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# New cluster algebras from old: integrability beyond Zamolodchikov periodicity

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**Abstract**
We consider discrete dynamical systems obtained as deformations of mutations in cluster algebras associated with finite-dimensional simple Lie algebras. The original (undeformed) dynamical systems provide the simplest examples of Zamolodchikov periodicity: they are affine birational maps for which every orbit is periodic with the same period. Following on from preliminary work by one of us with Kouloukas, here we present integrable maps obtained from deformations of cluster mutations related to the simple root systems $A_3$, $C_2$, $B_3$ and $D_4$. We further show how new cluster algebras arise, by considering Laurentification, that is, a lifting to a higher-dimensional map expressed in a set of new variables (tau functions), for which the dynamics exhibits the Laurent property. For the integrable map obtained by deformation of type $A_3$, which already appeared in our previous work, we show that there is a commuting map of Quispel–Roberts–Thompson (QRT) type which is built from a composition of mutations and a permutation applied to the same cluster algebra of rank 6, with an additional 2 frozen variables. Furthermore, both the deformed $A_3$ map and the QRT map correspond to translation by a generator in the Mordell-Weil group of a rational elliptic surface of rank two, and the underlying cluster algebra comes from a quiver that is mutation equivalent to the $q$-Painlevé III

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quiver found by Okubo. The deformed integrable maps of types $C_2$, $B_3$ and $D_4$ are also related to elliptic surfaces. From a dynamical systems viewpoint, the message of the paper is that special families of birational maps with completely periodic dynamics under iteration admit natural deformations that are aperiodic yet completely integrable.

Keywords: integrable map, cluster algebra, Laurent phenomenon, Zamolodchikov periodicity

## 1. Introduction

The recurrence relation of second order given by

$$x_{n+2}x_n = x_{n+1} + 1 \tag{1.1}$$

is commonly referred to by the name Lyness [26], after the British schoolteacher who observed that any pair of initial values $x_1, x_2$ produces the cycle of values

$$x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1x_2}, \frac{x_1+1}{x_2},$$

repeating with period 5. The Lyness five-cycle has many avatars, including the associahedron $K_4$ [8] and Abel's pentagon identity [29], while it also arises as one of the frieze patterns found by Coxeter [2], who revealed a much earlier connection with the results of Gauss on the *pentagramma mirificum* (see [3] for an interesting but somewhat idiosyncratic historical account of the latter). More recently, (1.1) was found by Zamolodchikov in the context of integrable quantum field theory, as one among many functional relations (Y-systems) that were observed to display periodic behavior; and a general axiomatic framework for describing such relations soon appeared in the shape of coefficient mutations in Fomin and Zelevinsky's theory of cluster algebras [6, 7]. Due to its relevance to Yangians, quantum affine algebras and solvable lattice models, the theory of Y-systems and other associated relations (T-systems and Q-systems) has now been extended considerably, and cluster algebras and other techniques have been applied extensively to resolve Zamolodchikov's periodicity conjecture for Y-systems [22, 23].

The starting point of the recent work [20] was the observation that (1.1) admits a 2-parameter deformation,

$$x_{n+2}x_n = ax_{n+1} + b, \tag{1.2}$$

with parameters $a, b$, which is also often referred to as the Lyness map [36]. This deformed map no longer has periodic orbits, except when the parameters are constrained to the case $b = a^2$ (which gives (1.1) when $a = 1$). Also, while (1.1) corresponds to a mutation in a cluster algebra, and the five-cycle corresponds to a sequence of seeds (in the cluster algebra of finite type $A_2$, to be precise), the deformed map does not have this property. However, (1.2) preserves the same log-canonical symplectic structure

$$\omega = \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{x_1x_2}$$

as (1.1), and has a rational first integral for any $a, b$, so it is an integrable map in the Liouville sense [1, 27, 39]. Moreover, the general Lyness map admits a lift from the plane to seven-dimensional affine space, defined by the transformation

$$x_n = \frac{\tau_n \tau_{n+5}}{\tau_{n+2}\tau_{n+3}}, \tag{1.3}$$

where the tau function $\tau_n$ satisfies the bilinear recurrence

$$\tau_{n+7}\tau_n = a\,\tau_{n+6}\tau_{n+1} + b\,\tau_{n+4}\tau_{n+3}, \tag{1.4}$$

and (as pointed out in [10]) this particular Somos-7 relation is generated by mutations in a cluster algebra of rank 7, with the parameters $a, b$ regarded as frozen variables.

In [20], we derived the most general deformation of a cluster mutation that preserves the same symplectic (or more generally, presymplectic) structure, and showed how other Liouville integrable maps arise as deformations of cluster maps that exhibit Zamolodchikov periodicity. In particular, in addition to (1.2), which is the deformation of the type $A_2$ cluster map, we found integrable maps from deformations of types $A_3$ and $A_4$. The purpose of this article is twofold: firstly, we aim to get a better understanding of the deformed $A_3$ map, together with another commuting map of the kind considered by Quispel, Roberts and Thompson (QRT) [34]; and secondly, we begin to investigate the result of applying an analogous deformation process to cluster algebras associated with other Dynkin types (including examples of the non-simply laced case). We are conducting a separate, parallel, investigation that is concerned with analyzing the deformations of higher rank cluster algebras of type A [12].

### 1.1. Zamolodchikov periodicity and cluster mutations

Some of the simplest examples of cluster algebras are provided by starting from the Cartan matrix $C$ of a finite-dimensional Lie algebra and then constructing an associated companion matrix $B$, called the exchange matrix, which is skew-symmetrizable (i.e. there is a diagonal integer matrix $D$ such that $BD$ is skew-symmetric). The exchange matrix $B$ is the raw combinatorial data that is needed to define a cluster algebra.

