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The Key Class in Networks[☆]

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ABSTRACT

This paper examines optimal targeting of multiple network players from a new perspective, focusing on classes of players holding similar network positions – and thus fulfilling similar network roles – as captured by the graph theoretic notion of equitable partition. Unlike existing centrality measures, we show that analysing the network game with local payoff complementarities under symmetry brings out new insights about the relative influence of classes of similarly positioned network players on the Nash equilibrium activity. Our analysis introduces two novel class-based centrality measures with broad theoretical and empirical applicability that geometrically characterise the key class whose removal results in the maximal reduction of aggregate and per-capita network activity, respectively.

1. Introduction

A key area of focus in the analysis of social networks concerns developing measures of network centrality to identify which agents are the most *important* in the network according to some desired criteria, owing to the ubiquity of social networks and the central role they play in influencing agents' behaviour. Beyond the traditional off-the-shelf centrality measures that derive from the network's topological properties (such as degree, betweenness, or eigenvector centrality), it is also of interest for the social planner to consider centrality measures that derive from strategic interactions among network players, especially when targeting the overall activity in the network in a Nash equilibrium. Studying network games with payoff externalities due to strategic complementarities among players, Ballester et al. (2006), in their seminal paper, develop the 'intercentrality' measure to characterise the key player whose removal results in the maximum disruption to the aggregate equilibrium activity.[1] In addition to targeting a *single* key player as in Ballester et al. (2006), it may be of interest for the social planner to identify *multiple* key players to be eliminated for maximally reducing the overall network activity, such as crime (or, equivalently, to be preserved for optimally increasing the network output, for example, in R&D or financial networks, through steps like bailouts). However, Ballester et al. (2010), while extending the key-player problem to its group analogue, prove that the key-group problem is NP-hard, meaning it cannot be solved by any possible algorithm in reasonable (polynomial) time, as stated in their Proposition 5.

In this paper, we propose an alternative strategy to the key group problem, for targeting multiple players in a principled way, by exploiting network symmetries and focusing on classes of players occupying similar network positions. Indeed, it is well-known in various organisational contexts, such as those characterised by hierarchical structures like crime networks or mafia organisations, that players' network positions determine the roles they fulfil within the organisation.[2] Consequently, rather than targeting the

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[1] Zenou (2016) provides a comprehensive review of empirical applications of the key player measure, including in criminal, R&D, educational, and financial networks, etc.

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key group of criminals, a sensible alternate policy for the social planner is to identify the key class of criminals – a set of similar players in terms of their network positions and ties, and in turn, their roles and contributions within the network – whose removal together results in the maximal reduction in the overall network activity. Targeting players of a particular class may require very different capabilities than going after a disparate group of network players, and potentially offer economies of scale, which may be preferred by the social planner. Perhaps a possible illustration of this alternative approach can be the recent and significant shift towards the *hugs, not bullets* security strategy by the Mexican government in its long-standing war on drugs. Instead of targeting the figureheads of cartels (the key group in the network), their focus turned to addressing the underlying roots by lifting low-ranking cartel associates (the key class of peripheral players) out of poverty, aiming to diminish the attractiveness for them to engage in criminal activities.

In order to develop our key class analysis, we employ a graph-theoretic approach exploiting network (graph) symmetries to construct the network's class structure, followed by a game-theoretic analysis to identify the key class of players whose removal maximally reduces the equilibrium network activity. Accounting for the symmetry in players' network positions in analysing their equilibrium behaviour is a special case of the more general set-up of symmetric games as considered in Plan (2023). Studying the relative influence of the symmetry-based classes on the equilibrium network outcome can be informative for several situations. An important area of application, for instance, is for the class of networks called "overlapping hierarchies", as defined in Sadler (2022), which displays an inherent hierarchical class structure. It includes the 'hierarchical communities' considered in Belhaj and Deroïan (2010) as well as the key network structure of nested-split graphs, which is well-recognised in economics. Indeed, nestedness covers various empirical applications like in criminal and R&D networks, for which key class identification can be informative, as we discuss in Section 2 (among other potential applications beyond nestedness). In fact, it is interesting to note that for nested split graphs, the key class, when made of players with the most links, also turns out to be the key group of players (see Remark 5 in Section 5). We now describe the graph-theoretic and game-theoretic components of our key class analysis in more detail.

Network classes are modelled via the concept of *equitable partition*: players are sorted into classes, wherein *class* refers to cells of the equitable partition of the network. As defined in Powers and Sulaiman (1982), equitable partition requires that all players in a class have the same number of links amongst themselves, and with members of other classes. It generalises the well-known feature of network symmetry, which characterises the structural invariance of networks when certain nodes are interchanged (see Remark 1 in Section 3 for an illustration). Indeed, Xiao et al. (2008) call symmetry a "universal structural property of complex networks" and develop the statistical framework for reproducing the symmetry found in real networks, along the lines of the random network model by Newman et al. (2001).[3] Furthermore, players in a class have identical Bonacich centrality, a measure of their network embeddedness.[4] In the context of network games, the Bonacich–Nash linkage obtained in Ballester et al. (2006) assumes importance: they prove that players' Bonacich centrality is proportional to their equilibrium strategic behaviour. Hence, an equitable partitioned network reflects a society divided into classes of players who enjoy the same influence in the society and adopt identical actions in equilibrium, thus, in a related sense, having similar network roles.

The network game is modelled using linear–quadratic utilities with bilateral externalities, as introduced in Ballester et al. (2006), such that there exists strategic complementarity of efforts between pairs of linked players.[5] Considering the network game under equitable partitioning, rather than the original network game, brings out new insights about the relative influence of groups formed by similarly positioned players, which is used to characterise two class-based centrality measures. To do this, we establish and exploit a relationship between the graph representing the overall network with the graph of its equitable partitioning, the so-called quotient graph, to show that the aggregate equilibrium activity of classes is related to their position within the network. This result is the class analogue of the key Bonacich–Nash linkage established in Ballester et al. (2006) and forms the basis for the two class-based centrality measures proposed in this paper. The first is the *class-centrality* index, to identify the most important class whose removal results in maximal disruption in the overall network outcome. At first glance, it may seem intuitive to think that this measure would select the class with the most players as the key class. However, this is not always the case, since class-centrality reflects two kinds of effects that removing a class has on the aggregate network outcome. The first is the direct effect due to lesser contributing members in the resulting network after removing a class. But in addition, there is also the indirect effect due to a change in the network architecture which alters the peer influences and their intensity, as the links get altered within and across classes. For instance, if the largest class has few direct links with other classes and most indirect links in the network do not pass through it, then it may not be the key class, especially if the indirect links in the network are strong (high attenuation factor). Moreover, the index is relevant if there are more than one class of the largest size.

[2] Role similarity in social networks arising from players' similar positions and interaction patterns is well-established; see, for instance, Wasserman and Faust (1994) or Jin et al. (2011) for a review.

[3] A well-noted source of network symmetry is the presence of tree-like regions common in large real networks, arising from the network growth process via identical branches growing from the same vertices (nodes). Such tree-like structures are known to have symmetry with almost absolute certainty, as proven in Erdős and Rényi (1963), thus, lending a certain degree of symmetry to most real-world networks. Symmetry is, however, not limited to networks with trees alone, as noted in Xiao et al. (2008).

[4] Bonacich centrality counts the total number of paths in the network originating from a node, discounted by their length. One of its most well-known applications is the "PageRank" algorithm of Google search engine for ranking webpages; see, for instance, the discussion on hub-based Katz centrality (another name for Bonacich centrality) in Sargent and Stachurski (2024).

[5] Linear–quadratic utilities are used to model various social and economic phenomena. See e.g., Calvó-Armengol et al. (2009) who study effect of peer influence on education outcomes in friendship network, Liu et al. (2012) for criminal networks, or Goyal and Moraga-Gonzalez (2001) for R & D collaboration among Cournot competitors.

Class size, which is a model primitive for any given class structure within a network, can still be an important factor for implementing targeting policies. In practice, this is not as restrictive as it may seem: any given network typically displays multiple class structures (in addition to the unique 'coarsest' equitable partition — see Remark 2 in Section 3.1), so that the planner has sufficient flexibility in targeting different-sized classes, as we discuss in Section 6.4. Additionally, we also propose a second measure aimed at reducing the average network activity, as a 'size-sensitive' alternative to class-centrality, which selects a class typically smaller than the one with the highest class-centrality, for any given class structure. This is the *per-capita class-centrality* that characterises the class whose removal reduces the per capita network activity by the most, which can be informative, say in presence of planner's resource limitations. A choice of the appropriate class-based centrality measure between the two will ultimately depend on the planner's preference.

The rest of the paper is organised as follows. Section 2 further motivates the relevance of targeting a network class, Section 3 describes the network model and its equitable partition, and Section 4 provides the Nash equilibrium analysis for class activity. Section 5 introduces the class-based centrality measures, which are illustrated through examples and real-world applications in Sections 6 and 7, respectively. Section 8 concludes the paper. All proofs are presented in Appendix A and Appendix B illustrates how equitable partition relates to role equivalence in networks.

