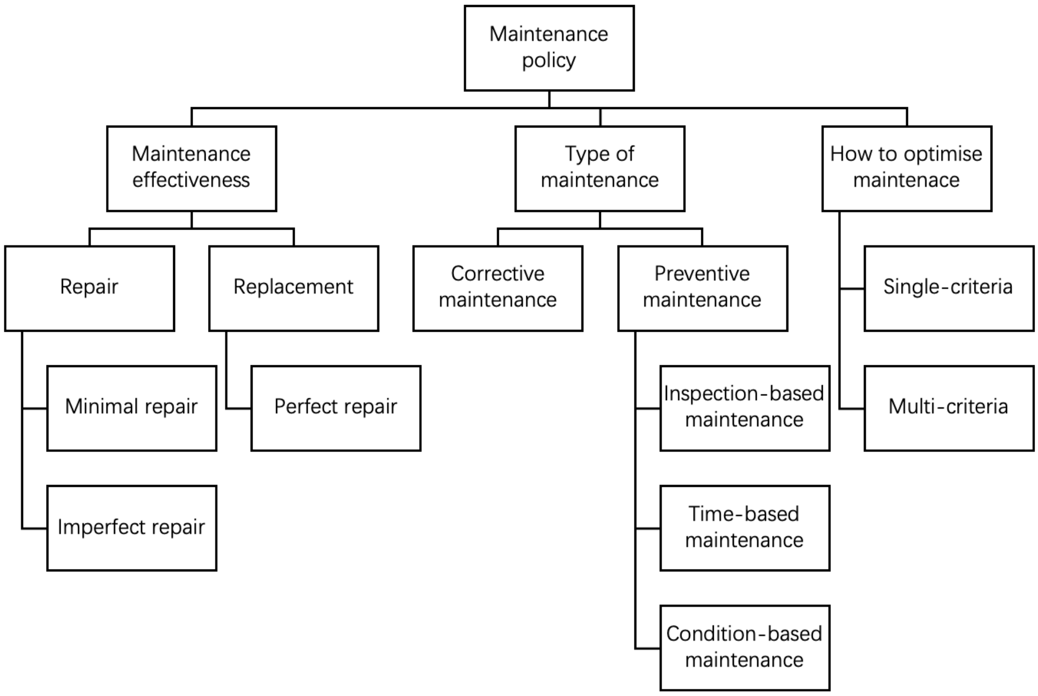

FIGURE 2.2: Framework of the maintenance policy

Perfect repair restores a system to the "as good as new" condition. Generally, the replacement of an item with a new identical one is a perfect repair. Therefore, the times-between-failures forms a renewal process in which the inter-arrival times are identical and independently distributed (Brown, 2005).

Minimal repair restores a failed item to the status immediately before the failure occurs, or the "as bad as old" state. Wu and Scarf (2017) indicate that the non-homogeneous Poisson process (NHPP) is the only model for describing minimal repair.

Imperfect repair restores a system to a state which is between "as good as new" and "as bad as old". Thus, perfect repair and minimal repair can be considered as two extreme conditions of imperfect repair. Models such as the arithmetic Reduction of Intensity model (ARI) and Arithmetic Reduction of Age model (ARA) can be used to model the effectiveness of imperfect repairs (Doyen and Gaudoin, 2004). Other models include the Brown-Proschan model (Brown and Proschan, 1983; Block, Borges, and Savits, 1985), Kijima's virtual age models (Kijima and Sumita, 1986; Kijima, 1989) and geometric process (Lam, 1988a; Wu and Clements-Croome, 2006; Wu and Scarf, 2017) also have been widely used for this type of maintenance action. (Tanwar, Rai, and Bolia, 2014).

2.4.2 Maintenance methods

In reliability, maintenance methods mainly determine when maintenance should be taken place. For a maintenance policy, the key point is whether the maintenance is implemented after a failure or before a failure. When maintenance is implemented upon failures, it is considered corrective maintenance. In comparison, preventive maintenance is a planned maintenance.

Corrective maintenance (CM) is taken upon failures (British Standard, 2017). With the economies of scale (Alaswad and Xiang, 2017), it is effective to save cost by reducing the probability of failure. In other words, CM can be the result of a deliberate run-to-failure strategy. Gan, Song, and Zhang (2022) proposed a CM policy under imperfect maintenance with two thresholds of the number of completed CM in two stages. The thresholds mean the allowed maximum number of CM in two stages, which are optimised by minimizing the expected cost. Hashemi, Asadi, and Tavangar (2022) proposed a warranty policy with two phases and the failed components of a system are replaced under a CM in the first phase. Similar applications of CM can also be seen in other warranty-related articles (Wang, Liu, and Liu, 2015; Changa, 2022; Gupta and Bhattacharya, 2022). Besides, CM is suitable for a system without high risk and can be performed quickly (Erkoyuncu et al., 2017; Pinciroli, Baraldi, and Zio, 2023).

Preventive maintenance (PM) aims at evaluating the risk of failure occurrence and minimizing the consequence of the failure (British Standard, 2017). Compared with CM, PM focuses more on preventing the system from stopping working. For the systems whose failures may induce severe consequences, PM can reduce the cost and time of maintenance. Moreover, it is helpful to avoid other losses due to the occurrence of failure. For example, traffic accidents and blockages caused by road potholes and accidents caused by cracks in oil pipelines.

Common types of PM include time-based preventive maintenance (TbM) and condition-based maintenance (CbM). One main difference between TbM and CbM is that the latter requires real-time monitoring data of physical factors related to the degradation of a system, such as temperature (Rasmekomen and Parlikad, 2016), vibration (Al-Najjar, 2012) and pressure (Kenda, Klobčar, and Bračun, 2021), etc. This makes data more difficult and costly to acquire. Therefore, in terms of the difficulty of data acquisition and scheduling time of maintenance, the implementation of TbM is usually simpler than that of CbM. The maintenance is scheduled by the usage time under TbM. However, precisely because of this scheduling maintenance time, TbM may encounter situations where the system is maintained while the state is still "healthy", and failures occur earlier than the expected maintenance time. This accumulation of waste will lead to a waste of long-term maintenance costs under TbM.

To sum up, TbM is suitable for a system with predictable failure patterns (Ahmad and Kamaruddin, 2012; Jonge, Teunter, and Tinga, 2017; Pinciroli, Baraldi, and Zio, 2023), and whose failures do not lead to severe consequences, such as the fire caused by system failure, or life safety. CbM is suitable for a system such systems whose status (pumps, turbines, and hydraulic system) can be monitored easily. Syan and Ramsoobag (2019) indicate that PM is widely used with a percentage of usage 23% and CM is 14.7%.

2.4.3 Optimisation objectives

How to optimise maintenance policies is determined by the optimisation objectives, which determine the ultimate result of the maintenance policy. Optimisation can be the long-term maintenance cost of pavement (Wu and Castro, 2020), maintenance cost for an electric utility in Algerian power industry (Yssaad, Khiat, and Chaker, 2014), profit-maximization of a deregulated power system (Mazidi et al., 2018), availability for multiple failure mode system (Qiu et al., 2021), etc.

According to Pinciroli, Baraldi, and Zio (2023), such objectives can be presented by

$$\begin{aligned} &\text{min(max)} && f(x) \\ &\text{subject to} && h_i(x) \leq s_i, i = 1, 2, \ldots, m \end{aligned}$$

where $f(x)$ is the objective function with q different criteria with $f(x) = [f_1(x), f_2(x), \ldots f_q(x)]$. The value of q determines the class of optimisation objectives:

- If $q = 1$, then it is a single-criteria policy;
- if $q > 1$, then it is a multi-criteria policy.

$h_i(x)$ is the constraint function with associated upper bound b_i, which limits the value on x. In general, $f(x)$ can be cost, profit, reliability, availability, safety, reliability or other factors in reliability, where cost is the most widely used. Syan and Ramsoobag (2019) indicated that the most common objective is cost with 61.3% of all possible objectives, followed by availability at 28% and reliability at 26%. $h_i(x)$ can be operation time, repair time, labor, or other factors relevant to the objective function. For example, if a maintenance policy involves two criteria: maintenance cost and risk of loss, which means $q = 2$, then, the constraints can be the downtime due to all failures in a given time. As people increasingly pursue the fitness of complex degradation models for simulating a more practical deterioration of a system, multi-criteria policies are increasingly common (Faghihinia and Mollaverdi, 2012; Cavalcante and Lopes, 2015).

2.5 Summary

In this section, we discussed two issues: degradation modelling and maintenance policy optimisation. The application of current degradation models will be summarized in three aspects: methodology, application field, and real-world cases. Then, we will propose knowledge gaps.

The degradation of a system is a process to describe how a system changes due to the environment and daily usage. This process leads to failure when the level of degradation exceeds a threshold for most systems. Even if the degradation does not cause the system to stop working, it will greatly reduce system performance.

According to the literature review in this thesis, we note that the scientific literature on maintenance policy optimisation mainly focuses on the development and extension of maintenance models.

2.5.1 Analysis of the research on combined degradation processes

Table 2.2 shows some papers that investigate methods to model the degradation process of a system through a combination of different approaches. Moreover, other research attempts to solve other problems for multi-component systems and systems with multi-failure modes. However, all of these models still have some limitations, which are analysed below.

Wu and Castro (2020) proposed a maintenance policy within a linear combination of degradation processes. Their model assumes that the effectiveness of repair is imperfect. However, when designing a long-term maintenance policy for a pavement network, the threshold may not have only one but multiple values for various failure modes, which is associated with different maintenance actions. For example, alligator cracking, consolidation of pavement layers, and longitudinal cracking may have different thresholds for a pavement network. This should be considered as the challenge for multiple degradation processes for multiple failure modes and will result in different maintenance activities with different costs. Besides, as the age of the road continuously grows, the rate of degradation may not be constant and is possibly increasing. Ma et al. (2020) propose a two-stage Wiener process for a warm standby cooling system. Their model is helpful in optimizing the total maintenance cost via the preventive control limit. However, a new idea for this model is to extend it to a multi-component system. Furthermore, developing this model with multi-failure modes is another possible direction for future study.

Feng et al. (2015) proposed a system-level reliability model for a multi-component system with two dominating failure modes. However, the independence between two failure modes in this study is an assumption. Many other papers on multi-failure modes also have the limitation of dependence analysis (Liu et al., 2013; Moghaddass and Zuo, 2012; Yang et al., 2018b). A different literature was proposed by Zhang et al. (2020), who combined a k-out-of-n system with a pure jump Levy process and failure dependence. The main difference of this model from other models is that the failure dependence among components is considered as a random factor in this model. Another advantage of this model is that it considers both the short-term and long-term horizons for maintenance costs to assess the effect of the failure dependence. Although the numerical example in this research shows a good result and this model can be applied for a three-wind turbine system, this model does not incorporate the factor of changeable cost and the multiple failure modes. Other research also has similar challenges, see in Yang et al. (2018a) and Ma et al. (2020)

In addition, the estimation of parameters for CbM models is still a big challenge. This problem is normally caused by the changeable environment. A common practice is to assume that the coefficient is a constant value and to bring in different fixed values to fit the model to the real-world data. However, the real environment is changeable, which leads to the rate of degradation will be different. Moreover, improving the estimation of parameters is helpful in increasing the performance of a model. Thus, to explore which factor will influence the degradation of a system and use dynamic value to replace a fixed value is meaningful. Cholette et al. (2019) indicated that a gamma process with a physic-based erosion model should be improved with respect to parameter uncertainty in the estimation and prediction. It can lead to underestimate the confidence intervals for the tube thickness.

Moreover, the problem of downtime or delay time should be considered. Wu and Castro (2020) assumed zero downtime when the pavement is repaired. However, the fact is that when road maintenance is carried out, problems such as traffic jams often occur at the same time. Thus, the negative impact of downtime or other opportunity cost is necessary to be considered in a CbM model, see in Nicolai, Dekker, and Van Noortwijk (2007) and Ma et al. (2020), for more discussions.

In addition to the above analyses, we summarise the real-world examples that the publications have used in Table 2.5. The purpose of this table is to illustrate the practical scope of various methods. These current challenges for CbM will be integrated into the framework of CbM. Some possible developments will be explained in Section 2.5.2.

2.5.2 Knowledge gaps

Based on the above literature review, we can identify the following knowledge gaps.

Multi-failure modes system

The existing literature has paid little consideration on the uncertainty of the modelling methods for the failure process of the system that needs preventive maintenance, which motivates us to consider this issue in this thesis.

In Chapter 3, we will develop a maintenance policy for a degrading k-out-of-n system with multi-criteria model-selecting in forms of geometric-like processes.

Maintenance policy and cost process

Although there is a large amount of existing research that has explored different maintenance policies for systems to optimise the expected maintenance cost, most

TABLE 2.5: Research on combined deterioration models and applications

Methodology	Application Field	Real-world cases
Gamma process	Modelling, reliability, diagnostics and prognostics	Cracks of pavement, Boiler heat exchangers, oil or gas tube
Wiener process	Modelling, reliability, diagnostics and prognostics, paraments estimation, RUL estimation	White-light LEDs, laser devices, fatigue crack growth, wear of structure, hard disk drives, bearings and rotating, batteries
Inverse Gaussian	Modelling, reliability, diagnostics and prognostics,	Cracks of pavement, fatigue crack growth, wear of structure, oil or gas tube
NHPP	Modelling, reliability, diagnostics and prognostics	Fatigue crack growth,
Hybrid models	Modelling, diagnostics and prognostics, k-out-of-n system, multi-components systems, single-components with multi-failure modes,	Nuclear reactor safety system and circuit breaker.

of them assume that the associated maintenance cost is constant. Such as the inspection cost, which may be adjusted due to inflation or economies of scale, is usually a fixed value for most maintenance policies. Besides, the cost is usually an objective function so that other variables such as the time interval of inspection and the number of inspections can be determined for an optimised result. The fact that cost itself is a very important constraint in reality is often forgotten: for a decision-maker, the budget of maintenance cost is limited in a given time. Besides, another fact that cannot be ignored is that as the remaining lifetime of the system becomes shorter and shorter, the corresponding maintenance costs and maintenance time may become larger and larger. Thus, considering different maintenance policies with unstable costs is meaningful. In this thesis, we indicate two possible developments to overcome such limitations of maintenance policy and cost:

- the change of cost can also be considered as a degradation process for a system. The definition of a cost process was first proposed by Wu and Castro (2020). Under this assumption, the threshold of cost has the physical meaning that is the budget limitation of total cost in a given time. However, more research is still needed to explore the potential applications of the cost process;

- the choice of maintenance policies is different from the multi-criteria model-selecting. Under the assumption of the cost process, it is possible that the degradation process of a system has not achieved its threshold but the cost process has achieved it first. Therefore, it is necessary to consider these four situations: a) under the degradation process, the pre-specified threshold of it has been achieved; b) under the cost process, the pre-specified threshold of it has been achieved; c) under the degradation process and the cost process, both pre-specified thresholds of them have been achieved; d) under the degradation process and the cost process, either one pre-specified threshold of them has been achieved.

Chapter 4 will develop a new maintenance policy to overcome the above-mentioned problems.

Model performance and statistical properties

According to the literature review, although the authors usually give numerical examples whenever a new degradation model or new maintenance policy is proposed, few people have conducted comparisons of the performance of models. In this thesis, we will compare the performance of models based on the classification in Section 2.2.1 and Section 2.3, which are related to extensions of the geometric process. Besides, existing literature rarely considers the sensitivity of the optimum policy to the values of parameters and systematic analysis on statistical properties of geometric-like processes. Therefore, Chapter 5 will investigate how the optimum policy is sensitive to the values of parameters and some statistical properties. Then, Chapter 6 will use 23 datasets to compare the model performance of several GeP-like models.

Chapter 3

Maintenance policies of a degrading k-out-of-n system

3.1 Introduction

In the reliability and maintenance literature, modelling the failure process of an engineered system is important so that people can estimate the probability of failures and then plan preventive maintenance policies to keep the system at a certain level of availability and prevent it from causing huge financial losses (Smith, 1993; Basri et al., 2017). However, such models are mathematical models approximating the reality.

A k-out-of-n system is a system that consists of n components. It requires at least k components workable for system operation (k-out-of-n: F system) or more than k components workable for system operation (k-out-of-n: G system). In other words, a k-out-of-n: G system fails when the number of failed components is greater than $(N - k)$.

The gamma process has been widely used in the area of reliability since it was first introduced by Abdel-Hameed (1975). According to Chapter 2, the gamma process is a stochastic process with independent and non-negative increments, which follows a gamma distribution. This means that the gamma process can describe a monotonically increasing degradation process, such as the corroded steel gates (Frangopol, Kallen, and Noortwijk, 2004), and pavement crack growth (Lawless and Crowder, 2004). Wu and Ding (2022) proposed a model of multiple dependent failure processes with the effect of a dynamic environment. They used the gamma process to estimate the natural wear behaviour of the system.

In terms of a k-out-of-n: F system, this chapter considers a multi-component system whose degradation process can be modelled with the gamma process and assumes that the scale parameter of the gamma process between failures becomes smaller after each repair.

3.2 Maintenance policies of multi-component systems: A framework

In this section, the common approaches to maintenance policy optimisation will be mainly discussed.

Maintenance can be defined as a series of actions which aim to restore an item to the state that it is able to continuously work as always (British Standard, 2017). To this end, corrective maintenance (CM) and preventive maintenance(PM) can be applied.

Corrective maintenance (CM) is taken upon failures (British Standard, 2017) which is conducted after a failure occurs. With the economies of scale (Alaswad and Xiang, 2017), it is effective to save cost by reducing the probability of failure. In other words, CM can be the result of a deliberate run-to-failure strategy. However, the downtime, which is caused by system failures, may increase the risk of other costly production or other losses. For instance, the producers must pay the penalty in time due to the failure of the equipment.

Preventive maintenance (PM) is done before a failure may occur and aims at evaluating the risk of failure occurrence and minimising the consequence of the failure (British Standard, 2017). It is basically done at regular intervals or when some components reach a specific level while the equipment is still functioning. Time-based preventive maintenance (TbM) and condition-based maintenance(CbM) are two typical types of PM. It is worth mentioning that age-based preventive maintenance(AbPM) is a special case of TbM. It is also developed to balance the trade-off between the time between failure information and the cost of maintenance.

Figure 3.1 shows the flow chart of the reliability and maintenance policy. Important steps in the flow chart will be explained as follows:

Understanding of deteriorating parameters and covariates: It is important to understand how the failure process of a system develops. This step will decide how to model the degradation progression of the system in Diagnostic & Prognostics.

Data collection & Data processing: These two steps are usually performed together. Commonly, these two types of data will be recorded and used in reliability: failure data and maintenance data. The former usually contains information about failure mode, failure cause, detection method, and operating time between failures. The latter contains the type of maintenance policy, maintenance activities, maintainable components, and downtime of maintenance.

Diagnostic & Prognostics: This is one of the most important steps in reliability. It aims to model the degradation progression of a system. Therefore, one core is

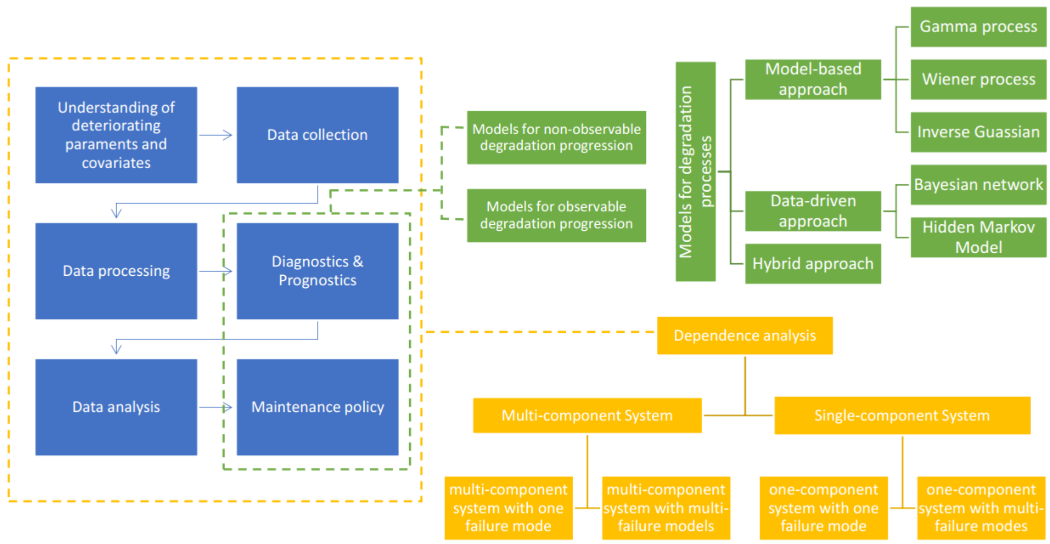

FIGURE 3.1: Flow chart of the reliability and maintenance policy

which approach should be used for modelling. Besides, it is necessary to consider the methods of parameter estimation. Such models for degradation progression normally can be categorized into systems with non-observable and observable degradation progression.

Data analysis & Model performance: This step aims to estimate the model performance based on the relevant data. Common methods such as the AIC, AICc, and BIC will be used.

Maintenance policy & optimisation: The high maintenance costs and the downtime due to system failure drive people to find effective maintenance to minimize the objective of a company. Therefore, this is also an important step in reliability which aims to design and optimise a maintenance policy for minimizing or maximizing the objective, which can be the long-term cost, the risk of loss, or the profitability of quality production (Sharma, Yadava, and Deshmukh, 2011).

In the last several decades, there is a lot of research about the reliability and maintenance that uses different models such as generalised renewal process (Kijima and Sumita, 1986; Kijima, 1989; Tanwar, Rai, and Bolia, 2014), NHPP (Duane, 1964), geometric process (Jain and Gupta, 2013) to assess the reliability of a system. However, a drawback is that the assumption of a model is over-simplified, which results in situations that the model and its associated results divorce from the ground truth.

3.3 Model development

3.3.1 Assumptions

This chapter makes the following assumptions:

- The system under discussion is composed of n components.
- Each component has its degradation level $X_i(t)$ at time t.
- All components have the same lifetime distribution.
- The system is new at time $t = 0$, all components develop from $X_i(0) = 0$.
- If at least k components exceed the pre-specified thresholds of the levels of their degradation, then all components will be replaced.
- The cost of a replacement cost is denoted by c_r. Time on replacement or repair can be neglected.
- The degradation process of component i is modelled by the GaP with shape parameters α and scale parameters β_j.
- PM is carried out at each T unit of time.
- The scale parameter of component i after the j-th maintenance is $\beta_j = a^{j-1}\beta$ with $a > 0$ and the shape parameter remains unchanged, where t is the time starting from the $(j-1)$-th repair.

3.3.2 Basic processes

Suppose a system is composed of n identical components, each of which degrades. The system fails once there are at least k components whose levels of degradation exceed the pre-specified thresholds.

Repair is carried out upon the failures of a component. As assumed, the scale parameter in the gamma distribution after the $(j-1)$-th repair is given by $\beta_j = a^{j-1}\beta$ with $a > 0$ and $j = \{1, 2, \dots\}$. That is, $X_{j,i}(t)$ has its cumulative probability function after the $(j-1)$-th repair is given by

$$F_j(x; \alpha(t), \beta_j) = \frac{\gamma\left(\alpha(t), \frac{x}{a^{j-1}\beta}\right)}{\Gamma(\alpha(t))} = F(a^{1-j}x; \alpha(t), \beta). \tag{3.1}$$

According to the pdf of GaP, its probability density function (pdf) after the $(j-1)$-th repair becomes

$$f_j(x; \alpha(t), \beta_j) = \frac{a^{1-}\beta^{\alpha(t)} x^{\alpha(t)-1} e^{-x/a^{j-1}\beta}}{\Gamma(\alpha(t))} 1_{(x>0)} = f(a^{1-j}x; \alpha(t), \beta_j). \tag{3.2}$$

Denote $\mathbb{A}_j$ as the set of random variable $X_{j,i}(t)$ in descending order, then

$$\mathbb{A}_j = \{X_{j,(1)}(t), X_{j,(2)}(t), \dots, X_{j,(k-1)}(t), X_{j,(k)}(t), X_{j,(k+1)}(t), \dots, X_{j,(n)}(t)\} \tag{3.3}$$

where $X_{j,(i)} \geq X_{j,(i+1)}$. Denote $Y_j(t)$ as the degradation level of the system after the $(j-1)$-th repair. That is,

$$Y_j(t) = X_{j,(k)}(t). \tag{3.4}$$

Denote the failtime distribution of $Y_j(t)$ as $G_j(x;\alpha(t),\beta_j)$, then we obtain

$$G_j(x;\alpha(t),\beta_j) = \sum_{i=k}^{n} \binom{n}{i} F_j(x;\alpha(t),\beta_j)^i (1 - F_j(x;\alpha(t),\beta_j))^{(n-i)} \tag{3.5}$$

where $F_j(x;\alpha(t),\beta_j)$ is the lifetime distribution of component i. The associated probability density function of the system is therefore given by

$$\begin{aligned} &g_j(x;\alpha(t),\beta_j) \\ =&\sum_{i=k}^{n} \binom{n}{i} (if(t)F(x;\alpha(t),\beta_j)^{i-1}(1 - F(x;\alpha(t),\beta_j))^{(n-i)} \\ &-\sum_{i=k}^{n}(n-1)f(x;\alpha(t),\beta_j)F(x;\alpha(t),\beta_j)^i(1 - F(x;\alpha(t),\beta_j))^{n-i-1}), \end{aligned} \tag{3.6}$$

where $f(t)$ is the pdf of a component. In this case, $G_j(t)$ is the cdf of the whole system at time t and $g_j(t)$ is the pdf of the whole system at time t.

3.3.3 First hitting time

To characterise the maintenance scheme of this system, the distribution of the first hitting time of the process $\{Y(t), t \geq 0\}$ should be obtained. Starting from $Y(0) = 0$ and for a fixed degradation level L for each component, the first hitting time σ_L is defined as the amount of time required for at least k components of the system in the process $Y_{X_{(n)}}$ to

$$\sigma_{j,L} = \inf(t > 0 : Y_j(t) \geq L).$$

The cdf of $\sigma_{j,L}$ is obtained by

$$\begin{aligned} F_{\sigma_{j,L}}(L;\alpha(t),\beta_j) &= P(Y_j(t) \geq L) \\ &= 1 - G_j(L;\alpha(t),\beta_j) \\ &= 1 - \sum_{i=k}^{n} \binom{n}{i} F_j(L;\alpha(t),\beta_j)^i (1 - F_j(L;\alpha(t),\beta_j))^{(n-i)} \\ &= \sum_{i=0}^{k-1} \binom{n}{i} F_j(L;\alpha(t),\beta_j)^i (1 - F_j(L;\alpha(t),\beta_j))^{(n-i)} \\ &= \sum_{i=0}^{k-1} \binom{n}{i} F_j(L;\alpha(t),\beta_j)^i (1 - F_j(L;\alpha(t),\beta_j))^{(n-i)} \\ &= \sum_{i=0}^{k-1} \binom{n}{i} F(a^{1-j}L;\alpha(t),\beta)^i (1 - F(a^{1-j}L;\alpha(t),\beta))^{(n-i)} \end{aligned} \tag{3.7}$$

Then the pdf of $\sigma_{j,L}$ is given by

$$
\begin{aligned}
f_{\sigma_{j,L}}(L;\alpha(t),\beta_j) =& \sum_{i=0}^{k-1}\binom{n}{i} ia^{1-j}f(a^{1-j}L;\alpha(t),\beta)F(a^{1-j}L;\alpha(t),\beta)^{i-1}(1-F(a^{1-j}L;\alpha(t),\beta))^{(n-i)} \\
&-\sum_{i=0}^{k-1}\binom{n}{i}(n-i)a^{1-j}f(a^{1-j}L;\alpha(t),\beta)F(a^{1-j}L;\alpha(t),\beta)^{i}(1 \\
&-F(a^{1-j}L;\alpha(t),\beta))^{(n-i)} \qquad (3.8)
\end{aligned}
$$

3.4 Optimisation of maintenance policy

The replacement costs of different components will be totally different. Denote c_i as the cost of replacing the ith failed component. Then, the expected repair cost for the whole system after the $(j-1)$-th repair can be obtained by:

$$
\begin{aligned}
U_j(t) &= \sum_{i=1}^{k} C_{j,i}(t|Y_j(t)\geq L) \\
&= \sum_{i=1}^{k} c_i E[X_{j,i}(t|Y_j(t)\geq L)] \\
&= \sum_{i=1}^{k} c_i \int_0^{+\infty} x f_{\sigma_{j,L}}(x;\alpha(t),\beta_j)dx \qquad (3.9)
\end{aligned}
$$

Assume that we replace the system after N failures have occurred. Then, with a pre-specified threshold L, the expected cost ratio is given by

$$
C(N|L) = \frac{c_r + \sum_{i=1}^{k}\sum_{j=1}^{N} c_i \int_0^{+\infty} t f_{\sigma_{j,L}}(L;\alpha(t),\beta_j)dt}{\sum_{j=1}^{N}\int_0^{+\infty} t f_{\sigma_{j,L}}(L;\alpha(t),\beta_j)dt}, \qquad (3.10)
$$

where c_r is the replacement cost of the system and c_i is the repair cost for each repair. The optimal N^* can be obtained from

$$
N^* = argmin_N C(N|L) \qquad (3.11)
$$

3.4.1 From a GeP perspective

As can be seen from the above discussion, $\{X_{1,i}, X_{2,i}, \ldots\}$ forms a GeP, based on which relevant maintenance policies and the probabilistic properties can be investigated. The parameter a in Eq. (3.1) can be estimated from observations collected from real applications. Apparently, assuming a stochastically decreasing or increasing time to failures is restrictive. This is especially the case for engineered systems whose failure behaviour may exhibit bathtub curves. To relax this assumption, one can use other models. For example, an extension of the geometric process introduced in Wu and Clements-Croome (2006) or the α-series process introduced in Braun, Li, and Zhao (2005), as defined in the followings.

Definition 15. *(Wu and Clements-Croome, 2006) Given a sequence of non-negative random variables* $\{X_j, j = 1, 2, ...\}$*, if they are independent and the cdf of* X_j *is given by* $F((\alpha a^{j-1} + \beta b^{j-1})x)$ *for* $j = 1, 2...,$ *where* $\alpha + \beta \neq 0$*,* $\alpha, \beta \geq 0$*,* $a \geq 0$ *and* $0 < b \leq 1$*, then* $\{X_j, j = 1, 2, ...\}$ *is called as an extended Possion process (EPP).*

Besides, an EPP has these two main properties:

- if $\alpha a^{j-1} \neq 0$ and $b = 1$,
  (a) if $a > 1$, then the EPP can model a failure process with decreasing failure intensity functions with respect to j;
  (b) if $a < 1$, then the EPP can model a failure process with increasing failure intensity functions with respect to j;
- if $a = 1, b < 1$ and $\beta b^{j-1} \neq 1$, then $\{X_j, j = 1, 2, ...\}$ can model the process with increasing increasing failure intensities over j.

Definition 16. *(Braun, Li, and Zhao, 2005) Given a sequence of non-negative random variables* $\{X_j, j = 1, 2, ...\}$*, if they are independent and the cdf of* X_j *is given by* $F(j^a x)$ *for* $j = 1, 2...,$ *where* $a \in \mathbb{R}$*, then* $\{X_j, j = 1, 2, ...\}$ *is called as an* α*-series process (*α*-series).*

Moreover, an α has these main properties:

- if $a < 0$ then, $\{X_j, j = 1, 2, ...\}$ is stochastically increasing,
- if $a > 0$, then $\{X_j, j = 1, 2, ...\}$ is stochastically decreasing,
- if $a = 0$, then $\{X_j, j = 1, 2, ...\}$ reduces to a renewal process.