Although the general definition of a cluster algebra is rather intricate, and appears somewhat complicated at first sight, one of the original motivations behind this definition was the remarkable phenomenon called Zamolodchikov periodicity. It was observed by Zamolodchikov in [40] that for a certain family of integrable quantum field theories, namely deformations of conformal field theories associated with simple Lie algebras, the thermodynamic Bethe ansatz allowed the form factors of correlation functions to be determined from systems of difference equations called Y-systems, and the solutions of these equations were conjectured to be periodic with the period being the same for any initial data (namely the Coxeter number plus 2).

A general cluster algebra involves two sets of variables: cluster variables, and coefficients. The generators of the algebra (given by clusters) are defined recursively by a process called mutation, which modifies the cluster variables and the coefficients at each step. The mutation formula for coefficients is modeled on Y-systems, while cluster mutation corresponds to so-called T-systems (see [23] for full details). One of Fomin and Zelevinsky's first important results was to classify cluster algebras of finite type: they showed that cluster algebras with finitely many cluster variables correspond precisely to $B$ matrices whose Cartan companions $C$ define Lie algebras of finite type [7]. So this part of the theory of cluster algebras mirrors

Dynkin's classification, and eventually this led to a method to prove Zamolodchikov's periodicity conjectures for Y-systems, and various generalizations thereof.

Below we will just be concerned with exchange relations for cluster variables without coefficients (coefficient-free cluster algebras). However, even without coefficients, T-systems provide another avatar of Zamolodchikov periodicity: sequences of mutations in coefficient-free cluster algebras associated with Dynkin diagrams of finite type produce periodic maps. To set the scene, a coefficient-free cluster algebra $\mathcal{A}(\mathbf{x},B)$ of rank[4] $N$ is constructed by starting from a seed $(\mathbf{x},B)$, which consists of an initial cluster $\mathbf{x}$, that is an $N$-tuple $\mathbf{x}=(x_1,\ldots,x_N)$, and an exchange matrix $B=(b_{ij})\in\mathrm{Mat}_N(\mathbb{Z})$, which is required to be skew-symmetrizable. Then the mutation $\mu_k$ in the direction $k$ produces the new seed $(\mathbf{x}',B')=\mu_k(\mathbf{x},B)$, where $B'=(b'_{ij})$ is obtained via matrix mutation, as specified by

$$b'_{ij}=\begin{cases}-b_{ij} & \text{if } i=k \text{ or } j=k,\\ b_{ij}+\mathrm{sgn}\,(b_{ik})\,[b_{ik}b_{kj}]_+ & \text{otherwise,}\end{cases}\tag{1.5}$$

and the new cluster $\mathbf{x}'=(x'_j)$ is defined by cluster mutation, that is

$$x'_j=\begin{cases}x_k^{-1}\left(\prod_{i=1}^N x_i^{[b_{ki}]_+}+\prod_{i=1}^N x_i^{[-b_{ki}]_+}\right) & \text{for } j=k\\ x_j & \text{for } j\neq k.\end{cases}\tag{1.6}$$

(In the above, $[r]_+=\max(r,0)$ for $r\in\mathbb{R}$.) The cluster algebra $\mathcal{A}(\mathbf{x},B)$ is the subalgebra of $\mathbb{Q}(\mathbf{x})$ generated by the union of all cluster variables obtained from arbitrary sequences of mutations applied to the initial seed. The cluster variables satisfy the Laurent property: they belong to $\mathbb{Z}[x_1^{\pm1},\ldots,x_N^{\pm1}]$, the ring of Laurent polynomials in the variables from the initial cluster $\mathbf{x}$.

T-systems are functional relations between variables that can be constructed from compositions of cluster mutations, as in (1.6). For Y-systems, one requires the more general setup of cluster algebras with coefficients [6], requiring the introduction of another $N$-tuple of coefficient variables $\mathbf{y}=(y_1,\ldots,y_N)$, which are subject to their own set of rules for coefficient mutation. However, our starting point in what follows will be periodic relations between cluster variables $x_j$ (essentially, T-systems rather than Y-systems), so we will omit any further discussion of coefficient mutation. Nevertheless, when we construct deformations we will be led to consider exchange relations between cluster variables involving additional frozen variables, which provide a natural way to treat coefficients appearing in these relations that are constant (they do not mutate). For further background material, the reader is referred to [20] and references.

### *1.2. Outline of the paper*

In the next section we briefly review the construction of the integrable deformation of the $\mathrm{A}_3$ map from [20], which reduces to a Liouville integrable map in the plane, depending on two arbitrary parameters, that preserves a pencil of biquadratic curves. As such, there is an associated QRT map associated with the same invariant pencil, and these two birational maps in the plane commute with one another. We proceed to show that the two maps admit a simultaneous Laurentification, in the sense defined in [14], meaning that they each admit a lift to maps with

[4] The reader unfamiliar with the terminology of cluster algebras should be advised that the word *rank* here refers to the number of cluster variables in each seed; in the finite type case, this happens to coincide with the rank of the root system of the associated Dynkin diagram.

the Laurent property, acting on the same six-dimensional space of tau functions. Moreover, the tau functions correspond to cluster variables in a skew-symmetric cluster algebra of rank 6, extended by an additional 2 frozen variables (associated with the two arbitrary parameters). The lift is such that each of the maps is obtained by composing a suitable sequence of cluster mutations with a permutation. The fact that the two maps commute means that the tau functions take values on the lattice $\mathbb{Z}^2$, and we are able to obtain an explicit formula for the degrees of all tau functions on the lattice, as Laurent polynomials in six initial data.