2. Relevance of targeting a network class

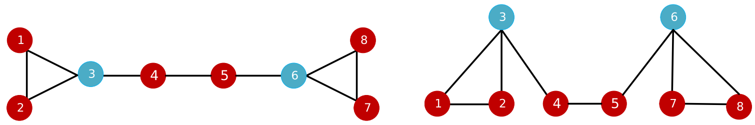
A natural question that may arise is why we should target all players in a class rather than, say, targeting the key group in which players are not constrained by any underlying class structure. In this Section, we provide more intuition for empirical and theoretical applications where our class-based centrality measures may be useful.

Note first that there is an obvious implementation advantage of identifying the key class over the key group, whenever the planner wishes to target multiple network players: the key group problem is classified as NP-hard and thus not computationally tractable, while key class identification is applicable for any generic network based on identifying its underlying (non-trivial) equitable partition, which can be found in polynomial-time using well-established algorithms (see Remark 2 in Section 3.1).

Beyond computational aspects, key class identification offers a principled alternative approach for targeting multiple network players, which can be informative especially since several economic and social networks naturally display class-based structures. As mentioned previously, an important class of networks in this regard is Sadler (2022)'s 'overlapping hierarchies'. Note that in terms of ordinal centralities, the 'weak centrality' measure of Sadler (2022) produces a total order of nodes for overlapping hierarchies graphs, and it ranks all players in a class the same. But beyond focusing on individual players, given the inherent class structure present in these networks, a class-based analysis for identifying the key class for optimally influencing the equilibrium network outcome becomes relevant for the social planner's targeting policy. In particular, overlapping hierarchies generalise the more familiar class of nested split graphs, found in several real-world networks like criminal and R&D organisations which display class-based hierarchies.[6] A key focus for criminal networks, in fact, has been on identifying the key player to be removed for maximally reducing criminal activity (see, for instance, Lee et al. (2021) or Liu et al. (2012)). Extending this to targeting the key class will, of course, lead to a larger disruption of the criminal activity. The ensuing disruption is also likely to be more stable, in the sense that replacing a single player such as the head of a criminal organisation may be easier than replacing an entire class of players who had well-established interaction patterns (and relatedly, roles) within the criminal network. Moreover, going after a homogeneous class of similarly-positioned criminals can be relevant for the social planner in that it offers a straightforward and methodical approach for targeting multiple criminals in the network, as against eliminating a disparate group of crime figureheads, especially if it is difficult to go after such a group. For instance, going after the leaders of a mafia organisation may require very different capabilities than, say, going after a class of similarly-connected, homogeneous lower level criminals, in addition to issues in identifying the disparate group of the organisation's figureheads arising from algorithmic considerations, as discussed previously. Other than 'eliminating' players, key class identification is also relevant for targeting the players to be 'preserved'. For instance, identifying the key class of firms who are most crucial to their industry – in the sense that a break-up of their well-established connections with other similarly-positioned firms will cause the maximum disruption in the total activity for the remaining firms – is relevant for deciding policies like bailout (similar to König et al. (2019)'s study on R&D networks).

The notion of equitable partition, however, is general enough to include grouping structures other than nestedness. An interesting example relates to epidemics diffusion in networks. Most studies for modelling epidemics diffusion use equitable partitioning for clustering individuals into communities made of homogeneous individuals who are exchangeable or indistinguishable among themselves, since epidemics typically spread over extremely large contact network of individuals, such that modelling the dynamics of the disease's spread at an individual level becomes computationally prohibitive (see, for instance, Bonaccorsi et al. (2015) and Ottaviano et al. (2018)). In addition to tractability in modelling achieved through a dimensionality reduction, given the sheer size of the networks over which infectious diseases spread, controlling its spread via targeting the key community of homogeneous and densely-connected individuals, instead of a solo individual, has appeal from a practically implementable policy perspective like isolation or lock-down measures. Other than diffusion of epidemics, Banerjee et al. (2013) note that Ballester et al. (2006)'s key player index analytically informs the choice of 'initial injection points' for information diffusion (in their case about microfinance).

[6] Nestedness in networks refer to neighbourhoods of players of lower degree to be contained in the neighbourhoods of higher degree players. See König et al. (2014) for an excellent discussion on nestedness, including its theoretical modelling and empirical evidence in banking and trade networks. Moreover, studying network formation with strategic complementarities in efforts, like in criminal activity or R&D expenditures, Hiller (2017) shows that Nash equilibrium networks are nested split graphs.



(a). Equitable, but not orbit partition

(b). Equitable partition: role equivalence

Fig. 1. McKay's Graph.

Additionally accounting for the community structure within a large population to identify the key community for maximally advancing the reach of information can be crucial, for instance in settings like studying voter behaviour; Ward (2021) notes the benefit for studying voting behaviour via symmetry based dimensionality reduction techniques.

Key class identification in equitable partitioned network can be significant for various theoretical applications as well. Examples include Allouch (2017), who considers segregated group membership-based interaction in studying welfare effects of income redistribution on the private provision of public goods in social networks. Moreover, it can contribute to the planner's problem of optimal network formation, similar to Belhaj et al. (2013)'s search for efficient networks, by suggesting which class to target so as to optimally alter the group-based structure for attaining desired network outcome.

3. The network model and graph theoretic concepts

We consider a network $\mathbf{g}$ of n players. The associated $(0,1)$-adjacency matrix is denoted by $\mathbf{G} = [g_{ij}]$, where g_{ij} represents unweighted and undirected connection between players i and j; for $i \neq j$, it takes value of 1 if there is a link between the corresponding two nodes in the network, and 0 otherwise. Further, $g_{ii} = 0$, meaning there are no loops in $\mathbf{g}$, and multiple links between any two nodes are ruled out by construction. Note that $\mathbf{G}^k$ represents the number of paths of length k between any two nodes in the network; its elements are denoted by $g_{ij}^{[k]}$.

3.1. Equitable partition

Consider an equitable partition of the network $\mathbf{g}$ into m classes $\{V_1, \dots, V_m\}$, $m \leq n$: for every $i, j \in 1, \dots, m$ there is a non-negative integer π_{ij} such that each node in V_i has exactly π_{ij} neighbours in V_j. An equitable partition results in a quotient graph $\boldsymbol{\pi}$ and the corresponding m-square quotient matrix is represented by $\boldsymbol{\Pi} = [\pi_{ij}]$. Note that unlike $\mathbf{G}$, the quotient matrix $\boldsymbol{\Pi}$ need not be symmetric. Denote the $(n \times m)$ indicator matrix by $\mathbf{X} = [X_{ij}]$, such that $X_{ij} = 1$ if vertex i is in the class V_j, and 0 otherwise. Let the number of members in a class V_i be denoted by r_i, such that, denoting the $(n \times 1)$ vector of ones by $\mathbf{1}_n$, the vector $\mathbf{r} = \mathbf{X}^T.\mathbf{1}_n$ lists the number of members in each class. The following property holds by definition:

$$\mathbf{G}\mathbf{X} = \mathbf{X}\boldsymbol{\Pi} \tag{3.1}$$

Also, the adjacency matrix $\mathbf{G}$ and the quotient matrix of its equitable partition, $\boldsymbol{\Pi}$, have the same spectral radius.[7] That is, if $\rho(\mathbf{A})$ denotes the largest absolute value of the eigenvalues of square matrix $\mathbf{A}$, then

$$\rho(\mathbf{G}) = \rho(\boldsymbol{\Pi}) = \rho.$$

Finally, denote $\boldsymbol{\Pi}^k = \left[\pi_{ij}^{[k]}\right]$ where $\pi_{ij}^{[k]}$ denotes the total paths of length k for any node in class V_i with its neighbours in class V_j.

Remark 1. A concept closely linked to equitable partition is that of automorphism partition (also called orbit partition), that is used to model symmetry in networks.[8] We illustrate via a simple example that equitable partition is a more flexible concept to specify the underlying class structure for any (connected) network than orbit partitions: the existence of a non-trivial equitable partition is less restrictive than the existence of a non-trivial orbit partition, since all orbit partitions are equitable as well, but the converse is not true in general. To see this consider McKay's graph shown in Fig. 1(a).

There are 6 orbit partitions (other than the trivial identity partition in which each node is an orbit in itself), σ_i, $i = 1, \dots, 6$, in the above example: σ_1 : $(7,8)$; σ_2 : $(1,7),(2,8),(3,6),(4,5)$; σ_3 : $(1,7,2,8),(3,6),(4,5)$; σ_4 : $(1,2)$; σ_5 : $(1,2),(7,8)$; σ_6 : $(1,8),(2,7),(3,6),(4,5)$.[9] These orbit partitions are equitable as well, but the equitable partition shown in Figure Fig. 1(a), which groups nodes $(3,6)$ in one class and nodes $(1,2,4,5,7,8)$ in another, is not an orbit partition, since no automorphism maps the outer nodes $1,2,7,8$ to the inner nodes 4,5 (automorphisms must preserve cycles — see Kudose (2009)). Thus, equitable partition can capture additional equivalences via preserving the linkage structure within and among classes which is not always captured by

[7] See Van Mieghem (2010), page 62, **art.** 62.

[8] Graph automorphism refers to adjacency preserving permutations of network vertices, which creates a network partition, each of whose cells are called orbits. Orbits contain the equivalent nodes which, if interchanged, preserve the network structure. For formal definitions, see, for instance, Xiao et al. (2008).