If one assumes that the parameter a in Eq. (3.1) is replaced with a more generic form, $h(j)$, for example, then different forms of $h(j)$ can be considered, including $h(j) = a^{j-1}$ in the GeP, $h(j) = \alpha a^{j-1} + \beta b^{j-1}$ in the EPP or $h(j) = j^a$ in the α-SP. To be more generic, we can assume M different forms of $h_m(j)$ with $m = 1, 2, , \ldots, M$. That is, Eq. (3.1) becomes

$$F_{m,j}(x; \alpha(t), \beta_j) = F((h_m(j))^{-1} x; \alpha(t), \beta), \tag{3.12}$$

and

$$f_{m,j}(x; \alpha(t), \beta_j) = f((h_m(j))^{-1} x; \alpha(t), \beta_j). \tag{3.13}$$

Similarly, Eq. (3.5) becomes

$$G_{m,j}(x; \alpha(t), \beta_j) = \sum_{i=k}^{n} \binom{n}{i} F((h_m(j))^{-1} x; \alpha(t), \beta)^i (1 - F((h_m(j))^{-1} x; \alpha(t), \beta))^{(n-i)}, \tag{3.14}$$

then Eq. (3.6) becomes

$$
\begin{aligned}
&g_{m,j}(x;\alpha(t),\beta_j)\\
&=\sum_{i=k}^{n}\binom{n}{i}(if(t)F((h_m(j))^{-1}x;\alpha(t),\beta)^{i-1}(1-F((h_m(j))^{-1}x;\alpha(t),\beta))^{(n-i)}\\
&-\sum_{i=k}^{n}(n-1)f((h_m(j))^{-1}x;\alpha(t),\beta)F((h_m(j))^{-1}x;\alpha(t),\beta)^{i}(1\\
&-F((h_m(j))^{-1}x;\alpha(t),\beta))^{n-i-1}),
\end{aligned}
\tag{3.15}
$$

besides, the Eq. (3.7) becomes

$$
\begin{aligned}
F_{\sigma_{j,L}}(L;\alpha(t),\beta_j)&=\sum_{i=0}^{k-1}\binom{n}{i}F((h_m(j))^{-1}L;\alpha(t),\beta)^{i}(1-F((h_m(j))^{-1}L;\alpha(t),\beta))^{(n-i)}\\
&=F_{\sigma_L}((h_m(j))^{-1}L;\alpha(t),\beta),
\end{aligned}
\tag{3.16}
$$

and Eq. (3.8) becomes

$$
\begin{aligned}
&f_{\sigma_{j,L}}(L;\alpha(t),\beta_j)=\\
&\sum_{i=0}^{k-1}\binom{n}{i}i(h_m(j))^{-1}f((h_m(j))^{-1}L;\alpha(t),\beta)F((h_m(j))^{-1}L;\alpha(t),\beta)^{i-1}(1\\
&-F((h_m(j))^{-1}L;\alpha(t),\beta))^{(n-i)}\\
&-\sum_{i=0}^{k-1}\binom{n}{i}(n-i)(h_m(j))^{-1}f((h_m(j))^{-1}L;\alpha(t),\beta)F((h_m(j))^{-1}L;\alpha(t),\beta)^{i}(1\\
&-F((h_m(j))^{-1}L;\alpha(t),\beta))^{(n-i)}\\
&=f_{\sigma_L}((h_m(j))^{-1}L;\alpha(t),\beta).
\end{aligned}
\tag{3.17}
$$

Obviously, in reality, estimating the different forms of $h_m(j)$ can have different model performances in the GaP in terms of measures such as the AIC or the BIC, which were discussed in the literature review.

One may also consider the uncertainty of the estimated parameters. Such uncertainty can be measured by various metrics, such as the Coefficient of Variation (CV), which shows the extent of variability in relation to the mean of the population. Without doubt, an estimator with a larger CV implies greater uncertainty in the parameter estimation than one with a smaller CV.

In our case, it is obvious that $C(N|L)$ in Eq. (3.10) is a function of $h_m(j)$. That is, if we want to understand the uncertainty of the expected cost ratio, which is the result of $h_m(j)$, we will need to transmit the uncertainty from $h_m(j)$ to $C(N|L)$, namely, we need to estimate the variability for functions of the estimated parameters.

In the literature, one may use some approaches to estimating the variability for functions of parameters, for example, the bootstrap method (Krinsky and Robb, 1986) and the δ method (see p. 172, (Weisberg, 2014)).

The δ method provides the approximate probability distribution for a function of an asymptotically normal statistical estimator from the limiting variance of that estimator. With the δ method, if we assume that we have obtained the CV of each estimator in an $h_m(j)$, we will be able to estimate the CV of $C_m(N|L)$. Once we obtain M values of the CV's due to different $h_m(j)$, we can use a method similar to the k-out-of-n system: That is, we sort $C_m(N|L)$ in ascending order and then choose the first m $C_m(N|L)$'s to optimise the N^*. Similarly, AIC$_m$ and CV$_m$ can be obtained.

Then, $C_m(N|L)$ is given by

$$C_m(N|L) = \frac{c_r + \sum_{i=1}^{k} \sum_{j=1}^{N} c_i \int_0^{+\infty} t f_{\sigma_{j,L}}(L; \alpha(t), \beta_j) dt}{\sum_{j=1}^{N} \int_0^{+\infty} t f_{\sigma_{j,L}}(L; \alpha(t), \beta_j) dt}, \tag{3.18}$$

where $f_{\sigma_{j,L}}(L; \alpha(t), \beta_j)$ has been obtained by Eq. (3.17). Based on the above discussion, we can adopt the following optimisation model to look for the optimal maintenance policy:

The optimal maintenance policy is a multi-criterion model-selecting problem, which minimizes the following three objective functions:

- to minimize the expected cost ratio $C(N|L)$;
- to minimize the AIC of $f_{\sigma_{j,L}}(L; \alpha(t), \beta_j)$;
- to minimize the CV.

Each objective function has its weight. Denote $H(N)$ as liner combination value of three objective functions, then it is given by

$$H(N) = w_1 C(N|L) + w_2 AIC + w_3 CV, \tag{3.19}$$

where w_1, w_2, and w_3 are the weights.

3.4.2 Calculation of AIC

Denote $f_j(x; \alpha(t), \beta_j)$ as the pdf associated with $Fj(x; \alpha(t), \beta_j)$. If all relevant data are available, we can obtain the parameters by maximising the following likelihood function

$$L(x; \alpha(t), \beta) = \prod_{j=1}^{J} f((h(j))^{-1} x; \alpha(t), \beta_j). \tag{3.20}$$

Then its AIC is given by

$$\text{AIC}_m = 2m_k - 2\ln(L_m), \tag{3.21}$$

where m_k is the total number of parameters in $f_{m,j}(x; \alpha(t), \beta_j)$.

We use the GeP, the EPP, and the α-SP to obtain the value of three objective functions and $H(N)$, respectively. Then, we can obtain three likelihood functions, respectively. The likelihood function in terms of the GeP is given by

$$L_{GeP}(x;\alpha(t),\beta) = \prod_{j=1}^{J} f(a^{1-j}x;\alpha(t),\beta), \tag{3.22}$$

and its log-likelihood function is given by

$$\ell_{GeP}(\alpha(t),\beta) = -J\alpha(t)\ln(\beta) + \frac{J(1-J)}{2}\ln(a) - J\ln(\Gamma(\alpha(t)) + (\alpha(t)-1)\sum_{j=1}^{J}\ln(x_i)$$
$$-\frac{1}{\beta}\sum_{j=1}^{J}(a^{j-1}x_j). \tag{3.23}$$

The likelihood function in terms of the EPP, is given by

$$L_{EPP}(x;\alpha(t),\beta) = \prod_{j=1}^{J} f((a_1a_2^{1-j} + b_1b_2^{1-j})x;\alpha(t),\beta), \tag{3.24}$$

and its log-likelihood function is given by

$$\begin{aligned}\ell_{EPP}(\alpha(t),\beta) = -J\alpha(t)\ln(\beta) + \sum_{j=1}^{J}\ln(a_1a_2^{1-j} + b_1b_2^{1-j}) - J\ln(\Gamma(\alpha(t))\\ + (\alpha(t)-1)\sum_{j=1}^{J}\ln(x_i) - \frac{1}{\beta}\sum_{j=1}^{J}(a_1a_2^{1-j} + b_1b_2^{1-j})x_j.\end{aligned} \tag{3.25}$$

The likelihood function in terms of the α-SP, is given by

$$L_{\alpha}(x;\alpha(t),\beta) = \prod_{j=1}^{J} f(j^{a}x;\alpha(t),\beta), \tag{3.26}$$

and its log-likelihood function is given by

$$\ell_{\alpha}(\alpha(t),\beta) = -J\alpha(t)\ln(\beta) - a\sum_{j=1}^{J}\ln(j) - J\ln(\Gamma(\alpha(t)) + (\alpha(t)-1)\sum_{j=1}^{J}\ln(x_i)$$
$$-\frac{1}{\beta}\sum_{j=1}^{J} j^{-a}x_j. \tag{3.27}$$

By differentiating the above log-likelihood functions, we can obtain the estimates of the parameters respectively.

3.4.3 Estimation of CV

Before the CV of $Y_j(t)$ is obtained, it is necessary to obtain the mean and variance of $Y_j(t)$. Therefore, the mean can be presented by

$$\mu_{g_j} = \int_{-\infty}^{\infty} x g_j(x; \alpha(t), \beta_j) \mathrm{d}x, \tag{3.28}$$

and the variance can be presented by

$$\sigma_{g_j} = \int_{-\infty}^{\infty} x^2 g_j(x; \alpha(t), \beta_j) \mathrm{d}x - \mu_{g_j}^2. \tag{3.29}$$

Then, the CV of $Y_j(t)$ under the m-th form can be presented by

$$CV_m = \frac{\int_{-\infty}^{\infty} x g_j(x; \alpha(t), \beta_j) \mathrm{d}x}{\int_{-\infty}^{\infty} x^2 g_j(x; \alpha(t), \beta_j) \mathrm{d}x - \mu_{g_j}^2}. \tag{3.30}$$

3.5 Numerical examples

This chapter investigated the impact of the uncertainty of the model, the uncertainty of the model parameters, and the expected cost of the maintenance policy optimisation. We use three indexes to evaluate the uncertainty: the $C(N|L)$, the AIC, and the CV.

For estimating the uncertainty under our maintenance policy, we simulate 4 datasets about time intervals between failures and degradation levels with β_j due to the $(j-1)$-th maintenance of a 2-out-of-4 system. For each component, the degradation level is determined by the gamma process in R with a pre-threshold L, which is a constant for all components. Then, the time to a failure can be estimated when the threshold is achieved. According to Eq. (3.3), the time intervals of failures of the system can be obtained for this 2-out-of-4 system. Figure 3.2 shows the time intervals between failures under the gamma process of a system.

According to Eq. (3.21), the AIC can be obtained under GeP, EPP and α-SP, Table 3.1 shows the AIC under such three GeP-like models.

TABLE 3.1: AIC of three GeP-like models for $N = 30$

AIC	**Dataset 1**	**Dataset 2**	**Dataset 3**	**Dataset 4**
GeP	159.96	**17.77**	221.68	**227.71**
EPP	139.24	21.24	241.77	286.58
α-SP	**129.14**	30.04	**189.91**	297.68

According to Table 3.1, we have the following findings:

1. For dataset 1, the minimal AIC = 129.14, which is the result based on the α-SP.
2. For dataset 2, the minimal AIC = 17.77, which is the result based on the GeP.
3. For dataset 3, the minimal AIC = 189.91, which is the result based on the α-SP.
4. For dataset 4, the minimal AIC = 227.71, which is the result based on the GeP.

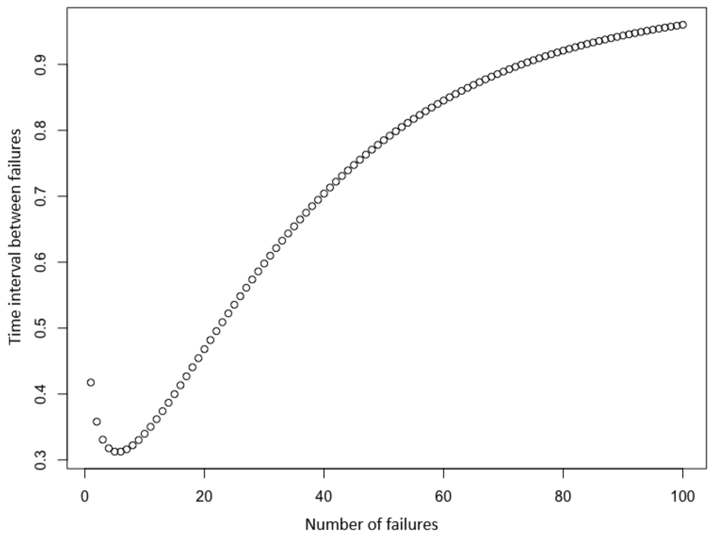

FIGURE 3.2: Dataset 1-Time intervals between failures under gamma process with L=0.5

Then, we assume $c_r = 10$, and $c_i = 5$, according to Eq. (3.18), the expected cost per unit can be obtained. Table 3.2 shows $C(N|L)$ under three GeP-like models.

TABLE 3.2: $C(N|L)$ of three GeP-like models for N=30

$C(N\|L)$	**Dataset 1**	**Dataset 2**	**Dataset 3**	**Dataset 4**
GeP	**70.84**	975.30	656.76	343.23
EPP	97.39	**841.79**	**454.23**	356.34
α-SP	85.10	918.24	456.98	**235.63**

We have the following findings:

1. For dataset 1, the minimal $C(N|L) = 70.84$, which is the result based on the GeP.
2. For dataset 2, the minimal $C(N|L) = 841.79$, which is the result based on the EPP.
3. For dataset 3, the minimal $C(N|L) = 454.23$, which is the result based on the EPP.
4. For dataset 4, the minimal $C(N|L) = 235.63$, which is the result based on the α-SP.

Therefore, the EPP is preferred according to the $C(N|L)$

Similarly, Table 3.3 shows the CV based on the three forms of GeP-like models. We have the following points:

1. For dataset 1, $CV = 38.36$, which is the result based on the α-SP.
2. For dataset 2, $CV = 31.14$, which is the result based on the GeP.
3. For dataset 3, $CV = 30.45$, which is the result based on the α-SP.
4. For dataset 4, $CV = 23.57$, which is the result based on the GeP.

TABLE 3.3: CV of three GeP-like models for N=30

CV	Dataset 1	Dataset 2	Dataset 3	Dataset 4
GeP	45.17	**31.14**	56.78	**23.57**
EPP	56.79	47.89	66.45	28.97
α-SP	**38.36**	57.82	**30.45**	27.47

Therefore, the α-SP and the GeP are preferred according to the $CV_{(m)}$.

To consider the impact of different weight values, we use three scenarios for comparison. Table 3.4 shows the value of w_1, w_2 and w_3 under such scenarios.

TABLE 3.4: Weight value of three scenarios

	Scenario 1	Scenario 2	Scenario 3
w1	0.5	0.3	0.2
w2	0.3	0.1	0.7
w3	0.2	0.6	0.1

TABLE 3.5: Scenario 1

$H(N)$	Dataset 1	Dataset 2	Dataset 3	Dataset 4
GeP	92.442	499.209	406.240	244.642
EPP	101.825	436.845	312.936	269.938
α-SP	88.964	479.696	291.553	212.613

TABLE 3.6: Scenario 2

$H(N)$	Dataset 1	Dataset 2	Dataset 3	Dataset 4
GeP	82.174	121.545	166.248	116.778
EPP	85.585	119.285	157.824	138.990
α-SP	70.268	135.528	120.941	129.349

TABLE 3.7: Scenario 3

$H(N)$	Dataset 1	Dataset 2	Dataset 3	Dataset 4
GeP	130.657	210.613	292.206	230.400
EPP	122.625	188.015	266.730	274.771
α-SP	111.254	210.458	227.378	258.249

We have the following findings:

1. Under scenario 1, the α-SP has the minimal $H(N)$ of dataset 1, dataset 3 and dataset 4.
2. Under scenario 1, the EPP has the minimal $H(N)$ of the dataset 2.
3. Under scenario 2, the α-SP has the minimal $H(N)$ of dataset 1 and dataset 3.
4. Under scenario 2, the EPP has the minimal $H(N)$ of the dataset 3.
5. Under scenario 2, the GeP has the minimal $H(N)$ of the dataset 4.

6. Under scenario 3, the α-SP has the minimal $H(N)$ of dataset 1 and dataset 3.
7. Under scenario 3, the EPP has the minimal $H(N)$ of the dataset 2.
8. Under scenario 3, the GeP has the minimal $H(N)$ of the dataset 4.

To sum up, for datasets 1 and 3, using the α-SP can obtain a lower uncertainty in terms of $C(N|L)$, AIC and CV.

3.6 Summary

This chapter considered a multi-component system, the degradation process of each component in the system is modelled by the gamma process. With a pre-specified threshold L, the time to a failure of each component for a k-out-of-N system can be estimated. We proposed a multi-criteria model-selecting maintenance policy with three indexes that are expected cost ratio, Akike information criterion, and the coefficient of variation to evaluate the uncertainty due to the parameter estimation of the system. The maintenance policy with the objective function $H(N)$, which is the weight value of three indexes, is the objective function of our maintenance policy.

We used the gamma process to simulate four datasets for the time intervals between failures and the degradation level of a 2-out-of-4 system. According to such datasets, we obtained the result of three indexes under the geometric process, the extended Poisson process, and the α-series process, respectively. Besides, the value of $H(N)$ under three scenarios (three groups of weight value) was obtained.

Chapter 4

Maintenance policies of multi-components systems

4.1 Introduction

The condition-based maintenance (CbM) is widely used as one type of preventive maintenance. It focuses on monitoring the state of a system or estimating the state of a system at each regular inspection and decides on maintenance activities, which can be repairs or replacements. Publications relating to CbM are enormous. For example, Li and Nilkitsaranont (2009) proposed a combined regression technique for CbM to assess the remaining useful life of gas turbine engines, which improves engine reliability and availability and reduces life cycle costs. Coraddu et al. (2016) used some machine approaches to effectively predict potential future failures of naval propulsion plants. Other research such as Zhu, Fouladirad, and Bérenguer (2015) presented a deterioration model that includes two system degradations: wear and shock and gives an optimal maintenance policy for the minimal cost criterion. These studies have pointed a direction for future research: how to build a model that is more suitable for a complex system with multiple components or failure types.

As aforementioned, in existing literature, stochastic processes such as the gamma process (Lawless and Crowder, 2004; Wu and Castro, 2020), the inverse Gaussian process (Li et al., 2017; Chen et al., 2015), and the WP (Ebrahimi, 2005; Wen, Gao, and Zhang, 2018) are widely used for different applications in CbM. Many pieces of research are concentrated on the combination approach to dealing with the increasingly complex system (see Galar et al. (2013), Feng et al. (2017), and Chang et al. (2019), for example). Caballé et al. (2015) proposed a condition-based maintenance policy by combining the non-homogeneous Poisson process(NHPP) and the gamma process(GaP). It modelled multiple degradation processes with dependent deterioration-threshold-shock models. This is a typical example of multi-failure modes. It carries out two incremental processes in two different methodologies, so as to achieve the situation where the decline mode of a single system changes. They also point out that the dependence analysis between the causes of failure is a potential development and the variability of the threshold should be considered in the future.

Liu et al. (2017) proposes a new CbM model based on three-state deterioration and the influence of external environmental shocks. The degradation process of the system is modelled by a two-state WP with a Doubly Stochastic Poisson Process (DSPP). It considers two different thresholds, namely normal threshold and defective threshold which depend on the system state. Other common methods such as the geometric process, regression, ANN and SVR can be seen in these examples: Dong et al. (2014), Liu, Jiang, and Zhang (2017), and Lo et al. (2019). Zhang et al. (2018) reviewed some developments and applications of the WP. Several challenges were pointed out in this paper, such as multiple time scales, inconsistent data, and recoveries of prognostics. Yang et al. (2019) proposed a two-phase preventive maintenance policy for a single-component system. The first stage is the imperfect maintenance phase which aims to keep the system working. The second stage is the postponed replacement phase which considers a preventive replacement. This means that this maintenance policy will be sufficient and flexible for resource allocation due to its phase variability. Zhao et al. (2021) proposed a multi-criteria mission abort policy that considers the normal and defective stages based on the time threshold. It also indicated that the performance of the optimal policy is compared in detail against several heuristic policies. Besides, the dynamic risk for controlling policy was also a possible extension for phased mission systems. Liu et al. (2021a) proposed a condition-based maintenance model in a finite-time horizon that considers a system with two heterogeneous dependent components with economic dependence. Moreover, this research pointed that the two-unit system in this chapter can be extended to multi-unit systems by generalizing the degradation process and Bellman equation, and the maintenance level can be extended to imperfect repair in the future. For a multi-component system, in which each component has an observable deteriorating process, Wu and Castro (2020) developed a weighted linear combination of degradation processes to optimise the time interval of maintenance for a pavement network.

Most existing maintenance policy optimisation approaches, such as Zhang et al. (2022), Shi et al. (2020b), Liu et al. (2021b), and Levitin, Xing, and Dai (2022), aim to minimise the relevant cost.

For a component in a system, it may have different failure modes. The degradation process of a system with different failure modes can be modelled by multiple degradation processes that can also be considered. Maintenance policies on such systems have been discussed in several papers. Zhu, Fouladirad, and Bérenguer (2016) studied the maintenance policies of a multi-component system with two independent failure modes. Qiu, Cui, and Gao (2017) considered an optimal maintenance policy by both maximizing steady-state availability and minimizing long-term average cost for a system with multiple failure modes. It assumed that failure modes are independent. Zheng and Makis (2020) considered the failure state of a system changes from a soft failure to a hard failure and assumes that under different state, different maintenance activities can be taken (such as corrective replacement for soft failure and minimal repair for hard failure).

In what follows, for convenience of expression, we regard the term *components* and *failure modes* exchangeable. That is, a system is composed of n components, or the degradation process of a system is composed of n failure modes.

4.1.1 Novelty and contributions

From the above review, it can be seen that there is a need to explore the problem of multiple failure modes for multiple-component systems. Typically, this chapter investigates the cost process relating to the linear combination, based on which maintenance policies are developed.

Hence, the contributions of this chapter include

- development of a cost process related to the linear combination of the degradation processes.;
- development of maintenance policies for a system whose cost process can be modelled by a linear combination of Wiener processes.

4.2 Assumption

This chapter makes the following assumptions.

- The system under discussion is composed of n components.
- Component k has its degradation level $X_k(t)$ at time t.
- All components have the same distribution.
- The system is new at time $t = 0$, all components develop from $X_k(0) = 0$.
- Replacement is carried out every T_a time units for an age replacement policy or T_b for a block replacement policy.
- Degradation processes of different failure modes are modelled by Wiener processes and gamma processes, respectively.
- The whole system is a linear combination of the magnitudes of the degradation that exceeds a pre-specified value.
- PM is carried out at T units of time and is performed at each PM.
- The degradation process of components will be modelled in two cases: the gamma process and the Wiener process.
- The degradation processes are independent of each other.

4.3 Model development

4.3.1 Basic process of the GaP

In the case of the GaP, let $X_k(t)$ be the degradation level of the kth component at time t. The properties of $X_k(t)$ can be known and its probability density function is given by

$$f_{GaP}(x;\alpha_k(t),\beta_k) = \frac{\beta_k^{-\alpha_k(t)}}{\Gamma(\alpha_k(t))} x^{\alpha_k(t)-1} e^{-x/\beta_k} 1_{\{x>0\}},$$

where $\Gamma(.)$ is the gamma function : $\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}\mathrm{d}u$. And $X_k(t)$ has mean $\beta_k\alpha_k(t)$ and variance $\beta_k^2\alpha_k(t)$, respectively. As this is a multi-component system with k components, the overall degradation of the system can be presented by

$$Y_{GaP}(t) = \sum_{i=1}^{k} w_k X_{k_{GaP}}(t), t \geqslant 0, b_i \geqslant 0,$$

where w_k is the weight of the kth component. Then $Y_{GaP}(t)$ has mean $\sum_{k=1}^{n} w_k\beta_k\alpha_k(t)$ and variance $\sum_{i=1}^{k} w_i^2\beta_i^2\alpha_i^2(t)$, respectively. Therefore, $Y_{GaP}(t)$ is a stochastic process with similar properties to $X(t)$. Besides, the density function of $Y_{GaP}(t)$ is presented by

$$f_{Y(t)_{GaP}}(y) = D(t)\sum_{k=1}^{\infty} \frac{\zeta_k(t)\beta_0^{-\rho(t)-k}}{\Gamma(\rho(t)+i)} y^{\rho(t)+k-1} e^{-y/\beta_0}, y > 0,$$

where $\beta_0 = min_{1\leqslant k\leqslant k} a_k\beta_k$. $D(t)$ and $\rho(t)$ can be presented by

$$D(t) = \prod_{k=1}^{n} (\frac{\beta_0}{w_k\beta_k})^{\alpha_k(t)},$$

and

$$\rho(t) = \sum_{k=1}^{n} \alpha_k(t), t \geqslant 0,$$

besides, it has $\zeta_{k+1}(t) = \frac{1}{k+1}\sum_{j=1}^{k} j\eta_j(t)\zeta_{k+1-j}(t)$ with $\zeta_0(t) = 1$ and $\eta_k(t) = \sum_{j=1}^{n} \alpha_j(1-\frac{\beta_0}{w_k\beta_k})^k/k$.

Basic process of the WP

In the case of the WP, the difference from the case of the GaP is that $X_k(t)$ is said to have drift coefficient $\mu_{k_{wp}}$ and variance parameter $\sigma_{k_{wp}}^2$, the stochastic process of it is:

$$X_{k_{wp}}(t) = \mu_{k_{wp}}t + \sigma_{k_{wp}}W_k(t), \tag{4.1}$$

where $\mu_{k_{wp}}$ and $\sigma_{k_{wp}}$ are the parameters of component k, respectively, $W_k(\cdot)$ is the standard WP, which also can be called as the Brownian motion. The estimation method of parameters for both GaP and WP can be seen in Shah et al. (2013).

4.3.2 Degradation processes

Now the degradation processes of a multi-components system with k components will be discussed separately in the case of the GaP and the WP.

Degradation process of the GaP

In this case, we assume that the time interval between failures becomes shorter and shorter, and it follows a GeP. Therefore, denote $X_{i_{gap}}(t)$ as the degradation level of ith component of a system with a shorter and shorter GeP-like time interval of failures, then, the pdf of $X_{k_{gap}}(t)$ is given by

$$f_{gap}(x;\alpha_k(t),\beta_k) = \frac{\beta_k^{-\alpha_k(t)}}{\Gamma(\alpha_k(t))} a x^{(n-1)(\alpha_k(t)-1)} e^{-ax^{n-1}/\beta_k} 1_{\{x>0\}}, a > 0. \tag{4.2}$$

Then, denote $Y_{gap}(t)$ as the overall degradation of this situation, it can be presented by

$$Y_{gap}(t) = \sum_{k=1}^{n} w_k X_k(at^{k-1}), t \geqslant 0, b_k \geqslant 0, a > 0, \tag{4.3}$$

and the probability density function $f_{gap}(x)$ of $Y_{gap}(t)$ can be presented by

$$f_{Y_{gap}}(y) = D(t) \sum_{k=1}^{\infty} \frac{\zeta_k(t)\beta_0^{-\rho(t)-k}}{\Gamma(\rho(t)+k)} a y^{(k-1)(\rho(t)+k-1)} e^{-ay^{n-1}/\beta_0}, y > 0, a > 0, \tag{4.4}$$

where $\beta_0 = \min_{1 \leq k \leq n} a_k \beta_k$. $D(t)$ and $\rho(t)$ are given by

$$D(t) = \prod_{k=1}^{n} (\frac{\beta_0}{a_k \beta_k})^{\alpha_k(t)}, \tag{4.5}$$

and

$$\rho(t) = \sum_{k=1}^{n} \alpha_k(t), t \geq 0. \tag{4.6}$$

The expected value and variance can be obtained, respectively,

$$\mathbb{E}[Y_{gap}(t)] = \mu_{Y_{gap}(t)} = \sum_{i=1}^{k} a w_k \beta_k \alpha_k(t), \tag{4.7}$$

and

$$\mathbb{V}[Y_{gap}(t)] = \sigma^2_{Y_{gap}(t)} = \sum_{k=1}^{n} a^2 w_k^2 \beta_k^2 \alpha_k^2(t). \tag{4.8}$$

Degradation process of the WP

In this case, we assume that the system has k degradation processes, each of which follows a WP.

Let $X_{k_{wp}}(t)$ be the degradation level of the kth component at time t, where $k = 1, 2, \ldots., n$. Then, $X_{k_{wp}}(t)$ have the following assumptions:

- $X_{1_{wp}}(0) = 0$, which also means that $\Delta W_k(0) = 0$.
- $W_k(t)$ has independent increment $\Delta W_k(t)$ that follows the normal distribution. That is, for $0 < s < t$, $W_k(t-s) - W_k(s)$ follows $N(0, (t-s))$.
- $W_k(t)$ is almost continuous in $(0, t)$.

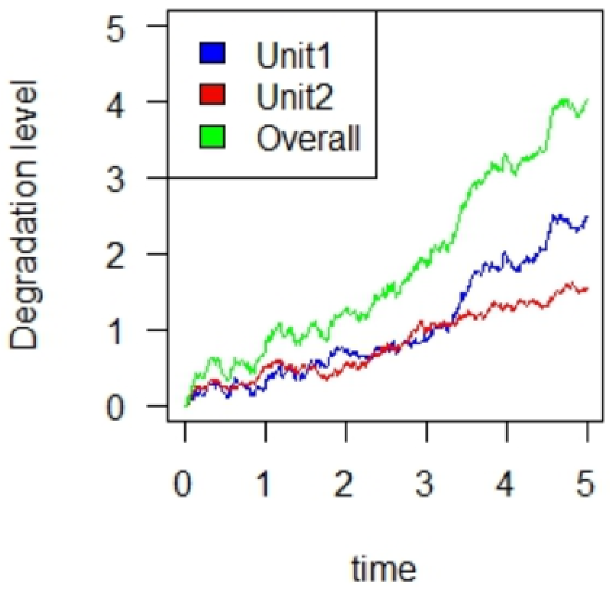

FIGURE 4.1: Realisation of two deterioration processes and a linear combination

The unconditional probability density function, which follows the normal distribution with mean = 0 and variance = t, at a fixed time t:

$$f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}.$$

We have $E[W_k(t)] = 0$ and $\text{Var}[W_k(t)] = t$.

These results follow immediately from the definition that increments have a normal distribution, centered at zero.

Thus, the expected value and the variance of $X_k(t)$ are given by: $E(X_{k_{wp}}(t)) = \mu_{k_{wp}} t$ and $V(X_{k_{wp}}(t) = \sigma^2_{k_{wp}} t$.

Now let us assume $Y_{wp}(t)$ is a linear combination of k WPs. The overall degradation $Y_{wp}(t)$ of the system is represented by

$$Y_{wp}(t) = \sum_{i=1}^{k} w_k X_{k_{wp}}(t), t \geqslant 0, a_k \geqslant 0, \tag{4.9}$$

where w_k is the weight of the kth component. Fig. 4.1 shows the realisation of a linear combination of two WPs.

Furthermore, the overall deterioration process $\{Y_{wp}(t), t > 0\}$ is a stochastic process with the following properties (without the skew-normal random effects):

- $Y_{wp}(0) = \sum_{k=1}^{n} W_k X_{k_{wp}}(0) = 0$,
- $\Delta Y Y_{wp}(t) = \sum_{k=1}^{n} W_k \Delta X_{k_{wp}}(t)$ is an independent increment as well.

Thus, $Y_{wp}(t)$ is given by

$$Y_{wp}(t) = t \sum_{k=1}^{n} w_k \mu_{k_{wp}} + \sum_{k=1}^{n} w_k \sigma_{k_{wp}} w_k(t). \tag{4.10}$$

Let $\mu_{Y_{wp}} = \sum_{k=1}^{n} w_k \mu_{k_{wp}}$ and $\sigma^2_{Y_{wp}} = \sum_{k=1}^{n} w_k^2 \sigma^2_{k_{wp}}$. Then $Y_{wp}(t)$ follows the normal distribution $N(\mu_{Y_{wp}} t, \sigma^2_{Y_{wp}} t)$.

4.4 Repair cost process

This section will propose a repair cost process in the case of the GaP and the WP. In this thesis, we consider that the repair costs of different components are different. We consider that the actual cost is dependent on the degradation level of the failure model. For example, the repair or replacement cost for a system with a longer usage time is normally higher than a system with a shorter usage time. Several literatures have considered this situation, see Liu et al. (2017) and Wu and Castro (2020), for example. It is worth noticing that, according to Wu and Castro (2020), the total cost $U_{gap}(t)$ or $U_{wp}(t)$, which are associated to $Y_{gap}(t)$ or $Y_{wp}(t)$, are also a stochastic process and do not have a linear correlationship with $Y_{gap}(t)$ or $Y_{wp}(t)$.