Section 3 concerns the construction of an integrable deformation of the $\mathrm{C}_2$ cluster map, which is given by a one-parameter family of symplectic maps in the plane, and we show that this lifts to a cluster algebra of rank 5 extended by a single frozen variable. Sections 4 and 5 are concerned with integrable deformations of the periodic cluster maps of types $\mathrm{B}_3$ and $\mathrm{D}_4$, respectively. The situation is more complicated for these latter two examples, as we find that in each case the periodic map admits more than one inequivalent deformation that is integrable. Nevertheless, we are able to find tau functions for each of the different deformations in these cases as well.

Overall, the message of the paper is the observation that (in all examples studied so far) the simplest cluster algebras of all, namely the finite type cluster algebras, hide within them cluster algebras that are much larger (in the sense that they are both higher rank, and infinite), whose more intricate structure is revealed via deformation. From the point of view of discrete dynamical systems, the message is that there are special families of birational maps with completely periodic dynamics under iteration, which admit natural deformations that are aperiodic yet completely integrable.

## 2. Integrable deformation of the $\mathrm{A}_3$ cluster map

The Cartan matrix for the $\mathrm{A}_3$ root system is

$$C=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

which is the companion of the skew-symmetric exchange matrix

$$B=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{2.1}$$

We begin with a seed $(\mathbf{x},B)$ and consider sequences of mutations in the associated cluster algebra.

### 2.1. Periodic map from the $\mathrm{A}_3$ cluster algebra

Starting from the exchange matrix (2.1), we consider the following sequence of three mutations acting on an initial cluster $\mathbf{x}=(x_1,x_2,x_3)$:

$$\begin{array}{llll} \mu_1: & (x_1,x_2,x_3)\mapsto(x_1',x_2,x_3), & & x_1'x_1=x_2+1, \\ \mu_2: & (x_1',x_2,x_3)\mapsto(x_1',x_2',x_3), & & x_2'x_2=x_1'x_3+1, \\ \mu_3: & (x_1',x_2',x_3)\mapsto(x_1',x_2',x_3'), & & x_3'x_3=x_2'+1. \end{array} \tag{2.2}$$

At each step, a prime is affixed only to the cluster variable that is mutated. The matrix (2.1) is cluster mutation-periodic with respect to this sequence of mutations, in the sense that

$$\mu_3\mu_2\mu_1(B) = B,$$

and the corresponding cluster map $\varphi$ given by the composition

$$(x_1,x_2,x_3) \overset{\mu_1}{\longmapsto} (x_1',x_2,x_3) \overset{\mu_2}{\longmapsto} (x_1',x_2',x_3) \overset{\mu_3}{\longmapsto} (x_1',x_2',x_3')$$

is periodic with period $6 = 4+2$ (two more than the Coxeter number of $\mathrm{A}_3$):

$$\varphi = \mu_3\mu_2\mu_1, \qquad \varphi^6(\mathbf{x}) = \mathbf{x}.$$

In order to interpret the above map $\varphi$ and its deformations in terms of Liouville integrability, we need to reduce it to a symplectic map in 2D. To begin with, we note that, by theorem 1.3 in [20], the log-canonical presymplectic 2-form $\omega$ associated with the matrix (2.1) is preserved by the action of $\varphi$, that is

$$\omega = \frac{1}{x_1x_2}\mathrm{d}x_1\wedge\mathrm{d}x_2 + \frac{1}{x_2x_3}\mathrm{d}x_2\wedge\mathrm{d}x_3, \qquad \varphi^*(\omega) = \omega. \tag{2.3}$$

The matrix $B$ has rank two, with the nullspace $\ker B$ spanned by $(1,0,1)^T$, and $\mathrm{im}\,B = (\ker B)^\perp = <\mathbf{v}_1,\mathbf{v}_2>$, spanned by the vectors

$$\mathbf{v}_1 = (0,1,0)^T, \qquad \mathbf{v}_2 = (-1,0,1)^T.$$

Then the monomial quantities

$$u_1 = \mathbf{x}_1^{\mathbf{v}} = x_2, \qquad u_2 = \mathbf{x}_2^{\mathbf{v}} = \frac{x_3}{x_1} \tag{2.4}$$

provide (local) coordinates for the leaves of the null foliation for $\omega$, transverse to the flow of the null vector field $x_1\partial_{x_1} + x_3\partial_{x_3}$, and, from computing $\varphi^*(u_1)$ and $\varphi^*(u_2)$, we find that the rational map

$$\begin{array}{llcll} \pi & : & \mathbb{C}^3 & \rightarrow & \mathbb{C}^2 \\ & & \mathbf{x} = (x_1,x_2,x_3) & \mapsto & \mathbf{u} = (u_1,u_2) \end{array} \tag{2.5}$$

intertwines $\varphi$ with the birational symplectic map $\hat{\varphi}$ given by

$$\begin{array}{llcll} \hat{\varphi} & : & \mathbb{C}^2 & \rightarrow & \mathbb{C}^2 \\ & & \mathbf{u} = (u_1,u_2) & \mapsto & \left(\frac{u_1u_2+u_2+1}{u_1}, \frac{u_2+1}{u_1u_2}\right) \end{array}, \tag{2.6}$$

that is

$$\hat{\varphi}\cdot\pi = \pi\cdot\varphi, \qquad \hat{\varphi}^*(\hat{\omega}) = \hat{\omega},$$

where $\pi^*(\hat{\omega}) = \omega$ is the pullback of the symplectic form

$$\hat{\omega} = \frac{1}{u_1u_2}\mathrm{d}u_1\wedge\mathrm{d}u_2 \tag{2.7}$$

under $\pi$.