[9] Using the usual representation for automorphisms, we write the non-identity orbits – cells which contain two or more equivalent nodes – inside a parenthesis.

orbits, and which also has a direct role interpretation obtained from the sociology literature (see Appendix B for an illustration of how equitable partition compares with classical notions of role equivalences in networks). This is better visualised in Fig. 1(b): for instance, considering it a notional supervisory network, nodes $(3, 6)$ have equivalent role, each supervising three candidates, and each node in the supervisee class of $(1, 2, 4, 5, 7, 8)$ is directly connected to the supervisor and with one other member among the supervisees. But, in general, our theory applies to all orbit partitions of a network as well.

Remark 2. There exists a unique coarsest equitable partition for any graph $\mathbf{g}$, as noted in Section 1.2 in McKay (1981).[10] From implementation perspective, polynomial-time algorithms exist in literature for finding the equitable partition for any network; in our examples and illustrative applications, we use Everett and Borgatti (1996)'s exact coloration procedure for finding the coarsest equitable partition, also known as 'exact coloration', for any simple graph (order n^3; see their '*Excatre*' algorithm in p.326).[11] Also, our theory is relevant for networks with certain symmetry, which is widespread for real-world networks. In particular, finding whether a network has non-trivial orbit partition is a polynomial-time problem for any finite network (see Luks (1982)), and well-established open-source software tools like *nauty* by McKay and Piperno (2014) can be used for finding the orbit partitions of networks, which are also equitable by definition.

3.2. Bonacich centrality

Here, we provide the definition of Bonacich centrality which is relevant for our analysis. Bonacich (1972)'s eigenvector-based centrality gives more importance to agents that have 'important' neighbours. The vector of Bonacich centralities, with a decay parameter a, in $\mathbf{g}$ is given by:

$$\mathbf{b}(\mathbf{g}, a) = [\mathbf{I}_n - a\mathbf{G}]^{-1}.\mathbf{1}_n = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k.\mathbf{1}_n \tag{3.2}$$

where $\mathbf{I}_n$ denotes an n-square identity matrix. Note that the above expression is well-defined for small values of a, specifically, if a is less than inverse of the largest absolute eigenvalue of $\mathbf{G}$. Recall that $\mathbf{G}^k$ represents the number of paths of length k between any two players in the network. Hence, Bonacich centrality counts the total number of paths emanating from player i in the network $\mathbf{g}$, weighted down by their length.

4. Network game: Nash equilibrium class activity

We consider the network game with local payoff complementarities as in Belhaj et al. (2013), which is a simplified version of the linear–quadratic utility function of Ballester et al. (2006). Players $\{1, \dots, n\}$ in a network engage in a non-cooperative game, where the strategy of each player is to decide the extent of efforts they exert. The utility of player i is given by

$$u_i(x_1, \dots, x_n) = x_i - \frac{1}{2}x_i^2 + \lambda \sum_{j=1}^{n} g_{ij} x_i x_j$$

where $x_i \geq 0$ denotes the effort of player i, and $\lambda > 0$ measures the intensity of interactions among pairs of players. Hence, the utility function consists of two components: an idiosyncratic component made up of own efforts and an interaction component reflecting strategic complementarities among connected players. Further, the linear–quadratic form implies that utility is strictly concave in one's own efforts.

In this setting of network game with linear–quadratic utility and payoff complementarities, Ballester et al. (2006) establish the proportionality between players' Nash equilibrium outcome and their Bonacich centrality. This is a key result which establishes the intuitive link between players' equilibrium behaviour with their positions within the network. Indeed, it can be shown that player i's unique Nash equilibrium outcome for the game described above, x_i^*, equals their Bonacich centrality $b_i(\mathbf{g}, \lambda)$.

We are interested in equilibrium analysis for determining class activity when the network of relative payoff complementarities has a partition structure as conceptualised by the symmetry-based notion of equitable partition. Recall that each member of a class has the same value of Bonacich centrality and, hence, adopts identical equilibrium strategy. As such, note that the idea of network game under equitable partitioning applies beyond the Ballester et al. (2006) framework and is a special case of a more general construction as considered in Plan (2023). Indeed, consider any non-cooperative game with complete information and a given set $\mathcal{N}$ of players, such that payoffs depend on a common vector θ of parameters ($\theta = \mathbf{G}$ in the Ballester et al. (2006) framework), and that players can be partitioned so that $\mathcal{N} = V_1 \cup V_2 \cup \dots \cup V_m$, where $i, k \in V_j$ iff the best-reply functions of players i and k are identical for any vector θ of parameters. Any such game has non-trivial symmetry groups, or classes, where reshuffling players is allowed within the classes, V_j, $j = 1, \dots, m$, but not across classes (see Plan, 2023).

For equilibrium analysis, we first present the following Lemma. Let $\mathbf{A}^T$ denote the transpose of matrix $\mathbf{A}$. Note that in what follows, in the Lemmas and Definitions that pertain to any general network structure and its equitable partition, we use the symbol 'a' to denote the attenuation factor, which plays the role of 'λ' for the results obtained in the context of the network game explained above.

[10] The binary relation 'coarser', as defined in McKay (1981), specifies a partial order over the set of all equitable partitions, which forms a finite, and hence, complete lattice. Therefore, a coarsest equitable partition exists from definition of a complete lattice.

[11] A simple graph is a graph without any loops or multiple edges.

Lemma 1. *Let $0 < a \leq 1/\rho$ so that $[\mathbf{I}_n - a\mathbf{G}]^{-1}$ and $[\mathbf{I}_m - a\boldsymbol{\Pi}]^{-1}$ are well-defined and nonnegative. Then, $[\mathbf{I}_n - a\mathbf{G}]^{-1}\mathbf{X} = \mathbf{X}[\mathbf{I}_m - a\boldsymbol{\Pi}]^{-1}$.*

Lemma 1 relates the overall network structure with its equitable partition. It enables to express the equilibrium activity of classes in relation to the network's partition structure. For this purpose, define the following matrix

$$\mathbf{N}(\boldsymbol{\pi}, \lambda) = \left[\mathbf{I}_m - \lambda \boldsymbol{\Pi}^T\right]^{-1} = \sum_{p=0}^{\infty} \lambda^p (\boldsymbol{\Pi}^p)^T,$$

which is well-defined and nonnegative for $\lambda \leq 1/\rho$. Its elements $N_{ij}(\boldsymbol{\pi}, \lambda) = \sum_{p=0}^{\infty} \lambda^p \pi_{ji}^{[p]}$ count the total number of paths of length p for any node in class V_j with the members in V_i, weighted down by λ^p. Let $\mathbf{y}^*(\boldsymbol{\pi}) = \left[y_i^*\right]$ denote the outcome vector for classes at equilibrium, where y_i^* is the sum of equilibrium outcomes of all players of class V_i, $i = 1, \dots, m$. Also, for a vector $\mathbf{z} \in \mathbb{R}^p$, we denote the sum of its entries as $z = z_1 + \cdots + z_p$.

Theorem 1. *The matrix $\mathbf{N}(\boldsymbol{\pi}, \lambda) = \left[\mathbf{I}_m - \lambda \boldsymbol{\Pi}^T\right]^{-1}$ is well-defined and nonnegative when $\lambda \leq 1/\rho$. Then, the unique and interior Nash equilibrium class activity for the network game characterised by u_i, $i = 1, \dots, n$, played over the quotient graph $\boldsymbol{\pi}$, is given by*

$$\mathbf{y}^*(\boldsymbol{\pi}) = \mathbf{N}(\boldsymbol{\pi}, \lambda).\mathbf{r} \equiv \mathbf{t}(\boldsymbol{\pi}, \lambda). \tag{4.1}$$

In the above, $\mathbf{N}(\boldsymbol{\pi}, \lambda).\mathbf{r}$ is the vector of sum of Bonacich centralities of players in each class. That the contribution of a class to the overall network activity is proportional to the sum of its members' Bonacich centralities is expected. But more importantly, Eq. (4.1) links the equilibrium activity of a class with its position in the network of local interactions between players of different classes through the matrix $\mathbf{N}(\boldsymbol{\pi}, \lambda)$. Indeed, since the matrix $\mathbf{N}(\boldsymbol{\pi}, \lambda)$ represents the interactions among members of the classes specified by the equitable partition network structure, Eq. (4.1) shows how the position of the classes in the network influence their equilibrium behaviour. Hence, Theorem 1 can be considered as the class analogue of the Bonacich–Nash linkage of Ballester et al. (2006).

5. The key class: Two measures

The above analysis shows that the class outcome at equilibrium is related to its position within the network when there exists payoff externalities among players. Removing a class alters the network structure of bilinear influences, in addition to reducing the number of players who contribute to the overall network activity, thus altering the equilibrium network outcome. In this section, we propose two geometric measures to characterise equilibrium outcome, in aggregate and in per-capita terms, upon removing classes. This informs simple criteria for targeting the optimal class if the planner wants to optimally alter the aggregate or the per-capita network activity, respectively.