4.4.1 Repair cost process of the GaP

If all degradation processes for k components follow the GaP, then, denote the maintenance cost for the ith component which is related to its degradation level, it is given by

$$U_{gap}(t) = \sum_{k=1}^{n} C_{k_{gap}}(t) = \sum_{k=1}^{n} a_k c_k X_{k_{gap}}(t), \tag{4.11}$$

where $U_{gap}(t)$ is a gamma process with a linear combination with weight w_k for the kth component, and its expected value and variance are given by

$$E(U_{gap}(t)) = \sum_{k=1}^{n} a_k c_k \mu_{k_{gap}} t = \mu_{U_{gap}}, \tag{4.12}$$

and

$$V(U_{gap}(t)) = \sum_{k=1}^{n} a_k^2 c_k^2 \sigma^2_{k_{gap}} t = \sigma_{U^2_{gap}}. \tag{4.13}$$

The covariance between $Y_{gap}(t)$ and $U_{gap}(t)$ is given by

$$Cov(Y_{gap}(t), U_{gap}(t)) = \sum_{k=1}^{n} a_k c_k \alpha_k(t) \beta_k^2(t).$$

The characteristic function of the bivariate normal distribution is given by

$$\phi_{(Y_{gap}(t), U_{gap}(t))}(t_1, t_2) = \prod_{k=1}^{n} \phi_{X_{k_{gap}}(t)}(a_k t_1 + c_k t_2).$$

Then the following can be obtained

$$f_{Y_{gap}(t), U_{gap}(t)}(y, u) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\prod_{k=1}^{n} \phi_{X_{k_{gap}}}(a_k t_1 + c_k t_2)) e^{-it_1 y - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2,$$

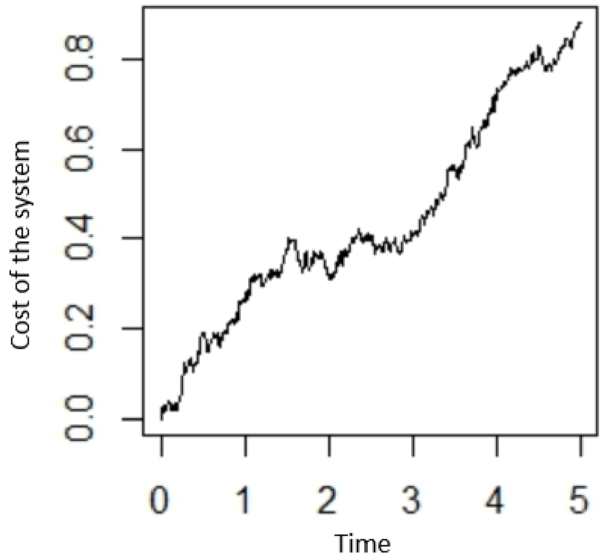

FIGURE 4.2: Cost process of C(t)

then conditional probability $f_{U_{gap}(t)|Y_{gap}(t)(y,u)}$ is given by

$$f_{U_{gap}(t)|Y_{gap}(t)(y,u)} = \frac{1}{4\pi^2 f_{Y_{gap}(t)}(y)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\prod_{k=1}^{n}(1-i(a_k t_1 + c_k t_2)\beta_k)^{-\alpha_k(t)}) e^{-it_1 y - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2. \tag{4.14}$$

4.4.2 Repair cost process of the WP

If all degradation processes for k components follow the WP, then, the maintenance cost for the ith component which is related to the degradation level is given by,

$$C_k(t) = w_k c_k X_k(t), \tag{4.15}$$

where c_k is the maintenance cost for the kth failure mode. Then, the total cost of the whole system with multiple components or failure modes is given by

$$U_{wp}(t) = \sum_{k=1}^{n} C_k(t) = \sum_{k=1}^{n} w_k c_k X_k(t), \tag{4.16}$$

where $U_{wp}(t)$ is a WP with a linear drift related to its degradation level.

As $X_{k_{wp}}(t)$ follows the normal distribution with mean = $\mu_{k_{wp}} t$ and variance = $\sigma^2_{k_{wp}} t$, the expected value and the variance of $C_k(t)$ are given by: $E(C_k(t)) = a_k c_k \mu_k t$ and $V(C_k(t)) = a_k c_k^2 \sigma^2_{k_{wp}} t$.

Then $U_{wp}(t)$ has expected value and variance,

$$E(U_{wp}(t)) = \sum_{k=1}^{n} w_k C_k \mu_{k_{wp}} t = \mu_{U_{wp}}, \tag{4.17}$$

and

$$V(U_{wp}(t)) = \sum_{k=1}^{n} w_k^2 C_k^2 \sigma_{k_{wp}}^2 t = \sigma_{U_{wp}}^2, \tag{4.18}$$

respectively.

Obviously, both of $Y_{wp}(t)$ and $U_{wp}(t)$ have the same values μ_i and σ_k, respectively, so the covariance between $Y_{wp}(t)$ and $U_{wp}(t)$ is given by

$$\begin{aligned}
\mathrm{Cov}(Y_{wp}(t), U_{wp}(t)) &= \mathrm{Cov}(\sum_{i=1}^{k} w_k X_k(t) \sum_{j=1}^{k} C_i X_j(t)) \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k} w_k C_i \mathrm{Cov}(X_k(t), X_j(t)) \\
&= \sum_{i=1}^{k} c_i C_i \mu_i^2 t. \qquad (4.19)
\end{aligned}$$

The characteristic function of the bivariate normal distribution is given by

$$\begin{aligned}
&\phi_{(Y_{wp}(t),U_{wp}(t))}(t_1, t_2) = \mathbb{E}[\exp(it_1 Y_{wp}(t) + it_2 U_{wp}(t))] \\
&= \mathbb{E}[\exp(it_1 \sum_{k=1}^{n} w_k X_k(t) + it_2 \sum_{k=1}^{n} q_k C_k X_k(t)] \\
&= \mathbb{E}[\exp(i \sum_{k=1}^{n} (a_k t_1 + a_k C_k t_2) X_k(t))] \\
&= \mathbb{E}[\exp(i \sum_{k=1}^{n} (a_k t_1 + a_k C_k t_2) X_k(t))] \\
&= \prod_{k=1}^{n} \mathbb{E}[\exp(i(a_k t_1 + a_k C_k t_2) X_k(t))] \\
&= \prod_{k=1}^{n} \phi_{X_k(t)}(a_k t_1 + a_k C_k t_2), \qquad (4.20)
\end{aligned}$$

then we can obtain

$$\begin{aligned}
&f_{Y_{wp}(t),U_{wp}(t)}(y, u) \qquad (4.21) \\
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{(Y_{wp}(t),U_{wp}(t))}(t_1, t_2) e^{-it_1 y - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2 \\
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\prod_{k=1}^{n} \phi_{X_k(t)}(a_k t_1 + C_i t_2))^{-it_1 y - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2, \\
&\qquad (4.22)
\end{aligned}$$

then conditional probability $f_{U_{wp}(t)|Y(t)_{wp}(y,u)}$ is given by

$$\begin{aligned} f_{U_{wp}(t)|Y_{wp}(t)(y,u)} &= \frac{f_{U_{wp}(t),Y_{wp}(t)(y,u)}}{f_{Y_{wp}(t)}(y)} \\ &= \frac{1}{4\pi^2 f_{Y_{wp}(t)}(y)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\prod_{k=1}^{n} \phi_{X_k(t)}(w_k t_1 + C_k t_2))^{-it_1 y - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2, \end{aligned} \tag{4.23}$$

where

$$\phi_{X_k(t)}(a_k t_1 + C_k t_2) = \exp\{\frac{\sigma_{k_{wp}}[1-(1-2i\mu_{k_{wp}}{}^2(a_k t_1 + a_k C_k t_2)\sigma_{k_{wp}}^{-1})^{1/2}]}{\mu_k}\}, \tag{4.24}$$

4.5 First hitting time of the threshold of the degradation process and the cost process

The first hitting time regards the time at which the random process achieves a specified target or threshold value from an initial value for the first time. It can be considered as a branch under the survival models in reliability. Since we assume that all components develop from $X_k(0) = 0$, the first hitting time in this section regards the time $t = 0$ to the time when $Y_{gap}(t)$ and $Y_{wp}(t)$ achieves their threshold value L_{gap} and L_{wp}, respectively.

4.5.1 First hitting time of the GaP

The distribution of the first hitting time of $Y_{gap}(t)$ should be obtained. The first hitting time $\omega_{L_{gap}}$ is defined when $Y_{gap}(t)$ research the degradation threshold level L_{gap}. Then $\omega_{L_{gap}}$ is normally presented by

$$\omega_{L_{gap}} = \inf(t > 0, Y_{gap}(t) \geqslant L_{gap}). \tag{4.25}$$

The distribution of $\omega_{L_{gap}}$ can be denoted as $F_{\omega_{L_{gap}}}(t)$, which should be

$$\begin{aligned} F_{\omega_{L_{gap}}}(t) &= P(Y_{gap}(t) \geqslant L_{gap}) \\ &= \int_{L_{gap}}^{\infty} D(t) \sum_{i=1}^{\infty} \frac{\zeta_k(t)\beta_0^{-\rho(t)-i}}{\Gamma(\rho(t)+i} a y^{(n-1)(\rho(t)+i-1)} e^{-ay^{n-1}/\beta_0} \mathbf{d}y \\ &= D(t) \sum_{i=1}^{\infty} \frac{\zeta_k(t)\beta_0^{-\rho(t)-i}}{\Gamma(\rho(t)+i} \int_{L_{gap}}^{\infty} a y^{(n-1)(\rho(t)+i-1)} e^{-ay^{n-1}/\beta_0} \mathbf{d}y. \end{aligned} \tag{4.26}$$

Besides, when the degradation level achieves the threshold L_{gap}, although the system is still working, in this thesis, we assume that a system has "failed" when it achieves its threshold L_{gap}. Maintenance actions will be taken after the threshold is achieved. Therefore, the time for achieving the threshold will be the first time of failure. Therefore, the first hitting time can be obtained

$$\begin{aligned} F_{\omega_{L_{gap}}}(t) &= D(t)\sum_{i=1}^{\infty}\frac{\zeta_k(t)\beta_0^{-\rho(t)-i}}{\Gamma(\rho(t)+i}e^{-a/\beta_0}\int_{L_{gap}}^{\infty} a\mathrm{d}y \\ &= D(t)\sum_{i=1}^{\infty}\frac{\zeta_k(t)\beta_0^{-\rho(t)-i}}{\Gamma(\rho(t)+i}e^{-a/\beta_0}(\lim_{u\to\infty} ay - 2a). \end{aligned} \tag{4.27}$$

4.5.2 First hitting time of the WP

The distribution of the first hitting time of the process $\{Y_{wp}(t), t \geq 0\}$, which starts from $Y_{wp}(0) = 0$ should be obtained. The first hitting time $\omega_{Y_{wp}(t)}$ is defined when $Y_{wp}(t)$ reaches the degradation level L_{wp}, according to the statistical characteristic of a WP, the first-passage-time, which is $\omega_{Y_{wp}(t)}$, follows an inverse Gaussian distribution (Ross, 1996; Pan et al., 2017; Ye and Chen, 2014), then

$$\omega_{L_{wp}} = \inf\{t > 0 : Y_{wp}(t) \geq L_{wp}\}, \tag{4.28}$$

Then, the pdf of $\omega_{L_{wp}}$ can be obtained by

$$\begin{aligned} f_{\omega_{L_{wp}}}(t) &= \frac{L_{wp}}{\sigma_{Y_{wp}}\sqrt{2\pi t^3}}\exp\left(\frac{-(L_{wp}-\mu_{Y_{wp}}t)^2}{2\sigma_{Y_{wp}}{}^2 t}\right) \\ &= \frac{L_{wp}}{\sigma_{Y_{wp}}\sqrt{\pi t^3}}\phi\left(\frac{-(L_{wp}-\mu_{Y_{wp}}t)}{\sigma_{Y_{wp}}\sqrt{t}}\right), \end{aligned} \tag{4.29}$$

where $\phi(\cdot)$ denotes the standard normal pdf. Then, the cdf of ω_L is obtained by

$$\begin{aligned} F_{\omega_{L_{wp}}}(t) &= P(Y_{wp}(t) \geq L_{wp}) \\ &= \Phi\left(\frac{-(L_{wp}-\mu_{Y_{wp}}t)}{\sigma_{Y_{wp}}\sqrt{t}}\right) - \exp\left(\frac{2\mu_{Y_{wp}}L_{wp}}{\sigma_{Y_{wp}}{}^2}\right) \end{aligned} \tag{4.30}$$

where $\Phi(\cdot)$ denotes the standard normal cdf.

However, if we consider a real situation: after a period of time, the overall cost, which can be the $U(t)$, becomes so high that using a new piece of equipment to replace the old one may be a better choice. Furthermore, the owner of the equipment may have an expectation of overall cost: when the overall cost is larger than this expectation, they will buy a new piece of equipment. For example, we assume this expectation cost is $L_{U_{wp}}$ in the case of the WP. Then, we can define

$$\omega_{U_{wp}} = \inf\{t > 0 : U(t) \geq L_{U_{wp}}\}, \tag{4.31}$$

Then, the pdf of $\omega_{U_{wp}}$ can be obtained as

$$f_{\omega_{U_{wp}}}(t) = \frac{L_{U_{wp}}}{\sigma_{U_{wp}}\sqrt{2\pi t^3}}\exp(\frac{-(L_{U_{wp}} - \mu^2_{U_{wp}}t)}{2\sigma^2_{U_{wp}}t})$$
$$= \frac{L_{U_{wp}}}{\sigma_{U_{wp}}\sqrt{\pi t^3}}\phi(\frac{-(L_{U_{wp}} - \mu_{U_{wp}}t)}{\sigma_{U_{wp}}\sqrt{t}}). \quad (4.32)$$

And the cdf of $\omega_{U_{wp}}$ is obtained by

$$F_{\omega_{L_{U_{wp}}}}(t) = P(U_{wp}(t) \geq L_{U_{wp}}) = \Phi(\frac{-(L_{U_{wp}} - \mu_{U_{wp}}t)}{\sigma_{U_{wp}}\sqrt{t}}) - \exp(\frac{2\mu_{U_{wp}}L_{U_{wp}}}{\sigma^2_{U_{wp}}}). \quad (4.33)$$

Similarly, denote $L_{U_{gap}}$ as the threshold of the cost process in the case of the GaP, then the cdf of $\omega_{U_{gap}}$ can be obtained as

$$F_{\omega_{U_{gap}}}(t) = \int_{L_{U_{gap}}}^{\infty} f_{\omega_{U_{gap}}}(u)\mathrm{d}u \quad (4.34)$$
$$= D(t)\sum_{k=0}^{\infty}\frac{\zeta_k(t)}{\Gamma(\rho(t)+k}\Gamma_{up}((\rho(t)+k), L_{U_{gap}/\beta_0}), \quad (4.35)$$

where $\Gamma_{up}((\rho(t)+k), \frac{L_{U_{gap}}\beta_0}{)}$ is the upper incomplete gamma function and can be presented by

$$\Gamma_{up}((\rho(t)+k), L_{U_{gap}/\beta_0}) = \int_{L_{U_{gap}/\beta_0}}^{\infty} m^{\rho(t)+k-1}e^{-m}\mathrm{d}m. \quad (4.36)$$

4.6 Maintenance policies

In this section, we will consider the maintenance policy under age replacement and block replacement policies. Then the following four maintenance policies will be considered:

- *Maintenance Policy A:* Under the degradation process, when the degradation level achieves the pre-specified threshold L, then maintenance activities will be taken. We denote this event as A_1.
- *Maintenance Policy B:* Under the cost process, when the cost level achieves the pre-specified threshold L_U, then maintenance activities will be taken. We denote this event as A_2.
- *Maintenance Policy C:* Only if both A_1 and A_2 have occurred, the age replacement will be conducted. Denote this event as $A_3 = A_1 \cap A_2$.
- *Maintenance Policy D:* If one of the two events, A_1 and A_2, occurs, the age replacement will be conducted. Denote this event as $A_4 = A_1 \cup A_2$.

Therefore, $G_1(t) := P(A_1) = F_{\omega_L}(t)$ and $G_2(t) := P(A_2) = F_{\omega_{L_U}}(t)$ and these functions can be obtained

$$\begin{aligned} G_3(t) &:= P(A_3) \\ &= P(A_1 \cap A_2) \\ &= P(A_1)P(A_2|A_1) \\ &= F_{\omega_L}(t)F_{\omega_{L_U}}(t|\omega_L), \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} G_4(t) &:= P(A_4) \\ &= P(A_1 \cup A_2) \\ &= P(A_1) + P(A_2) - P(A_1 \cap A_2) \\ &= P(A_1) + P(A_2) - P(A_3), \end{aligned} \tag{4.38}$$

We have already obtained the conditional probability $f_{U(t)|Y(t)(y,u)}$, using $f_{\omega_L}(t)$ and $f_{\omega_{L_U}}(t)$ to replace $f_{Y(t)}$ and $f_{U(t)}$, respectively, then

$$\begin{aligned} f_{\omega_{L_U}|\omega_L(y,u)} &= \frac{f_{\omega_{L_U},\omega_L(y,u)}}{f_{\omega_L}(y)} \\ &= \frac{1}{4\pi^2 f_{\omega_L}(y)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\prod_{k=1}^{n} \phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 y - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2, \end{aligned} \tag{4.39}$$

where

$$\phi_{X_k(t)}(a_k t_1 + c_k t_2) = \exp\{\frac{\sigma_k[1-(1-2i\mu_k^2(a_k t_1 + a_k c_k t_2)\sigma_k^{-1})^{1/2}]}{\mu_k}\}, \tag{4.40}$$

and

$$\begin{aligned} F_{\omega_{L_U}}(t|\omega_L) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\omega_L}^{-1}(t)(\prod_{k=1}^{n} \phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 t - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t \\ &= \frac{\ln f_{\omega_L}(t)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\prod_{k=1}^{n} \phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 t - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2. \end{aligned} \tag{4.41}$$

Therefore, the distribution of both $G_3(t)$ and $G_4(t)$ can be obtained.

The distribution of $G_3(t)$ is given by

$$G_3(t) := \frac{F_{\omega_L}(t)\ln f_{\omega_L}(t)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\prod_{k=1}^{n} \phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 t - it_2 u} \mathrm{d}t_1 \mathrm{d}t_2, \tag{4.42}$$

and $G_4(t)$ now can be presented by

$$G_4(t) := F_{\omega_L}(t) + F_{\omega_{L_U}}(t) - \frac{F_{\omega_L}(t)\ln f_{\omega_L}(t)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\prod_{k=1}^{n}\phi_{X_k(t)}(a_k t_1 + c_k t_2))^{-it_1 t - it_2 u}\mathrm{d}t_1\mathrm{d}t_2. \tag{4.43}$$

4.6.1 Age replacement policy

For the age replacement policy, a preventive replacement is conducted after a continuous working time T_a when there is no failure occurs (Barlow and Hunter, 1960).

Let T_a be a replacement age, then the mean time between replacements $M(T_a)$ will be

$$\begin{aligned} M(T_a) &= \int_0^{T_a} tf(t)dt + t_0 P(X > T_a) \\ &= \int_0^{T_a} tf(t)dt + t_0(t - F(T_a)) \\ &= \int_0^{T_a}(1 - F(t))dt. \end{aligned} \tag{4.44}$$

Then, the expected cost per time unit is given by

$$C_{A,i}(T_a) = \frac{c_r + c_m G_i(T_a)}{\int_0^{T_a}(1 - G_i(t))dt}, \tag{4.45}$$

where $i = 1, 2, 3, 4$, corresponding to maintenance policies A, B, C, and D, respectively and T_a is the decision variable, c_r is the expected replacements cost and c_m is the expected repair cost incurred due to failures.

Property 1. *For given t, if $G_1(t) \geq G_3(t)$, $G_2(t) \geq G_3(t)$, $G_1(t) \leq G_4(t)$ and $G_2(t) \leq G_4(t)$, then $C_{A,1}(T_a) \geq C_{A,3}(T_a)$, $C_{A,2}(T_a) \geq C_{A,3}(T_a)$, $C_{A,1}(T_a) \leq C_{A,4}(T_a)$, and $C_{A,2}(T_a) \leq C_{A,4}(T_a)$.*

Proof. Since $G_1(t) \geq G_3(t)$, $c_r + c_m G_1(T_a) \geq c_r + c_m G_3(T_a)$ and $\int_0^{T_a}(1 - G_1(t))dt \leq \int_0^{T_a}(1 - G_3(t))dt$. Hence, $C_{A,1}(T_a) = \dfrac{c_r + c_m G_1(T_a)}{\int_0^{T_a}(1 - G_i(t))dt} \geq \dfrac{c_r + c_m G_3(T_a)}{\int_0^{T_a}(1 - G_3(t))dt} = C_{A,3}(T_a)$.

Similar proofs can be established on the other inequality. ■

Moreover, the maintenance policy follows the following principles.

- The replacement time interval is T_a.
- Immediately after preventive or corrective maintenance, the system rests its age to 0.
- Both c_r and c_m are constants.

By minimising $C_{A,i}(T_a)$, we can obtain the optimum T_a^* for the age replacement policy based on maintenance policies A, B, C, and D, respectively.

4.6.2 Block replacement policy

For the block replacement policy, which is introduced by Barlow and Hunter (1960), a unit is replaced at a scheduled time regardless of time since its last repair. Any failure between replacements will be repaired with the minimal repair, which restores the failed system to the status just before the failure occurred. The expected cost per time unit for the block replacement policy is given by

$$C(T) = \frac{c_r + c_m M(T)}{T}, \tag{4.46}$$

where $M(t)$ is a renewal functions. To approximate this renewal function, given a

$$C_{B,i}(T_b) = \frac{c_r + c_m M_{\omega_L}(T_b)}{T_b}, \tag{4.47}$$

where $M_{\omega_L}(T_b)$ is the expected number of failed units with the CDF (cumulative distribution function) $F_{\omega_L}(t)$, during the interval $(0, T_b]$, c_r is the replacement cost and c_m is the maintenance cost. Assume that the replacement interval is so short that the probability of two or more failures occurring within $(0, T_b)$ is zero. Denote that $N(T_b)$ is the number of failures within an interval of length T_b, then

$$B(T_b) = E[M_{\omega_L}(T_b)], \tag{4.48}$$

then the expected cost per time unit is given by

$$C_{B,i}(T_b) = \frac{c_r + c_m B(T_b)}{T_b}, \tag{4.49}$$

According to our four maintenance policies, then

$$C_{B,i}(T_b) = \frac{c_r + c_m B_i(T_b)}{T_b}, \tag{4.50}$$

where $i = 1, 2, 3, 4$, corresponding to maintenance policies A, B, C, and D, the optimal scheduled replacement time T_b could be obtained by minimizing the $C_{B,i}(T_b)$. Similarly, we can obtain this property.

Property 2. *For given t, $G_1(t) \geq G_3(t)$, $G_2(t) \geq G_3(t)$, $G_1(t) \leq G_4(t)$, and $G_2(t) \leq G_4(t)$, then $C_{B,1}(T_b) \geq C_{B,3}(T_b)$, $C_{B,2}(T_b) \geq C_{B,3}(T_b)$, $C_{B,1}(T_b) \leq C_{B,4}(T_b)$, and $C_{B,2}(T_b) \leq C_{B,4}(T_b)$.*

Therefore, within the block replacement policy, maintenance activities have the following principles.

- The inspection will be taken every T_b.
- Immediately after a preventive or corrective maintenance, the system rests its age to 0.
- Both c_r and c_m are constants.

4.7 Numerical examples

We consider a system with two different failure modes. The deterioration process of the two failure modes is modelled with two WPs, respectively, each of which has different parameters α, β and σ. We assume that two modes have weights as following $a_1 = 0.3$ and $a_2 = 0.7$. α_1 , β_1 and σ_1 are 0.8, 0.5 and 0.2 for the first failure mode, respectively. α_2, β_2 and σ_2 are 0.7,1 and 0.5, respectively. We also assume that $c_r = 100$ and $c_m = 50$, then we can obtain the following result.

Thus, the linear combination of the two processes is given by

$$Y(t) = 0.3X_1 + 0.7X_2.$$

We assume that the system needs to be repaired when the deterioration levels exceed the threshold L_{w_L} and the threshold $L_{w_{L_c}}$, respectively. Replacement activities will be taken and the deterioration level will be restored to zero when the component is completely replaced. We obtain the result under $L_{w_L} = \{3, 3.5, 2\}$ and $L_{w_{L_c}} = \{1.5, 1, 2.5\}$ under policies A, B, C and D, respectively. It is worth noticing that all parameters can be estimated based on historical data or expert elicitation (Shah et al., 2013).

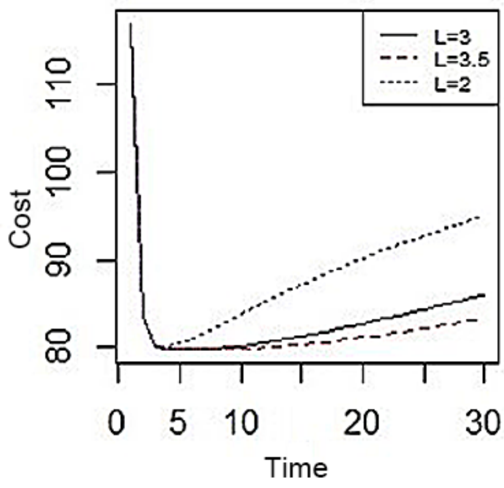

FIGURE 4.3: Maintenance Policy A

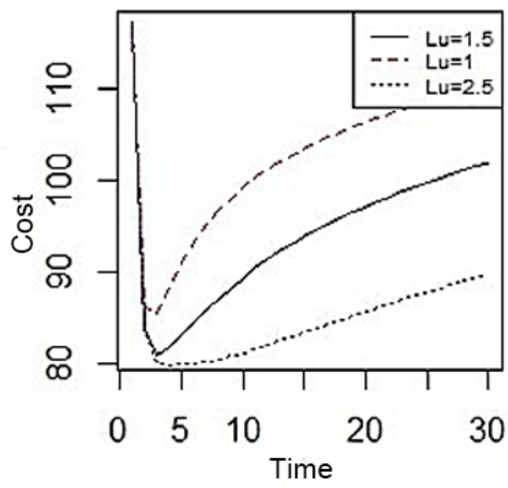

FIGURE 4.4: Maintenance Policy B

Figure 4.3 shows the expected cost per unit time under the maintenance policy A.

- When the threshold L_{w_L} is 3, the optimised time interval is ($T_{opt} = 4.318$) and the expected unit cost per time is 79.793.
- When the threshold L_{w_L} is 3.5, the optimised time interval is ($T_{opt} = 4.745$) and the expected unit cost per time is 79.789.
- When the threshold L_{w_L} is 3, the optimised time interval is ($T_{opt} = 3.410$) and the expected unit cost per time is 79.966

Figure 4.4 shows the expected cost per unit time under the maintenance policy B.

- When the threshold $L_{w_{L_c}}$ is 1.5, the optimised time interval is ($T_{opt} = 2.934$) and the expected unit cost per time is 80.787.
- When the threshold $L_{w_{L_c}}$ is 1, the optimised time interval is ($T_{opt} = 2.483$) and the expected unit cost per time is 84.731.
- When the threshold $L_{w_{L_c}}$ is 2.5, the optimised time interval is ($T_{opt} = 3.874$) and the expected unit cost per time is 79.817.

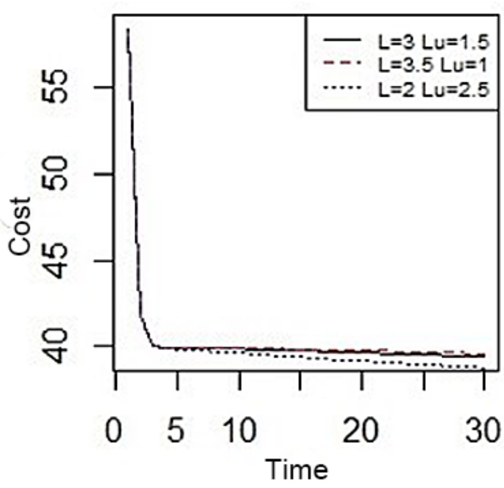

FIGURE 4.5: Maintenance Policy C

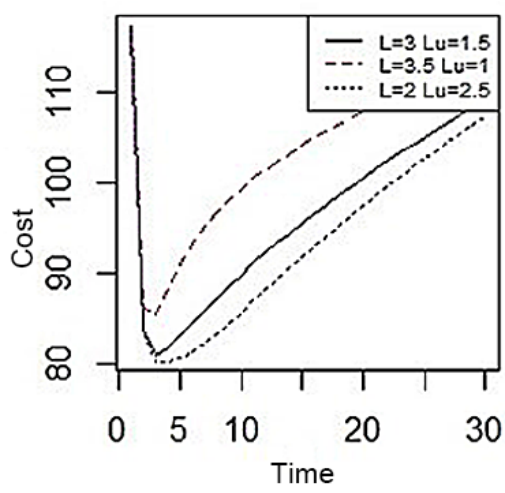

FIGURE 4.6: Maintenance Policy D

Figure 4.5 shows the expected cost per unit time under the maintenance policy C.

- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 3 and 1.5, respectively, the expected unit cost per time is 39.85863.
- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 3.5 and 1, respectively, the expected unit cost per time is 39.88409.
- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 2 and 2.5, respectively, the expected unit cost per time is 39.62653.

Figure 4.6 shows the expected cost per unit time under the maintenance policy D.

- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 3 and 1.5, respectively, the optimised time interval is ($T_{opt} = 2.934$) and the expected unit cost per time is 80.787.
- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 3.5 and 1, respectively, the optimised time interval is ($T_{opt} = 2.483$) and the expected unit cost per time is 84.731.
- When the thresholds for L_{w_L} and $L_{w_{L_c}}$ are 2 and 2.5, respectively, the optimised time interval is ($T_{opt} = 3.360$) and the expected unit cost per time is 79.994.

Figure 4.7 shows the comparison among policies A, B, C, and D. Table 4.1 is the optimised result which is related to Figure 4.7.

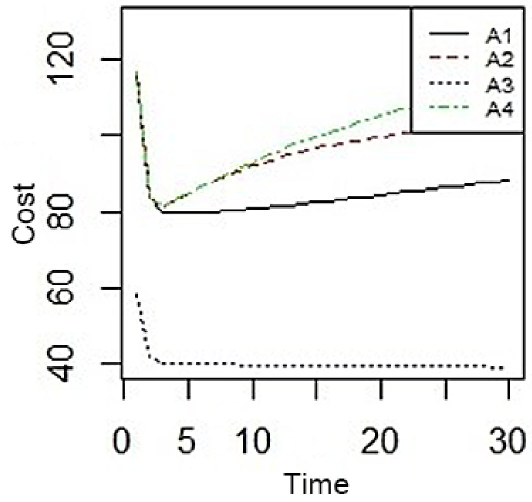

FIGURE 4.7: Comparison among policy A, B, C and D

Optimised result	A1	A2	A3	A4
Optimised expected unit cost per time	79.803	81.518	39.819	81.518
Time interval	4.024	2.777	-	2.777

TABLE 4.1: Comparison result among policy A, B, C and D

Then, we set 10 scenarios. Table 4.2 shows the parameters we used for these 10 scenarios. Table 4.2 shows the parameters we used for 10 scenarios.

Parameters	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10
c_r	100	100	100	100	80	85	90	100	120	100
c_m	50	50	50	50	40	80	70	80	90	120
L	3.00	2.00	2.50	3.50	3.00	3.00	2.00	2.50	3.50	3.00
L_u	1.50	1.80	2.00	2.50	1.50	1.50	1.80	2.00	2.50	1.50
a_1	0.30	0.30	0.40	0.50	0.30	0.30	0.30	0.40	0.50	0.30
a_2	0.70	0.70	0.60	0.50	0.70	0.70	0.70	0.60	0.50	0.70
α_1	0.80	0.60	0.70	0.65	0.80	0.80	0.60	0.70	0.65	0.80
α_2	0.70	0.50	0.80	0.55	0.70	0.70	0.50	0.80	0.55	0.70
β_1	0.50	0.60	0.80	1.20	0.50	0.50	0.60	0.80	1.20	0.50
β_2	1.00	0.90	0.80	0.70	1.00	1.00	0.90	0.80	0.70	1.00
σ_1	0.20	0.40	0.60	0.80	0.20	0.20	0.40	0.60	0.80	0.20
σ_2	0.50	0.55	0.65	0.45	0.50	0.50	0.55	0.65	0.45	0.50

TABLE 4.2: Parameters for 10 scenario

- S1, S5, S6, and S10 have the same parameters excluding the replacement cost and repair cost.
- S2 and S7 have the same parameters excluding the replacement cost and repair cost.
- S3 and S8 have the same parameters excluding the replacement cost and repair cost.

- S4 and S9 have the same parameters excluding the replacement cost and repair cost.
- S1, S2, S3, S4 have same replacement cost and repair cost. However, other parameters are different.

Table 4.3 shows the expected cost per time unit with its time interval based on our 10 scenarios. The value outside the brackets is the optimised expected cost per time unit and the value inside the brackets is the time interval.

Scenario	**A1**	**A2**	**A3**	**A4**
S1	80.787(2.934)	80.787(2.934)	39.859	80.787(2.934)
S2	80.197(3.186)	80.534(3.018)	39.498	80.776(2.894)
S3	80.006(3.358)	80.621(2.988)	39.600	80.715(2.929)
S4	79.801(4.067)	80.019(3.354)	39.827	80.012(3.348)
S5	63.835(4.280)	64.769(2.870)	31.880	64.770(2.870)
S6	67.826(4.210)	69.154(2.752)	33.853	69.154(2.752)
S7	72.312(3.074)	72.731(2.897)	35.351	73.009(2.775)
S8	80.091(3.247)	80.956(2.856)	39.424	81.073(2.801)
S9	95.766(3.993)	96.099(3.258)	47.752	96.102(3.253)
S10	79.796(4.169)	81.644(2.682)	39.809	81.644(2.682)

TABLE 4.3: Numerical examples for 10 scenario

According to Table 4.3, we can find that the result is satisfied with property 1, $C_{A,1}(T_a) \geq C_{A,3}(T_a), C_{A,2}(T_a) \geq C_{A,3}(T_a), C_{A,1}(T_a) \leq C_{A,4}(T_a)$, and $C_{A,2}(T_a) \leq C_{A,4}(T_a)$.