It is clear that the reduced map $\hat{\varphi}$ in (2.6) must also be periodic, because it arises from the pullback of $\varphi$ on the monomials (2.4), but in fact, its period is 3 (half that of $\varphi$): $\hat{\varphi}^3(\mathbf{u}) = \mathbf{u}$. Thus any symmetric function averaged over the period of an orbit is an invariant (first integral) for the map $\hat{\varphi}$. In particular, the functions

$$K_1 = \prod_{j=1}^{3} (\hat{\varphi}^*)^{j-1}(u_1) = 2 + \sum_{j=1}^{3} (\hat{\varphi}^*)^{j-1}(u_1), \quad K_2 = \sum_{j=1}^{3} (\hat{\varphi}^*)^{j-1}(u_2)$$

provide two independent invariants. Both of the latter are Laurent polynomials in $u_1, u_2$, with the first being given by

$$\begin{aligned} K_1 &= u_1 + u_2 + \frac{u_2}{u_1} + \frac{2}{u_1} + \frac{1}{u_2} + \frac{1}{u_1 u_2} + 2 \\ &= \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 + 2\,\mathfrak{m}_4 + \mathfrak{m}_5 + \mathfrak{m}_6 + \text{const}, \end{aligned} \tag{2.8}$$

where we have labeled the non-constant Laurent monomials appearing above by

$$\mathfrak{m}_1 = u_1, \quad \mathfrak{m}_2 = u_2, \quad \mathfrak{m}_3 = \frac{u_2}{u_1}, \quad \mathfrak{m}_4 = \frac{1}{u_1}, \quad \mathfrak{m}_5 = \frac{1}{u_2}, \quad \mathfrak{m}_6 = \frac{1}{u_1 u_2}.$$

An integrable deformation of the map $\hat{\varphi}$ can be obtained by introducing parameters into the mutations $\mu_1, \mu_2, \mu_3$ and finding conditions on the parameters such that an analogue of one of these invariants survives under the deformation.

### 2.2. Deformed $A_3$ map

In the $A_3$ case, the particular deformed mutations previously considered in [20] are of the same form as (2.2), but with constant parameters $a_j, b_j$ ($j = 1, 2, 3$) introduced into the exchange relations as follows:

$$\begin{aligned} \mu_1 &: \quad x_1' x_1 = a_1 x_2 + b_1, \\ \mu_2 &: \quad x_2' x_2 = a_2 x_1' x_3 + b_2, \\ \mu_3 &: \quad x_3' x_3 = a_3 x_2' + b_3. \end{aligned} \tag{2.9}$$

The deformed mutations above, which we denote by the same symbols $\mu_j$ as in the undeformed case, destroy the Laurent property: they do not generate Laurent polynomials in $x_1, x_2, x_3$ and the parameters $a_j, b_j$. Nevertheless, as shown in [20], the map $\varphi = \mu_3\mu_2\mu_1$ formed from the composition of the exchange relations (2.9) still preserves the same presymplectic form, that is $\varphi^*(\omega) = \omega$ with $\omega$ as in (2.3). Moreover, the reduction to the leaves of the null foliation defined by $\omega$ produces a birational map in 2D, which can be written in terms of the same coordinates $u_1, u_2$ as in (2.4).

Before considering the deformed 2D map, there is a further simplification to be made, by considering the freedom to rescale the cluster variables, so that $x_j \to \lambda_j x_j, x_j' \to \lambda_j x_j'$ with arbitrary $\lambda_j \neq 0$ for $j = 1, 2, 3$. Exploiting this freedom means that, given generic non-zero parameters $a_j, b_j$ in (2.9), we can always make a choice of coordinates so that 3 of the 6 parameters in the deformed map $\varphi$ can be removed. Following [20], we scale the parameters in (2.9) so that

$$b_2 \to c, \qquad a_2 \to d, \qquad b_3 \to e, \qquad a_1, b_1, a_3 \to 1.$$

With this choice of scale for the parameters of the deformation, we can reduce the composition $\varphi = \mu_3\mu_2\mu_1$ to a 2D map on the leaves of the null foliation for $\omega$, given by

$$\begin{array}{cccc} \hat{\varphi} & : & \mathbb{C}^2 & \to & \mathbb{C}^2 \\ & & \mathbf{u} = (u_1, u_2) & \mapsto & \left( \frac{du_1u_2+du_2+c}{u_1}, \frac{du_2+c}{u_1u_2} + \frac{e-c}{u_2(u_1+1)} \right) . \end{array} \tag{2.10}$$

According to theorem 1.3 in [20], the above map preserves the same rational symplectic form on $\mathbb{C}^2$ as before, that is $\hat{\varphi}^*(\hat{\omega}) = \hat{\omega}$, with

$$\hat{\omega} = \mathrm{d}\log u_1 \wedge \mathrm{d}\log u_2. \tag{2.11}$$

For the deformed map to be integrable in the Liouville sense, we require that at least one of the invariants $K_1, K_2$ identified for the periodic map must survive the deformation. Thus we begin by considering a deformed version of $K_1$, given by taking arbitrary linear combinations of the Laurent monomials $\mathfrak{m}_j$ appearing in (2.8), so that

$$K_1 = \kappa_1\mathfrak{m}_1 + \kappa_2\mathfrak{m}_2 + \kappa_3\mathfrak{m}_3 + \kappa_4\mathfrak{m}_4 + \kappa_5\mathfrak{m}_5 + \kappa_6\mathfrak{m}_6 + \text{const}, \tag{2.12}$$

where $\kappa_j$ are coefficients. Without loss of generality, we can fix the leading coefficient $\kappa_1 = 1$. Then a direct calculation shows that this rational function is invariant if and only if the following conditions hold:

$$c = e, \qquad \kappa_2 = \kappa_3 = \kappa_5 = d\kappa_1, \qquad \kappa_4 = \left(c + d^2\right)\kappa_1, \qquad \kappa_6 = cd\kappa_1.$$

If we impose these conditions, then the reduced map becomes

$$\hat{\varphi}: \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_1^{-1}\left(c + d(u_1+1)u_2\right) \\ (u_1u_2)^{-1}\left(c + du_2\right) \end{pmatrix}, \tag{2.13}$$

and by fixing $\kappa_1 = 1$ and adding the constant $c+1$ to the given linear combination of $\mathfrak{m}_j$, we find that $\hat{\varphi}^*(K_1) = K_1$, with the invariant $K_1$ taking the factorized form

$$K_1 = \frac{(u_1 + du_2 + c)\left((u_1+1)u_2 + d\right)}{u_1u_2}. \tag{2.14}$$

The conclusion of this calculation with the rational function $K_1$ is stated in theorem 2.1 in [20]: when $a_1 = a_3 = b_1 = 1$ and $b_2 = b_3 = c$, $a_2 = d$ (for arbitrary $c, d$) the deformed $\mathrm{A}_3$ map (2.9) reduces to the map (2.13) in the plane, which is integrable in the Liouville sense.