Consider the game of Section 4 being played over the network $\mathbf{g}$ with symmetric square adjacency matrix $\mathbf{G} = [g_{ij}]$, where $g_{ij} \in \{0, 1\}$ for $i \neq j$ and g_{ii} is set to 0; its corresponding quotient network is $\boldsymbol{\pi}$ with quotient matrix $\boldsymbol{\Pi} = [\pi_{ij}]$. Let a class j be removed from the network. The corresponding partition matrix is denoted by $\boldsymbol{\Pi}^{-j}$, by setting the jth row and jth column of $\boldsymbol{\Pi}$ to zero. Also, $\mathbf{r}^{-j}$ is the class size vector associated with removing class j by setting jth coordinate of $\mathbf{r}$ to 0. The overall network activity upon removing class j is the sum of the activities due to all remaining classes $y^*(\boldsymbol{\pi}^{-j}) = \sum_{i=1,\ i\neq j}^{m} y_i^*(\boldsymbol{\pi}^{-j})$.

The derivations of class-based centrality measures make use of the following Lemma, which characterises all path changes in the quotient network when a class is removed.

Lemma 2. *Let $0 \leq a \leq 1/\rho$ such that $\mathbf{N}(\boldsymbol{\pi}, a) = \left[\mathbf{I}_m - a\boldsymbol{\Pi}^T\right]^{-1}$ is well-defined and non-negative. Let $\mathbf{N}(\boldsymbol{\pi}^{-j}, a) = [\mathbf{I}_m - a(\boldsymbol{\Pi}^{-j})^T]^{-1}$. Then:*

$$N_{ik}(\boldsymbol{\pi}, a) - N_{ik}(\boldsymbol{\pi}^{-j}, a) = \frac{N_{ij}(\boldsymbol{\pi}, a).N_{jk}(\boldsymbol{\pi}, a)}{N_{jj}(\boldsymbol{\pi}, a)}. \tag{5.1}$$

The above result will be crucial for the key class problem, as it indicates the changes in the number of paths between classes if a class were removed, as specified by $N_{ik}(\boldsymbol{\pi}) - N_{ik}(\boldsymbol{\pi}^{-j})$, in terms of the intra-class and inter-class paths within the quotient network, $\boldsymbol{\pi}$.

5.1. Class-centrality

Class-centrality is concerned with identifying the class removing which results in an optimal reduction in the aggregate network outcome. Formally, the planner's objective is to:

$$\min\left\{y^*(\boldsymbol{\pi}^{-j})\right\} \quad \text{or} \quad \max\left\{y^*(\boldsymbol{\pi}) - y^*(\boldsymbol{\pi}^{-j})\right\}, \quad j = 1, \dots, m. \tag{5.2}$$

Definition 1. Let there be a quotient network $\boldsymbol{\pi}$ that divides the network $\mathbf{g}$ into m classes, with the associated partition matrix $\boldsymbol{\Pi}$ and a decay factor $a > 0$ such that $[\mathbf{I}_m - a\boldsymbol{\Pi}]^{-1}$ is well-defined and non-negative. The class-centrality measure of class j is given by:

$$e_j(\boldsymbol{\pi}, a) = \frac{t_j(\boldsymbol{\pi}, a).s_j(\boldsymbol{\pi}, a)}{N_{jj}(\boldsymbol{\pi}, a)}, \tag{5.3}$$

where $\mathbf{N}(\boldsymbol{\pi}, a) = \left[\mathbf{I}_m - a\boldsymbol{\Pi}^T\right]^{-1}$, $\mathbf{s}(\boldsymbol{\pi}, a) = \mathbf{1}_m^T.\mathbf{N}(\boldsymbol{\pi}, a)$, and $\mathbf{t}(\boldsymbol{\pi}, a) = \mathbf{N}(\boldsymbol{\pi}, a).\mathbf{r}$.

The above index informs a simple criterion to characterise the key class j^* to optimally reduce (or increase) network outcome, as presented in the following Theorem.

Theorem 2. *If $\lambda \leq 1/\rho$, the class that solves* $\max\left\{y^*(\boldsymbol{\pi}) - y^*(\boldsymbol{\pi}^{-j})\right\}$ *is the j^* for which the class-centrality measure is the highest, that is, $e_{j^*}(\boldsymbol{\pi}, \lambda) \geq e_j(\boldsymbol{\pi}, \lambda)$ for all $j = 1, \ldots, m$.*

Note that removing a class has a direct and an indirect effect on network activity. Direct effect is by virtue of a reduction in the number of players who contribute to network activity. Indirect effect is due to the fact that removing a class alters the network structure such that the remaining classes adopt different equilibrium actions, thereby again altering the aggregate network activity. Hence, the class with the most players need not be the key class for reducing the aggregate network activity.

5.2. Per-capita class-centrality

Other than bilinear influences, the size of a class, indeed, plays an important role in determining the key class using the class-centrality index, which can have implications for targeting policies, say, in presence of the planner's budget constraints.[12] In this Section, we provide a relatively 'size-sensitive' alternative to class-centrality, which selects a class typically smaller than the key class, for any given class-structure.

The per-capita class centrality provides a geometric measure for identifying the class removing which results in maximum per-capita reduction in network activity. The planner's objective is:

$$\min\left\{\frac{y^*(\boldsymbol{\pi}^{-j})}{n - r_j}\right\} \quad \text{or} \quad \max\left\{\frac{y^*(\boldsymbol{\pi})}{n} - \frac{y^*(\boldsymbol{\pi}^{-j})}{n - r_j}\right\}, \quad j = 1, \ldots, m. \tag{5.4}$$

Definition 2. For the quotient network and decay factor a as specified in Definition 1, the per-capita class-centrality measure of class j is given by:

$$h_j(\boldsymbol{\pi}, a) = \frac{n.\left(t_j(\boldsymbol{\pi}, a)/N_{jj}(\boldsymbol{\pi}, a)\right).s_j(\boldsymbol{\pi}, a) - r_j.t(\boldsymbol{\pi}, a)}{n(n - r_j)}, \tag{5.5}$$

where $\mathbf{N}(\boldsymbol{\pi}, a) = \left[\mathbf{I}_m - a\boldsymbol{\Pi}^T\right]^{-1}$, $\mathbf{s}(\boldsymbol{\pi}, a) = \mathbf{1}_m^T.\mathbf{N}(\boldsymbol{\pi}, a)$, $\mathbf{t}(\boldsymbol{\pi}, a) = \mathbf{N}(\boldsymbol{\pi}, a).\mathbf{r}$, and $t(\boldsymbol{\pi}, a)$ denotes the sum of the coordinates of $\mathbf{t}(\boldsymbol{\pi}, a)$.

Per-capita class-centrality $h_j(\boldsymbol{\pi}, a)$ characterises the per-capita network activity upon removing class j, in terms of the position that its players occupy within the partitioned network. This informs a simple criterion for selecting the key class for optimally lowering per-capita network activity from the planner's perspective, as given by the following Theorem.

Theorem 3. *If $\lambda \leq 1/\rho$, the class that solves* $\max\left\{\frac{y^*(\boldsymbol{\pi})}{n} - \frac{y^*(\boldsymbol{\pi}^{-j})}{n-r_j}\right\}$ *is the j^* for which the per-capita class-centrality measure is the highest, that is, $h_{j^*}(\boldsymbol{\pi}, \lambda) \geq h_j(\boldsymbol{\pi}, \lambda)$ for all $j = 1, \ldots, m$.*

Similar to class-centrality, the per-capita measure also reflects the dual effects of lesser contributing members as well as changes in the network structure of peer-effects, in determining the network activity of the resultant network upon removing a class. This interplay of the direct and indirect effects is, in fact, at the heart of both the class-based centrality measures: the basic idea is to remove a class and analyse how the ensuing alterations in network ties and their intensities impact the equilibrium outcome level, repeating this for all classes, such that the class that can maximally reduce the outcome, in aggregate or per-capita terms, is the key class.

We provide a general result comparing the class sizes of the per-capita key class and the key class in Proposition 1 below, which places an upper bound on the size of the former.

Proposition 1. *The per-capita key class is less than or equal to the key class in size.*

Since per-capita class centrality typically selects a class smaller in size than the key class, it can be informative as a size-sensitive alternative to class-centrality.

Remark 3. Since there is a unique coarsest quotient network $\boldsymbol{\pi}$ associated with every network $\mathbf{g}$, the class-based centrality measures are generic indices applicable to any generic network structure.

Remark 4. At the extreme case, when the cardinality of each class is 1 and equitable partitioning is trivial, the key-class problem is equivalent to Ballester et al. (2006)'s key player problem and both yield the same result. But, additionally to their intercentrality measure for identifying the key player, the class-based centrality measures proposed here capture the relative influence of players occupying symmetric network positions, to inform choice for targeting multiple players. The choice between the two measures, however, will depend on the specific application and the planner's objective.

[12] One way to deal with this limitation, as we discuss in Section 6.4, is to identify various equitable partitions of a network with varying class sizes, and apply the class-centrality index for identifying the key class for all the resulting partitions, choosing the one that best suits the planner's objectives.