We compare these results from two aspects: the influence of cost and the influence of other parameters excluding cost. According to Table 4.3, we use results of S1, S5, S6, and S10 for the first aspect and S1, S2, S3, S4 for the second aspect.

4.7.1 Comparison among S1, S5, S6, S10

We focus on the influence of cost in this part.

- According to Figures 4.8, 4.9 and 4.10, with the increase of cost, all of policies A, B, and D have increasing expected costs.
- Among them, maintenance policy D is the most sensitive to price changes. The expected cost of S6 is gradually higher than that of S5.

4.7.2 Comparison among S1, S2, S3, S4

We focus on the influence of other parameters excluding cost in this part.

- According to Figure 4.11, the ratios of cost changing from the highest to the lowest are: $S5 > S6 > S1 > S10$.
- According to Figure 4.12, the ratios of cost changing from the highest to the lowest are: $S1 > S5 > S6 > S10$.

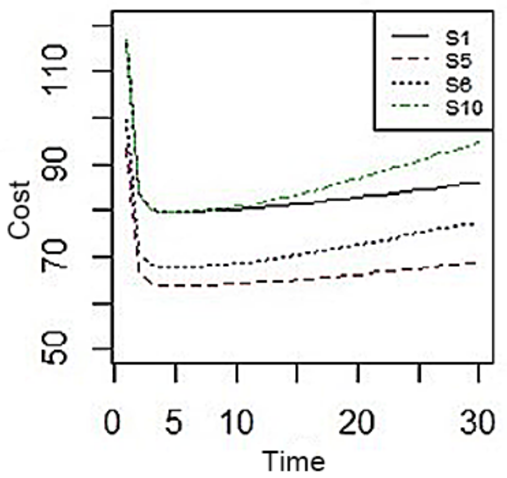

FIGURE 4.8: Policy A for S1, S5, S6 and S10

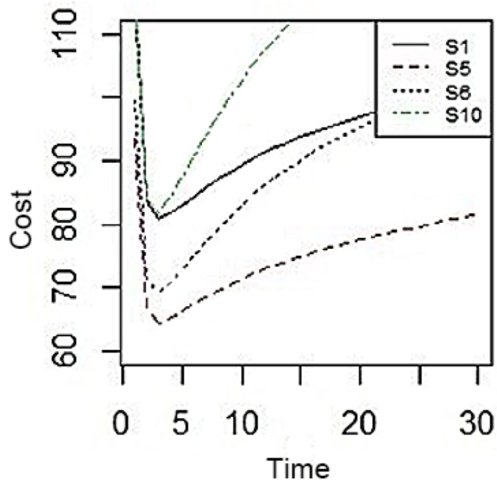

FIGURE 4.9: Policy B for S1, S5, S6 and S10

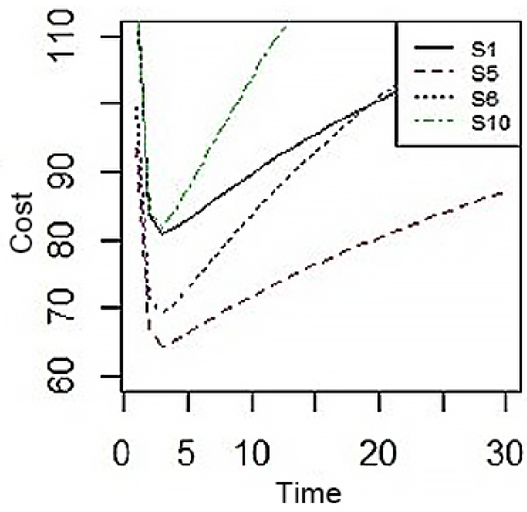

FIGURE 4.10: Policy D for S1, S5, S6 and S10

- According to Figure 4.13, the ratios of cost changing from the highest to the lowest are: $S1 > S5 > S6 > S10$ before the turning point $t = 10$ and $S5 > S1 > S6 > S10$ after the turning point.

4.8 Summary

This chapter investigated maintenance policies for a system whose deterioration process is a linear combination of Wiener processes. It proposed four maintenance policies with both degradation and cost thresholds for a multi-component system and then compared them. It also discussed two properties based on these four maintenance policies. Numerical examples were given to illustrate the optimisation process.

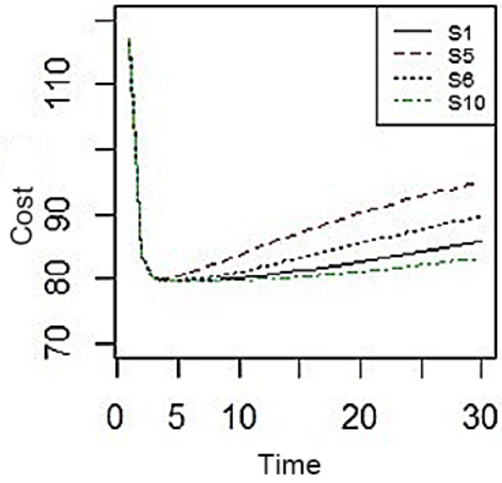

FIGURE 4.11: Policy A for S1, S2, S3 and S4

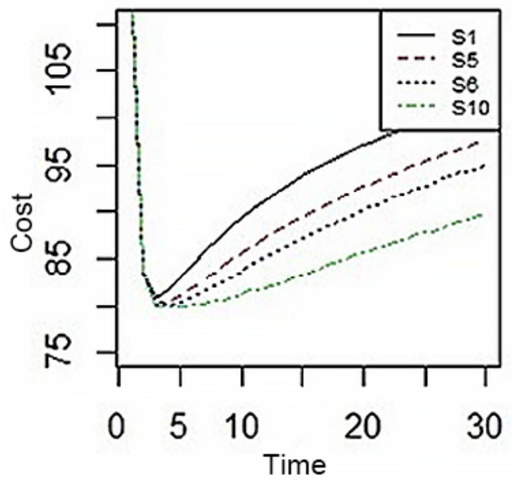

FIGURE 4.12: Policy B for S1, S2, S3 and S4

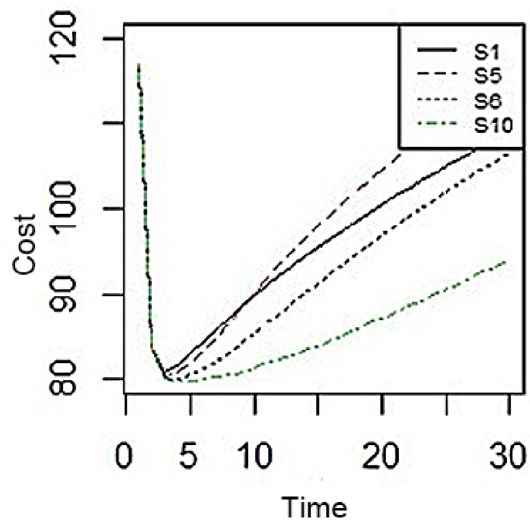

FIGURE 4.13: Policy D for S1, S2, S3 and S4

However, there are several limitations in our research. Firstly, the deterioration process of a system may be a non-linear combination of deterioration processes. A non-linear combination of deterioration processes based on other models, such as the GPs and the geometric process can be considered in the future.

Secondly, the dependence amongst failure modes or failure components has not been considered in this thesis. For example, economic dependence is a possible problem for designing the maintenance policy. Such problems will be investigated in our future work.

Chapter 5

Uncertainty of maintenance policies for items modelling by the Extended Poisson Process

5.1 Introduction

The geometric process (GeP) was introduced by Lam (1988a) and has been widely applied to model the failure process of items for maintenance policy optimisation. In this chapter, we analyse the GeP and its extensions in two aspects: the trend analysis and the influence due to the uncertainty of turning points.

The trend and relationship among the GeP and GeP-like models that have not been explored in existing research. Several authors have extended the GeP to overcome its various drawbacks, see in Section 2.3. One of the extensions, the extended Poisson process (EPP), can be used to model the repair process for items with the trend of their working times being non-monotonic. Therefore, an interesting question in the application of the EPPs in maintenance policy optimisation is to find the turning point at which the trends vary. this thesis derives the turning point of the EPP, analyses the uncertainties of the maintenance interval, and the associated cost. Numerical examples are provided to illustrate the proposed maintenance policies. All extensions will be called GeP-like models in this thesis.

5.2 Common properties and special cases of the GeP and its extensions

The GeP and its extensions have been introduced in Chapter 2. Wu, Peng, and Wu (2020) indicated that a GeP-like model consists of two functions: a positive function of δ_1 and k and a positive function of δ_2 and k, where δ_1 and δ_2 are estimable parameter vectors. They proposed that given a sequence of non-negative random variables $\{X_k, k = 1, 2, ...\}$, if they are independent and the cdf of X_k is given by $F(g(\delta_1, k)x^{h(\delta_2,k)})$ for $k = 1, 2...$, where $g(\delta_1, k)$ is positive function of ratio and $h(\delta_2, k)$

is a positive function of k, and δ_1 and δ_2 are parameter vectors, then $\{X_k, k = 1, 2, ...\}$ is a generalized GP.

However, the above description cannot explain all GeP-like models, for example, the GePAP and the BGP. Therefore, we classify the GeP-like models into two types: Type I extension that can be summarized by $g(\delta_1, k)$, and Type II extension that cannot. Table 5.1 shows the generalized GeP with $g(\delta_1, k)$.

TABLE 5.1: Type I GeP-like models

$g(\delta_1, k)$	$h(\delta_2, k)$	Model	Reference
1	1	Renewal Process (RP)	Lam (1988b)
a^{k-1}	1	Geometric process (GeP)	Lam (1988b)
a^{k-1}	1	Arithmetic geometric process (ArGeP)	Francis (2001)
k^a	1	α-series process (α)	Braun, Li, and Zhao (2005)
a^{i-M_k}	1	Threshold geometric process (TGeP)	Chan et al. (2006)
$\alpha a^{k-1} + \beta b^{k-1}$	1	Extended Possion process (EPP)	Wu and Clements-Croome (2006)
a^{b_k}	1	Extended geometric process (EEGeP)	Zhang and Wang (2016)
a^{k-1}	1	Semi-geometric process (SGeP)	Wu and Wang (2018)
a^{k-1}	$(1 + \log(k))^b$	Doubly geometric process (DGeP)	Wu (2018)
a_k	b_k	Double ratio geometric process (DRGeP)	Wu (2022)
Z^{k-1}	1	Rate randomized geometric process (RRGeP)	Asadi and Wu (2024)

According to the Table 5.1 and the basic properties of the GeP, all common GeP-like models have the following properties:

- if $h(\delta_2, k) = 1$ and $g(\delta_1, k)$ increases over k, then $\{X_k, k = 1, 1 \dots\}$ is stochastically decreasing,
- f $h(\delta_2, k) = 1$ and $g(\delta_1, k)$ decreases over k, then $\{X_k, k = 1, 1 \dots\}$ is stochastically decreasing.

Among them, the RRGeP is a special case as it can be non-monotonic if $h(\delta_2, k) = 1$. According to the above table, it can be seen that the main characteristic of the generalized GeP-like model is the difference of the parametric equation of x, which is denoted by $g(\delta_1, k)$. For example, for the GeP, $g(\delta_1, k)$ can be considered as an exponential function. The α changes this point and assumes that $g(\delta_1, k)$ becomes a power function.

Other GeP-like models such as the BGP, the EGeP, the DGeP, the GePAP, the DRGeP and the AIGeP cannot be summarized by $g(\delta_1, k)$ and $h(\delta_2, k)$. Table 5.2 shows the cdf of such special cases of the GeP-like models

TABLE 5.2: Type II GeP-like models

Survival function	Model	Reference
$1 - F(a^{k-1}x)$	Binary geometric process (BGP)	Chan and Leung (2010)
$F_k = pF_{k-1}(x) + qF_{k-1}(ax)$	Extended geometric process (EGeP)	Zhang and Wang (2016)
$1 - (1-\rho)e^{\mu a^{k-1}}x$	Gemotric-Pólya-Aeppli process (GePAP)	Chukova and Minkova (2020)
$1 - (1 - F(a_k x))^{b_k/a_k}$	Double ratio geometric process (DRGeP)	Wu (2022)
$\{(X_1, Y_1), (X_2, Y_2), \dots,\}$	Alternating geometric process (AIGeP)	Arnold et al. (2022)

The AIGeP is slightly different from others in this group. Since it is not simply assumed that the survival function of a system follows the GeP, it also considers the downtime or repair time follows the GeP. The AIGeP can be considered a compound cumulative geometric process, and its expression should be the sum of two GePs.

Then, Table 5.3 shows a summary of the cdf, the pdf, and the monotonicity or non-monotonicity of the GeP-like models.

TABLE 5.3: Summary of monotonicity/non-monotoniciy for GeP and its extensions

Model	**cdf** $F_k(x)$	**pdf** $f_k(x)$	**Monotonicity**	$E(X_k)$	$V(X_k)$
GeP (Lam, 1988b)	$F(a^{k-1}x)$	$a^{k-1}f(a^{k-1}x)$	Monotonicity	$\frac{E(X_1)}{a^{k-1}}$	$\frac{Var(X_1)}{a^{2(k-1)}}$
GRP (Wang and Zhang, 2013)	$F_k(x) = (g(k)x)$	$g(k)xf(g(k)x)$	Monotonicity	$\frac{E(X_1)}{g(k)}$	$\frac{Var(X_1)}{(g(k))^2}$
ArGeP (Francis, 2001)	$F(a^{k-1}x + a^{k-1}(k-1)d)$	$a^{k-1}f(a^{k-1}x + a^{k-1}(k-1)d)$	Monotonicity	$\frac{E(X_1)}{a^{k-1}} - (k-1)d$	$\frac{Var(X_1)}{a^{2(k-1)}}$
α-SP (Braun, Li, and Zhao, 2005)	$F(k^a x)$	$k^a f(k^a x)$	Monotonicity	$\frac{E(X_1)}{k^\alpha}$	$\frac{Var(X_1)}{k^{2\alpha}}$
TGeP (Chan et al., 2006)	$F(a^{k-M_i}x)$	$a^{k-M_i}f(a^{k-M_i}x)$	Non-monotonicity	$\frac{E(X_1)}{a^{k-M_i}}$	$\frac{Var(X_1)}{a^{2(k-M_i)}}$
EPP (Wu and Clements-Croome, 2006)	$F((\alpha a^{k-1} + \beta b^{k-1})x)$	$(\alpha a^{k-1} + \beta b^{k-1})f((\alpha a^{k-1} + \beta b^{k-1})x)$	Non-monotonicity	$\frac{E(X_1)}{\alpha a^{k-1} + \beta b^{k-1}}$	$\frac{Var(X_1)}{(\alpha a^{k-1} + \beta b^{k-1})^2}$
BGeP (Chan and Leung, 2010)	$1 - F(a^{k-1}x)$	$1 - a^{k-1}f(a^{k-1}x)$	Monotonicity	$\frac{E(X_1)}{}$	$\frac{Var(X_1)}{}$
EEGeP (Bordes and Mercier, 2013)	$F(a^{b_k}x)$	$a^{b_k}f(a^{b_k}x)$	Monotonicity	$\frac{E(X_1)}{a^{b_k}}$	$\frac{Var(X_1)}{a^{2b_k}}$
EGeP (Zhang and Wang, 2016)	$F_k = pF_{k-1}(x) + qF_{k-1}(ax)$	$pf_{k-1}(x) + qaf_{k-1}(ax)$	Non-monotonicity	$E(X_1)(p + \frac{q}{a})^{k-1}$	$Var(X_1)(p + \frac{q}{a})^{k-1}$
SGeP (Wu and Wang, 2018)	$F(a^{k-1}x)$	$a^{k-1}f(a^{k-1}x)$	Monotonicity	$\frac{E(X_1)}{a^{k-1}}$	$\frac{Var(X_1)}{a^{2(k-1)}}$
DGeP (Wu, 2018)	$F(a^{k-1}x^{(1+\log(k))^b})$	$a^{k-1}(1+\log(k))^b x^{(1+\log(k))^b - 1} f(a^{k-1}x^{(1+\log(k))^b})$	Non-monotonicity	$E((\frac{X_1}{a^{k-1}})^{-(1+\log(k))^b})$	$Var((\frac{X_1}{a^{k-1}})^{-(1+\log(k))^b})$
RRGeP (Asadi and Wu, 2024)	$F(Z^{k-1}x)$	$Z^{k-1}f(Z^{k-1}x)$	Non-monotonicity	$\frac{E(X_1)}{Z^{k-1}}$	$\frac{Var(X_1)}{Z^{2(k-1)}}$

5.2.1 Comments on the GeP and its extensions

An interesting characteristic of the GeP is that the history of a GeP is defined by the time since the last renewal and additionally. Then, one interesting problem is that what leads to the change from $F(a^{k-1}x)$ in cycle kth to $F(ax)$ in cycle $(k+1)$th? We will discuses in two respects

- When the GeP is used to describe the deterioration nature of a system, then the change from $F(a^{k-1}x)$ in cycle kth to $F(ax)$ in cycle $(k+1)$th is due to the nature of deterioration.
- When the GeP is used to describe a maintenance effect process, then the change from $F(a^{k-1}x)$ in the kth to $F(ax)$ in cycle $k+1$th is due to the effect of maintenance.

According to Lam (2007), the author stated the justification of introducing the new concept as follows.

Nevertheless, in practice, the repair of a failure system will usually yield a functioning system; its successive survival times are decreasing and eventually dying out. This is often the case if a system is deteriorative. For example, consider a tyre or a pneumatic tyre; the successive survival times after repair will be decreasing and will tend to zero, because of the deterioration (pages 366-367 in Lam (2007)).

The above statement explains that the change of the survival distribution from $F(a^{k-1}t)$ to $F(a^k x)$, which is a monotonous trend, is due to the nature of deterioration.

However, Fig. 5.1 and Fig. 5.2 show failure rates of the GeP where X_k follows an exponential distribution and a Weibull distribution, respectively. The initial failure intensities are drawn in dashed lines. In Fig. 5.1, the initial failure intensity is constant, however, the result of the GeP shows that the failure intensity is increasing after repairs. In Fig. 5.2, the initial failure intensity is decreasing, but the GeP shows that the failure intensities are increasing after repairs. Such examples show that the changes of failure intensities are due to the effectiveness of repair, rather then the nature of deterioration.

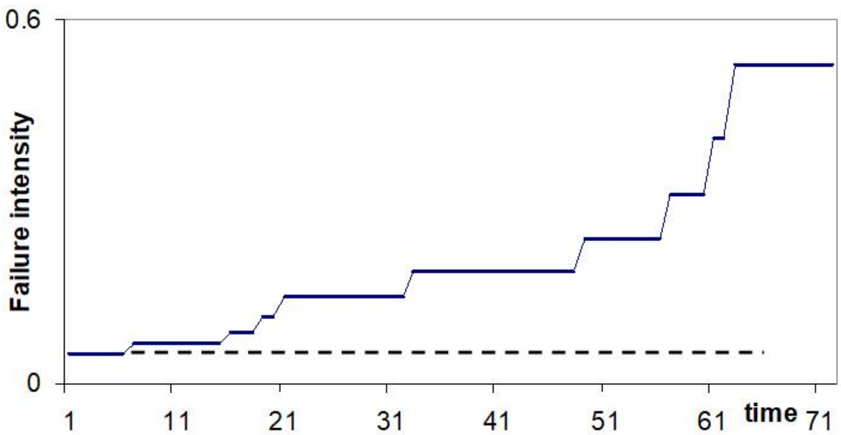

FIGURE 5.1: A constant failure rate cast

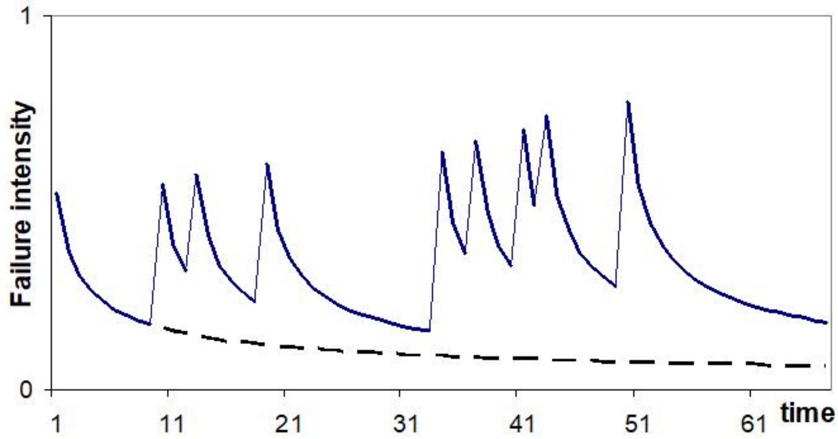

FIGURE 5.2: A decreasing failure rate case

Commonly, with different levels of maintenance effect, maintenance can be broken down into three categories: perfect, normal, and imperfect maintenance. A perfect maintenance can restore a system to an "as good as new" state, a normal maintenance is assumed to bring the system to any condition, and an imperfect maintenance can restore the system to the exact state it was before failure. It should be noted, however, that the changes of survival distributions between two adjacent cycles are assumed to be due to an imperfect maintenance conflicts with the assumption in the definition of the threshold GeP. The threshold GeP is introduced to overcome the weakness that a GeP can not be used to depict the well-known bathtub curve, with which the failure rate of a system becomes smaller in the early failure period, is constant in the intrinsic failure period and becomes increasing in the wear-out failure period. This is because the threshold GeP can have increasing and decreasing Zn. Nevertheless, it is hard to imagine that the effect of maintenance might follow as a bathtub curve

5.2.2 Trend analysis

For the GP, AGP, EPP, TGP DGP, EGP, and DRGP, the trend analysis has already been studied, as shown in preceding section. However, the trend analysis of the other extensions has not been properly investigated.

Trend analysis of the alternating EEGP

The definition of the EEGP has been introduced, therefore, we find that

- if $b(k) = k - 1$,
    1. and $a > 1$, then $X_k, k = 1, 2, \ldots$ is stochastically decreasing,
    2. and $0 < a < 1$, then $X_k, k = 1, 2, \ldots$ is stochastically increasing,
    3. and $a = 1$, then $X_k, k = 1, 2, \ldots$ reduces to a RP.
- if $b(k) = (k - 1)^\alpha$,
    1. then α should greater than 0,

2. and $\alpha = 0$, then $X_k, k = 1, 2, \ldots$ reduces to a GP.

The expectation of X_k of the EEGP has not been mentioned by Bordes and Mercier (2013), it should be given by

$$\mathrm{E}[X_k] = \frac{\mathrm{E}[Y_k]}{a^{b_k}}, \tag{5.1}$$

and the variance is given by

$$\mathrm{Var}[X_k] = \frac{\mathrm{Var}[Y_k]}{a^{2b_k}}. \tag{5.2}$$

5.2.3 Relationships between existing GeP-like models

Fig. 5.3 illustrates the relationship between the GeP-like models.

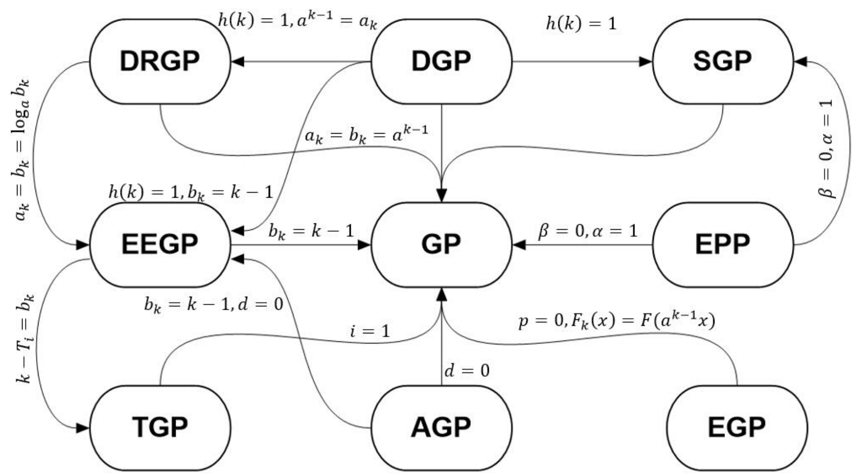

FIGURE 5.3: Relationship of GeP-likes model

Under certain conditions, these GeP-like models will reduce to the GeP or other extensions.

- If $\alpha = 1, \beta = 0$, then the EPP reduces to the GeP;
- If $d = 0$, then the AGeP reduces to the GeP;
- If $i = 1$ and $a_i = a_1 = a$, then the TGeP reduces to the GeP. That is because TGeP can be narrowly considered as the piecewise GeP based on different ratio a_i;
- If $b_k = k - 1, d = 0$, then the AGeP recues to the TGeP;
- If $k - T_i = b_k$, then the EEGeP reduces to the TGeP,;
- If $b_k = k - 1$, the EEGeP reduces to GeP;
- If $b = 0$, then the DGeP reduces to the GeP;
- If $\alpha = 1, \beta = 0$, then the EPP reduces to the SGeP;
- If $p = 0, F_k(x) = F(a^{k-1}x)$, then the EGeP reduces to the GeP;
- If $b = 0, b_k = k - 1$, then the DGeP reduces to the EEGeP;
- If only $b = 0$, then the DRP reduces to the SGeP;
- If $a^{k-1} = a_k$, $b = 0$, then the $h(k)$ for the DGeP will equal to 1, and the DGeP reduces to the DRGeP;

- If $a_k = b_k = a^{k-1}$, the DRGeP reduces to the GeP;
- If $a_k = b_k = \log_a b_k$, then the DRGeP reduces to the EEGeP;

5.3 Uncertainty of the turning points of the EPP

Several authors have extended the GeP to overcome its various drawbacks. One of the extensions, the extended Poisson process (EPP), can be used to model the repair process for items with the trend of their working times being non-monotonic. Therefore, an interesting question is to understand the properties of the extensions of the GeP.

5.3.1 Sensitivity analysis and maintenance policies

Sensitivity analysis (SA) is a helpful tool to know how the uncertainty of parameters, such as measurement errors and sampling errors, impacts the model's overall uncertainty under a given set of conditions. It provides another aspect to understand how the model responds to the changes in the parameters of a model. Castillo, Mínguez, and Castillo (2008) summarized a sensitivity analysis of the objective function for reliability in the cases of non-linear programming. Razavi et al. (2021) indicated that sensitivity analysis should be promoted as an independent discipline. In reliability, it is normally useful to

- understand which values/parameters are the most influential on the degradation process model.
- understand how changes in components influence the state of the whole system.
- understand the costs and reliability implications associated with the changes in failure probabilities and degradation process.
- understand the component implications associated with the whole cost in a long-term period.
- understand the implications of the objective in a set of given assumptions.

Table 5.4 shows how sensitivity analysis is used in reliability

Therefore, the application of the sensitivity analysis can be classified into several aspects:

- Using the SA for parameters analysis. From this aspect, how the changes in parameters will influence the whole system will be explored. For example, an SA of components may find which component is the most implicated in the system.
- Using the SA for component dependence. From this aspect, the dependence between components can be explored.
- Using the SA for structural dependence. From this aspect, the impact of differences on system structure can be compared.
- Using the SA for cost analysis. From this aspect, the relationship between parameters will be explored. Changes in different parameters may have different

TABLE 5.4: Sensitivity analysis in reliability

Relevant application	Relevant reference
Sensitivity of costs	Castillo, Mínguez, and Castillo (2008)
Sensitivity indices of parameters	Guo and Du (2009)
Sensitivity of structural performance of arch dam	Khaneghahi, Alembagheri, and Soltani (2019)
Sensitivity of risk matrix uncertainty	Sarazin et al. (2019)
Sensitivity of component dependence	Lee and Tien (2019)
Sensitivity of peak power and radiating frequency	Almansoori et al. (2020)
Sensitivity of complex non-linear dynamic model	Li-Sha et al. (2020)
Sensitivity in seismic of bridge	Xia et al. (2022)
Sensitivity of component dependence	Zhang, Zhang, and Shen (2023)
Sensitivity of critical design factors	Chen, Wang, and Zhang (2019)
Sensitivity of failure rate and maintenance policy	Wang et al. (2020)
Sensitivity of threshold	Li, Yuan, and Fu (2019)

effects on the cost models/cost processes than on the degradation processes. This will influence the final maintenance policy optimisation.

5.3.2 Introduction of the turning points

With the development of the GeP, some extensions can show both stochastically increasing and decreasing trends. Therefore, such GeP-like models can correspond to both the first step and third step of the bathtub curve such as the EPP.

When the changing trend of hazard rate changes (such as changing from random failure to wear-out failure in the bathtub curve), we call such time points turning points. In the coming section, we will use the EPP as a case to introduce the turning point in detail. Before we discuss the turning point, the definition and the properties of the EPP will be provided again.

Definition 17. *(Wu and Clements-Croome, 2006) Given a sequence of non-negative random variables* $\{X_k, k = 1, 2, ...\}$*, if they are independent and the cdf of* X_k *is given by* $F((\alpha a^{k-1} + \beta b^{k-1})x)$ *for* $k = 1, 2...$*, where* $F(x)$ *is an exponential cdf,* $\alpha + \beta \neq 0$*,* $\alpha, \beta \geq 0$*,* $a \geq 0$ *and* $0 < b \leq 1$*, then* $\{X_k, k = 1, 2, ...\}$ *is called as an extended Poisson process (EPP).*

The expectation and the variance of the $X_k(t)$ is given by

$$\mathrm{E}[X_k] = \mathrm{E}[X_1]((\alpha a^{k-1} + \beta b^{k-1}))^{-1},$$

and

$$\mathrm{Var}[X_k] = \sigma^2((\alpha a^{k-1} + \beta b^{k-1}))^{-1}$$

where $\mathrm{E}(X_1)$ is the expectation of X_1 and σ^2 is the variance of X_1.

The EPP has the following properties:

- if $a = b = 1$, then the EPP reduces to the HPP with the cdf $F((\alpha + \beta)x)$;
- if $\alpha a^{k-1} \neq 0$ and $\beta b^{k-1} = 0$ (or $\alpha a^{k-1} = 0$ and $\beta b^{k-1} \neq 0$), then the EPP reduces to a GeP, in this case,

(1) $\alpha a > 1$ (or $\beta b^{k-1} > 1$), the EPP is stochastically decreasing;
(2) $0 < \alpha a < 1$ (or $0 < \beta b^{k-1} < 1$), the EPP is stochastically increasing;
(3) $\alpha a = 1$ (or $\beta b^{k-1} = 1$), the EPP reduces to the GeP.

- if $\alpha a^{k-1} \neq 0$ and $\beta b^{k-1} \neq 0$, in this case,
  (a) if $a = 1$ and $b < 1$ (or $a > 1$ and $b = 1$), then the EPP can model a failure process with decreasing failure intensity functions with respect to k;
  (b) if $a > 1$ and $b < 1$, then the EPP can model a failure process with a more complicated failure intensity functions with respect to k;

It can be seen that one can set $a = 1$ or $b = 1$ in applications when the EPP is used to model recurrent event data with non-monotonous trends.

If the EPP is used to describe a non-monotonic failure trend, it must satisfy the conditions: $\alpha a^{k-1} \neq 0$, a<1, 0<b<1, and $\beta b^{k-1} \neq 1$.

Then, the definition of the turning point is introduced following

Definition 18. *Given a sequence of non-negative random variables $\{Z_k, k = 1, 2, ...n\}$, if they are i.i.d and $\{Z_k, k = 1, 2, ...k^*\}$ is stochastically increasing (or decreasing), and $\{Z_k, k = k^*, k^* + 1, ...n\}$ is stochastically decreasing (or increasing), then $k = k^*$ can be called as the turning point.*

A turning point may be a peak turning point or a valley turning point. For the former, the hazard rate is stochastically increasing before the turning point and is stochastically decreasing after the turning point. For the latter, it is totally converse. Therefore, now the problem is how to estimate the turning point. The coming section will describe it.

5.3.3 Turning points of EPP

The invariance of the shape parameter has been explained by Wu (2018). Therefore, for the EPP, denote $g(k^*) = \alpha a^{k^*-1} + \beta b^{k^*-1}$, if we treat k as a real number, then the derivation of $g(k^*)$ is given by

$$g'(k^*) = \alpha a^{k^*-1} \ln a + \beta b^{k^*-1} \ln b, \tag{5.3}$$

then let $g'(k^*)$ be zero as the gradient at the turning point is zero, the following result can be obtained

$$k^* = \frac{\ln(-\beta \ln b) - \ln(\alpha \ln a)}{\ln a - \ln b} + 1. \tag{5.4}$$

This suggests k^* be a turning point, Then, the following lemma can be introduced

Lemma 1. *For a a sequence of non-negative random variables $\{X_k, k = 1, 2, ...n\}$, which follows a EPP, then $\{Z_k, k = 1, 2, ...n\}$ is the sequence of hazard rate. The turning point of the hazard rate is $k = k^*$ where $k^* = \dfrac{\ln(-\beta \ln b) - \ln(\alpha \ln a)}{\ln a - \ln b} + 1$.*

Definition 18 explains what is the turning point and the Lemma 1 explains how to estimate the turning point of an EPP. The coming section will discuss the properties of the turning point.