### *2.3. Compatible QRT map*

The family of level sets of the invariant $K_1$, given by fixing the value $K_1 = \kappa$, defines a pencil of biquadratic curves, of which a generic member has genus 1, that is

$$(u_1 + du_2 + c)\left((u_1+1)u_2 + d\right) = \kappa u_1u_2. \tag{2.15}$$

Each curve in such a pencil admits the pair of involutions

$$\iota_h:\ (u_1, u_2) \mapsto \left(u_1^\dagger, u_2\right), \qquad \iota_v:\ (u_1, u_2) \mapsto \left(u_1, u_2^\dagger\right), \tag{2.16}$$

referred to as the horizontal switch and the vertical switch [4], defined by mapping each point on the curve to the other intersection point with a horizontal/vertical line, respectively. Using the Vieta formula for the product of the roots of a quadratic, it is clear that each of these involutions is a birational map, and is given by a formula that is independent of the parameter value $\kappa$. Then their composition

$$\hat{\psi} = \iota_v \cdot \iota_h$$

can be written as

$$\begin{array}{rccc} \hat{\psi} & : & \mathbb{C}^2 & \to & \mathbb{C}^2 \\ & & \mathbf{u} = (u_1, u_2) & \mapsto & (\bar{u}_1, \bar{u}_2) = \left( \frac{(du_2+c)(u_2+d)}{u_1 u_2}, \frac{\bar{u}_1 + c}{u_2(\bar{u}_1+1)} \right) . \end{array} \tag{2.17}$$

By construction, the map (2.17) preserves the same first integral (2.14) as (2.13) does (so $\hat{\psi}^*(K_1) = K_1$), as well as leaving the same symplectic form (2.11) invariant (so $\hat{\psi}^*(\hat{\omega}) = \hat{\omega}$). Hence $\hat{\psi}$ is a Liouville integrable map in the plane. From standard arguments about QRT maps (see e.g. [4]), we can infer a lot more: on each generic fiber of the pencil, either of the two maps $\hat{\psi}$ and $\hat{\varphi}$ acts as an automorphism without fixed points, and since a generic fiber has genus 1, this implies that both maps must act as translation by a point in the abelian group law of the corresponding elliptic curve; hence it follows that the two maps commute, that is

$$\hat{\psi} \cdot \hat{\varphi} = \hat{\varphi} \cdot \hat{\psi}, \tag{2.18}$$

a fact which is readily verified by direct calculation. An interesting question is whether this commutativity is trivial, as would be the case if the two maps were both iterated powers of the same (possibly simpler) birational map. Neither map can be a power of the other, because $\hat{\psi}$ becomes an involution (period 2) when $c = d = 1$, while in that case $\hat{\varphi}$ is the reduction of the $A_3$ cluster map, hence has period 3. However, we will see below that in fact the maps $\hat{\psi}$ and $\hat{\varphi}$ are independent of one another, in the sense that (for generic values of the parameters $c, d$) the Mordell-Weil group of the corresponding elliptic surface has rank 2, and they correspond to independent translations in this group.

### *2.4. Laurentification and cluster structure of commuting maps*

To make contact with the notation used in [20], and to avoid excessive use of indices, we will use $(y, w)$ to denote the coordinates of $\mathbf{u} \in \mathbb{C}^2$, so that

$$\mathbf{u} = (u_1, u_2) = (y, w) .$$

It will further be convenient to adopt the convention that iterates of $\hat{\varphi}$ are denoted by a lower index $n$, so that the coordinates along an orbit are labeled thus:

$$\hat{\varphi}^n(\mathbf{u}) = (y_n, w_n) .$$

One of the main results on the deformed $A_3$ map in our previous work was that it admits Laurentification, in the sense that we can lift $\hat{\varphi}$ to a map in 6 dimensions that has the Laurent property, related via the monomial map

$$\tilde{\pi} : \qquad y_n = \frac{\tau_{n-1}\tau_{n+2}}{\tau_n \tau_{n+1}}, \quad w_n = \frac{\sigma_{n+1}\tau_n}{\sigma_n \tau_{n+1}}, \tag{2.19}$$

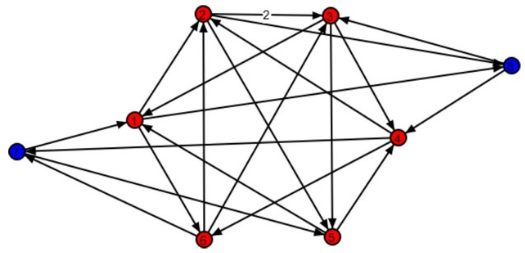


**Figure 1.** The initial quiver $\tilde{Q}$ associated with the extended exchange matrix (2.22).

where $\tilde{\pi}:\mathbb{C}^6\to\mathbb{C}^2$, and the sequence of tau functions $\sigma_n,\tau_n$ satisfy the pair of bilinear equations

$$\begin{aligned}\sigma_{n+2}\,\tau_{n-1} &= d\,\sigma_{n+1}\tau_n + c\,\sigma_n\tau_{n+1},\\ \tau_{n+3}\,\sigma_n &= \tau_{n+1}\sigma_{n+2} + d\,\tau_{n+2}\sigma_{n+1}.\end{aligned} \tag{2.20}$$

(Note that the index on $\tau_n$ has been shifted compared with [20].)