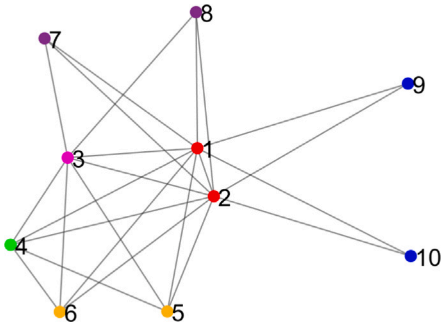


Fig. 2. Nested split graph: this figure is the network representation of König et al. (2014)'s connected nested split graph illustration (see their Fig. 1 and the description therein). The nodes of same colour represent cells of the equitable partition, which is also the degree partition of nested split graph. The key class, as well as the key group of 2 players, consists of nodes 1 and 2 (shown in red), who have the highest degree of 9. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

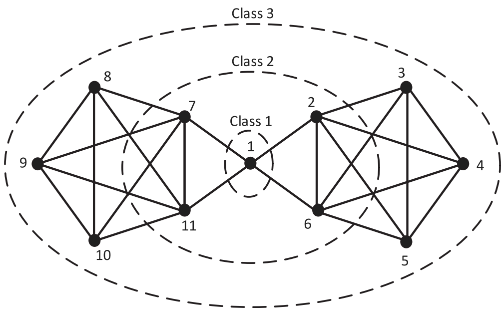


Fig. 3. Example network: class-based centrality vs intercentrality.

Remark 5.

In the presence of nestedness, which postulates that neighbourhoods of every player is contained in the neighbourhoods of higher degree players, the degree partition that characterises the nested split graph is the same as its equitable partition. Also, under nestedness, the key class (which, say has a size of q) is also the key group of q players, whenever the former is made of the players with the highest number of links, as shown for the nested split graph example of König et al. (2014) in Fig. 2. Noting that players in a class have the same degree, this equivalence follows from Hiller (2022), who shows that optimal targeting for eliminating any q players in nested split graphs boils down to the simple criteria of ranking players by their number of links and targeting the first q players. However, it is important to note that the key class may not always consist of the highest degree players; for instance, a different class having more lower-degree players with important direct or indirect links may, instead, turn out to be the key class. In that case, the key group will be different from the key class. Indeed, our class-based centralities bring out new insights about the joint behaviour of similarly positioned players with well-defined linkages, unlike the key group of players. Finally, note that equitable partition is a concept general enough to incorporate other kinds of grouping structures in networks beyond nestedness.

6. Examples and discussion

In this section, we illustrate some important points related to the proposed class-based centrality measures on four example networks. In the first three examples, we consider the unique coarsest equitable partition of the given networks, and compare our two proposed class-based centrality measures with other common centrality measures. In a fourth example, we have considered various underlying class structures of the network (equitable partitions other than the coarsest one) to illustrate how to approach targeting key classes of varying sizes using our theory.

6.1. Example 1: Class-based centrality vs intercentrality

Fig. 3 considers the 11-player network **g** with three classes, as used in Ballester et al. (2006), and compares the proposed centrality measures with their intercentrality index, another centrality metric from planner's optimality concerns to identify the key player type.

Table 1
Class-based centrality vs intercentrality.

Class type	$a = 0.1$			$a = 0.2$		
	e_i	h_i	c_i	e_i	h_i	c_i
1	2.92	0.11	2.92	41.67	3.33	41.67 ∗
2	11.09	0.57 ∗	3.28 ∗	80.67	6.76 ∗	40.33
3	12.96 ∗	0.46	2.79	81.67 ∗	6.33	32.67

e_i and h_i denote class-centrality and per-capita class-centrality, respectively. c_i denotes intercentrality measure of Ballester et al. (2006). The highest values are indicated by '∗'.

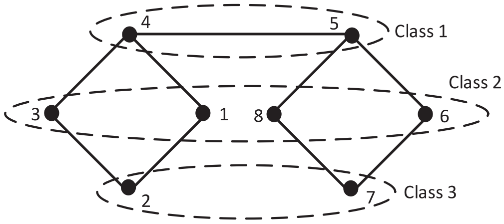


Fig. 4. Example network: class-based centrality vs Bonacich centrality.

Table 2
Class-based centrality vs Bonacich centrality.

Class type	$a = 0.1$			$a = 0.2$		
	e_i	h_i	b_i	e_i	h_i	b_i
1	3.41	0.14	1.39 ∗	6.50	0.47	2.13 ∗
2	6.13 ∗	0.24 ∗	1.26	10.26 ∗	0.72 ∗	1.77
3	3.08	0.08	1.25	5.31	0.27	1.71

e_i, h_i and b_i denote class-centrality, per-capita class-centrality and Bonacich centrality, respectively. The highest values are indicated by '∗'.

Table 1 computes centralities for two values of the decay factor a.[13] We find that the largest class (class 3) is also the key class for reducing overall equilibrium activity, for both values of a. This is because along with having most members, this class is also quite well-connected. It has direct links with class 2 (which, by being the link between the other two classes, is the most central class — its players have the highest Bonacich centrality), and indirect links with class 1. Hence, removing class 3 alters the network structure in a way to cause maximal disruption in equilibrium contribution by remaining players. However, in terms of per-capita network activity, class 2 becomes the most important one since it is smaller than class 3 but has direct links with both classes 1 and 3, removing which causes most damage to the network activity of the altered network, measured in per-capita terms.

We also note that the key class, both for total and per-capita outcome reduction, mostly differs from the player type with the highest intercentrality value. This is expected as intercentrality depends on an individual level analysis of peer-effects between pairs of players for characterising their importance, while class-based centrality internalises the group-level dynamics among the members within a class as well, in addition to studying the peer-effects across members of different classes. For the class with only one member (class 1), there is no such intra-group dynamics per se, and its intercentrality c_i as well as class-centrality e_i are the same, as also noted in Remark 4.

6.2. Example 2: Class-based centrality vs bonacich centrality

In the above example, the class which was topologically most central was also the key class for optimally reducing per-capita activity. It is, however, not necessary that removing the most central class in terms of position alone, that is, whose players have the highest Bonacich centrality, will result in an optimal change in the structure of bilinear influences so as to minimise the per-capita network activity. This is evident in the example considered in Fig. 4, borrowed from Allouch (2017) who considers segregation in social networks.

The class-based centrality values for the three classes, along with the Bonacich centrality for players in those classes is reported in Table 2, for two different values of a.[14] For this simple network where two of the classes are of same size, the key class for total and per-capita activity reductions turns out to be the same (class 2). Note that while class 1 is most centrally located, since its players, who have the highest Bonacich centrality, form a bridge through which the other players are connected, it is not the key

[13] Here, the maximum value of a in line with our centrality definitions is 0.227.

[14] The largest value for a compatible with our definitions is 0.427.

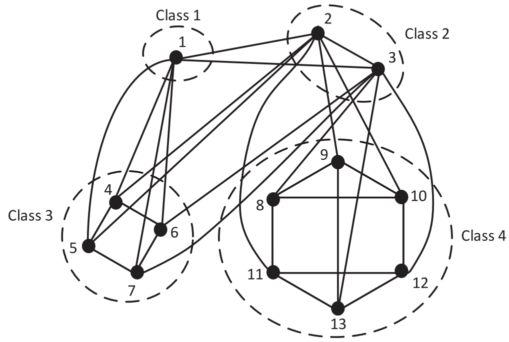


Fig. 5. Example network: analysing class-centrality.

Table 3
Analysing class-centrality.

Class type	$a = 0.1$		$a = 0.2$	
	e_i	h_i	e_i	h_i
1	4.40	0.21	514.35	39.08
2	8.81	0.45 ∗	561.62 ∗	42.79 ∗
3	9.59	0.22	540	39.80
4	12.40 ∗	0.14	535.71	37.85

class, for either optimally reducing total or per-capita network activity. This is because for network activity, how removing a class alters the peer-effects within and across classes matter. Taking this into account makes class 2 the key class.

6.3. Example 3: Class-centrality need not be highest for the largest class

In the above two examples, we find that the key class for inducing maximal disruption in total network activity is the one that has most members. While this was true for the simplistic network structures considered in Figs. 3–4, it will not, in general, be the case. We consider the example in Fig. 5, from Bonaccorsi et al. (2015)'s study of epidemic outbreaks in networks with equitable partitions. Unlike the previous examples, this network displays more complexity and variations in the indirect links between members of various classes (features which are likely to be present in realistic networks). For instance, it can be seen that even though all players in class 4 have the same number of links amongst themselves and with class 2, the indirect links for players 8 and 11 are different from others in their class: 8 has a direct link with node 3 of class 2, while all its neighbours – 9, 10, 11 – have direct links with node 2, whereas all other nodes in class 4 (except 11) have one of their neighbours linking with the same node in class 2 as they do, and the other two neighbours link with the remaining class 2 node. Node 11 also has an analogous linkage pattern.

Table 3 reports the centrality values for the aggregate and per-capita indices, for $a = 0.1$ and 0.2.[15] We focus on the class-centrality e_i which is informative for our purpose. Notice that for the lower value of a, the largest class (class 4) is also the key class, while with $a = 0.2$, class 2 (which is much smaller in size than class 4) becomes key for optimally decreasing overall network activity. This is because, with smaller value of a, the direct effect due to class size is the dominant factor in determining the key class. But when the indirect links become stronger, removing class 2, through which most of the indirect links are formed, has the highest combined direct and indirect effects in determining the aggregate network activity.

6.4. Example 4: Targeting classes of varying sizes

In the above three examples, we considered the coarsest equitable partition, which yielded a key class (or its per-capita analogue) of a given size, for targeting purposes. While class size is a model primitive (as determined by the underlying class structure for any given network), there is still some amount of flexibility for the planner in deciding what size of key class to target. The planner can do so by considering different equitable partitions within the network, other than the coarsest one, each of which will have a different class structure with varying class sizes. Applying our key class theory to the different class structures will yield a key class (or its per-capita analogue) for all such partitions, providing the planner with a range of different options for optimal classes with varying sizes to choose from, depending on their preferences.