5.3.4 Properties of turning point

According to Definition 18, the following situation will be discussed: if the time interval between failures before the turning point becomes longer and longer and the time interval after the turning point becomes shorter and shorter.

Then, the following proposition can be introduced.

Theorem 1. *If $\{Z_k, k = 1, 2, ...k^*\}$ is stochastically increasing and $\{Z_k, k = k^*, k^* + 1, ...\}$ is stochastically decreasing, then the expectation $E[Z_k]$ will be the maximum value. The value of turning point k^* satisfies* $\frac{\ln(\beta(1-b)) - \ln(\alpha(a-1))}{\ln a - \ln b} + 1 < k^* < \frac{\ln(\beta(1-b)) - \ln(\alpha(a-1))}{\ln a - \ln b} + 2$.

The following shows the proof.

Proof. For the first situation, denote the turning point k^* at Z_{k^*}, then

$$E[Z_{k^*-1}] < E[Z_{k^*}], \tag{5.5}$$

and

$$E[Z_{k^*}] > E[Z_{k^*+1}]. \tag{5.6}$$

According to Wu and Clements-Croome (2006), the expectation of the EPP is $E[Z_{k^*}] = ((\alpha a^{k^*-1} + \beta b^{k^*-1})\mathrm{E}(X_1))^{-1}$. Then based on the Eq. (5.5) and Eq. (5.6), the value of k^* satisfies

$$\frac{\ln(\beta(1-b)) - \ln(\alpha(a-1))}{\ln a - \ln b} + 1 < k^* < \frac{\ln(\beta(1-b)) - \ln(\alpha(a-1))}{\ln a - \ln b} + 2. \tag{5.7}$$

As we know the expectation of X_k,

$$E[X_k] = \mathrm{E}(X_1)((\alpha a^{k-1} + \beta b^{k-1}))^{-1},$$

then according to Eq. (5.5)

$$\mathrm{E}(X_1)((\alpha a^{k^*-2} + \beta b^{k^*-2}))^{-1} < \mathrm{E}(X_1)((\alpha a^{k^*-1} + \beta b^{k^*-1}))^{-1}, \tag{5.8}$$

then

$$\alpha a^{k^*-1} - \alpha a^{k^*-2} < \beta b^{k^*-2} - \beta b^{k^*-1}, \tag{5.9}$$

then

$$\alpha a^{k^*-2}(a-1) < \beta b^{k^*-2}(1-b), \tag{5.10}$$

then

$$(\frac{a}{b})^{k^*-2} < \frac{\beta(1-b)}{\alpha(a-1)}, \tag{5.11}$$

then

$$k^* - 2 < \frac{\ln(\beta(1-b)) - \ln(\alpha(a-1))}{\ln a - \ln b}, \tag{5.12}$$

and we can obtain

$$k^* < \frac{\ln(\beta(1-b)) - \ln(\alpha(a-1))}{\ln a - \ln b} + 2. \tag{5.13}$$

Similarly, according to Eq. (5.6)

$$\mathrm{E}(\mathrm{X}_1)((\alpha a^{k^*-1} + \beta b^{k^*-1}))^{-1} > \mathrm{E}(\mathrm{X}_1)((\alpha a^{k^*} + \beta b^{k^*}))^{-1}, \tag{5.14}$$

then

$$\alpha a^{k^*} - \alpha a^{k^*-1} > \beta b^{k^*-1} - \beta b^{k^*}, \tag{5.15}$$

then

$$\alpha a^{k^*-1}(a-1) > \beta b^{k^*-1}(1-b), \tag{5.16}$$

then

$$(\frac{a}{b})^{k^*-1} > \frac{\beta(1-b)}{\alpha(a-1)}, \tag{5.17}$$

then

$$k^* - 1 > \frac{\ln(\beta(1-b)) - \ln(\alpha(a-1))}{\ln a - \ln b}, \tag{5.18}$$

and we can obtain

$$k^* > \frac{\ln(\beta(1-b)) - \ln(\alpha(a-1))}{\ln a - \ln b} + 1. \tag{5.19}$$

This completes the proof. □

Then, we have the following remark.

Remark 2.

(1) As all parameters α, a, β and b are positive value, $(1-b)$ and $(a-1)$ should greater than 0. Therefore, $0 < b < 1$ and $a > 1$;

(2) Because a is greater than b in this situation, therefore, $\ln a - \ln b$ is a positive value. This means that $\ln(\beta(1-b)) - \ln(\alpha(a-1))$ is a positive value. Then, we can obtain $\frac{\beta}{\alpha} > \frac{(a-1)}{(1-b)}$;

(3) If $\frac{(a-1)}{(1-b)} \geq 1$, then $a + b \geq 2$, and β is greater than α;

(4) If $\frac{(a-1)}{(1-b)} < 1$, then $a + b < 2$ and β is not necessarily smaller than α.

Besides, we discuss another situation: if the time interval between failures before the turning point becomes shorter and shorter and the time interval after the turning point becomes longer and longer. However, this situation does not exist. The following shows the proof. According to the assumption of this situation, $E[Z_{k^*}]$ should be a minimum value, therefore,

$$E[Z_{k^*}] < E[Z_{k^*-1},], \tag{5.20}$$

and

$$E[Z_{k^*+1}] > E[Z_{k^*}]. \tag{5.21}$$

Then interval of k^* now is given by

$$\frac{\ln\beta(b-1) - \ln\alpha(1-a)}{\ln a - \ln b} + 1 < k^* < \frac{\ln(\beta(b-1)) - \ln(\alpha(1-a))}{\ln a - \ln b} + 2. \tag{5.22}$$

Because all parameters are positive values, therefore, $b > 1$ and $1 < a < 1$. However, according to the definition of the EPP, the parameters should satisfy $\alpha + \beta \neq 0$, $a \geq 1$, and $0 < b \leq 1$. Therefore, this situation does not exist. And we have the following theorem.

Remark 3. *The expected value of X_k in the EPP does not present a convex pattern with respect to k.*

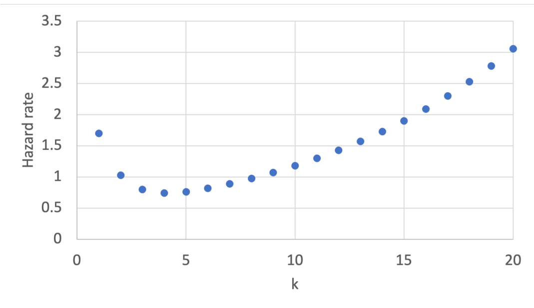

FIGURE 5.4: Hazard rate with $a = 1.1$, $\alpha = 0.5$, $\beta = 1.2$, and $b = 0.4$

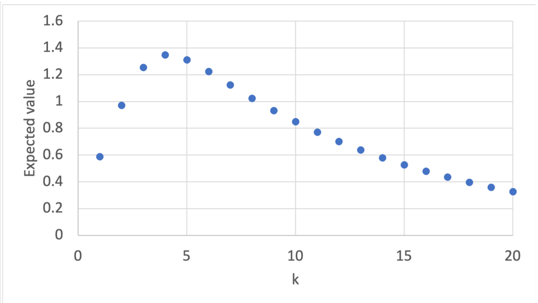

FIGURE 5.5: Expected value $E(X_k)$ with $a = 1.1$, $\alpha = 0.5$, $\beta = 1.2$,and $b = 0.4$

Figure 5.4 shows the change of hazard rate over k when $\alpha = 0.5$, $\beta = 1.2$, $a = 1.1$ and $b = 0.4$. Figure 5.5 shows the change of the expected value of $E(X_k)$ over k with the same parameters. The turning point is $k = k^* = 4$ with the highest expected value of X_k and the minimum hazard rate. It is worth noticing that the expected value is the reciprocal of the hazard rate.

5.3.5 Parameter estimation

According to the properties of EPP, if X_1 follows the Weibull distribution, the pdf of X_1 will be,

$$f(a,b,\alpha,\beta;x_1) = (\alpha a^{k-1} + \beta b^{k-1})\frac{\gamma}{\rho}(\frac{(\alpha a^{k-1} + \beta b^{k-1})x_1}{\rho})^{\gamma-1}\exp(-\frac{(\alpha a^{k-1} + \beta b^{k-1})x_1}{\rho})^{\gamma}, \tag{5.23}$$

where γ is the shape parameter and ρ is the scale parameter.

Then, the likelihood function is given by

$$L(a,b,\alpha,\beta,\gamma,\rho;x)=\prod_{i=1}^{k} f(a,b,\alpha,\beta,\gamma,\rho;x_i).$$

And the log-likelihood function is given by

$$\begin{aligned}\ell(a,b,\alpha,\beta,\gamma,\rho;x)=\sum_{i=1}^{k}&\ln(\alpha a^{k-1}+\beta b^{k-1})+\ln\gamma+(\gamma-1)(\ln(\alpha a^{k-1}+\beta b^{k-1})+\ln x_i-\ln\rho)\\&-\ln\rho-(\frac{(\alpha a^{k-1}+\beta b^{k-1})x_i}{\rho})^{\gamma-1}.\end{aligned}\tag{5.24}$$

Differentiating with respect to $a,b,\alpha,\beta,\gamma,\rho$, we have

$$\begin{aligned}\frac{\partial\ell(\theta)}{\partial a}&=\sum_{i=1}^{k}\frac{\alpha a^{k-2}}{\alpha a^{k-1}+\beta b^{k-1}}-(k-1)(\gamma-1)\frac{\alpha a^{k-2}(\alpha a^{k-1}+\beta b^{k-1})^{\gamma-2}}{\rho},\\\frac{\partial\ell(\theta)}{\partial b}&=\sum_{i=1}^{k}\frac{\beta b^{k-2}}{\alpha a^{k-1}+\beta b^{k-1}}-(k-1)(\gamma-1)\frac{\alpha a^{k-2}(\alpha a^{k-1}+\beta b^{k-1})^{\gamma-2}}{\rho},\\\frac{\partial\ell(\theta)}{\partial \alpha}&=\sum_{i=1}^{k}\frac{a^{k-1}}{\alpha a^{k-1}+\beta b^{k-1}}-(\gamma-1)\frac{a^{k-1}(\alpha a^{k-1}+\beta b^{k-1})^{\gamma-2}}{\rho}\\\frac{\partial\ell(\theta)}{\partial \beta}&=\sum_{i=1}^{k}\frac{b^{k-1}}{\alpha a^{k-1}+\beta b^{k-1}}-(\gamma-1)\frac{b^{k-1}(\alpha a^{k-1}+\beta b^{k-1})^{\gamma-2}}{\rho}\\\frac{\partial\ell(\theta)}{\partial \gamma}&=\sum_{i=1}^{k}\frac{1}{\gamma}+\ln(\alpha a^{k-1}+\beta b^{k-1})+\ln x_i-\ln\rho-(\frac{(\alpha a^{k-1}+\beta b^{k-1})x_i}{\rho})^{\gamma-1}\\&\quad+\ln(\frac{(\alpha a^{k-1}+\beta b^{k-1})x_i}{\rho})\\\frac{\partial\ell(\theta)}{\partial \rho}&=\sum_{i=1}^{k}\frac{(\gamma-2)}{\rho}+\frac{(a-1)((\alpha a^{k-1}+\beta b^{k-1})x_i)^{\gamma-1}}{\rho^{\gamma}},\end{aligned}\tag{5.25}$$

where parameter vector $\theta=(a,b,\alpha,\beta,\gamma,\rho)$ as $\theta=(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)$, and the first derivative of it will be

$$\nabla L(\theta)=\begin{bmatrix}\frac{\partial\ell(\theta)}{\partial\theta_1}\\\frac{\partial\ell(\theta)}{\partial\theta_2}\\\frac{\partial\ell(\theta)}{\partial\theta_3}\\\frac{\partial\ell(\theta)}{\partial\theta_4}\\\frac{\partial\ell(\theta)}{\partial\theta_5}\\\frac{\partial\ell(\theta)}{\partial\theta_6}\end{bmatrix},\tag{5.26}$$

and the second partial derivatives will be

$$\nabla^2 L(\theta) = \begin{bmatrix} \frac{\partial^2 \ell(\theta)}{\partial {\theta_1}^2} & \frac{\partial^2 \ell(\theta)}{\partial \theta_1 \partial \theta_2} & \cdots & \frac{\partial^2 \ell(\theta)}{\partial \theta_1 \partial \theta_d} \\ \frac{\partial^2 \ell(\theta)}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ell(\theta)}{\partial {\theta_2}^2} & \cdots & \frac{\partial^2 \ell(\theta)}{\partial \theta_2 \partial \theta_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \ell(\theta)}{\partial \theta_d \partial \theta_1} & \frac{\partial^2 \ell(\theta)}{\partial \theta_d \partial \theta_2} & \cdots & \frac{\partial^2 \ell(\theta)}{\partial {\theta_d}^2} \end{bmatrix} \tag{5.27}$$

where $d = 1, 2, 3, 4, 5, 6$.

According to the theory of the Fisher information matrix, the expected Fisher information can be presented by

$$\mathrm{I}(\theta) = \mathrm{Var}\{\nabla L(\theta)\} = -\mathrm{E}\{\nabla^2 L(\theta)\}, \tag{5.28}$$

where $\mathrm{I}(\theta)$ is the Fisher information matrix.

Then, the asymptotic distribution of the MLE $\hat{\theta}$ is $(\hat{\theta} - \theta) \xrightarrow{D} N(0, I^{-1}(\theta))$, where $I^{-1}(\theta)$ is the inverse of the Fisher information matrix $I(\theta)$. It can be presented by

$$I(\hat{\theta}) = (-\frac{\partial^2 L(\theta)}{\partial \theta_d \partial \theta_d})_{6\times 6}\Big|_{\theta=\hat{\theta}} \tag{5.29}$$

Therefore, the $100(1-\sigma)\%$ CI of θ_d is $(\hat{\theta} \pm z_{\sigma/2}\sqrt{I^{-1}(\hat{\theta})_{dd}})$.

5.3.6 Parameter estimation to k

Denote the lower bound of k^* as $h(a, b, \alpha, \beta, \gamma, \rho)$ and the upper bound as $g(a, b, \alpha, \beta, \gamma, \rho)$, then, due to the uncertainty of each parameter, the estimation of lower bound and upper bound can be obtained, respectively

$$\hat{h}(\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\rho}) = \frac{\ln \hat{\beta}(1-\hat{b}) - \ln \hat{\alpha}(\hat{a}-1)}{\ln \hat{a} - \ln \hat{b}} + 1, \tag{5.30}$$

and

$$\hat{g}(\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\rho}) = \frac{\ln \hat{\beta}(1-\hat{b}) - \ln \hat{\alpha}(\hat{a}-1)}{\ln \hat{a} - \ln \hat{b}} + 2, \tag{5.31}$$

The variance of $\hat{h}(\cdot)$ and $\hat{g}(\cdot)$ can be obtained by the $I(\hat{\theta})$.

5.3.7 Methods for parameter estimation

Denote θ as the vector of parameters a, b, α and β, which can be presented as $\theta = (a, b, \alpha, \beta)$. Then the asymptotic distribution of $\widehat{\theta}$ can be obtained

$$\sqrt{q}(\theta - \widehat{\theta}) \xrightarrow{D} N(0, A), \tag{5.32}$$

where A is the covariance matrix of vector θ which can be obtained by the Fisher information matrix, and q is the number of observations. Furthermore, $\widehat{k^*}$ can be presented by

$$\widehat{k^*} = g(\widehat{\theta}), \tag{5.33}$$

then,

$$\sqrt{q}(k^* - \widehat{k^*}) \xrightarrow{D} N(0, \nabla g(\theta)^T \times A \times \nabla g(\theta)). \tag{5.34}$$

Then, the error is estimated by the δ-method which approximates the standard errors of transformations of random variables using a first-order Taylor approximation. Regression coefficients are themselves random variables. Thus, it can be used to approximate the standard errors of their transformations.

Denote G be the transformation function and μ_X be the mean vector of random variable ($X = x_1, x_2, ...$), then

$$G(X) \approx G(\mu_X) + \triangledown G(\mu_X)^T (X - \mu_X) \tag{5.35}$$

where $\triangledown G(\mu_X)$ is the gradient of $G(X)$ when $X = \mu_X$. Then the variance of this approximation to estimate the variance $G(X)$ and the standard error of a transformed parameter can be taken. The variance is approximated by

$$Var(G(X)) \approx \triangledown G(\mu_X)^T Cov(X) \triangledown G(\mu_X) \tag{5.36}$$

or it can be implied

$$\sqrt{n_0}(G(X) - G(\mu_X)) \to^D N(0, \triangledown G(\mu_X)^T Cov(X) \triangledown G(\mu_X)) \tag{5.37}$$

where n_0 is the number of observation, and $\to^D$ denotes convergence in distribution.

5.4 Impact of the uncertainty on maintenance policy

A general maintenance policy: N replacement policy is introduced by Lam (2007). Several assumptions are proposed for the maintenance model, which will be followed in this chapter. Then, the EPP under this maintenance policy will have the following assumptions.

Assumption 1. *At $t = 0$, a new system is installed. When the system fails, it will be repaired. A replacement policy N is applied by which the system is replaced by a new and identical one at the time following the N-th failure.*

Assumption 2. *Let X_1 be the system operating time after the installation or a replacement. In general, for $k > 1$, let X_k be the system operating time after the $(k-1)$th repair, then $\{X_k, k = 1, 2, ...\}$ form a EPP with $E[X_1] = \mu_X^{-1} > 0$ and $E[X_k] = [(\alpha a^{k-1} + \beta b^{k-1})\mu_X]^{-1}$*

Assumption 3. *Denote the inspection cost as c_r, the repair cost as c_X, the basic replacement cost as R, and the expected replacement cost relevant to replacement time Z as c_Z.*

According to such assumptions, the lifetime of the system before the replacement of the whole system is defined as a life cycle. Then, the long-run average cost per unit time is given by

$$\text{Average cost per unit time} = \frac{\text{Expected cost incurred in a life cycle}}{\text{Expected length of a cycle}}.$$

The average cost for the long-term maintenance policy in the case of the EPP is given by

$$A_{EPP}(N) = \frac{c_r(N-1) + c_X\mu_X \sum_{k=1}^{N}(\alpha a^{1-k} + \beta b^{1-k}) + R + c_Z\mu_Z}{\mu_X \sum_{k=1}^{N}(\alpha a^{1-k} + \beta b^{1-k}) + \mu_Z}. \tag{5.38}$$

where $(N-1)$ is the time of inspection.

Due to the impact of the turning point, we also discuss the maintenance policy according to Section 5.3.4: if the value of the turning point k^* satisfies:
$\frac{\ln(\beta(1-b)) - \ln(\alpha(a-1))}{\ln a - \ln b} + 1 < k^* < \frac{\ln(\beta(1-b)) - \ln(\alpha(a-1))}{\ln a - \ln b} + 2.$

The hazard rate decreases until the k-th failure and increases after the k-th failure. Therefore, the failures of the system will be focused seriously due to the rapidly increasing failure rate. The maintenance policy is normally operated during this period (associated with the wear-out failures period of the bathtub curve). This means that the N replacement policy will be operated from the $(k = k^* + 1)$-th failure to the $(k = k^* + N)$-th failure. Then the average cost is presented by

$$A^-_{EPP}(N^-) = \frac{c_r(N^- - 1) + c_X\mu_X \sum_{k=k^*+1}^{k^*+N^-}(\alpha a^{1-k} + \beta b^{1-k}) + R + c_Z\mu_Z}{\mu_X \sum_{k=k^*+1}^{k^*+N^-}(\alpha a^{1-k} + \beta b^{1-k}) + \mu_Z}. \tag{5.39}$$

5.5 Sensitive analysis and numerical examples

We consider that the random variables follow the Weibull distribution with shape parameter λ and scale parameter δ under the EPP. Then, we make the following assumptions:

S1: α=0.5, 0.7, 0.9 and 1.1, $\beta = 1.2$, $a = 1.1$, $b = 0.4$.

S2: β=1.2, 1.4, 1.6, and 1.8, $\alpha = 0.5$; $a = 1.1$, $b = 0.4$.

S3: a=1.1, 1.3, 1.5, and 1.7, $\alpha = 0.5$, $\beta = 1.2$, $b = 0.4$.

S4: b=0.4, 0.6, 0.8 and 0.85, $\alpha = 0.5$, $\beta = 1.2$, $b = 0.85$.

Tables 5.5–5.8 show the interval of k^*, which will be the uncertainty of the turning point.

Table 5.5 shows the interval of k^* of scenario 1 and Table 5.6 shows the interval of k^* of scenario 2.

Table 5.7 shows the interval of k^* of scenario 3 and Table 5.8 shows the interval of k^* of scenario 4.

According to Tables 5.5–5.8, we have the following findings:

- The value of β has the smallest impact on the estimation of k^*.

TABLE 5.5: Lower and upper bounds of k^* in S1

S1	Lower	k^*	Upper
$\alpha = 0.5$	3.6329	4	4.6329
$\alpha = 0.7$	3.3003	3	4.3003
$\alpha = 0.9$	3.0518	3	4.0518
$\alpha = 1.1$	2.8535	3	3.8535

TABLE 5.6: Lower and upper bounds of k^* in S2

S2	Lower	k^*	Upper
$\beta = 1.2$	3.6329	4	4.6329
$\beta = 1.2$	3.7870	4	4.7870
$\beta = 1.2$	3.9206	4	4.9206
$\beta = 1.2$	4.0383	4	5.0383

TABLE 5.7: Lower and upper bounds of k^* in S3

S3	Lower	k^*	Upper
$a = 1.1$	3.6329	4	4.6329
$a = 1.3$	2.2811	2	3.2811
$a = 1.5$	1.7203	2	2.7203
$a = 1.7$	1.3971	2	2.3971

TABLE 5.8: Lower and upper bounds of k^* in S4

S4	Lower	k^*	Upper
$b = 0.4$	3.6329	4	4.6329
$b = 0.6$	5.2084	5	6.2084
$b = 0.8$	8.9800	9	9.9800
$b = 0.85$	10.9043	11	11.9043

- The value of b has the most significant impact on the estimation of k^*.
- With the increase of α and a, the estimation of k^* decrease.

Besides, we would like to investigate the impact of k^* on the average cost $A^-_{EPP}(N^-)$. We set k^* = 2, 3, 4, 5 and 6. We assume that the random variable X_1 follows the Weibull distribution with $\lambda = 3.5$ and $\delta = 20$. Then, Table 5.9 shows the average cost with different k^*.

TABLE 5.9: $A^-_{EPP}(N^-)$ with $\lambda = 3.5$, $\delta = 20$, $c_r = 2$, $\mu_x = 1.5$, $c_x = 3$, $c_z = 5$, $\mu_z = 0.5$, and $R = 100$

N^-	$k^* = 2$	$k^* = 3$	$k^* = 4$	$k^* = 5$	$k^* = 6$
1	203.609	202.464	201.075	199.527	197.910
2	205.031	202.510	199.627	196.568	193.504
3	205.027	201.042	196.719	192.327	188.099
4	203.504	198.121	192.546	187.105	182.055
5	200.519	193.939	187.402	181.249	175.729
6	196.269	188.788	181.625	175.101	169.433
7	191.044	183.002	175.551	168.967	163.421
8	185.182	176.917	169.481	163.095	157.880
9	179.018	170.832	163.663	157.673	152.937
10	172.856	164.996	158.283	152.827	148.668
11	166.945	159.594	153.467	148.634	145.106
12	161.471	154.755	149.296	145.130	142.254
13	156.566	150.559	145.806	142.322	140.093
14	152.309	147.045	143.005	140.191	138.584
15	148.740	144.220	140.876	138.702	137.680
16	145.867	142.068	139.385	137.809	**137.323**

17	143.674	140.556	138.485	**137.454**	137.450
18	142.126	139.638	**138.123**	137.578	137.995
19	141.177	**139.258**	138.235	138.113	138.891
20	**140.772**	139.354	138.759	138.996	140.073
21	140.847	139.863	139.628	140.162	141.482
22	141.339	140.718	140.779	141.552	143.064
23	142.179	141.856	142.154	143.112	144.773
24	143.305	143.218	143.698	144.798	146.572
25	144.657	144.750	145.367	146.572	148.430
26	146.180	146.407	147.123	148.406	150.328
27	147.829	148.152	148.939	150.278	152.249
28	149.567	149.956	150.794	152.174	154.184
29	151.365	151.799	152.671	154.083	156.127
30	153.201	153.665	154.562	156.000	158.074
31	155.060	155.544	156.461	157.921	160.023
32	156.934	157.432	158.364	159.843	161.973
33	158.815	159.323	160.268	161.767	163.923
34	160.700	161.216	162.174	163.691	165.873
35	162.587	163.110	164.080	165.616	167.824
36	164.475	165.005	165.986	167.540	169.775
37	166.363	166.899	167.892	169.464	171.725
38	168.251	168.794	169.799	171.389	173.676
39	170.140	170.689	171.705	173.313	175.626
40	172.029	172.584	173.611	175.238	177.577
41	173.917	174.479	175.517	177.162	179.528
42	175.806	176.373	177.424	179.087	181.478
43	177.694	178.268	179.330	181.011	183.429
44	179.583	180.163	181.236	182.936	185.379
45	181.472	182.058	183.142	184.860	187.330
46	183.360	183.952	185.049	186.784	189.280
47	185.249	185.847	186.955	188.709	191.231
48	187.137	187.742	188.861	190.633	193.182
49	189.026	189.637	190.768	192.558	195.132
50	190.914	191.532	192.674	194.482	197.083
Min $A^{-}_{EPP}(N^{-})$	140.772	139.258	138.123	137.454	137.323
Optimised N^{-}	20	19	18	17	16

Then we have the following findings:

- At $k^* = 2$, the minimal $A^{-}_{EPP}(N^{-}) = 140.772$ with the optimised $N^{-} = 20$;
- At $k^* = 3$, the minimal $A^{-}_{EPP}(N^{-}) = 139.258$ with the optimised $N^{-} = 19$;
- At $k^* = 4$, the minimal $A^{-}_{EPP}(N^{-}) = 138.123$ with the optimised $N^{-} = 18$;

- At $k^* = 5$, the minimal $A^-_{EPP}(N^-) = 137.454$ with the optimised $N^- = 17$;
- At $k^* = 6$, the minimal $A^-_{EPP}(N^-) = 137.323$ with the optimised $N^- = 16$.

Figure 5.6 shows the trend of $A^-_{EPP}(N^-)$ at $k^* = 2$, $k^* = 3$, $k^* = 4$, $k^* = 5$ and $k^* = 6$.

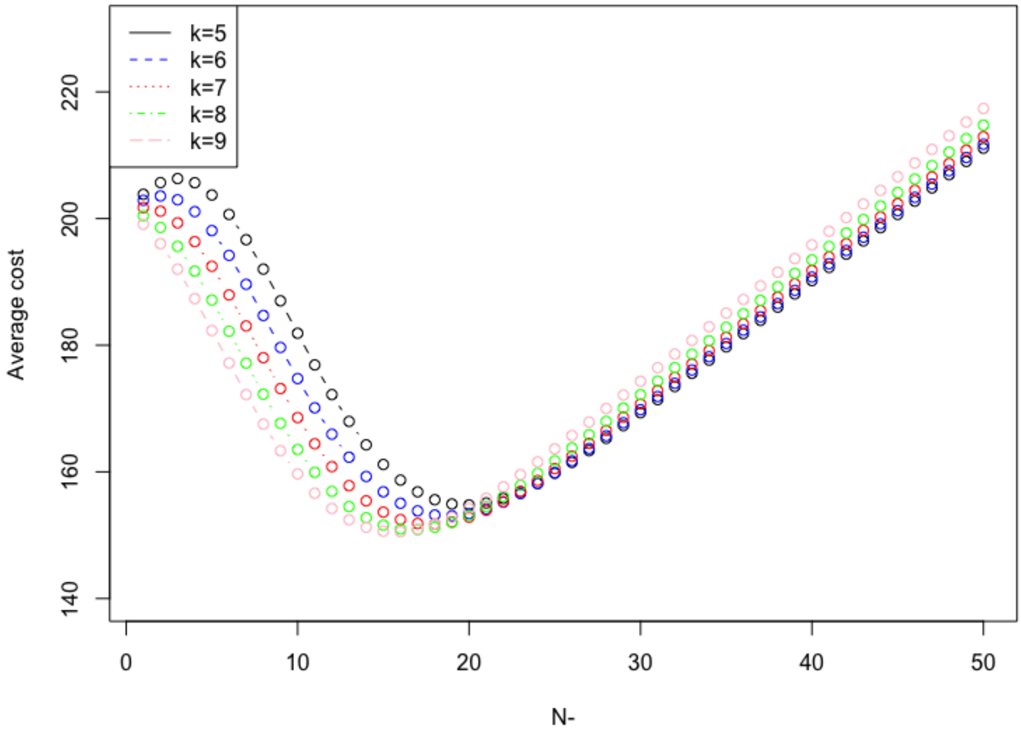

FIGURE 5.6: $A^-_{EPP}(N^-)$ with $\lambda = 3.5$, $\delta = 20$, $c_r = 2$, $\mu_x = 1.5$, $c_x = 3$, $c_z = 5$, $\mu_z = 0.5$, and $R = 100$

According to Table 5.9, if $k^* \to \infty$ and $N^- \to \infty$, the optimal N^- is not sensitive to the change of the turning point k^* as the increment of N is too small (compared with the whole operating time). Besides, the change of minimum A^-_{EPP} is not significant with the associated N^- under different k^*. This is due to the infinite operating time of the system, which is not available in the real world (Nakagawa and Mizutani, 2009). Therefore, we assume the following constraint holds

$$\mu_X \sum_{k=1}^{k^*+N^-} (\alpha a^{1-k} + \beta b^{1-k})^{-1} \leq S,$$

which is the constraint of N^-.

Besides, if the time is finite, then the value of both k^* and N^- should not be large. Secondly, every change of increasing or decreasing by 1 will be sensitive for both k^* and N^- as the interval is finite. Then, we consider that the value of N^- should not be greater than the value of k^*, which is because the hazard rate of a system will increase rapidly after the occurrence of the turning point (the period of wear-out failures of the bathtub curve). Therefore, the failure of the system (but not the failure of the component) will occur soon after the turning point. This means that the wear-out stage will not be longer than the stage before the occurrence of the turning point. In other words, the number of failures after the turning point (but before the failure of the system) should be less than the number of failures before the turning point, which can be presented by $N^- < k^*$. This is another constraint of N^-.

Table 5.10 shows the average cost with different k^* under two constraints.

TABLE 5.10: $A^-_{EPP}(N^-)$ with $\lambda = 1.5$, $\delta = 3$, $c_r = 2$, $\mu_x = 1.5$, $c_x = 3$, $c_z = 5$, $\mu_z = 0.5$, $R = 100$ and $S = 10$

Optimised N^-	$k^* = 5$	$k^* = 6$	$k^* = 7$	$k^* = 8$	$k^* = 9$
1	139.549	157.704	173.669	185.930	194.254
2	118.404	141.320	162.960	180.520	192.860
3	109.674	134.782	159.249	179.581	194.061
4	106.239	132.718	158.882	180.834	196.537
5	105.393	132.874	160.180	183.168	199.627
6	105.879	134.154	162.305	186.026	203.004
7	107.052	135.998	164.831	189.127	206.511
8	108.573	138.116	167.544	192.339	210.074
9	110.263	140.363	170.343	195.598	213.659
10	112.032	142.667	173.177	198.878	217.255
11	113.838	144.997	176.027	202.165	220.853
12	115.659	147.337	178.884	205.455	224.454
13	117.487	149.682	181.742	208.746	228.054
14	119.317	152.028	184.602	212.038	231.655
15	121.148	154.374	187.461	215.330	235.256
16	122.980	156.721	190.321	218.622	238.857
17	124.811	159.068	193.181	221.914	242.458
18	126.643	161.415	196.041	225.206	246.058
19	128.475	163.761	198.901	228.497	249.659
20	130.306	166.108	201.760	231.789	253.260
Min $A^-_{EPP}(N^-)$	105.393	132.718	158.882	179.581	192.860
Optimised N^-	5	4	4	3	2

Then we have the following findings:

- At $k^* = 5$, the minimal $A^-_{EPP}(N^-) = 105.393$ with the optimised $N^- = 5$;
- At $k^* = 6$, the minimal $A^-_{EPP}(N^-) = 132.718$ with the optimised $N^- = 4$;
- At $k^* = 7$, the minimal $A^-_{EPP}(N^-) = 158.882$ with the optimised $N^- = 4$;
- At $k^* = 8$, the minimal $A^-_{EPP}(N^-) = 179.581$ with the optimised $N^- = 3$;
- At $k^* = 9$, the minimal $A^-_{EPP}(N^-) = 192.860$ with the optimised $N^- = 2$.

Figure 5.7 shows the trend of $A^{-}_{EPP}(N^{-})$ at $k^* = 5$, $k^* = 6$, $k^* = 7$, $k^* = 8$ and $k^* = 9$ with $S = 10$. With the constraints of N^{-}, the change of the cost is more sensitive to the change of both k^* and N^{-}.