A set of initial data for (2.20) is provided by a set of six tau functions, namely

$$\mathbf{x}^\dagger = (\tau_{-1},\tau_0,\tau_1,\tau_2,\sigma_0,\sigma_1) = (\tilde{x}_j)_{1\leqslant j\leqslant 6}.$$

It is possible to prove directly that under iteration, the bilinear system (2.20) generates Laurent polynomials in these initial data, belonging to the ring $\mathbb{Z}[c,d,\tau_{-1}^{\pm1},\tau_0^{\pm1},\tau_1^{\pm1},\tau_2^{\pm1}]$. However, this is more easily seen as a consequence of the fact that each of the bilinear equations correspond to mutations in a cluster algebra, whose coefficient-free part is given by an exchange matrix $B^\dagger$ obtained by pulling back the symplectic form (2.11) by $\tilde{\pi}$, as in (2.19), to find

$$\tilde{\omega} = \tilde{\pi}^*(\hat{\omega}) = \sum_{i<j}\tilde{b}_{ij}\frac{\mathrm{d}\tilde{x}_i\wedge\mathrm{d}\tilde{x}_j}{\tilde{x}_i\tilde{x}_j}, \tag{2.21}$$

where all the indices above run from 1 to 6. The matrix $B^\dagger = (\tilde{b}_{ij})_{1\leqslant i,j\leqslant 6}$ is skew-symmetric, and is the $6\times 6$ square submatrix appearing at the top of the extended $8\times 6$ exchange matrix $\tilde{B}$ given by

$$\tilde{B} = \begin{pmatrix} 0 & 1 & -1 & 0 & -1 & 1\\ -1 & 0 & 2 & -1 & 1 & -1\\ 1 & -2 & 0 & 1 & 1 & -1\\ 0 & 1 & -1 & 0 & -1 & 1\\ 1 & -1 & -1 & 1 & 0 & 0\\ -1 & 1 & 1 & -1 & 0 & 0\\ 1 & 0 & 0 & -1 & 1 & -1\\ -1 & -1 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{2.22}$$

The quiver $\tilde{Q}$ associated with $\tilde{B} = (\tilde{b}_{ij})$ is shown in figure 1: the convention is that $|b_{ij}|$ is the number of arrows between node $i$ and node $j$, with the sign fixed according to whether $|b_{ij}|$ arrows run $i\to j$ or vice versa.

**Remark 2.1.** Note that the subquiver of $\tilde{Q}$ in figure 1 containing the 6 unfrozen nodes is mutation equivalent to the quiver that was shown by Okubo to produce the $q$-Painlevé III equation via an appropriate sequence of coefficient mutations (cf. figure 21 in [32]).

To specify the cluster algebra that contains the appropriate sequence of tau functions, we proceed to extend the initial data, in order to get an extended cluster

$$\tilde{\mathbf{x}} = (\tau_{-1}, \tau_0, \tau_1, \tau_2, \sigma_0, \sigma_1, c, d) = (\tilde{x}_j)_{1 \leqslant j \leqslant 8},$$

which includes the coefficients $c = \tilde{x}_7$ and $d = \tilde{x}_8$ as additional frozen variables (which are not allowed to be mutated). Then we see that the pair of successive mutations

$$\begin{aligned} \tilde{\mu}_1: \quad & \tilde{x}_1' \tilde{x}_1 = \tilde{x}_8 \tilde{x}_2 \tilde{x}_6 + \tilde{x}_7 \tilde{x}_3 \tilde{x}_5, \\ \tilde{\mu}_5: \quad & \tilde{x}_5' \tilde{x}_5 = \tilde{x}_8 \tilde{x}_4 \tilde{x}_6 + \tilde{x}_1' \tilde{x}_3 \end{aligned} \tag{2.23}$$

together correspond to iterations of (2.20), provided that these two mutations are composed with the inverse of the cyclic permutation $\rho = (123456)$ (which only acts on the non-frozen variables). More precisely, we have

$$\tilde{\mu}_5 \tilde{\mu}_1 \left(\tilde{B}\right) = \rho \left(\tilde{B}\right), \qquad \tilde{\pi} \cdot \tilde{\varphi} = \hat{\varphi} \cdot \tilde{\pi}, \tag{2.24}$$

where the lifted map $\tilde{\varphi}$ acts as the shift $n \to n+1$ on the tau functions, that is, it acts as

$$\tilde{\varphi} = \rho^{-1} \tilde{\mu}_5 \tilde{\mu}_1: \quad (\tau_{n-1}, \tau_n, \tau_{n+1}, \tau_{n+2}, \sigma_n, \sigma_{n+1}, c, d) \mapsto (\tau_n, \tau_{n+1}, \tau_{n+2}, \tau_{n+3}, \sigma_{n+1}, \sigma_{n+2}, c, d) \tag{2.25}$$

for all $n$, but leaves $\tilde{B}$ invariant, since $\tilde{\varphi}(\tilde{B}) = \rho^{-1} \tilde{\mu}_5 \tilde{\mu}_1(\tilde{B}) = \tilde{B}$ from (2.24).