To make this idea precise, suppose the planner wants to target a class of size k (or a smaller, but more optimal choice, in the sense that targeting the smaller class leads to a larger disruption of the network activity). The planner must then consider such

[15] For this example, the maximum permissible value of a to satisfy our centrality definitions is 0.204.

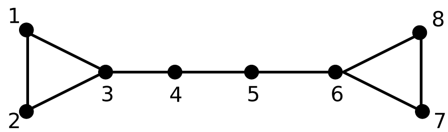


Fig. 6. Example network: Analysing all class structures.

Table 4
Analysing all class structures.

Class structure	e_i^*	h_i^*	Key class	Per-capita key class
$(7,8)$, $(1,2)$	2.81	0.08	$(1,2)$ or $(7,8)$	3 or 6
$(1,7)$, $(2,8)$, $(3,6)$, $(4,5)$ $(1,8)$, $(2,7)$, $(3,6)$, $(4,5)$	3.68	0.18	$(3,6)$	$(3,6)$
$(1,7,2,8)$, $(3,6)$, $(4,5)$	5.63	0.18	$(1,7,2,8)$	$(3,6)$
$(1,2,4,5,7,8)$, $(3,6)$	8.34	0.29	$(1,2,4,5,7,8)$	$(1,2,4,5,7,8)$

Each row under 'Class structure' indicates a distinct class structure: the nodes inside parenthesis denote a class, all remaining nodes being identity classes (that is, the respective node is also the 'class' in itself). e_i^* and h_i^* denote highest class-centrality and per-capita class-centrality values, respectively, among all classes in the corresponding class structure.

Table 5
Optimal class: Varying size.

Size	Key class	Per-capita key class
2	$(3,6)$	$(3,6)$
4	$(1,7,2,8)$	–
6	$(1,2,4,5,7,8)$	$(1,2,4,5,7,8)$

equitable partitions which are coarsest *up to size* k, which we call as the k*-coarsest equitable partitions* of the network. Note that while the coarsest equitable partition of any network is unique, the k-coarsest equitable partitions need not be so. For instance, consider McKay's graph in Fig. 6, whose underlying equitable partitions are listed in Table 4. This network has three 2-coarsest equitable partitions as shown by the first three class structures in Table 4, while the next two class structures show the 4-coarsest and 6-coarsest equitable partition (which is also the coarsest equitable partition for this network), respectively.

Each of the partitions in Table 4 represents a distinct underlying class-structure. Table 4 further reports the corresponding key class and per capita key class for each of these stratifications, along with the class-based centrality values, e_i^* and h_i^* (maximum value of the indices, among all classes in the concerned stratification). Decay factor, a, for all computations is taken as 0.1. Note that implementing this algorithmically is straightforward for any connected network, using existing algorithms from literature.[16]

The above analysis suggests which class should the planner target, based on their capacity for going after different sized classes, as summarised in Table 5 below.[17] Focusing on the key class for causing an optimal disruption to the overall network activity is insightful. For targeting a class of, say, size 2, the planner must consider the 2-coarsest equitable partitions (the first three class structures in Table 4). Selecting the largest e_i^* from among all such class structures yields $(3,6)$ as the optimal key class of size 2. The planner can similarly target larger classes as shown in Table 5.

Note that while considering the k-coarsest equitable partitions, it is not necessary that the key class will be one of size k, as some other smaller but well-connected class can turn out to be the key. In this case, since targeting the smaller-sized class should also be within the planner's budget and yields a greater reduction in total (or per-capita) output, this class should obviously be targeted. There is still some restriction though, as the planner will be bound by the underlying class structures present in the network. So, for example, if the planner wants to target a class of size 3 for the network in Fig. 6, there being no such class, the planner must go for the next best and target the class of size 2 or 4. A cost–benefit analysis from comparing the gain in e_i^* from targeting larger classes, with the planner's specific associated costs, can be informative in this regard.

7. Two illustrative applications

In this section, we present two illustrative applications to show the applicability of equitable partition and key class identification in real-world networks. For both the applications, we find the coarsest equitable partitions for the respective networks using Everett

[16] As a simple way to implement this algorithmically, the planner can find all orbit partitions within a network, each of which are also equitable by definition, in addition to the coarsest equitable partition. Remark 2 mentions some relevant algorithms for finding these partitions. Note that the computational cost for implementing our theory to all orbit partitions is of the same order as that of finding those partitions, for which highly efficient C language based procedures exist (for example, the open-source package *nauty* by McKay and Piperno, 2014).

[17] Note that a key class of size 1 is simply the key player, as obtained from applying Ballester et al. (2006)'s intercentrality measure. This comes out to be node 3 in Fig. 6, which coincides with per-capita key class of size 1 obtained from considering the first class structure in Table 4.

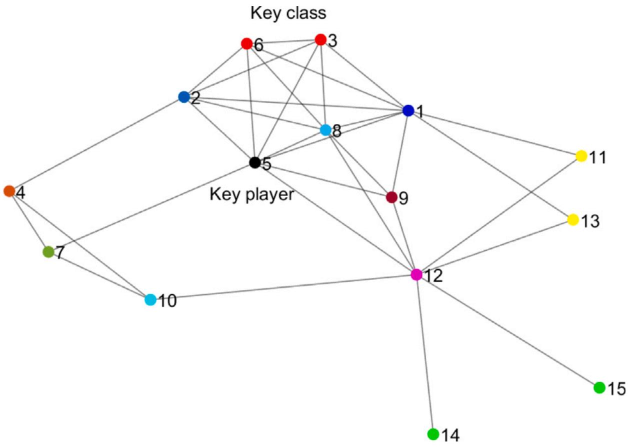


Fig. 7. Key players in Thurman (1979)'s office communication network.

and Borgatti (1996)'s exact coloration algorithm. Also, the value of the decay factor a, in computing the centrality measures, is taken as 0.1, which lies within the permissible range for our centrality definitions.

7.1. Informal office communication network

Social relationships among individuals in informal networks play a crucial role in shaping individuals' opinions and actions. Studying this in the context of an office setting, Thurman (1979) conducted a 16 month study of informal communications among employees in an international organisation, taking a network approach to analyse social relationships in the office and its role in influencing the office's internal workings. During this time, a few major disputes broke out in a sub-group of 15 employees. We revisit Thurman (1979)'s informal office communication network to identify the most important members in the network, who have the maximal impact on communications (and, in a related sense, disputes) among the employees. As shown in Fig. 7, the network contains 15 nodes, with 33 undirected links, which represent interactions between individuals. Further, it has 12 classes, which are highlighted by the different colours in the network.

We compute the class-centralities of all classes to identify the key class in this sample network, and compare our findings with Ballester et al. (2006)'s intercentrality measure to identify the key player. The key player comes out to be node 5 (highlighted in black), who is, therefore, central to the communication flow within this network. This corroborates Thurman (1979)'s observation that node 5 was "the center of (the) social circle". The key class (highlighted in red), however, is made of two different individuals, nodes 3 and 6 (who are, unsurprisingly, directly linked with the key player). This suggests that both individuals 3 and 6 together have a greater influence on their peers, rather than node 5 alone (otherwise node 5 would also have been the key class). This can be informative for setting dispute resolution tactics in the office: the application suggests that both individuals 3 and 6 should be consulted as they are together likely to play a bigger role than node 5, if the objective is to resolve the office disputes in an optimal manner.

7.2. PhD network

The application of office communication network, being small in size, allowed a convenient visualisation and interpretation of the key players problem. But it is of interest to investigate if our equitable partition setting, and consequently the class centrality measures, are applicable to more complex real-world networks, which is the aim of this illustrative exercise. For the purpose of this exercise, we consider the large-scale PhD network as analysed in MacArthur et al. (2008) who study the symmetry features of this network. The network consists of 1025 nodes with 1043 edges, which represent connections among Ph.D. students and their supervisors in Theoretical Computer Science over several years (see MacArthur et al. (2008) or De Nooy et al. (2018) for details). The network is displayed in Fig. 8.

The coarsest equitable partition of this network has 511 classes; classes with size 9 or more are highlighted with different colours in Fig. 8. Note that finding the equitable partition even for large networks is easy; Everett and Borgatti (1996)'s algorithm required a few seconds to obtain the partition using personal laptop with configuration i7-7500U CPU and 16 GB RAM. Here too, we find that the key player is different from the members of the key class: the key player is highlighted by the colour black, while the key class consists of 35 different players highlighted in red. Associations between Ph.D. students and their advisors can be important for creating and advancing scientific knowledge, and the set of symmetrically-positioned players who play a similar role within the academic network as specified by the key class can be informative in that regard, with the exact utility depending on the planner's specific objective.