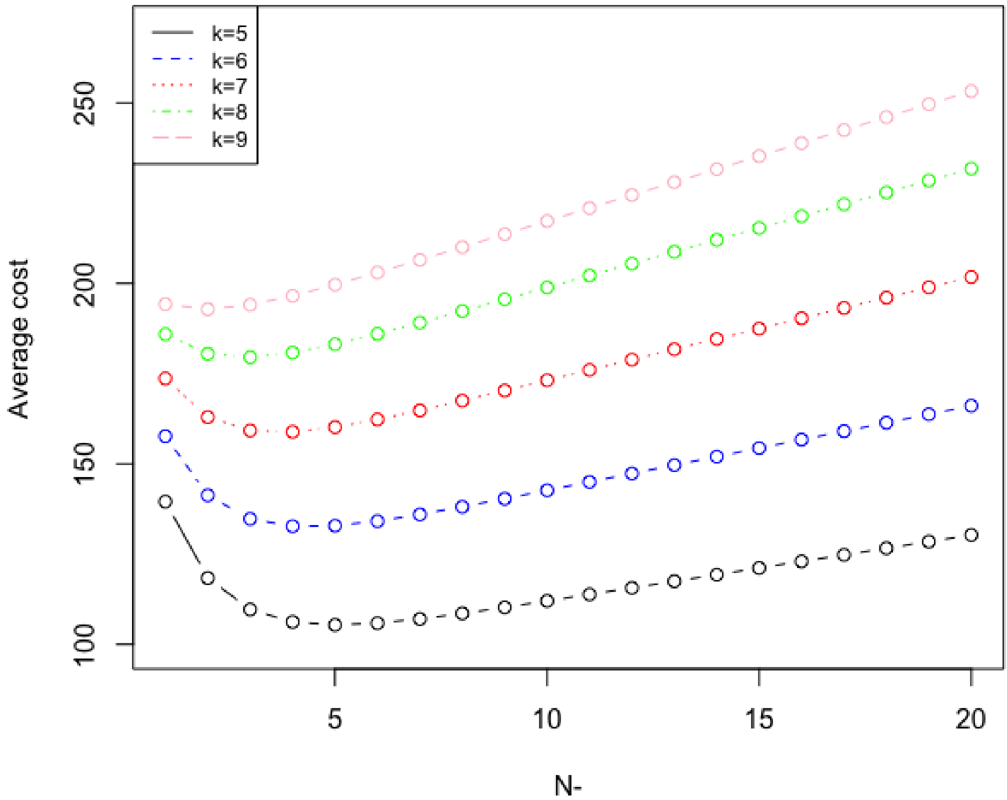

FIGURE 5.7: $A^{-}_{EPP}(N^{-})$ with $\lambda = 1.5$, $\delta = 3$, $c_r = 2$, $\mu_x = 1.5$, $c_x = 3$, $c_z = 5$, $\mu_z = 0.5$, $R = 100$ and $S = 10$

5.6 Summary

This chapter analysed the GeP-like models in several aspects. The main contributions of this chapter include:

- It explored the relationship between the various extensions of the GeP;
- It discussed a drawback in the definition of the GeP; and
- It investigated the impact of the uncertainty of the parameter estimation on the expected maintenance cost.

The definition of the turning point under the extended Poisson process is first proposed in this chapter. The estimation and the interval of the turning point k^* are proved. Several properties of the turning point are discussed. With several numerical examples, we found that under the extended Poisson process, parameter b has the most significant impact on the estimation of the turning point k^*. Parameter β

has the smallest impact on the estimation of the turning point. Besides, we gave examples to find the optimal N^-, which is the number of failures before replacing the whole system in different cases. We found that if the time is infinite, then the optimal N^- is not sensitive to k^*. Then we add two constraints to N^-: the value of optimal N^- is smaller than k^- and the total operating time of the system is smaller than S. Under these two constraints, the optimal N^- is more sensitive to k^- and the average cost per unit time. The relationship between the various extensions of the GeP can be developed further to obtain a generalized function of all GeP-like models which can describe both Type I and Type II extensions.

Chapter 6

Comparison of the performance of GeP-like models

6.1 Introduction

The survival distribution and basic properties of GeP-like models have been discussed in Chapter 2. In this Chapter, we discuss and demonstrate their properties and statistics inference in different aspects.

Modelling the failure process of a system is an important research topic in the reliability and maintenance research community and is normally done by using stochastic processes. The geometric process (GeP), is one of such stochastic processes. The GeP has been widely used for modelling system failure processes. It is able to describe a stochastically increasing or decreasing trend, which is a characteristic in many practical applications in different fields (Zhang, 1999; Lam et al., 2004; Wang and Yam, 2017; Pekalp and Aydoğdu, 2021). In fact, the GeP has received a lot of attention and has been extended into different variants, such as the general repair geometric process (Finkelstein, 1993), the α-series process (Braun, Li, and Zhao, 2005), the threshold geometric process (Chan et al., 2006), the extended Possion process (Wu and Clements-Croome, 2006), the doubly geometric process (Wu, 2018) and the doubly-ratio geometric process (Wu, 2022). However, little research specifically compares the performance of these models based on real-world datasets. This chapter aims to compare the performance of the GeP and its extensions.

6.2 Least square estimation

6.2.1 Least square estimation for the GeP

In Section 4.3 in Lam (2007), the least squares estimators of a GeP are given. Let

$$Y_i = a^{i-1} X_i, i = 1, 2, \ldots, n.$$

As X_i follows a GeP, then Y_i is a sequence of i.i.d random variables. Therefore, the mean and variance of Y_i can be denoted by $\mu = E[\ln Y_i]$ and $\tau^2 = Var[\ln Y_i]$ respectively. Then the result of taking the logarithm is given by

$$\ln Y_i = (i-1)\ln a + \ln X_i, i = 1, 2, \ldots, n,$$

and

$$\ln X_i = \mu - (i-1)\ln a + e_i, i = 1, 2, \ldots, n,$$

where e_i are random variables with mean 0 and variance τ^2. Then, following least squares estimators of μ, $\beta = \ln a$ and τ^2 can be obtained

$$\hat{\mu} = \frac{2}{n(n+1)} \sum_{i=1}^{n} (2n - 3i + 2) \ln X_i,$$

and

$$\hat{\beta} = \frac{6}{(n-1)(n+1)n} \sum_{i=1}^{n} (n - 2i + 1) \ln X_i,$$

and

$$\hat{\tau^2} = \frac{1}{n-1} \{ \sum_{i=1}^{n} (\ln X_i)^2 - \frac{1}{n} (\sum_{i=1}^{n} \ln X_i)^2 - \frac{\hat{\beta}}{n} \sum_{i=1}^{n} (n - 2i + 1) \ln X_i \}.$$

The result of a is given by

$$\hat{a} = \exp(\hat{\beta}),$$

and the moment estimators for λ and σ^2 are given by

$$\hat{\lambda} = \begin{cases} \bar{\hat{Y}}, a \neq 1, \\ \bar{X}, a = 1, \end{cases}$$

and

$$\hat{\sigma^2} = \begin{cases} \frac{1}{n-1} \sum_{i=1}^{n} (\hat{Y}_i - \bar{\hat{Y}}_i)^2, a \neq 1 \\ \frac{1}{n-1} \sum_{i=1}^{n} (\hat{X}_i - \bar{\hat{X}}_i)^2, a = 1. \end{cases}$$

Therefore, estimators $\hat{a}$, $\hat{\lambda}$ and $\hat{\sigma^2}$ are non-parametric estimators. They can be obtained by minimizing the sum squares of errors

$$Q = \sum_{i=1}^{n} [\ln X_i - \mu + (i-1)\ln a]^2.$$

Similarly, the least square method can be applied to other GeP's extensions.

6.2.2 Least square estimation of the α-series process

For the α-series, let

$$Y_k = k^a X_k, k = 1, 2, ...n. \tag{6.1}$$

Then $\{Y_k, k = 1, 2, ...\}$ is a sequence of i.i.d random variables, so is $\{\ln Y_k, k = 1, 2, ...\}$. Taking logarithm on both sides of Eq. (6.1) give

$$\ln Y_k = \alpha \ln k + \ln X_k, k = 1, 2, ...n. \tag{6.2}$$

Let

$$\ln Y_k = \mu_\alpha + e_k, k = 1, 2, ...n, \tag{6.3}$$

then,

$$\ln X_k = \mu_\alpha - \alpha \ln k + e_k, k = 1, 2, ...n. \tag{6.4}$$

where Eq. (6.4) can be considered as a simple regression equation with the slope $-a$ and the intercept μ_α.

Lemma 1. *Denote Q_α as the sum squares of errors*

$$Q_\alpha = \sum_{k=1}^{n} (\ln x_k - \mu_\alpha + \alpha \ln k)^2, \tag{6.5}$$

by minimising Q_α, the least square estimators of α and μ_α are given by

$$\hat{\alpha} = \frac{\sum_{k=1}^{n} \ln x_k \sum_{k=1}^{n} \ln k - n \sum_{k=1}^{n} \ln x_k \ln k}{n \sum_{k=1}^{n} (\ln k)^2 - (\sum_{k=1}^{n} \ln k)^2}. \tag{6.6}$$

and

$$\hat{\mu}_\alpha = \frac{\sum_{k=1}^{n} \ln x_k \sum_{k=1}^{n} (\ln k)^2 - \sum_{k=1}^{n} \ln x_k \ln k \sum_{k=1}^{n} \ln k}{n \sum_{k=1}^{n} (\ln k)^2 - (\sum_{k=1}^{n} \ln k)^2}. \tag{6.7}$$

Proof. Set the partial derivatives $\frac{\partial Q_\alpha}{\partial \hat{\mu}_\alpha}$ and $\frac{\partial Q_\alpha}{\partial \hat{\alpha}}$ of Q_α to 0, Then

$$\frac{\partial Q_\alpha}{\partial \hat{\mu}_\alpha} = 0, \tag{6.8}$$

then

$$-2 \sum_{k=1}^{n} (\ln(x_k) - \hat{\mu}_\alpha + \alpha \ln(k)) = 0, \tag{6.9}$$

then

$$\sum_{k=1}^{n} \ln(x_k) - \sum_{k=1}^{n} \hat{\mu}_\alpha + \sum_{k=1}^{n} \ln(k) = 0, \tag{6.10}$$

and we can obtain

$$\hat{\mu}_\alpha = \frac{1}{n} (\sum_{k=1}^{n} \ln(x_k) + \alpha \sum_{k=1}^{n} \ln(k)). \tag{6.11}$$

Then

$$\frac{\partial Q_\alpha}{\partial \hat{\alpha}} = 0, \tag{6.12}$$

then

$$2\sum_{k=1}^{n}(\ln(x_k)-\mu_\alpha+\hat{\alpha}\ln(k))\ln(k)=0, \tag{6.13}$$

then

$$\sum_{k=1}^{n}\ln(x_k)\ln(k)-\mu_\alpha\ln(k)+\hat{\alpha}\ln^2(k)=0, \tag{6.14}$$

and we can obtain

$$\hat{\alpha}=-\frac{\sum_{k=1}^{n}\ln x_k\ln(k)-\mu_\alpha\sum_{k=1}^{n}\ln(k)}{\sum_{k=1}^{n}\ln^2(k)}. \tag{6.15}$$

Plug Eq. (6.15) in Eq. (6.11),

$$\hat{\alpha}=-\frac{\sum_{k=1}^{n}\ln x_k\ln(k)-\frac{1}{n}(\sum_{k=1}^{n}\ln(x_k)+\alpha\sum_{k=1}^{n}\ln(k))\sum_{k=1}^{n}\ln(k)}{\sum_{k=1}^{n}\ln^2(k)}, \tag{6.16}$$

then

$$\hat{\alpha}=-\frac{\sum_{k=1}^{n}\ln x_k\ln k}{\sum_{k=1}^{n}(\ln k)^2}+\frac{\sum_{k=1}^{n}\ln k}{n\sum_{k=1}^{n}(\ln k)^2}(\sum_{k=1}^{n}\ln x_k+\alpha\sum_{k=1}^{n}\ln k), \tag{6.17}$$

then

$$\hat{\alpha}=-\frac{\sum_{k=1}^{n}\ln x_k\ln k}{\sum_{k=1}^{n}(\ln k)^2}+\frac{\sum_{k=1}^{n}\ln x_k\sum_{k=1}^{n}\ln k}{n\sum_{k=1}^{n}(\ln k)^2}+\frac{\alpha(\sum_{k=1}^{n}\ln k)^2}{n\sum_{k=1}^{n}(\ln k)^2}, \tag{6.18}$$

then

$$\hat{\alpha}(1-\frac{(\sum_{k=1}^{n}\ln k)^2}{n\sum_{k=1}^{n}(\ln k)^2})=\frac{\sum_{k=1}^{n}\ln x_k\sum_{k=1}^{n}\ln k-n\sum_{k=1}^{n}\ln x_k\ln k}{n\sum_{k=1}^{n}(\ln k)^2}, \tag{6.19}$$

then

$$\hat{\alpha}(\frac{n\sum_{k=1}^{n}(\ln k)^2-(\sum_{k=1}^{n}\ln k)^2}{n\sum_{k=1}^{n}(\ln k)^2})=\frac{\sum_{k=1}^{n}\ln x_k\sum_{k=1}^{n}\ln k-n\sum_{k=1}^{n}\ln x_k\ln k}{n\sum_{k=1}^{n}(\ln k)^2}, \tag{6.20}$$

and we can obtain

$$\hat{\alpha}=\frac{\sum_{k=1}^{n}\ln x_k\sum_{k=1}^{n}\ln k-n\sum_{k=1}^{n}\ln x_k\ln k}{n\sum_{k=1}^{n}(\ln k)^2-(\sum_{k=1}^{n}\ln k)^2}. \tag{6.21}$$

Similarly, the estimator $\hat{\mu}_\alpha$ then is presented by

$$\hat{\mu}_\alpha=\frac{\sum_{k=1}^{n}\ln x_k\sum_{k=1}^{n}(\ln k)^2-\sum_{k=1}^{n}\ln x_k\ln k\sum_{k=1}^{n}\ln k}{n\sum_{k=1}^{n}(\ln k)^2-(\sum_{k=1}^{n}\ln k)^2}. \tag{6.22}$$

This completes the proof. □

Theorem 2. *If $E[(\ln Y_k)^2]<\infty$, then*

$$n(\hat{\alpha}-\alpha)\xrightarrow{L}N(0,\sigma_Y^2), \tag{6.23}$$

where σ_Y^2 is the variance of $\ln Y_k$.

Proof. First of all, denote $\mu_Y = \mathrm{E}[\ln Y_k]$ and $\sigma_Y^2 = \mathrm{Var}[\ln Y_k]$, then, because $\ln Y_k = \alpha \ln k + \ln X_k$, so

$$\ln x_k = \ln Y_k - \alpha \ln k.$$

According to the result of $\hat{\alpha}$, we can obtain

$$\begin{aligned}
\hat{\alpha} - \alpha &= \frac{\sum_{k=1}^n \ln x_k \sum_{k=1}^n \ln k - n \sum_{k=1}^n \ln x_k \ln k}{n \sum_{k=1}^n (\ln k)^2 - (\sum_{k=1}^n \ln k)^2} - a \\
&= \frac{1}{n \sum_{k=1}^n (\ln k)^2 - (\sum_{k=1}^n \ln k)^2} [\sum_{k=1}^n \ln x_k \sum_{k=1}^n \ln k - n \sum_{k=1}^n \ln x_k \ln k - \\
&\quad (n \sum_{k=1}^n (\ln k)^2 - (\sum_{k=1}^n \ln k)^2)\alpha] \\
&= \frac{1}{n \sum_{k=1}^n (\ln k)^2 - (\sum_{k=1}^n \ln k)^2} T,
\end{aligned}$$

where

$$\begin{aligned}
T &= \sum_{k=1}^n (\ln Y_k \sum_{k=1}^n \ln k - \alpha (\sum_{k=1}^n \ln k)^2 - n (\sum_{k=1}^n ((\ln Y_k)(\ln k) - \alpha k (\ln k)^2) - a \\
&= \sum_{k=1}^n \ln Y_k \sum_{k=1}^n \ln k - \alpha (\sum_{k=1}^n \ln k)^2 - n \sum_{k=1}^n (\ln Y_k)(\ln k) + n\alpha \sum_{k=1}^n (\ln k)^2 + \alpha (\sum_{k=1}^n \ln k)^2 \\
&= \sum_{k=1}^n \ln Y_k \sum_{k=1}^n \ln k - n \sum_{k=1}^n (\ln Y_k)(\ln k) \\
&= \sum_{k=1}^n [\ln Y_k (\sum_{k=1}^n \ln k - n \ln k)].
\end{aligned}$$

Then, we have

$$\hat{\alpha} - \alpha = \frac{1}{n \sum_{k=1}^n (\ln k)^2 - (\sum_{k=1}^n \ln k)^2} \sum_{k=1}^n [\ln Y_k (\sum_{k=1}^n \ln k - n \ln k)].$$

Moreover, we denote

$$\beta_{nk} = (\ln Y_k - \mu_Y)(\sum_{k=1}^n \ln k - n \ln k), k = 1, 2, \ldots, n, \tag{6.24}$$

which is independent, let

$$\begin{aligned}
B_n &= \sum_{k=1}^n \mathrm{Var}[\beta_{nk}] = \sigma_Y^2 \sum_{k=1}^n (\sum_{k=1}^n \ln k - n \ln k)^2, \\
&= \sigma_Y^2 \sum_{k=1}^n (\ln(n!) - n \ln k)^2.
\end{aligned} \tag{6.25}$$

According to Stirling's approximation $\sum_{k=1}^{n} \ln k \approx n \ln n - n$ for $n \to \infty$, Eq. (6.25) becomes

$$\begin{aligned}
B_n &\approx \sigma_Y^2 \sum_{k=1}^{n} (n \ln n - n - n \ln k)^2, \\
&= \sigma_Y^2 \sum_{k=1}^{n} [n^2 (\ln n - 1 - \ln k)^2], \\
&= n^2 \sigma_Y^2 \sum_{k=1}^{n} [(\ln n)^2 - 2(\ln n)(\ln k) - 2\ln n + 2\ln k + 1 + (\ln k)^2], \\
&= \sigma_Y^2 n^2 (n(\ln n)^2 - 2n \ln n + n) + \sigma_Y^2 n^2 \sum_{k=1}^{n} ((\ln k)^2 - 2(\ln n)(\ln k) + 2\ln k), \\
&= \sigma_Y^2 n^2 (n(\ln n)^2 - 2n \ln n + n) + \sigma_Y^2 n^2 (\sum_{k=1}^{n} (\ln k)^2 - 2(\ln n)(\ln n!) + 2\ln n!), \\
&\approx \sigma_Y^2 n^2 (n(\ln n)^2 - 2n \ln n + n) + \sigma_Y^2 n^2 (\sum_{k=1}^{n} (\ln k)^2 - 2(\ln n)(n \ln n - n) + 2(n \ln n - n)), \\
&= \sigma_Y^2 n^3 ((\ln n)^2 - 2\ln n + 1) + \sigma_Y^2 n^2 (\sum_{k=1}^{n} (\ln k)^2 - 2n(\ln n - 1)^2) \\
&= \sigma_Y^2 n^2 \sum_{k=1}^{n} (\ln k)^2 - \sigma_Y^2 n^3 ((\ln n)^2 + 1).
\end{aligned} \tag{6.26}$$

Denote the distribution of β_{nk} by G_{nk} and the common distribution of $(\ln Y_k - \mu_Y)$ by G. Then we check the Lindeberg condition and obtain

$$\begin{aligned}
L_n &= \frac{1}{B_n} \sum_{k=1}^{n} \int_{|z| \geq \varepsilon B_n^{1/2}} z^2 \mathrm{d}G_{nk}(z) \\
&= \frac{1}{B_n} \sum_{k=1}^{n} \int_{|w(n \ln k - \sum_{k=1}^{n} \ln k)| \geq \varepsilon B_n^{1/2}} w^2 (n \ln k - \sum_{k=1}^{n} \ln k)^2 \mathrm{d}G(w), \\
&\leq \frac{1}{B_n} \sum_{k=1}^{n} \int_{|w| \geq \varepsilon B_n^{1/2} / [n(\ln^2 - \ln n + 1)]} w^2 \mathrm{d}G(w) \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned} \tag{6.27}$$

where Eq. (6.27) is due to Eq. (6.26). Thus, by the Lindeberg-Feller theorem, we obtain

$$n(\hat{\alpha} - \alpha) \xrightarrow{L} N(0, \sigma_Y^2).$$

6.2.3 Independent and identically distributed test

One important assumption of the GeP and its extension is that $\{X_n, n = 1, 2, \ldots\}$ are independent and identically distributed (i.i.d) random variables. According to Theorem 4.2.1, Eqs. (4.2.1) and (4.2.2) in Lam (2007), if denote

$$Y_k = \frac{X_{2k}}{X_{2k-1}}, k = 1, 2 \ldots,$$

and

$$Y'_k = \frac{X_{2k+1}}{X_{2k}}, k = 1, 2...,$$

if $\{X_k, k = 1, 2, ...\}$ is a GeP, then $\{Y_k, k = 1, 2, ...\}$ and $\{Y'_k, k = 1, 2, ...\}$ are respectively two sequence of i.i.d random variable.

The statistical R package "spgs" is used in this chapter for the i.i.d test (download at https://cran.r-project.org/web/packages/spgs/index.html).

6.2.4 Evaluation of the model performance

In this section, the AIC and AICc, which have been discussed in Chapter 2, will be used as indexes to compare the fitness among different GeP-like models. Among the candidate models with the smallest AIC and AICc is regarded as the best model.

The ML will be used to estimate the performance of a model. The model with the largest ML score regarded the best. The statistical R package "GenSA" is used in this chapter for optimisation strategy of parameters (download at https://cran.r-project.org/web/packages/GenSA/index.html).

The estimation of the parameters of the GeP-like models, and the performance of the GeP-like models are important. Besides, the distribution of the time to the first failure is another important issue. The choice of such a distribution influences the number of parameters in a GeP-like model. Therefore, the statistical inference of the GeP-like models will be influenced by the different distributions (for the first occurrence time of failure) and the estimation methods of parameters. This section reviews several methods for parameter estimation, probability distributions for the time to first failure, and methods for assessing model performance in the existing research.

Tables 6.1 and 6.2 show the abbreviation of several parameter estimations, the type of GeP-like models, and the distribution of the time to first failure.

TABLE 6.1: Abbreviation of parameter estimation methods

Abbreviation	Parameter estimation methods	Times
ML	Maximum likelihood estimation	15
MML	Modified maximum likelihood estimation	2
MM	Modified moment estimation	7
MLS	Modified least square	5
MLM	Modified L-moments estimation	3
LSE	Least squared estimation	3
MMS	Maximum spacing estimation	3
BE	Bayesian estimation	2

The ML estimation is the most frequently used for estimating parameters of the GeP-like models. Then, the MM estimation method is the second frequently used.

TABLE 6.2: Distributions and parameter estimation methods

Model	First occurrence time	Parameter estimation method	Reference
GeP	Weibull distribution	MM & MML	Aydoğdu, Şenoğlu, and Kara (2010)
GeP	Gamma distribution	ML & MML	Kara et al. (2022)
GeP	Weibull distribution	BE	Usta (2022)
GeP	Rayleigh distribution	ML	Biçer et al. (2019)
GeP	Lindley distribution	ML & BE	Yilmaz, Kara, and Kara (2022)
GeP	Hjorth marginal distribution	ML & MM & LSE & MMS	Demirci Bicer, Bicer, and Bakouch (2022)
GeP	Scaled Muth	ML & MM & LSE & MMS	Biçer, Bakouch, and Biçer (2021)
GeP	Exponential distribution	ML & MM & MLS	Idrees and Sulaiman (2021)
GeP	Lindley distribution	ML & MLS & MM & MLM	Demirci Biçer (2020)
GeP	Inverse gaussian distribution	ML & MM	Kara, Aydoğdu, and Türkşen (2015)
GeP	Power lindley distribution	ML & MLS & MM & MLM	Bicer (2018)
GeP	Exponential distribution	ML & MM & MLS	Idrees and Sulaiman (2021)
DGeP	Weibull distribution	ML	Pekalp, Eroğlu İnan, and Aydoğdu (2022)
DPG	Exponential distribution	ML	Pekalp, İnan, and Aydoğdu (2021)
α-SP	Rayleigh Distribution	ML & MMS & MLS MM & MLM	Demirci Biçer (2019)
α-SP	Truncated normal distribution	ML & LSE	Yan et al. (2012)
α-SP	Lognormal distribution	ML & MM	Kara et al. (2019)

Bias and the MSE are most frequently used for discussing the statistical inference of the GeP-like models.

In addition to the problem of parameter estimation, there is also a core point of how to evaluate the performance of models between different GeP-like models. Table 6.3 shows some common methods to estimate the model performance.

Akaike information criterion (AIC) is a method for evaluating how well a model fits the original data. The AIC of a model is given by

$$AIC = 2k - 2\ln(\hat{L}),$$

where k is the number of estimated parameters in the model and $\hat{L}$ is the likelihood function. Normally, the more parameters the model has, the larger the AIC will be. If the AIC is used for model selection, the model with a smaller AIC is better.

Corrected AIC (AICc) is AIC with a correction for small sample sizes. Therefore, it is given by

$$AICc = AIC + \frac{2k^2 + 2k}{n - k - 1},$$

where n is the number of observations of the sample and k is the number of parameters. If the AICc is used for model selection, the model with a smaller AICc is better.

Maximum likelihood estimation (MLE) assumes that the parameters follow a specified probability distribution with some observed data, and estimates the parameters by maximizing the likelihood function of the model. Denote $f_n(y;\theta)$ is the probability distribution with the random variable x and parameter θ. Then

$$\hat{\theta} = arg\max L_n(\theta; y),$$

TABLE 6.3: Methods for model performance

Methods	Relevant application	Relevant reference
Akaike information criterion	Failure data fitting	Wu (2018)
Corrected AIC	Failure data fitting	Wu (2018)
Bayesian information criterion	Financial markets	Chan et al. (2012) and Chan, Wan, and Yu (2014)
Bias	Reliability analysis, monotonicity analysis	Biçer et al. (2019), Demirci Bicer, Bicer, and Bakouch (2022), Usta (2022), Biçer, Bakouch, and Biçer (2021), Demirci Biçer (2020), Bicer (2018), Pekalp, Eroğlu İnan, and Aydoğdu (2022), Pekalp, İnan, and Aydoğdu (2021), and Kara et al. (2019)
Mean squared error	Disease prediction, reliability analysis	Lam et al. (2004), Chan, Wan, and Yu (2014), Chan, Choy, and Lam (2014), Aydoğdu, Şenoğlu, and Kara (2010), Biçer et al. (2019), Kara et al. (2022), Usta (2022), Demirci Bicer, Bicer, and Bakouch (2022), Biçer, Bakouch, and Biçer (2021), Idrees and Sulaiman (2021), Demirci Biçer (2020), Bicer (2018), Pekalp, Eroğlu İnan, and Aydoğdu (2022), Pekalp, İnan, and Aydoğdu (2021), Yan et al. (2012), and Kara et al. (2019)
Mean percentage error	Reliability analysis, optimisation	Lam et al. (2004) and Chan, Choy, and Lam (2014)
Maximum likelihood estimation	Reliability analysis, optimisation	Chan and Leung (2010) and Demirci Biçer (2020)
Least squared error	Reliability analysis	Chan and Leung (2010)
Adjusted mean squared error	Disease prediction	Chan et al. (2006)
Deviance information criterion	Reliability analysis, disease prediction	Chan and Wan (2014), Chan, Choy, and Lam (2014), and Chan et al. (2012)
Absolute percentage bias	Disease prediction	Chan et al. (2012)
Root mean square error	Disease prediction	Chan et al. (2012)
Coverage percentage	Disease prediction	Chan et al. (2012)

where $\hat{\theta}$ is the maximum likelihood estimator of θ.

6.3 Real World failure datasets

Table 6.4 shows the datasets used in this thesis and their sources.

TABLE 6.4: Summary of datasets

Dataset	n	Description	Reference
1	23	Failures of a load-haul-dump (LHD) machine	Kumar and Klefsjö (1992)
2	25	Failures of a load-haul-dump (LHD) machine	Kumar and Klefsjö (1992)
3	27	Failures of a load-haul-dump (LHD) machine	Kumar and Klefsjö (1992)
4	28	Failures of a load-haul-dump (LHD) machine	Kumar and Klefsjö (1992)
5	26	Failures of a load-haul-dump (LHD) machine	Kumar and Klefsjö (1992)
6	23	Failures of a load-haul-dump (LHD) machine	Kumar and Klefsjö (1992)

7	69	Failures of the Armoured Flexible Conveyor (AFC)	Gupta et al. (2009)
8	33	Failures of the Powertrain System of bus	Guida and Pulcini (2009)
9	54	Failures of the Powertrain System of bus	Guida and Pulcini (2009)
10	30	Failures of the air conditioning system of aircraft	Proschan (2000)
11	29	Failures of the air conditioning system of aircraft	Proschan (2000)
12	24	Failures of the air conditioning system of aircraft	Proschan (2000)
13	24	Failures of the air conditioning system of aircraft	Proschan (2000)
14	65	Failures of the main pumps at oil refineries	Percy and Alkali (2007)
15	51	Failures of the main pumps at oil refineries	Percy and Alkali (2007)
16	18	Failures of material fatigue or component aging	Xie and Lai (1996)
17	128	Failures of computer break down	Chakraborty and Chakravarty (2012)
18	135	Failures of software reliability	Musa (1979)
19	190	Failures of the coal-mining disaster	Andrews and Herzberg (2012)
20	40	Failures of the car	Hand et al. (1993)
21	92	Spread of SARS epidemic	Chan et al. (2006)
22	68	Spread of SARS epidemic	Chan et al. (2006)
23	30	Arrival and inter-arrival for the valve	Modarres (2006)
24	71	Failure data related to the U.S.S. Halfbeak motor	Ascher and Feingold (1984)
25	56	Failure data of diesel engine	Lee (1980)

Before applying the GeP-like process, it is necessary to test if the data are independent. Therefore, Table 6.5 shows the independence test for each dataset. A dataset with a p-value lower than 0.05 means that it does not follow the independence assumption, therefore, such datasets will not be used for further comparisons.

TABLE 6.5: Summary of independence test

Dataset	independence p-value	uniformity p-value
1	0.29	1.00
2	0.35	1.00
3	0.53	1.00
4	0.92	0.99
5	0.25	1.00
6	0.87	1.00
7	0.98	1.00
8	0.01	1.00
9	0.40	1.00
10	0.91	0.99
11	0.80	0.99
12	0.68	1.00
13	0.61	1.00
14	0.19	1.00
15	0.19	1.00
16	0.00	0.94
17	0.84	0.99
18	0.87	1.00
19	0.08	1.00
20	0.97	0.99
21	0.00	0.94
22	0.00	0.99
23	0.28	1.00
24	0.50	1.00
25	0.28	1.00

According to the above result, the following findings can be obtained:

- The size of our most dataset is not huge enough ($n \leqq 30$) so that it is impossible to use Chi-square test.
- Datasets 8, 16, 21 and 22 are not satisfied with the conditions of independence as the independence p-value is less than 0.05. Thus, there datasets will be ignored in this chapter.

It is worth noticing that the Section 5 in Lam (2007) briefly studies the model performance by the GeP, the RP or the homogeneous Poisson process (HPP) model, the Cox-Lewis model and the Weibull process (WP) model based on ten real datasets. Among them, six datasets, which are 11, 13, 19, 21, 22 and 24, will be used in this chapter. To compare the fitness of models, the mean squared error (MSE) and the maximum percentage error (MPE) will be used to evaluate the model performance. In Lam (2007), the GeP model is the best model among these four models.

In this chapter, more different datasets, which have been shown in Table 6.4 will be used for comparison of the performance among more GeP's extensions in this chapter. AIC, AICc and ML will be used to evaluate the model performance. The next section describes numerical results in this chapter.

6.4 Case studies

Table 6.6 shows the model performance based on 25 datasets. The order of data in each sheet from top to bottom is AIC, AICc and ML. It is worth noticing that, the results are shown in the sequence in the rank of model performance.