As described above, the map $\hat{\varphi}$ is the reduction to the plane of the integrable deformation of the $\mathrm{A}_3$ cluster map. The preceding observations about the map $\tilde{\varphi}$ in (2.25), which is the Laurentification of $\hat{\varphi}$, were summarized in theorem 2.3 in [20]. To go beyond the latter, we now explain how the same ideas can be extended to the QRT map (2.17) that commutes with (2.13). To do this, it will initially be convenient to abuse notation by adding an index $m$ to the pair of coordinates $(y, w)$, and write the sequence of points on an orbit as

$$\hat{\psi}^m(\mathbf{u}) = (y_m, w_m).$$

To avoid creating any confusion, we henceforth take the convention that the letters $m, n$ are used exclusively to label iterates of $\hat{\psi}$, $\hat{\varphi}$, respectively. Then a lift of $\hat{\psi}$ to a map in 6 dimensions is defined by

$$\bar{\pi}: \qquad y_m = \frac{\eta_m \chi_m}{\xi_m \theta_m}, \quad w_m = \frac{\xi_{m+1} \theta_m}{\xi_m \theta_{m+1}}, \tag{2.26}$$

where the tau functions $\eta_m, \chi_m, \xi_m, \theta_m$ satisfy the bilinear system

$$\begin{aligned} \eta_{m+1} \chi_m &= d \xi_{m+1} \theta_m + c \xi_m \theta_{m+1}, \\ \chi_{m+1} \eta_m &= \xi_{m+1} \theta_m + d \xi_m \theta_{m+1}, \\ \xi_{m+2} \theta_m &= c \xi_{m+1} \theta_{m+1} + \chi_{m+1} \eta_{m+1}, \\ \theta_{m+2} \xi_m &= \xi_{m+1} \theta_{m+1} + \chi_{m+1} \eta_{m+1}. \end{aligned} \tag{2.27}$$

The above bilinear system for the QRT map has the Laurent property, as summarized in the following result, which is a direct analogue of theorem 2.3 in [20].

**Theorem 2.2.** *The tau function sequences $(\eta_m)$, $(\chi_m)$, $(\xi_m)$, $(\theta_m)$ for the integrable map (2.17) consist of elements of the ring of Laurent polynomials with positive coefficients, lying in $\mathbb{Z}_{>0}[c, d, \chi_0^{\pm 1}, \theta_0^{\pm 1}, \xi_0^{\pm 1}, \eta_0^{\pm 1}, \theta_1^{\pm 1}, \xi_1^{\pm 1}]$, being generated by the action of a permutation composed with a sequence of mutations in the cluster algebra defined by the quiver in figure 1.*

**Proof.** Setting $m = 0$ in (2.26), the pullback of the symplectic form (2.11) is

$$\tilde{\omega} = \tilde{\pi}^* \left( \mathrm{d}\log\left(\frac{\eta_0\chi_0}{\xi_0\theta_0}\right) \wedge \mathrm{d}\log\left(\frac{\xi_1\theta_0}{\xi_0\theta_1}\right)\right) = \sum_{1\leqslant i<j\leqslant 6} \tilde{b}_{ij}\frac{\mathrm{d}\tilde{x}_i \wedge \mathrm{d}\tilde{x}_j}{\tilde{x}_i\tilde{x}_j},$$

which coincides precisely with (2.21), for the same $6\times 6$ skew-symmetric submatrix $B^\dagger = (\tilde{b}_{ij})$, if we identify the unfrozen variables as

$$\mathbf{x}^\dagger = (\chi_0, \theta_0, \xi_0, \eta_0, \theta_1, \xi_1) = (\tilde{x}_j)_{1\leqslant j\leqslant 6}.$$

Extending this to the full matrix $\tilde{B}$, as in (2.22), with the same frozen variables as before, namely $c = \tilde{x}_7$ and $d = \tilde{x}_8$, we see that repeatedly applying the sequence of four successive mutations

$$\begin{array}{ll} \tilde{\mu}_1: & \tilde{x}_1'\tilde{x}_1 = \tilde{x}_8\tilde{x}_2\tilde{x}_6 + \tilde{x}_7\tilde{x}_3\tilde{x}_5, \\ \tilde{\mu}_4: & \tilde{x}_4'\tilde{x}_4 = \tilde{x}_2\tilde{x}_6 + \tilde{x}_8\tilde{x}_3\tilde{x}_5, \\ \tilde{\mu}_2: & \tilde{x}_2'\tilde{x}_2 = \tilde{x}_7\tilde{x}_5\tilde{x}_6 + \tilde{x}_1'\tilde{x}_4', \\ \tilde{\mu}_3: & \tilde{x}_3'\tilde{x}_3 = \tilde{x}_5\tilde{x}_6 + \tilde{x}_1'\tilde{x}_4' \end{array} \tag{2.28}$$

is equivalent to iterating the bilinear system (2.27), provided that these four mutations are composed with the inverse of the cyclic permutation $\bar{\rho} = (14)(2536)$ (which only affects the non-frozen variables). To be precise we have

$$\tilde{\mu}_3\tilde{\mu}_2\tilde{\mu}_4\tilde{\mu}_1\left(\tilde{B}\right) = \bar{\rho}\left(\tilde{B}\right), \qquad \tilde{\pi}\cdot\tilde{\psi} = \hat{\psi}\cdot\tilde{\pi}, \tag{2.29}$$

where the lifted map $\tilde{\psi}$ acts as the shift $m \to m+1$ on the tau functions, that is, it acts as

$$\begin{aligned} \tilde{\psi} = \bar{\rho}^{-1}\tilde{\mu}_3\tilde{\mu}_2\tilde{\mu}_4\tilde{\mu}_1: \quad & (\chi_m, \theta_m, \xi_m, \eta_m, \theta_{m+1}, \xi_{m+1}, c, d) \\ \mapsto & (\chi_{m+1}, \theta_{m+1}, \xi_{m+1}, \eta_{m+1}, \theta_{m+2}, \xi_{m+2}, c, d) \end{aligned} \tag{2.30}$$

for all $m$, but leaves $\tilde{B}$ invariant, since $\tilde{\psi}(\tilde{B}) = \bar{\rho}^{-1}\tilde{\mu}_3\tilde{\mu}_2\tilde{\mu}_4\tilde{\mu}_1(\tilde{B}) = \tilde{B}$ from (2.29). The fact that the sequences of tau functions generated by (2.27) consist of Laurent polynomials in the initial cluster variables, with integer coefficients, is just the Laurent phenomenon for the cluster algebra [6], while the coefficients being in $\mathbb{Z}_{>0}$ is a consequence of the positivity conjecture, proved for skew-symmetric exchange matrices in [25]. □