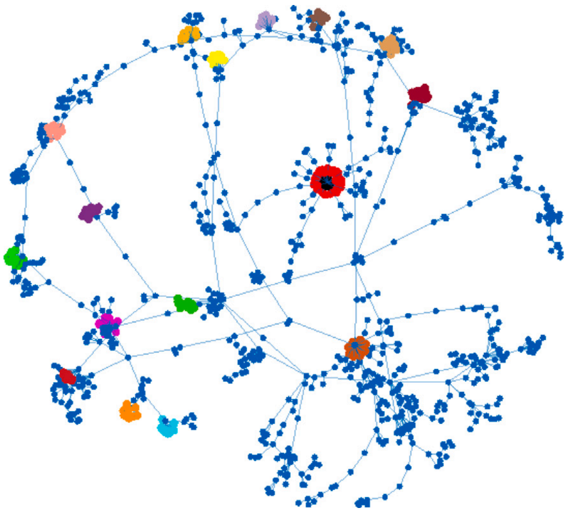


Fig. 8. Computer Science PhD network.
Classes of size 9 or more are shown by different colours. Key player is in black, key class in red. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Thus, while the exact applicability will, of course, depend on the specific setting under investigation, the preceding two illustrative applications show that detection of classes conceptualised by the idea of equitable partition is not restrictive for application in complex real world networks, and the subsequent identification of the key class can be informative for the planner's purposes.

8. Concluding remarks

This paper approaches the multiple players targeting problem from a new perspective of focusing on network classes made of players who occupy symmetrical network positions and have well-defined linkage structures in the network, as captured via the notion of equitable partition. Studying the network game with local payoff complementarities under equitable partitioning, we bring out new insights about the relative influence of network classes in determining the overall activity in equilibrium. This analysis informs two novel centrality measures to geometrically characterise the key class for the social planner who wishes to optimally increase (or decrease) the aggregate or the per-capita network activity.

The class-based centrality measures can be informative for the social planner in several scenarios, like in criminal or R& D networks. An interesting future work can be to develop such applications for particular scenarios, for instance, for crime reduction in criminal networks displaying hierarchical interaction patterns, or for devising bail-out policies in R&D networks with well-established linkage structure among similarly positioned firms, or for containing the spread of epidemics by isolating a community of similarly positioned and linked individuals in the society. Also, in a recent work, Parise and Ozdaglar (2023) have developed Graphon games, which can be thought of as the limits of sequences of finite network games, providing a richer statistical framework for targeting over large scale networks. It may also be worth exploring, in a future work, the key class problem for Graphon games, especially since some Graphon counterpart concepts of equitable partition have been provided (see, for example, Grebík and Rocha (2019)). Finally, while class, here, has been defined through the symmetry-based notion of equitable partition, an interesting and challenging future work would be to expand that idea and consider other general partitioning of networks, for instance, as defined in Van Mieghem (2010), in order to find the key class for general grouping structure in networks.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proof section

Proof of Lemma 1. Both the inverse matrices are well-defined and non-negative for $0 \leq a \leq 1/\rho$. Then, since from (3.1) $\mathbf{G}^k\mathbf{X} = \mathbf{X}\boldsymbol{\Pi}^k$, we have

$$[\mathbf{I}_n - a\mathbf{G}]^{-1}\mathbf{X} = \left[\sum_{k=0}^{\infty} a^k\mathbf{G}^k\right]\mathbf{X} = \sum_{k=0}^{\infty} a^k\mathbf{X}\boldsymbol{\Pi}^k = \mathbf{X}\left[\sum_{k=0}^{\infty} a^k\boldsymbol{\Pi}^k\right]$$

which proves the Lemma. ■

Proof of Theorem 1. The pure Nash equilibrium strategies $\mathbf{x}^* \in \mathbb{R}^n_+$ for the network game in Section 4 solves $\partial u_i/\partial x_i(\mathbf{x}^*) = 0$, such that it satisfies the first order conditions:

$$\left[\mathbf{I}_n - \lambda\mathbf{G}\right]\mathbf{x}^* = \mathbf{1}_n$$

As shown in Ballester et al. (2006), the Nash equilibrium exists and is unique if the inverse $\left[\mathbf{I}_n - \lambda\mathbf{G}\right]^{-1}$ exists, that is, when $\lambda \leq 1/\rho$. Then, from definition of $\mathbf{b}(\mathbf{g}, \lambda)$ in (3.2),

$$\mathbf{x}^* = \mathbf{b}(\mathbf{g}, \lambda).$$

Hence, from Lemma 1, we have

$$\mathbf{y}^*(\boldsymbol{\pi}) = \mathbf{X}^T.\mathbf{x}^* = \left[\mathbf{I}_m - a\boldsymbol{\Pi}^T\right]^{-1}.\mathbf{X}^T.\mathbf{1}_n$$

Noting that $\mathbf{X}^T.\mathbf{1}_n = \mathbf{r}$ then proves the Theorem. ■

Proof of Lemma 2. Recall that the elements of $\boldsymbol{\Pi}^p$, $\pi^{[p]}_{ik}$, denotes the total paths of length p for any v in class V_i with its neighbours in V_k. Let $\pi^{[p]}_{i(j^0)k}$ denote the total number of such paths not containing the class j. Similarly, $\pi^{[p]}_{i(j)k}$ denotes only such p-length paths that contain class j. Then, denoting the ik-th element of $(\boldsymbol{\Pi}^p)^T$ as $\pi^{[p,T]}_{ik}$ and setting $\pi^{[0]}_{jj} = 1$, for $0 \leq a \leq 1/\rho$, we have

$$N_{ik}(\boldsymbol{\pi}, a) - N_{ik}(\boldsymbol{\pi}^{-j}, a) = \sum_{p=1}^{\infty} a^p(\pi^{[p,T]}_{ik} - \pi^{[p,T]}_{i(j^0)k}).$$

Note that

$$\begin{aligned}\pi^{[p,T]}_{ik} - \pi^{[p,T]}_{i(j^0)k} &= \pi^{[p,T]}_{i(j)k} = \pi^{[p,T]}_{i(j)k}.\pi^{[0,T]}_{jj} \qquad (\text{since } \pi^{[0]}_{jj} = 1)\\ &= \sum_{\substack{r'+s'=p\\ r'\geq 1,\ s'\geq 1}} \pi^{[r',T]}_{ij}.\pi^{[s',T]}_{jk} - \sum_{\substack{r+s=p\\ r\geq 2,\ s\geq 1}} \pi^{[r,T]}_{i(j)k}.\pi^{[s,T]}_{jj}.\end{aligned}$$

The above identity specifies that total p-length paths from class i to k passing through class j (that is, $\pi^{[p]}_{i(j)k}$) equals the sum of all paths from class i to j and from class j to k of lengths r' and $s'(\geq 1)$, respectively, such that $r' + s' = p$ (that is, $\sum_{\substack{r'+s'=p\\ r'\geq 1,\ s'\geq 1}} \pi^{[r']}_{ij}.\pi^{[s']}_{jk}$), excluding any double counting due to paths involving class j to j loops (which is given by $\sum_{\substack{r+s=p\\ r\geq 2,\ s\geq 1}} \pi^{[r]}_{i(j)k}.\pi^{[s]}_{jj}$). To clarify on this further, note that paths from class i to k passing through j, of any given length p, can in part pass through two nodes belonging to class j itself — which we call the j to j loop part of the overall path. Equating all p-length paths from i to k via j as sum of paths from class i to j and from j to k (of total length p), can result in a double counting of paths with j to j loops as these loops can either be counted in the i to j portion (that is, i to j to j; overall i to j) or in the j to k portion (that is, j to j to k; overall j to k). Hence, we need to subtract one set of such p-length paths from i to k via j involving j to j loops, which is given by $\left(\sum_{\substack{r+s=p\\ r\geq 2,\ s\geq 1}} \pi^{[r]}_{i(j)k}.\pi^{[s]}_{jj}\right)$, to ensure that there is no double counting.

Hence, we have

$$a^p \sum_{\substack{r+s=p\\ r\geq 2,\ s\geq 0}} \pi^{[r,T]}_{i(j)k}.\pi^{[s,T]}_{jj} = a^p \sum_{\substack{r'+s'=p\\ r'\geq 1,\ s'\geq 1}} \pi^{[r',T]}_{ij}.\pi^{[s',T]}_{jk}.$$

This equates to $[N_{ik}(\boldsymbol{\pi}, a) - N_{ik}(\boldsymbol{\pi}^{-j}, a)].N_{jj}(\boldsymbol{\pi}, a) = N_{ij}(\boldsymbol{\pi}, a).N_{jk}(\boldsymbol{\pi}, a)$ which proves the Lemma. ■

Proof of Theorem 2. Note that from Theorem 1, $y^*(\boldsymbol{\pi})$ and $y^*(\boldsymbol{\pi}^{-j})$ are increasing in $t(\boldsymbol{\pi}, \lambda)$ and $t(\boldsymbol{\pi}^{-j}, \lambda)$, respectively. Hence, the planner's objective function (5.2) can be re-written as follows:

$$\sum_{i=1,\ i\neq j}^{m} (t_i(\boldsymbol{\pi}, \lambda) - t_i(\boldsymbol{\pi}^{-j}, \lambda)) + t_j(\boldsymbol{\pi}, \lambda).$$