TABLE 6.6: The values of AIC, AICc, and ML of the models

Dataset	RP	GeP	EPP	α-SP	DGeP	TGeP	DRGeP
	263.97	263.03	267.02	263.51	265.01	**260.12**	265.03
1	264.57	264.29	270.55	264.78	267.23	**261.38**	267.25
	-129.99	-128.52	-128.51	-128.75	-128.51	**-124.06**	-128.52
	301.45	303.04	307.04	301.28	299.35	**297.33**	298.23
2	301.99	304.18	310.20	302.42	301.35	**298.47**	300.23
	-148.72	-148.52	-148.52	-147.64	-145.67	**-142.66**	-145.11
	337.10	334.33	338.32	334.18	336.21	**324.12**	336.08
3	337.60	335.37	341.18	335.23	338.03	**325.16**	337.89
	-166.55	-164.16	-164.16	-164.09	-164.10	**-156.06**	-164.04
	320.10	322.06	326.06	321.24	321.97	320.27	**319.57**
4	**320.58**	323.06	328.79	322.24	323.71	321.27	321.31
	-158.05	-158.03	-158.03	-157.62	-156.99	**-154.13**	-155.79
	306.41	306.65	310.65	**306.24**	308.38	308.63	308.23
5	**306.92**	307.74	313.65	307.32	310.28	309.72	310.14
	-151.20	-150.33	-150.33	-150.12	-150.19	**-148.31**	150.11
	278.53	280.52	284.52	280.35	281.93	279.29	281.43
6	**279.13**	281.79	288.05	281.61	284.15	280.56	283.65
	-137.27	-137.26	-137.26	-137.17	-136.96	**-133.65**	-136.71
	783.58	785.21	787.59	785.13	787.06	785.18	787.13
7	**783.76**	785.58	788.54	785.50	787.69	785.55	787.76
	-389.79	-389.61	-388.80	-389.57	-389.53	**-386.59**	-389.57
	716.36	717.20	**715.78**	718.05	716.88	720.44	717.28
9	716.36	717.20	**715.78**	718.05	716.88	720.44	717.28
	-356.18	-355.60	**-352.89**	-356.03	-354.44	-354.22	-354.64
	307.87	**306.42**	309.81	306.53	308.42	308.35	308.52
10	308.32	**307.35**	312.31	307.46	310.02	309.27	310.12
	-151.94	-150.21	-149.91	-150.27	-150.21	**-148.18**	-150.26
	313.39	315.00	318.75	315.00	317.0014	320.31	**295.31**
11	313.85	315.96	321.36	315.96	318.67	321.27	**297.13**
	-154.69	-154.50	-154.38	-154.50	-154.50	-154.15	**-143.65**

12	251.70	253.62	255.47	253.69	253.76	**242.03**	246.84
	252.27	254.82	258.80	254.89	255.86	**243.24**	249.06
	-123.85	-123.81	-122.73	-123.84	-122.88	**-115.02**	-119.42
13	**254.73**	256.32	260.04	256.53	258.25	261.65	258.11
	255.30	257.52	263.38	257.73	260.35	262.85	260.21
	-125.37	-125.16	-125.02	-125.26	-125.12	**-124.82**	-125.05
14	607.28	606.41	610.01	**605.46**	607.94	606.46	606.92
	607.47	606.80	611.03	**605.85**	608.61	607.70	607.59
	-301.64	-300.20	-300.01	-299.73	-299.97	**-297.23**	-299.46
15	**497.91**	628.00	631.65	628.69	630.00	628.09	500.95
	498.16	628.51	632.99	629.21	630.87	628.60	501.82
	-246.96	-311.00	-310.83	-311.35	-311.00	-308.05	**-246.48**
17	**615.81**	617.43	616.58	617.77	618.75	617.89	617.73
	615.90	617.63	617.08	617.97	619.07	618.09	618.06
	-305.90	-305.72	-303.29	-305.89	-305.37	**-299.95**	-304.87
18	1981.17	**1925.02**	1928.17	1935.15	1925.49	1927.94	1926.09
	1981.26	1925.21	1928.64	1935.33	**1925.80**	1928.12	1926.40
	-988.58	-959.51	-959.09	-964.58	-958.74	**-957.97**	-959.04
19	2347.00	2324.06	2328.06	2334.13	2322.68	**2320.82**	2324.50
	2347.06	2324.19	2328.38	2334.26	2322.89	**2320.95**	2324.72
	-1171.50	-1159.03	-1159.03	-1164.07	-1157.34	**-1151.41**	-1158.25
23	385.73	381.66	384.23	**381.21**	382.340	379.50	383.39
	386.17	382.59	386.73	382.13	384.00	**380.42**	384.99
	-190.86	-187.83	-187.12	-187.60	-187.20	**-180.75**	-187.70
24	949.35	926.44	930.44	919.76	925.66	**918.82**	922.19
	386.17	382.59	386.73	382.13	384.00	**380.42**	384.99
	-472.67	-460.22	-460.22	-456.88	-458.83	**-450.41**	-457.09
25	742.58	743.53	746.61	**743.48**	745.50	744.95	745.44
	742.80	743.99	747.81	743.94	746.28	745.41	746.23
	-369.29	-368.77	-368.31	-368.74	-368.75	**-366.47**	-368.72
Average	650.12	651.80	654.88	652. 33	652.65	**643.50**	644.9943
	589.06	590.09	594.07	591.04	589.53	589.23	623.80
	-323.06	-322.90	-322.44	-323.17	-322.32	**-315.18**	-318.50

According to the above table, the following result can be obtained:

- In terms of the AIC result
  - Based on the rankings of model performance, the result shows that $AIC_{TGeP} > AIC_{RP} > AIC_{\alpha\text{-series}} > AIC_{GeP} = AIC_{DRGeP} > AIC_{EPP} > AIC_{DGeP}$.
  - Based on the rank of the average AIC score, $AIC_{TGeP} > AIC_{DRGeP} > AIC_{RP} > AIC_{GeP} > AIC_{\alpha\text{-series}} > AIC_{DRGeP} > AIC_{EPP}$.

  - The TGeP has the best performance (6 best scores), the second one is the RP (5 best scores), then the α-SP (4 best scores), EPP (2 best scores) and DRGeP (2 best scores) have similar performance.
  - The EPP works better than the DGeP in these 23 datasets. However, the DGeP model has lower average AIC score than the EPP.
  - The DRGeP has the second lowes t average AIC score, its combined performance is better than other five models and is second only to the TGeP.
- In terms of the AICc result
  - Based on the rankings of model performance, the result shows that $\text{AICc}_{\text{RP}} > \text{AICc}_{\text{TGeP}} > \text{AICc}_{\alpha\text{-series}} = \text{AICc}_{\text{GeP}} = \text{AICc}_{\text{DGeP}} = \text{AICc}_{\text{DRGeP}} > \text{AICc}_{\text{EPP}}$.
  - Based on the rank of the average AICc score, the result shows that $\text{AICc}_{\text{RP}} > \text{AICc}_{\text{TGeP}} > \text{AICc}_{\text{DGeP}} > \text{AICc}_{\text{GeP}} > \text{AICc}_{\text{EPP}} > \text{AICc}_{\alpha\text{-series}} > \text{AICc}_{\text{DRGeP}}$.
  - The RP has the best performance (9 best scores), second one is the TGeP (7 best scores). The performance of these two models is more pronounced in terms of AICc.
  - In terms of AICc, TGeP has the best performance, which is different from the result of AIC.
  - The performance of the GeP is better in terms of AICc (comparing to the AIC).
  - The performance of the DRGeP has obvious disadvantage in terms of AICc. It is 623.80 which is much higher than other six models.
- In terms of the ML result
  - The TGeP has significant advantage on model performance.
  - Based on the rank of model performance, the result shows that $\text{ML}_{\text{TGeP}} > \text{ML}_{\text{DRGeP}} > \text{ML}_{\text{EPP}} > \text{ML}_{\text{RP}} = \text{ML}_{\text{GeP}} = \text{ML}_{\alpha\text{-series}} = \text{ML}_{\text{DGeP}}$.
  - Based on the rankings of the average ML score, the result shows that $\text{ML}_{\text{TGeP}} > \text{ML}_{\text{DRGeP}} > \text{ML}_{\text{DGeP}} > \text{ML}_{\text{EPP}} > \text{ML}_{\text{GeP}} > \text{ML}_{\text{RP}} > \text{ML}_{\alpha\text{-series}}$.
  - The performance of both DRGeP and DGeP has significant increase comparing with the result in terms of AIC and AICc.

Table 6.7 shows the best model for each dataset in terms of different indices.

TABLE 6.7: Result of the performance for datasets

Dataset	AIC	AICc	ML
1	TGeP	TGeP	TGeP
2	TGeP	TGeP	TGeP
3	TGeP	TGeP	TGeP
4	DRGeP	RP	TGeP
5	α-SP	RP	TGeP
6	RP	RP	TGeP
7	RP	RP	TGeP

9	EPP	EPP	EPP
10	GeP	GeP	TGeP
11	DRGeP	DRGeP	DRGeP
12	TGeP	TGeP	TGeP
13	RP	RP	TGeP
14	α-SP	α-SP	TGeP
15	RP	RP	DRGeP
17	RP	RP	TGeP
18	GeP	DGeP	TGeP
19	TGeP	TGeP	TGeP
20	α-SP	TGeP	TGeP
24	α-SP	RP	TGeP
25	TGeP	RP	α-SP
Best	TGeP	RP	TGeP
Second Best	RP	TGeP	DRGeP

6.5 Fisher information matrix

Denote $I(\theta)$ as the fisher information, then

$$I(\theta) = E[S(X;\theta)^2] \tag{6.28}$$

where

$$S(X;\theta) = \sum_{i=1}^{n} \frac{logf(X_i;\theta)}{\theta} \tag{6.29}$$

which is the score function according to the MLE. It measures the amount of information of an observable random variable about an unknown parameter θ. Tables 6.8–6.29 show the estimated parameters and their standard errors based on the GeP and its extension. As the negative Hessian matrix evaluated at the MLE is the same as the observed Fisher information matrix evaluated at the MLE, we use R package of Hessian matrix and then get the square root of the matrix diagonal for the variance of each parameter.

Then Tables 6.8–6.29 show the Fisher information matrix from Dataset 1 to Dataset 25. The value of each column means the parameter estimator and parameter variance (in brackets).

TABLE 6.8: Parameter estimations for Dataset 1 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{\lambda}$=0.82	$\hat{\delta}$=96.96				
	(0.13)	(26.01)				
GeP	$\hat{a}$=1.07	$\hat{\lambda}$=0.89	$\hat{\delta}$=188.02			
	(0.04)	(0.15)	(85.34)			
EPP	$\hat{\alpha}$=0.07	$\hat{a}$=0.75	$\hat{\beta}$=1.07	$\hat{b}$=0.89		$\hat{\delta}$=191.63
	(3.71)	(3.34)	(0.12)	(0.15)		(355.66)
α-SP	$\hat{a}$=0.48	$\hat{\lambda}$=0.87	$\hat{\delta}$=276.09			
	(0.32)	(0.14)	(211.21)			
TGeP	$\hat{a}_1$=1.02	$\hat{\lambda}_1$=0.88	$\hat{\delta}_1$=178.01	$\hat{a}_2$=0.81	$\hat{\lambda}_2$=0.78	$\hat{\delta}_2$=162.14
	(0.02)	(0.11)	(79.25)	(0.01)	(0.15)	(68.24)
DGeP	$\hat{a}$=1.06	$\hat{b}$=0.05	$\hat{\lambda}$=0.86	$\hat{\delta}$=206.15		
	(0.05)	(0.33)	(0.25)	(163.00)		

TABLE 6.9: Parameter estimations for Dataset 2 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.01	$\hat{\lambda}$=141.80				
	(0.15)	(29.63)				
GeP	$\hat{a}$=1.01	$\hat{\lambda}$=1.03	$\hat{\delta}$=168.92			
	(0.02)	(0.16)	(57.87)			
EPP	$\hat{\alpha}$=2.71	$\hat{a}$=1.01	$\hat{\beta}$=1.01	$\hat{b}$=1.03		$\hat{\delta}$=627.00
	(5.31)	(0.03)	(0.07)	(0.16)		(870.39)
α-SP	$\hat{a}$=0.26	$\hat{\lambda}$=1.08	$\hat{\delta}$=258.23			
	(0.18)	(0.17)	(119.28)			
TGeP	$\hat{a}_1$=0.89	$\hat{\lambda}_1$=1.02	$\hat{\delta}_1$=166.87	$\hat{a}_2$=0.96	$\hat{\lambda}_2$=1.02	$\hat{\delta}_2$=171.89
	(0.01)	(0.12)	(59.21)	(0.01)	(0.15)	(79.04)
DGeP	$\hat{a}$=0.88	$\hat{b}$=0.64	$\hat{\lambda}$=0.78	$\hat{\delta}$=420.53		
	(0.09)	(0.35)	(0.22)	(332.29)		

TABLE 6.10: Parameter estimation for Dataset 3 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.01	$\hat{\lambda}$=176.38				
	(0.15)	(35.52)				
GeP	$\hat{a}$=1.05	$\hat{\lambda}$=1.12	$\hat{\delta}$=322.57			
	(0.02)	(0.17)	(106.42)			
EPP	$\hat{\alpha}$=2.95	$\hat{a}$=1.05	$\hat{\beta}$=0.10	$\hat{b}$=1.12		$\hat{\delta}$=990.00
	(2.52)	(0.02)	(0.73)	(0.17)		(752.12)
α-SP	$\hat{a}$=0.47	$\hat{\lambda}$=1.13	$\hat{\delta}$=520.98			

	(0.20)	(0.17)	(265.65)			
TGeP	$\hat{a}_1$=1.06	$\hat{\lambda}_1$=1.14	$\hat{\delta}_1$=311.54	$\hat{a}_2$=1.02	$\hat{\lambda}_2$=1.09	$\hat{\delta}_2$=312.65
	(0.01)	(0.09)	(88.78)	(0.01)	(0.13)	(98.5)
DGeP	$\hat{a}$=1.04	$\hat{b}$=0.10	$\hat{\lambda}$=1.05	$\hat{\delta}$=401.61		
	(0.04)	(0.30)	(0.26)	(308.22)		

TABLE 6.11: Parameter estimation for Dataset 4 regarding RP, GeP, EPP, TGeP and DGeP

	1	**2**	**3**	**4**	**5**	**6**
RP	$\hat{a}$=1.00	$\hat{\lambda}$=103.95				
	(0.15)	(20.72)				
GeP	$\hat{a}$=1.00	$\hat{\lambda}$=1.00	$\hat{\delta}$=109.84			
	(0.02)	(0.15)	(36.99)			
EPP	$\hat{\alpha}$=1.04	$\hat{a}$=1.00	$\hat{\beta}$=1.00	$\hat{b}$=1.00		$\hat{\delta}$=223.62
	(9.07)	(0.04)	(0.05)	(0.15)		(993.06)
α-SP	$\hat{a}$=0.17	$\hat{\lambda}$=1.01	$\hat{\delta}$=155.34			
	(0.19)	(0.15)	(76.57)			
TGeP	$\hat{a}_1$=1.01	$\hat{\lambda}_1$=1.02	$\hat{\delta}_1$=300.42	$\hat{a}_2$=1.00	$\hat{\lambda}_2$=1.19	$\hat{\delta}_2$=302.52
	(0.02)	(0.08)	(32.66)	(0.01)	(0.13)	(33.26)
DGeP	$\hat{a}$=0.94	$\hat{b}$=0.39	$\hat{\lambda}$=0.80	$\hat{\delta}$=194.59		
	(0.06)	(0.33)	(0.22)	(139.66)		

TABLE 6.12: Parameter estimation for Dataset 5 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=0.91	$\hat{\lambda}$=119.27				
	(0.14)	(26.94)				
GeP	$\hat{a}$=1.03	$\hat{\lambda}$=0.96	$\hat{\delta}$=180.59			
	(0.03)	(0.16)	(68.07)			
EPP	$\hat{\alpha}$=1.22	$\hat{a}$=1.03	$\hat{\beta}$=1.03	$\hat{b}$=0.96		$\hat{\delta}$=401.00
	(4.46)	(0.06)	(0.07)	(0.16)		(789.99)
α-SP	$\hat{a}$=0.35	$\hat{\lambda}$=0.97	$\hat{\delta}$=269.39			
	(0.24)	(0.16)	(163.74)			
TGeP	$\hat{a}_1$=1.05	$\hat{\lambda}_1$=0.94	$\hat{\delta}_1$=191.21	$\hat{a}_2$=1.01	$\hat{\lambda}_2$=0.89	$\hat{\delta}_2$=188.93
	(0.02)	(0.21)	(67.16)	(0.03)	(0.15)	69.57
DGeP	$\hat{a}$=1.02	$\hat{b}$=0.14	$\hat{\lambda}$=0.87	$\hat{\delta}$=238.43		
	(0.05)	(0.30)	(0.23)	(181.82)		

TABLE 6.13: Parameter estimation for Dataset 6 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.03	$\hat{\lambda}$=145.67				
	(0.18)	(30.79)				
GeP	$\hat{a}$=1.00	$\hat{\lambda}$=1.03	$\hat{\delta}$=149.49			
	(0.03)	(0.18)	(54.64)			
EPP	$\hat{\alpha}$=0.02	$\hat{a}$=1.00	$\hat{\beta}$=1.00	$\hat{b}$=1.03		$\hat{\delta}$=151.76
	(7.83)	(0.51)	(0.03)	(0.18)		(1168.73)
α-SP	$\hat{a}$=0.10	$\hat{\lambda}$=1.04	$\hat{\delta}$=180.76			
	(0.23)	(0.18)	(99.49)			
TGeP	$\hat{a}_1$=1.01	$\hat{\lambda}_1$=1.05	$\hat{\delta}_1$=151.08	$\hat{a}_2$=0.98	$\hat{\lambda}_2$=1.10	$\hat{\delta}_2$=138.29
	(0.02)	(0.21)	(61.90)	(0.01)	(0.18)	(62.61)
DGeP	$\hat{a}$=0.95	$\hat{b}$=0.31	$\hat{\lambda}$=0.85	$\hat{\delta}$=253.28		
	(0.10)	(0.49)	(0.32)	(261.71)		

TABLE 6.14: Parameter estimation for Dataset 7 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.18	$\hat{\lambda}$=113.70				
	(0.11)	(12.21)				
GeP	$\hat{a}$=1.00	$\hat{\lambda}$=1.19	$\hat{\delta}$=125.50			
	(0.01)	(0.11)	(24.54)			
EPP	$\hat{\alpha}$=2.96	$\hat{a}$=0.10	$\hat{\beta}$=1.00	$\hat{b}$=1.20		$\hat{\delta}$=132.53
	(0.64)	(0.16)	(0.01)	(0.11)		(26.61)
α-SP	$\hat{a}$=0.08	$\hat{\lambda}$=1.19	$\hat{\delta}$=145.98			
	(0.12)	(0.11)	(57.45)			
TGeP	$\hat{a}_1$=1.01	$\hat{\lambda}_1$=1.12	$\hat{\delta}_1$=126.13	$\hat{a}_2$=1.12	$\hat{\lambda}_2$=1.17	$\hat{\delta}_2$=113.78
	(0.02)	(0.15)	(19.56)	(0.01)	(0.12)	(18.21)
DGeP	$\hat{a}$=1.00	$\hat{b}$=0.08	$\hat{\lambda}$=1.11	$\hat{\delta}$=159.03		
	(0.01)	(0.23)	(0.24)	(115.20)		

TABLE 6.15: Parameter estimation for Dataset 9 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=7.49	$\hat{\lambda}$=3.79				
	(0.77)	(0.07)				
GeP	$\hat{a}$=1.00	$\hat{\lambda}$=4.50	$\hat{\delta}$=3.72			
	(0.00)	(0.52)	(0.22)			
EPP	$\hat{\alpha}$=0.04	$\hat{a}$=0.59	$\hat{\beta}$=1.00	$\hat{b}$=4.50		$\hat{\delta}$=3.73
	(0.20)	(0.82)	(0.01)	(0.52)		(0.25)
α-SP	$\hat{a}$=0.02	$\hat{\lambda}$=7.59	$\hat{\delta}$=3.97			

	(0.02)	(0.79)	(0.26)			
TGeP	$\hat{a}_1$=1.02	$\hat{\lambda}_1$=4.16	$\hat{\delta}_1$=3.24	$\hat{a}_2$=1.05	$\hat{\lambda}_2$=3.67	$\hat{\delta}_2$=3.51
	(0.01)	(0.38)	(0.18)	(0.01)	(0.49)	(0.28)
DGeP	$\hat{a}$=0.99	$\hat{b}$=0.31	$\hat{\lambda}$=4.50	$\hat{\delta}$=4.64		
	(0.01)	(0.16)	(0.74)	(0.67)		

TABLE 6.16: Parameter estimation for Dataset 10 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.24	$\hat{\lambda}$=76.82				
	(0.18)	(13.40)				
GeP	$\hat{a}$=0.98	$\hat{\lambda}$=1.26	$\hat{\delta}$=64.37			
	(0.02)	(0.19)	(20.31)			
EPP	$\hat{\alpha}$=0.01	$\hat{a}$=1.18	$\hat{\beta}$=0.96	$\hat{b}$=1.25		$\hat{\delta}$=57.38
	(0.04)	(0.24)	(0.04)	(0.19)		(18.50)
α-SP	$\hat{a}$=0.09	$\hat{\lambda}$=1.25	$\hat{\delta}$=63.03			
	(0.19)	(0.19)	(28.86)			
TGeP	$\hat{a}_1$=0.88	$\hat{\lambda}_1$=1.35	$\hat{\delta}_1$=87.21	$\hat{a}_2$=0.92	$\hat{\lambda}_2$=1.54	$\hat{\delta}_2$=66.20
	(0.01)	(0.24)	(18.91)	(0.01)	(0.31)	(21.06)
DGeP	$\hat{a}$=0.97	$\hat{b}$=0.09	$\hat{\lambda}$=1.19	$\hat{\delta}$=72.34		
	(0.06)	(0.36)	(0.34)	(42.11)		

TABLE 6.17: Parameter estimation for Dataset 11 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=0.85	$\hat{\lambda}$=54.61				
	(0.12)	(12.36)				
GeP	$\hat{a}$=1.04	$\hat{\lambda}$=0.91	$\hat{\delta}$=96.85			
	(0.02)	(0.13)	(36.22)			
EPP	$\hat{\alpha}$=0.58	$\hat{a}$=1.05	$\hat{\beta}$=0.10	$\hat{b}$=0.92		$\hat{\delta}$=61.48
	(0.31)	(0.02)	(0.26)	(0.13)		(10.86)
α-SP	$\hat{a}$=0.47	$\hat{\lambda}$=0.91	$\hat{\delta}$=169.83			
	(0.26)	(0.13)	(115.72)			
TGeP	$\hat{a}_1$=1.12	$\hat{\lambda}_1$=0.88	$\hat{\delta}_1$=89.48	$\hat{a}_2$=1.0	$\hat{\lambda}_2$=1.54	$\hat{\delta}_2$=98.56
	(0.01)	(0.15)	(33.44)	(0.02)	(0.21)	(34.79)
DGeP	$\hat{a}$=1.04	$\hat{b}$=0.02	$\hat{\lambda}$=0.90	$\hat{\delta}$=100.32		
	(0.03)	(0.38)	(0.28)	(84.90)		

TABLE 6.18: Parameter estimation for Dataset 12 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.12	$\hat{\lambda}$=79.92				
	(0.18)	(14.37)				
GeP	$\hat{a}$=0.99	$\hat{\lambda}$=1.13	$\hat{\delta}$=66.01			
	(0.02)	(0.18)	(22.74)			
EPP	$\hat{\alpha}$=0.01	$\hat{a}$=1.15	$\hat{\beta}$=0.96	$\hat{b}$=1.14		$\hat{\delta}$=58.49
	(0.16)	(0.65)	(0.08)	(0.18)	(23.93)	
α-SP	$\hat{a}$=0.13	$\hat{\lambda}$=1.14	$\hat{\delta}$=58.81			
	(0.20)	(0.18)	(29.44)			
TGeP	$\hat{a}_1$=0.89	$\hat{\lambda}_1$=1.15	$\hat{\delta}_1$=67.18	$\hat{a}_2$=0.98	$\hat{\lambda}_2$=1.26	$\hat{\delta}_2$=98.56
	(0.02)	(0.15)	(21.76)	(0.01)	(0.12)	(16.43)
DGeP	$\hat{a}$=0.99	$\hat{b}$=0.01	$\hat{\lambda}$=1.15	$\hat{\delta}$=64.77		
	(0.04)	(0.30)	(0.31)	(35.54)		

TABLE 6.19: Parameter estimation for Dataset 13 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.07	$\hat{\lambda}$=68.67				
	(0.18)	(14.14)				
GeP	$\hat{a}$=1.00	$\hat{\lambda}$=1.07	$\hat{\delta}$=65.56			
	(0.03)	(0.18)	(26.28)			
EPP	$\hat{\alpha}$=0.01	$\hat{a}$=1.23	$\hat{\beta}$=0.94	$\hat{b}$=1.11		$\hat{\delta}$=52.02
	(0.04)	(0.23)	(0.05)	(0.19)		(23.47)
α-SP	$\hat{a}$=0.09	$\hat{\lambda}$=1.08	$\hat{\delta}$=56.34			
	(0.25)	(0.18)	(33.46)			
TGeP	$\hat{a}_1$=1.01	$\hat{\lambda}_1$=1.15	$\hat{\delta}_1$=61.05	$\hat{a}_2$=1.10	$\hat{\lambda}_2$=1.19	$\hat{\delta}_2$=64.58
	(0.02)	(0.16)	(22.97)	(0.01)	(0.11)	(24.90)
DGeP	$\hat{a}$=1.04	$\hat{b}$=0.40	$\hat{\lambda}$=1.46	$\hat{\delta}$=38.25		
	(0.03)	(0.30)	(0.39)	(16.80)		

TABLE 6.20: Parameter estimation for Dataset 14 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.07	$\hat{\lambda}$=39.30				
	(0.11)	(4.78)				
GeP	$\hat{a}$=1.01	$\hat{\lambda}$=1.11	$\hat{\delta}$=52.81			
	(0.01)	(0.11)	(11.18)			
EPP	$\hat{\alpha}$=0.85	$\hat{a}$=0.25	$\hat{\beta}$=1.01	$\hat{b}$=1.11		$\hat{\delta}$=54.91
	(1.64)	(0.47)	(0.01)	(0.11)		(12.20)
α-SP	$\hat{a}$=0.24	$\hat{\lambda}$=1.12	$\hat{\delta}$=84.60			
	(0.13)	(0.11)	(35.76)			
TGeP	$\hat{a}_1$=1.02	$\hat{\lambda}_1$=1.16	$\hat{\delta}_1$=48.65	$\hat{a}_2$=1.01	$\hat{\lambda}_2$=1.21	$\hat{\delta}_2$=50.73
	(0.01)	(0.11)	(10.67)	(0.01)	(0.12)	(8.96)
DGeP	$\hat{a}$=1.00	$\hat{b}$=0.21	$\hat{\lambda}$=0.93	$\hat{\delta}$=86.97		
	(0.01)	(0.33)	(0.28)	(77.15)		

TABLE 6.21: Parameter estimation for Dataset 15 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.06	$\hat{\lambda}$=48.02				
	(0.11)	(6.68)				
GeP	$\hat{a}$=1.01	$\hat{\lambda}$=1.08	$\hat{\delta}$=58.74			
	(0.01)	(0.12)	(15.52)			
EPP	$\hat{\alpha}$=1.55	$\hat{a}$=1.01	$\hat{\beta}$=0.89	$\hat{b}$=1.09		$\hat{\delta}$=117.25
	(3.35)	(0.02)	(0.17)	(0.12)		(196.76)
α-SP	$\hat{a}$=0.10	$\hat{\lambda}$=1.07	$\hat{\delta}$=64.90			
	(0.15)	(0.11)	(31.31)			
TGeP	$\hat{a}_1$=1.02	$\hat{\lambda}_1$=1.07	$\hat{\delta}_1$=53.64	$\hat{a}_2$=1.01	$\hat{\lambda}_2$=1.12	$\hat{\delta}_2$=55.29
	(0.02)	(0.14)	(18.96)	(0.01)	(0.11)	(16.33)
DGeP	$\hat{a}$=1.01	$\hat{b}$=0.04	$\hat{\lambda}$=1.05	$\hat{\delta}$=63.50		
	(0.01)	(0.24)	(0.24)	(37.53)		

TABLE 6.22: Parameter estimation for Dataset 16 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.24	$\hat{\lambda}$=17.92				
	(0.23)	(3.59)				
GeP	$\hat{a}$=1.16	$\hat{\lambda}$=2.23	$\hat{\delta}$=51.94			
	(0.03)	(0.41)	(13.09)			
EPP	$\hat{\alpha}$=0.01	$\hat{a}$=1.58	$\hat{\beta}$=1.05	$\hat{b}$=3.02		$\hat{\delta}$=38.62
	(0.01)	(0.11)	(0.03)	(0.52)		(6.51)
α-SP	$\hat{a}$=0.63	$\hat{\lambda}$=1.86	$\hat{\delta}$=53.00			

	(0.17)	(0.37)	(17.66)			
TGeP	$\hat{a}_1$=1.14	$\hat{\lambda}_1$=2.15	$\hat{\delta}_1$=56.52	$\hat{a}_2$=1.15	$\hat{\lambda}_2$=2.02	$\hat{\delta}_2$=51.29
	(0.01)	(0.35)	(11.64)	(0.02)	(0.28)	(10.18)
DGeP	$\hat{a}$=1.18	$\hat{b}$=0.26	$\hat{\lambda}$=2.70	$\hat{\delta}$=40.42		
	(0.03)	(0.17)	(0.55)	(9.23)		

TABLE 6.23: Parameter estimation for Dataset 17 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.16	$\hat{\lambda}$=4.32				
	(0.08)	(0.35)				
GeP	$\hat{a}$=1.00	$\hat{\lambda}$=1.17	$\hat{\delta}$=3.40			
	(0.01)	(0.08)	(0.52)			
EPP	$\hat{\alpha}$=1.27	$\hat{a}$=0.83	$\hat{\beta}$=1.00	$\hat{b}$=1.18		$\hat{\delta}$=3.90
	(1.16)	(0.15)	(0.01)	(0.08)		(0.79)
α-SP	0.18	1.18	2.14			
	(0.08)	(0.08)	(0.70)			
TGeP	$\hat{a}_1$=1.01	$\hat{\lambda}_1$=1.12	$\hat{\delta}_1$=3.16	$\hat{a}_2$=1.02	$\hat{\lambda}_2$=1.14	$\hat{\delta}_2$=3.6
	(0.01)	(0.09)	(0.25)	(0.01)	(0.12)	(0.42)
DGeP	1.00	0.07	1.09	3.62		
	(0.01)	(0.39)	(0.43)	(1.38)		

TABLE 6.24: Parameter estimation for Dataset 18 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=0.67	$\hat{\lambda}$=494.87				
	(0.04)	(65.26)				
GeP	$\hat{a}$=0.98	$\hat{\lambda}$=0.86	$\hat{\delta}$=91.89			
	(0.01)	(0.06)	(18.71)			
EPP	$\hat{\alpha}$=1.87	$\hat{a}$=0.10	$\hat{\beta}$=0.98	$\hat{b}$=0.87		$\hat{\delta}$=94.47
	(6.88)	(1.89)	(0.01)	(0.06)		(20.34)
α-SP	$\hat{a}$=0.88	$\hat{\lambda}$=0.83	$\hat{\delta}$=13.95			
	(0.09)	(0.06)	(5.31)			
TGeP	$\hat{a}_1$=0.91	$\hat{\lambda}_1$=0.75	$\hat{\delta}_1$=82.87	$\hat{a}_2$=0.89	$\hat{\lambda}_2$=0.88	$\hat{\delta}_2$=90.42
	(0.01)	(0.05)	(17.04)	(0.01)	(0.02)	(13.18)
DGeP	$\hat{a}$=0.99	$\hat{b}$=0.26	$\hat{\lambda}$=1.12	$\hat{\delta}$=43.81		
	(0.01)	(0.18)	(0.21)	(21.38)		

TABLE 6.25: Parameter estimation for Dataset 19 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=0.91	$\hat{\lambda}$=168.69				
	(0.05)	(14.18)				
GeP	$\hat{a}$=0.99	$\hat{\lambda}$=0.97	$\hat{\delta}$=85.54			
	(0.01)	(0.06)	(12.64)			
EPP	$\hat{\alpha}$=2.39	$\hat{a}$=0.99	$\hat{\beta}$=0.99	$\hat{b}$=0.97		$\hat{\delta}$=290.19
	(7.40)	(0.01)	(0.01)	(0.06)		(631.46)
α-SP	$\hat{a}$=0.30	$\hat{\lambda}$=0.94	$\hat{\delta}$=46.24			

	(0.07)	(0.05)	(14.33)			
TGeP	$\hat{a}_1$=0.98	$\hat{\lambda}_1$=0.96	$\hat{\delta}_1$=78.38	$\hat{a}_2$=0.95	$\hat{\lambda}_2$=0.98	$\hat{\delta}_2$=86.36
	(0.02)	(0.05)	(12.26)	(0.01)	(0.04)	(11.92)
DGeP	$\hat{a}$=0.98	$\hat{b}$=0.36	$\hat{\lambda}$=0.68	$\hat{\delta}$=342.23		
	(0.02)	(0.50)	(0.34)	(800.09)		

TABLE 6.26: Parameter estimation for Dataset 20 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=1.07	$\hat{\lambda}$=8.07				
	(0.13)	(1.27)				
GeP	$\hat{a}$=1.02	$\hat{\lambda}$=1.10	$\hat{\delta}$=11.29			
	(0.01)	(0.13)	(3.35)			
EPP	$\hat{\alpha}$=1.02	$\hat{a}$=1.03	$\hat{\beta}$=0.79	$\hat{b}$=1.11		$\hat{\delta}$=15.11
	(1.76)	(0.03)	(0.50)	(0.13)		(20.71)
α-SP	$\hat{a}$=0.17	$\hat{\lambda}$=1.08	$\hat{\delta}$=12.84			
	(0.19)	(0.13)	(7.02)			
TGeP	$\hat{a}_1$=1.01	$\hat{\lambda}_1$=1.08	$\hat{\delta}_1$=8.90	$\hat{a}_2$=1.02	$\hat{\lambda}_2$=1.11	$\hat{\delta}_2$=10.46
	(0.01)	(0.12)	(2.89)	(0.01)	(0.13)	(1.76)
DGeP	$\hat{a}$=1.02	$\hat{b}$=0.23	$\hat{\lambda}$=1.32	$\hat{\delta}$=8.78		
	(0.01)	(0.36)	(0.40)	(3.83)		

TABLE 6.27: Parameter estimation for Dataset 23 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=0.73	$\hat{\lambda}$=189.68				
	(0.10)	(49.67)				
GeP	$\hat{a}$=1.06	$\hat{\lambda}$=0.86	$\hat{\delta}$=451.36			
	(0.02)	(0.12)	(179.63)			
EPP	$\hat{\alpha}$=0.21	$\hat{a}$=1.07	$\hat{\beta}$=0.11	$\hat{b}$=0.89		$\hat{\delta}$=111.66
	(0.30)	(0.02)	(0.22)	(0.13)		(149.54)
α-SP	$\hat{a}$=0.74	$\hat{\lambda}$=0.87	$\hat{\delta}$=1159.79)			
	(0.25)	(0.13)	(762.11)			
TGeP	$\hat{a}_1$=1.02	$\hat{\lambda}_1$=0.82	$\hat{\delta}_1$=468.34	$\hat{a}_2$=1.08	$\hat{\lambda}_2$=0.96	$\hat{\delta}_2$=467.93
	(0.02)	(0.11)	(181.40)	(0.01)	(0.13)	(179.28)
DGeP	$\hat{a}$=1.01	$\hat{b}$=0.50	$\hat{\lambda}$=0.62	$\hat{\delta}$=1895.00		
	(0.06)	(0.25)	(0.15)	(1232.09)		

TABLE 6.28: Parameter estimation for Dataset 24 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=0.63	$\hat{\lambda}$=250.93				
	(0.06)	(49.75)				
GeP	$\hat{a}$=1.04	$\hat{\lambda}$=0.77	$\hat{\delta}$=776.57			
	(0.01)	(0.07)	(201.83)			
EPP	$\hat{\alpha}$=2.85	$\hat{a}$=1.04	$\hat{\beta}$=1.04	$\hat{b}$=0.77		$\hat{\delta}$=2988.49
	(1.29)	(0.01)	(0.02)	(0.07)		(589.19)

α-SP	$\hat{a}$=0.82	$\hat{\lambda}$=0.80	$\hat{\delta}$=3113.00			
	(0.10)	(0.07)	(1211.48)			
TGeP	$\hat{a}_1$=1.09	$\hat{\lambda}_1$=0.65	$\hat{\delta}_1$=780.26	$\hat{a}_2$=1.08	$\hat{\lambda}_2$=0.69	$\hat{\delta}_2$=773.32
	(0.01)	(0.05)	(201.67)	(0.01)	(0.03)	(191.36)
DGeP	$\hat{a}$=1.03	$\hat{b}$=0.33	$\hat{\lambda}$=0.59	$\hat{\delta}$=3211.00		
	(0.01)	(0.06)	(0.07)	(635.06)		

TABLE 6.29: Parameter estimation for Dataset 25 regarding RP, GeP, EPP, TGeP and DGeP

RP	$\hat{a}$=0.97	$\hat{\lambda}$=265.78				
	(0.10)	(38.30)				
GeP	$\hat{a}$=1.01	$\hat{\lambda}$=0.98	$\hat{\delta}$=339.01			
	(0.01)	(0.10)	(94.35)			
EPP	$\hat{\alpha}$=0.12	$\hat{a}$=1.05	$\hat{\beta}$=0.98	$\hat{b}$=0.99		$\hat{\delta}$=320.35
	(0.25)	(0.03)	(0.04)	(0.11)		(144.12)
α-SP	$\hat{a}$=0.15	$\hat{\lambda}$=0.98	$\hat{\delta}$=418.54			
	(0.15)	(0.10)	(204.16)			
TGeP	$\hat{a}_1$=1.02	$\hat{\lambda}_1$=0.99	$\hat{\delta}_1$=326.74	$\hat{a}_2$=1.01	$\hat{\lambda}_2$=0.98	$\hat{\delta}_2$=322.20
	(0.01)	(0.11)	(83.77)	(0.01)	(0.12)	(81.12)
DGeP	$\hat{a}$=1.01	$\hat{b}$=0.03	$\hat{\lambda}$=0.96	$\hat{\delta}$=365.97		
	(0.01)	(0.16)	(0.16)	(202.32)		

6.6 Summary

This chapter compared the performance of the goodness-of-fit of the GeP-like models on 25 real-world datasets. Three of the datasets are not used because they do not meet the conditions of iid. GeP-like models used in this chapter include the renewal process, the geometric process, the extended possion process (EPP), the α-series process, and the threshold geometric process (TGeP). Three performance metrics, AIC (Akaike Information Criterion), AICc (correction of AIC) and ML (Maximum Likelihood) are used to rank the model with the best performance. According to the comparison results, based on the AIC, the best model is the TGeP and the second best is RP. Then in terms of the AICc, the best model is the DRGeP and the second best is the TGeP. Based on the result of the ML, the best model is the TGeP and the second best is the DRGeP. These results will give support for choosing the best model in practical applications. Tables 6.8—6.29 shown the parameter estimation and standard error of each parameter for each dataset.

Besides, this chapter discusses the LSE (least square estimation) as an alternative method to estimate the model parameters. We use the α-series process to be an example and estimate its parameters by the LSE. The asymptotic distributions of its estimators are given to understand its unknown statistical inference, which has not been given by the original authors. The relevant proof of α-series process is given in this chapter. It is worth mentioning that the proofs of asymptotic distributions of other GeP-like models resemble other such proofs of the α-series process. Thus, we do not provide the proofs of other GeP-like models in this thesis. However, this will be one future work.

Chapter 7

Conclusions and future work

This thesis reviewed related existing research and identified three knowledge gaps:

- Multi-failure modes system;
- Maintenance policy and cost process;
- Model performance and statistical properties.

To overcome such gaps, several maintenance policies are proposed for multi-component system. Besides, the statistical properties and impact of the uncertainty of the parameter estimation are investigated in cases of geometric-like processes. This chapter summarizes the objectives and contributions of all previous chapters of this thesis.

7.1 Conclusions

Chapter 1 introduced the significance of the project. Chapter 2 reviewed existing literature and identified knowledge gaps.

Chapter 3 proposed a maintenance policy based on multi-criteria model-selecting for a k-out-of-n. The gamma process was used to fit the degradation process of each component. Besides, a framework of maintenance policies was developed.

Chapter 4 developed maintenance policies with consideration of a cost process with a cost threshold regarding the degradation process of a system. This chapter considered the cost threshold as a constraint of the cost budget for a decision-maker. In fact, in many real-world scenarios, an unlimited budget for repairs or replacements is impossible. Generally, when a system has been used for many years, considering the problems of work efficiency, maintenance cost (which will become more and more expensive as the system is used), and depreciation, decision-makers usually tend to purchase a new system to replace the old one. Therefore, our maintenance policies consider four situations when the degradation process and the cost process reach their threshold at different times. Such maintenance policies solve the problem that as repairs become more frequent, it will cost less to replace the old system with a new one than to continue to maintain it. Under such maintenance policies, the policy that achieves the threshold first (policy based on the degradation process and policy based on the cost process) will be given priority.

The main contribution of Chapter 5 was that it investigated the impact of the estimation uncertainty of the parameters in a model on the related maintenance policy. With the development of geometric process-like models, some of them overcome the drawback that the geometric process can only randomly increase or decrease. For non-monotonic geometric process-like models, the "peak" of the expected value is defined as a turning point. The "valley" of the expected value does not exist, which had been explained in Chapter 5. Parameter estimation will lead to different results of turning points. This leads to the uncertainty of the turning points. Besides, we used the N^* replacement policy to investigate the optimal N^- with minimum average cost per unit time. With the sensitive analysis, two constraints about the N^* were added due to the finite time of operating time for a system in the real world. Therefore, this chapter analysed the impact of the uncertainty of the turning points in the case of the extended Poisson process and evaluated the influence of cost due to the turning point under some conditions of the real world. The sensitive analysis was provided based on six scenarios of different parameters.

The main contribution of Chapter 6 was that comparison of model performance for several geometric process-like models on 25 datasets was performed. It used three performance metrics, which are the Akaike Information Criterion, the corrected Akaike Information Criterion, and the likelihood to show the rank of model performance. Besides, this chapter derived the least square estimation to evaluate the parameters of the α-series process and then discuss its asymptotic distributions.

7.2 Future work

The following research work is needed in our future research:

- The maintenance policy based on the multi-criteria model-selecting of a k-out-of-N system can be improved. In Chapter 3, we optimised the maintenance policy using different criteria for model selection. This maintenance policy can be improved to a m-out-of-M multi-criteria decision making maintenance policy. Under this policy, it has m forms of model where $m = 1, 2, \ldots, M$ and q forms of index where $q = 1, 2, \ldots, Q$. Then, there would be a matrix for multi-criteria decision-making, which can be presented by

	$m=1$	$m=2$	$\ldots$	$m=M$
$q=1$	z_{11}	z_{12}	$\ldots$	z_{1m}
$q=2$	z_{21}	z_{22}	$\ldots$	z_{2m}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$q=Q$	z_{q1}	z_{q2}	$\ldots$	z_{qm}

m-out-of-M means if we sort the matrix in ascending order of m (similar to k-out-of-n), then, the matrix becomes

	$(m)=1$	$(m)=2$	$\dots$	$(m)=M$
$q=1$	$z_{1(m)}$	$z_{1(2)}$	$\dots$	$z_{1(m)}$
$q=2$	$z_{2(1)}$	$z_{2(2)}$	$\dots$	$z_{2(m)}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$q=Q$	$z_{q(1)}$	$z_{q(2)}$	$\dots$	$z_{q(m)}$

The objective function $H_q(x)$ becomes

$$\begin{aligned} \min \quad & H_q(x) = w_1 z_{1(m)} + w_2 z_{2(m)} + \cdots + w_q z_{q(m)} \\ \text{subject to} \quad & h_q(x) \leq s_q, q = 1, 2, \dots, m \end{aligned}$$

where $w_1, w_2, \dots, w_q$. are weight values of each index and q can be maintenance cost, risk of loss or downtime due to all failures in a given time.

- We pointed out there is a lack of analysis the dependence between the components in a system, which is a knowledge gap and needs further development. Similarly, the assumptions of different failure modes are simple. It is necessary to overcome such disadvantages. To overcome this drawback, the following work should be focused:
  1. To review existing research on dependence analysis between the components in a system.
  2. To investigate other methods for risk analysis.
  3. To develop current maintenance policies (from this thesis) and consider the influence of dependence analysis on the long-term maintenance cost.

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Appendices

Other performance measures

Bayesian information criterion (BIC) is a method for evaluating the model performance based on the likelihood function. It is given by

$$BIC = k\ln(n) - 2\ln(\hat{L}),$$

where $\hat{L}$ is the likelihood function of the model, k is the number of parameters and n is the number of observations. A model with lower BIC is preferred.

Bias is a basic statistical method that shows the difference between expectation and the 'true' value of a parameter. Denote $\hat{\theta}$ as an estimator of parameter θ, then the bias is given by

$$bias(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta.$$

If $\mathbb{E}[\hat{\theta}] = 0$, then $\hat{\theta}$ is unbiased for θ.

Mean squared error (MSE) shows the average of the squares of the error for a model. Normally, it is given by

$$MSE = \frac{1}{n}\sum_{k=1}^{n}(X_k - \hat{X}_k)^2.$$

Normally, a model with a smaller MSE is preferred.

Mean percentage error (MPE) is the average of percentage errors between the actual value and the estimated value, it is given by

$$MPE = \frac{100\%}{n}\sum_{k=1}^{n}\frac{a_k - f_k}{a_k},$$

where a_k is the actual value of the kth observation and f_k is the estimated value of the kth observation. A model with a smaller absolute of MPE is better.

Least squared error is also a common method for estimating the model performance which aims to minimize the sum of the squares of errors between the observed value and the estimated value. The sum of the squares of errors (SSE) is given by

$$SSE = min\sum_{k=1}^{n}(X_k - \hat{X}_k)^2,$$

where $\hat{X}_k$ is the estimator of the observed value X_k.

Adjusted mean squared error is the sum of the MSE and a penalized term of a scalar multiple of the number based on the number of parameters. It is given by

$$AMSE = \frac{1}{2}\ln\frac{S_n}{n} + \frac{1}{n}\sum_{k=1}^{n}(X_k - \hat{X}_k)^2,$$

where S_n is the sum of observations.

Deviance information criterion (DIC) is a generalization of the AIC, which is also normally used for a large sample. It can be presented by

$$DIC = p_D + \overline{D(\theta)},$$

where $D(\theta)$ is the logarithm of likelihood function and p_D is one half of the variance of $D(\theta)$. Similar to the AIC, a model with a smaller DIC is preferred.

Absolute percentage bias

$$APB = \frac{\sum_{k=1}^{n}|\frac{X_k - \hat{X}_k}{X_k}|}{\sum^n k = 1X_k},$$

where X_k is the observed value and $\hat{X}_k$ is the forecasted value.

Root mean square error (RMSE) is the standard deviation of the prediction errors. It is given by

$$RMSE = \sqrt{\frac{\sum_{k=1}^{n}(X_k - \hat{X}_k)^2}{n}}. \quad (7.1)$$

A model with a lower RMSE is preferred.

Fisher information matrix

TABLE 7.3: Fisher information matrix for RP

Dataset 1	1	2	Dataset 13	1	2
1	0.02	1.13	1	0.03	0.79
2	1.13	676.43	2	0.79	199.98
Dataset 2	1	2	Dataset 14	1	2
1	0.02	1.48	1	0.01	0.16
2	1.48	877.64	2	0.16	22.82
Dataset 3	1	2	Dataset 15	1	2
1	0.02	1.67	1	0.01	0.25
2	1.67	1261.58	2	0.25	44.56
Dataset 4	1	2	Dataset 16	1	2
1	0.02	0.98	1	0.05	0.26
2	0.98	429.13	2	0.26	12.87
Dataset 5	1	2	Dataset 17	1	2
1	0.02	1.20	1	0.01	0.01

2	1.20	725.65	2	0.01	0.12
Dataset 6	1	2	Dataset 18	1	2
1	0.03	1.69	1	0.00	0.90
2	1.69	948.01	2	0.90	4259.00
Dataset 7	1	2	Dataset 19	1	2
1	0.01	0.43	1	0.00	0.23
2	0.43	149.11	2	0.23	201.15
Dataset 9	1	2	Dataset 20	1	2
1	0.59	0.02	1	0.02	0.05
2	0.02	0.01	2	0.05	1.60
Dataset 10	1	2	Dataset 23	1	2
1	0.03	0.82	1	0.01	1.55
2	0.82	179.58	2	1.55	2467.24
Dataset 11	1	2	Dataset 24	1	2
1	0.01	0.48	1	0.00	0.92
2	0.48	152.77	2	0.92	2475.07
Dataset 12	1	2	Dataset 25	1	2
1	0.03	0.77	1	0.01	1.20
2	0.77	206.39	2	1.20	1467.09

TABLE 7.4: Fisher information matrix for GeP

Dataset 1	1	2	3	Dataset 13	1	2	3
1	0.00	0.00	2.64	1	0.00	0.00	0.71
2	0.00	0.02	2.41	2	0.00	0.03	0.52
3	2.64	2.41	7282.82	3	0.71	0.52	690.47
Dataset 2	1	2	3	Dataset 14	1	2	3
1	0.00	0.00	1.06	1	0.00	0.00	0.05
2	0.00	0.03	2.20	2	0.00	0.01	0.22
3	1.06	2.20	3349.09	3	0.05	0.22	124.92
Dataset 3	1	2	3	Dataset 15	1	2	3
1	0.00	0.00	1.99	1	0.00	0.00	0.12
2	0.00	0.03	2.76	2	0.00	0.01	0.39
3	1.99	2.76	11324.59	3	0.12	0.39	240.79
Dataset 4	1	2	3	Dataset 16	1	2	3
1	0.00	0.00	0.61	1	0.00	0.00	0.37
2	0.00	0.02	0.91	2	0.00	0.17	-0.12
3	0.61	0.91	1368.36	3	0.37	-0.12	171.39
Dataset 5	1	2	3	Dataset 17	1	2	3
1	0.00	0.00	1.44	1	0.00	0.00	0.00
2	0.00	0.02	2.00	2	0.00	0.01	0.01

3	1.44	2.00	4634.09	3	0.00	0.01	0.27
Dataset 6	1	2	3	Dataset 18	1	2	3
1	0.00	0.00	1.22	1	0.00	0.00	0.04
2	0.00	0.03	1.68	2	0.00	0.00	0.19
3	1.22	1.68	2985.27	3	0.04	0.19	350.10
Dataset 7	1	2	3	Dataset 19	1	2	3
1	0.00	0.00	0.10	1	0.00	0.00	0.01
2	0.00	0.01	0.52	2	0.00	0.00	0.16
3	0.10	0.52	602.05	3	0.01	0.16	159.69
Dataset 9	1	2	3	Dataset 20	1	2	3
1	0.00	0.00	0.00	1	0.00	0.00	0.04
2	0.00	0.27	0.01	2	0.00	0.02	0.08
3	0.00	0.01	0.05	3	0.04	0.08	11.25
Dataset 10	1	2	3	Dataset 23	1	2	3
1	0.00	0.00	0.39	1	0.00	0.00	3.54
2	0.00	0.04	0.50	2	0.00	0.02	5.82
3	0.39	0.50	412.51	3	3.54	5.82	32265.70
Dataset 11	1	2	3	Dataset 24	1	2	3
1	0.00	0.00	0.66	1	0.00	0.00	1.02
2	0.00	0.02	0.80	2	0.00	0.01	1.92
3	0.66	0.80	1312.07	3	1.02	1.92	40736.64
Dataset 12	1	2	3	Dataset 25	1	2	3
1	0.00	0.00	0.44	1	0.00	0.00	0.72
2	0.00	0.03	0.60	2	0.00	0.01	1.20
3	0.44	0.60	516.92	3	0.72	1.20	8902.50

TABLE 7.5: Fisher information matrix for EGeP

Dataset 1	1	2	3	4	5
1	13.73	11.35	0.42	0.13	1276.00
2	11.35	11.14	0.36	0.11	1076.57
3	0.42	0.36	0.01	0.00	42.03
4	0.13	0.11	0.00	0.02	15.07
5	1276.00	1076.57	42.03	15.07	126495.82
Dataset 2	1	2	3	4	5
1	28.18	0.02	0.02	0.05	4484.23
2	0.02	0.00	0.00	0.00	0.01
3	0.02	0.00	0.01	0.00	0.01
4	0.05	0.00	0.00	0.03	0.01
5	4484.23	0.01	0.01	0.01	757580.30
Dataset 3	1	2	3	4	5

1	6.33	0.02	0.04	0.03	1733.46
2	0.02	0.00	0.00	0.00	0.93
3	0.04	0.00	0.53	0.00	37.21
4	0.03	0.00	0.00	0.03	0.41
5	1733.46	0.93	37.21	0.41	565677.80
Dataset 4	1	2	3	4	5
1	82.20	0.01	0.01	0.02	8977.90
2	0.01	0.00	0.00	0.00	0.00
3	0.01	0.00	0.00	0.00	0.01
4	0.02	0.00	0.00	0.02	0.01
5	8977.90	0.00	0.01	0.01	986172.20
Dataset 5	1	2	3	4	5
1	19.85	0.02	0.02	0.03	3455.92
2	0.02	0.00	0.00	0.00	0.00
3	0.02	0.00	0.01	0.00	0.00
4	0.03	0.00	0.00	0.02	0.01
5	3455.92	0.00	0.00	0.01	624084.40
Dataset 6	1	2	3	4	5
1	61.26	0.01	0.01	0.01	9137.37
2	0.01	0.26	0.00	0.00	0.08
3	0.01	0.00	0.00	0.00	0.00
4	0.01	0.00	0.00	0.03	0.02
5	9137.37	0.08	0.00	0.02	1365939.00
Dataset 7	1	2	3	4	5
1	0.41	0.57	0.00	0.05	23.49
2	0.57	0.03	0.00	0.00	0.15
3	0.00	0.00	0.00	0.00	0.11
4	0.05	0.00	0.00	0.01	0.52
5	23.49	0.15	0.11	0.52	708.24
Dataset 9	1	2	3	4	5
1	0.04	0.00	0.00	0.00	0.02
2	0.00	0.67	0.00	0.00	0.03
3	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.28	0.01
5	0.02	0.03	0.00	0.01	0.06
Dataset 10	1	2	3	4	5
1	0.00	0.01	0.00	0.00	0.26
2	0.01	0.06	0.00	0.01	1.27
3	0.00	0.00	0.00	0.00	0.61
4	0.00	0.01	0.00	0.03	0.75

5	0.26	1.27	0.61	0.75	342.34
Dataset 11	1	2	3	4	5
1	0.10	0.01	0.25	0.02	2.92
2	0.01	0.00	0.00	0.00	0.22
3	0.25	0.00	0.07	0.01	26.21
4	0.02	0.00	0.01	0.02	2.58
5	2.92	0.22	26.21	2.58	118.01
Dataset 12	1	2	3	4	5
1	0.02	0.10	0.01	0.00	0.02
2	0.10	0.42	0.04	0.00	0.51
3	0.01	0.04	0.01	0.00	0.88
4	0.00	0.00	0.00	0.03	0.62
5	0.02	0.51	0.88	0.62	572.58
Dataset 13	1	2	3	4	5
1	0.00	0.01	0.00	0.00	0.00
2	0.01	0.05	0.01	0.00	0.15
3	0.00	0.01	0.00	0.00	1.16
4	0.00	0.00	0.00	0.03	0.16
5	0.00	0.15	1.16	0.16	551.02
Dataset 14	1	2	3	4	5
1	2.68	0.08	0.00	0.01	6.16
2	0.08	0.23	0.00	0.00	0.65
3	0.00	0.00	0.00	0.00	0.06
4	0.01	0.00	0.00	0.01	0.24
5	6.16	0.65	0.06	0.24	148.72
Dataset 15	1	2	3	4	5
1	11.22	0.04	0.13	0.00	645.51
2	0.04	0.00	0.00	0.00	2.24
3	0.13	0.00	0.03	0.00	3.20
4	0.00	0.00	0.00	0.01	0.59
5	645.51	2.24	3.20	0.59	38713.08
Dataset 16	1	2	3	4	5
1	0.00	0.00	0.00	0.00	0.00
2	0.00	0.01	0.00	0.02	0.10
3	0.00	0.00	0.00	0.00	0.17
4	0.00	0.02	0.00	0.27	0.57
5	0.00	0.10	0.17	0.57	42.32
Dataset 17	1	2	3	4	5
1	1.35	0.06	0.00	0.00	0.32
2	0.06	0.02	0.00	0.00	0.05
3	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.01	0.01

5	0.32	0.05	0.00	0.01	0.62
Dataset 18	1	2	3	4	5
1	47.31	11.41	0.00	0.03	19.99
2	11.41	3.56	0.00	0.01	10.57
3	0.00	0.00	0.00	0.00	0.05
4	0.03	0.01	0.00	0.00	0.18
5	19.99	10.57	0.05	0.18	413.88
Dataset 19	1	2	3	4	5
1	54.74	0.00	0.00	0.01	4661.15
2	0.00	0.00	0.00	0.00	0.01
3	0.00	0.00	0.00	0.00	0.02
4	0.01	0.00	0.00	0.00	0.01
5	4661.15	0.01	0.02	0.01	398742.70
Dataset 20	1	2	3	4	5
1	3.09	0.03	0.14	0.00	33.36
2	0.03	0.00	0.01	0.00	0.16
3	0.14	0.01	0.25	0.00	2.03
4	0.00	0.00	0.00	0.02	0.03
5	33.36	0.16	2.03	0.03	428.94
Dataset 23	1	2	3	4	5
1	0.09	0.00	0.01	0.00	43.36
2	0.00	0.00	0.00	0.00	0.40
3	0.01	0.00	0.05	0.00	6.78
4	0.00	0.00	0.00	0.02	0.42
5	43.36	0.40	6.78	0.42	22363.47
Dataset 24	1	2	3	4	5
1	1.67	0.01	0.01	0.01	446.97
2	0.01	0.00	0.00	0.00	0.00
3	0.01	0.00	0.00	0.00	0.00
4	0.01	0.00	0.00	0.01	0.00
5	446.97	0.00	0.00	0.00	347141.40
Dataset 25	1	2	3	4	5
1	0.06	0.01	0.00	0.00	18.55
2	0.01	0.00	0.00	0.00	2.26
3	0.00	0.00	0.00	0.00	2.05
4	0.00	0.00	0.00	0.01	1.45
5	18.55	2.26	2.05	1.45	20770.22

TABLE 7.6: Fisher information matrix for Alfa

Dataset 1	1	2	3	Dataset 13	1	2	3

1	0.10	0.00	64.09	1	0.06	0.00	7.90
2	0.00	0.02	3.22	2	0.00	0.03	0.11
3	64.09	3.22	44611.63	3	7.90	0.11	1119.73
Dataset 2	1	2	3	Dataset 14	1	2	3
1	0.03	0.00	19.55	1	0.02	0.00	4.33
2	0.00	0.03	2.82	2	0.00	0.01	0.38
3	19.55	2.82	14227.50	3	4.33	0.38	1278.68
Dataset 3	1	2	3	Dataset 15	1	2	3
1	0.04	0.00	50.36	1	0.02	0.00	4.63
2	0.00	0.03	4.87	2	0.00	0.01	0.46
3	50.36	4.87	70570.65	3	4.63	0.46	980.56
Dataset 4	1	2	3	Dataset 16	1	2	3
1	0.04	0.00	13.23	1	0.03	0.00	2.69
2	0.00	0.02	0.68	2	0.00	0.14	1.25
3	13.23	0.68	5863.35	3	2.69	1.25	311.73
Dataset 5	1	2	3	Dataset 17	1	2	3
1	0.06	0.00	37.28	1	0.01	0.00	0.05
2	0.00	0.02	2.60	2	0.00	0.01	0.01
3	37.28	2.60	26809.37	3	0.05	0.01	0.49
Dataset 6	1	2	3	Dataset 18	1	2	3
1	0.05	0.00	20.98	1	0.01	0.00	0.47
2	0.00	0.03	1.55	2	0.00	0.00	0.02
3	20.98	1.55	9898.17	3	0.47	0.02	28.25
Dataset 7	1	2	3	Dataset 19	1	2	3
1	0.01	0.00	6.38	1	0.00	0.00	0.97
2	0.00	0.01	0.68	2	0.00	0.00	0.11
3	6.38	0.68	3300.46	3	0.97	0.11	205.47
Dataset 9	1	2	3	Dataset 20	1	2	3
1	0.00	0.00	0.01	1	0.04	0.00	1.28
2	0.00	0.63	0.04	2	0.00	0.02	0.10
3	0.01	0.04	0.07	3	1.28	0.10	49.30
Dataset 10	1	2	3	Dataset 23	1	2	3
1	0.04	0.01	5.00	1	0.06	0.00	179.65
2	0.01	0.03	0.40	2	0.00	0.02	20.85
3	5.00	0.40	832.96	3	179.65	20.85	580813.98
Dataset 11	1	2	3	Dataset 24	1	2	3
1	0.07	0.00	28.64	1	0.01	0.00	133.11
2	0.00	0.02	1.46	2	0.00	0.01	12.90
3	28.64	1.46	13392.15	3	133.11	12.90	1467693.00

Dataset 12	1	2	3	Dataset 25	1	2	3
1	0.04	0.00	5.40	1	0.02	0.00	29.56
2	0.00	0.03	0.49	2	0.00	0.01	1.09
3	5.40	0.49	866.76	3	29.56	1.09	41680.39

TABLE 7.7: Fisher information matrix for DGeP

Dataset 1	1	2	3	4
1	0.00	-0.01	0.01	-1.36
2	-0.01	0.11	-0.07	42.90
3	0.01	-0.07	0.06	-23.72
4	-1.36	42.90	-23.72	26568.57
Dataset 2	1	2	3	4
1	0.01	-0.03	0.02	-18.13
2	-0.03	0.12	-0.06	90.95
3	0.02	-0.06	0.05	-42.95
4	-18.13	90.95	-42.95	110416.08
Dataset 3	1	2	3	4
1	0.00	-0.01	0.01	-7.17
2	-0.01	0.09	-0.06	82.90
3	0.01	-0.06	0.07	-54.32
4	-7.17	82.90	-54.32	95000.61
Dataset 4	1	2	3	4
1	0.00	-0.02	0.01	-4.71
2	-0.02	0.11	-0.06	36.52
3	0.01	-0.06	0.05	-19.32
4	-4.71	36.52	-19.32	19504.23
Dataset 5	1	2	3	4
1	0.00	-0.01	0.01	-2.85
2	-0.01	0.09	-0.06	45.16
3	0.01	-0.06	0.05	-25.13
4	-2.85	45.16	-25.13	33060.30
Dataset 6	1	2	3	4
1	0.01	-0.05	0.03	-18.51
2	-0.05	0.24	-0.14	114.39
3	0.03	-0.14	0.10	-62.14
4	-18.51	114.39	-62.14	68489.92
Dataset 7	1	2	3	4
1	0.00	0.00	0.00	-0.84
2	0.00	0.05	-0.05	24.82
3	0.00	-0.05	0.06	-23.43

4	-0.84	24.82	-23.43	13271.48
Dataset 9	1	2	3	4
1	0.00	-0.01	0.00	-0.01
2	-0.01	0.03	-0.09	0.10
3	0.00	-0.09	0.55	-0.29
4	-0.01	0.10	-0.29	0.45
Dataset 10	1	2	3	4
1	0.01	-0.02	0.02	-1.41
2	-0.02	0.13	-0.11	12.47
3	0.02	-0.11	0.12	-9.48
4	-1.41	12.47	-9.48	1773.21
Dataset 11	1	2	3	4
1	0.00	-0.01	0.01	-0.99
2	-0.01	0.15	-0.09	28.97
3	0.01	-0.09	0.08	-17.66
4	-0.99	28.97	-17.66	7207.29
Dataset 12	1	2	3	4
1	0.00	-0.01	0.01	-0.59
2	-0.01	0.09	-0.07	8.35
3	0.01	-0.07	0.09	-6.28
4	-0.59	8.35	-6.28	1263.45
Dataset 13	1	2	3	4
1	0.00	-0.01	0.01	-0.14
2	-0.01	0.09	-0.09	3.87
3	0.01	-0.09	0.15	-3.64
4	-0.14	3.87	-3.64	282.12
Dataset 14	1	2	3	4
1	0.00	0.00	0.00	-0.78
2	0.00	0.11	-0.09	24.53
3	0.00	-0.09	0.08	-19.23
4	-0.78	24.53	-19.23	5952.27
Dataset 15	1	2	3	4
1	0.00	0.00	0.00	-0.20
2	0.00	0.06	-0.05	8.00
3	0.00	-0.05	0.06	-6.71
4	-0.20	8.00	-6.71	1408.35
Dataset 16	1	2	3	4
1	0.00	0.00	0.00	0.09
2	0.00	0.03	-0.04	0.90
3	0.00	-0.04	0.31	-1.03
4	0.09	0.90	-1.03	85.27

Dataset 17	1	2	3	4
1	0.00	0.00	0.00	0.00
2	0.00	0.16	-0.17	0.49
3	0.00	-0.17	0.18	-0.52
4	0.00	0.49	-0.52	1.90
Dataset 18	1	2	3	4
1	0.00	0.00	0.00	-0.10
2	0.00	0.03	-0.04	3.66
3	0.00	-0.04	0.05	-3.97
4	-0.10	3.66	-3.97	457.28
Dataset 19	1	2	3	4
1	0.00	0.01	0.01	13.66
2	0.01	0.25	0.17	402.82
3	0.01	0.17	0.12	276.36
4	13.66	402.82	276.36	640139.18
Dataset 20	1	2	3	4
1	0.00	0.00	0.00	0.00
2	0.00	0.13	0.13	1.14
3	0.00	0.13	0.16	1.12
4	0.00	1.14	1.12	14.66
Dataset 23	1	2	3	4
1	0.00	0.01	0.01	28.79
2	0.01	0.06	0.03	225.47
3	0.01	0.03	0.02	91.27
4	28.79	225.47	91.27	1518048.00
Dataset 24	1	2	3	4
1	0.00	0.00	0.00	0.69
2	0.00	0.00	0.00	24.70
3	0.00	0.00	0.00	12.51
4	0.69	24.70	12.51	403303.60
Dataset 25	1	2	3	4
1	0.00	0.00	0.00	0.85
2	0.00	0.02	0.02	26.76
3	0.00	0.02	0.03	19.56
4	0.85	26.76	19.56	40932.73