In what follows, we drop arguments in function for simplicity of notation wherever convenient, and write ik-th element of $\mathbf{N}(\boldsymbol{\pi}^{-j}, \lambda)$ as N^{-j}_{ik}. Since $\mathbf{t}(\boldsymbol{\pi}, \lambda) = \mathbf{N}(\boldsymbol{\pi}, \lambda).\mathbf{r}$, we re-write the above expression as:

$$\sum_{i=1,\ i\neq j}^{m}\left[\sum_{k=1}^{m}N_{ik}r_k-\sum_{k=1,\ k\neq j}^{m}N_{ik}^{-j}r_k\right]+\sum_{k=1}^{m}N_{jk}r_k$$
$$=\sum_{i=1,\ i\neq j}^{m}\left[N_{ij}r_j+\sum_{k=1,\ k\neq j}^{m}\left\{(N_{ik}-N_{ik}^{-j})r_k\right\}\right]+\sum_{k=1}^{m}N_{jk}r_k.$$

Using Lemma 2, this becomes

$$\sum_{i=1,\ i\neq j}^{m}\left[N_{ij}r_j+\sum_{k=1,\ k\neq j}^{m}\left\{\frac{N_{ij}.N_{jk}}{N_{jj}}r_k\right\}\right]+\sum_{k=1}^{m}N_{jk}r_k$$
$$=\sum_{i=1,\ i\neq j}^{m}\left[\sum_{k=1}^{m}\frac{N_{ij}.N_{jk}}{N_{jj}}r_k\right]+\sum_{k=1}^{m}N_{jk}r_k=\sum_{i=1,\ i\neq j}^{m}\left[\frac{N_{ij}}{N_{jj}}t_j\right]+t_j\frac{N_{jj}}{N_{jj}}=\frac{t_j}{N_{jj}}\sum_{i=1}^{m}N_{ij}$$

where the last line uses the equality $t_j=\sum_{k=1}^{m}N_{jk}r_k$. Noting that $\sum_{i=1}^{m}N_{ij}=s_j$ proves the Theorem. ■

Proof of Theorem 3. As in proof for Theorem 2, the problem statement translates to:

$$\max\left\{\frac{\sum_{i=1}^{m}(t_i(\boldsymbol{\pi},\lambda)}{n}-\frac{\sum_{i=1,\ i\neq j}^{m}t_i(\boldsymbol{\pi}^{-j},\lambda))}{n-r_j}\equiv h_j(\boldsymbol{\pi},\lambda)\right\},\ j=1,\dots,m.$$

Dropping arguments in function for simplicity of notation and denoting the ik-th element of $\mathbf{N}(\boldsymbol{\pi}^{-j},\lambda)$ as N_{ik}^{-j}, from $\mathbf{t}(\boldsymbol{\pi},\lambda)=\mathbf{N}(\boldsymbol{\pi},\lambda).\mathbf{r}$ such that $t_i=\sum_{k=1}^{m}N_{ik}r_k$, we have

$$h_j=\sum_{i=1,\ i\neq j}^{m}\left\{\frac{(n-r_j)\sum_{k=1}^{m}N_{ik}r_k-n\sum_{k=1,\ k\neq j}^{m}N_{ik}^{-j}r_k}{n(n-r_j)}\right\}+\frac{\sum_{k=1}^{m}N_{jk}r_k}{n}$$
$$=\sum_{i=1,\ i\neq j}^{m}\left\{\frac{nN_{ij}r_j-r_j\sum_{k=1}^{m}N_{ik}r_k+n\sum_{k=1,\ k\neq j}^{m}\left(N_{ik}-N_{ik}^{-j}\right)r_k}{n(n-r_j)}\right\}+\frac{\sum_{k=1}^{m}N_{jk}r_k}{n}$$

Using Lemma 2, this becomes

$$h_j=\sum_{i=1,\ i\neq j}^{m}\left\{\frac{nN_{ij}r_j-r_jt_i+n\sum_{k=1,\ k\neq j}^{m}\left(\frac{N_{ij}.N_{jk}}{N_{jj}}\right)r_k}{n(n-r_j)}\right\}+\frac{t_j}{n}$$
$$=\sum_{i=1,\ i\neq j}^{m}\left\{\frac{nN_{ij}r_j-r_jt_i+n\left\{\sum_{k=1}^{m}\left(\frac{N_{ij}.N_{jk}}{N_{jj}}\right)r_k-N_{ij}r_j\right\}}{n(n-r_j)}\right\}+\frac{t_j}{n}$$
$$=\sum_{i=1,\ i\neq j}^{m}\left\{\frac{n(N_{ij}/N_{jj})t_j-r_jt_i}{n(n-r_j)}\right\}+\frac{t_j}{n}=\sum_{i=1}^{m}\left\{\frac{n(N_{ij}/N_{jj})t_j-r_jt_i}{n(n-r_j)}\right\}.$$

Noting that $\sum_{i=1}^{m}N_{ij}=s_j$, then, proves the Theorem. ■

Proof of Proposition 1. The proposition is established via proof by contradiction.

Let class j and k ($j,k\in 1,\dots,m$) denote the key class and the per-capita key class, respectively, with corresponding class sizes r_j and r_k. Since removing class k minimises the per-capita network outcome, we have, for any $k\neq j$

$$\frac{y^*(\boldsymbol{\pi}^{-k})}{n-r_k}<\frac{y^*(\boldsymbol{\pi}^{-j})}{n-r_j}. \tag{A1.1}$$

Assume the following hypothesis:

$$r_k>r_j;\quad k\neq j. \tag{A1.2}$$

Since j is the key class removing which minimises the overall network activity and, given that a non-trivial class-structure exists, $r_j,r_k<n$, we have, for any $j\neq k$:

$$y^*(\boldsymbol{\pi}^{-j})<y^*(\boldsymbol{\pi}^{-k})\implies\frac{y^*(\boldsymbol{\pi}^{-j})}{n-r_j}<\frac{y^*(\boldsymbol{\pi}^{-k})}{n-r_j}\implies\frac{y^*(\boldsymbol{\pi}^{-j})}{n-r_j}<\frac{y^*(\boldsymbol{\pi}^{-k})}{n-r_k}$$

where the last inequality comes from noting that $(n-r_k)<(n-r_j)$ from hypothesis (A1.2). This contradicts (A1.1), thus, invalidating the hypothesis and proving the Proposition. ■

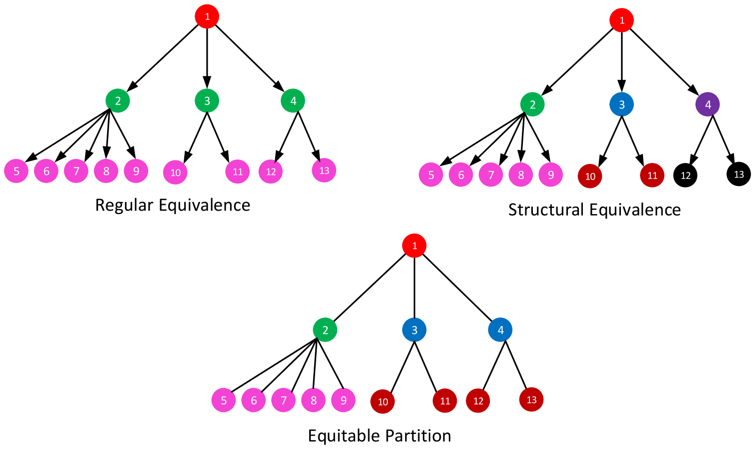


Fig. B.1. Equitable partition vs classical role equivalence notions.

Appendix B. How equitable partition relates to role-equivalence in networks

The intuitive relationship between the structural positions occupied by network players and their network roles is well-established in the sociology literature; see, for instance, Wasserman and Faust (1994). In this Appendix, we illustrate how equitable partition relates to the notion of role-equivalence in networks, in comparison with other classical role-equivalence concepts in literature.

Two of the classical notions of role-equivalence in networks are structural and regular equivalences: while formal definitions can be found in Wasserman and Faust (1994), simply put, structural equivalence stipulates equivalent actors to have *identical* ties to and from *identical* nodes in the network, and regular equivalence theorises that actors who have the same *kind of ties* with others are equivalent in terms of their roles in the network. These notions of equivalences in networks are illustrated in the example below. Equitable partition strikes a balance between these two ideas of role-equivalence, as we illustrate in this Appendix, via a simple example.

Fig. B.1 depicts a notional supervisory network: node 1 denotes an upper-level manager who supervises three mid-level managers (nodes 2,3 and 4), node 2 has five supervisees assigned to her (nodes 5 to 9), while nodes 3 and 4 supervise nodes (10,11) and (12,13), respectively. As is common practice, equivalence is visualised by 'coloration' of nodes in the graph; equivalent nodes have the same colour. The notion of regular equivalence considers all mid-level managers to be equivalent, even though node 2 supervises five individuals as against two each for nodes 3 and 4. Structural equivalence, on the other hand, requires agents to be connected with the same to/from actors in order to be equivalent. As opposed to these, the symmetry-preserving notion of equitable partition takes into consideration the structural differences among actors occupying the same social position, in determining role-equivalent classes, while relaxing the strict identical ties condition of structural equivalence: among the middle-level managers, nodes (3,4) are assumed to play the same role, differently from that of node 2.

Hence, while the structural equivalence clearly presents a severely restrictive definition of equivalence, regular equivalence suffers from the limitation that it does not distinguish between the structural differences among actors occupying the same social position. In comparison, equitable partition presents a reasonable level of compromise free from the limitations of both.

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