### *2.5. Somos relations for tau functions*

A sequence $(\mathrm{x}_n)$ generated by a quadratic recurrence relation of the form

$$\mathrm{x}_{n+k}\mathrm{x}_n = \sum_{j=1}^{\lfloor k/2\rfloor} \alpha_j\,\mathrm{x}_{n+j}\mathrm{x}_{n+k-j} \tag{2.31}$$

is known as a Somos-$k$ sequence. There are particular cases of Somos-type recurrences that fit into the framework of cluster algebras, namely those that have a sum of two monomials on the right-hand side of (2.31)[5]. One such case is that of Somos-5 sequences, which satisfy a

[5] Somos recurrences with three monomials on the right can be constructed in the more general framework of LP algebras [24].

bilinear relation given in the form

$$\mathrm{x}_{n+5}\mathrm{x}_n = \tilde{\alpha}\,\mathrm{x}_{n+4}\mathrm{x}_{n+1} + \tilde{\beta}\,\mathrm{x}_{n+3}\mathrm{x}_{n+2}.$$

It turns out that the Somos-5 recurrence can be reduced to a certain QRT map, equivalent to the recurrence

$$u_{n+1}u_nu_{n-1} = \tilde{\alpha}\,u_n + \tilde{\beta}, \tag{2.32}$$

where $u_n$ is given by a certain ratio composed of tau functions $\tau_n$ (see [15], for instance). We refer to the above iteration as the Somos-5 QRT map.

The Laurentification of the map (2.13) in the plane produces two sequences of tau functions $\sigma_n, \tau_n$. The form of the substitution for $y_n$ in (2.19) is the same as was used for Somos-5 in [15], which suggests a connection with Somos sequences. It turns out that, along each orbit of the cluster map $\tilde{\varphi}$, the tau functions $\sigma_n$ and $\tau_n$ both satisfy the same Somos-5 relation.

**Theorem 2.3.** *Along each orbit of the cluster map $\tilde{\varphi}$, given by iteration of the bilinear system (2.20), the sequence of tau functions $(\tau_n)$ satisfies the Somos-5 relation*

$$\tau_{n+5}\tau_n = \tilde{\alpha}\,\tau_{n+4}\tau_{n+1} + \tilde{\beta}\,\tau_{n+3}\tau_{n+2}, \tag{2.33}$$

*with constant coefficients given by*

$$\tilde{\alpha} = d^2 - c \qquad \tilde{\beta} = cK_1 + (c+1)\left(d^2 - c\right),$$

*where $K_1$ is the corresponding value of the first integral for the map (2.13), obtained from the initial tau functions by setting*

$$u_1 = \frac{\tau_{-1}\tau_2}{\tau_0\tau_1}, \quad u_2 = \frac{\sigma_1\tau_0}{\sigma_0\tau_1}, \tag{2.34}$$

*in the formula (2.14). Similarly, on each orbit of $\tilde{\varphi}$, the sequence of tau functions $(\sigma_n)$ satisfies the Somos-5 relation (2.33) with the same coefficients $\tilde{\alpha}, \tilde{\beta}$.*

**Proof.** As a recursion relation, the first component of the map $\hat{\varphi}$ given by (2.13) is equivalent to

$$y_{n+1} = \frac{c + d\,(y_n + 1)\,w_n}{y_n},$$

while the first component of the inverse map $\hat{\varphi}^{-1}$ gives

$$y_{n-1} = \frac{cd + cw_n + dy_n}{w_ny_n}.$$

Then a direct calculation shows that

$$y_{n+1}y_ny_{n-1} = \left(d^2 - c\right)\,y_n + cK_1 + (c+1)\left(d^2 - c\right),$$

where $K_1$ is the first integral for the map $\hat{\varphi}$ with $u_1 = y_n$, $u_2 = w_n$. Note that as $K_1$ is invariant under $\hat{\varphi}$, from (2.24) we have $\tilde{\varphi}^*(\tilde{\pi}^*(K_1)) = \tilde{\pi}^*(\hat{\varphi}^*(K_1)) = \tilde{\pi}^*(K_1)$, so $K_1$ pulls back to a first integral for the cluster map $\tilde{\varphi}$, obtained by making the substitutions (2.34) in (2.14). Thus we have shown that $u_n = y_n$ satisfies the Somos-5 QRT map, in the form of the recurrence (2.32) with the required values of the coefficients $\tilde{\alpha}, \tilde{\beta}$, and this implies immediately that $\tau_n$ is a solution of the Somos-5 relation (2.33), by substituting for $y_n$ as in (2.19). Now from the other sequence of tau functions $(\sigma_n)$ we can define the ratio

$$y_n^* := \frac{\sigma_{n-1}\sigma_{n+2}}{\sigma_n\sigma_{n+1}} = \frac{y_n w_{n+1}}{w_{n-1}},$$

where the final expression on the right-hand side above is obtained from (2.19). Using

$$w_{n+1} = \frac{c + dw_n}{w_n y_n}, \quad w_{n-1} = \frac{y_n}{d + w_n}$$

from the second components of $\hat{\varphi}$ and $\hat{\varphi}^{-1}$, we can rewrite $y_n^*$ in terms of $y_n, w_n$ alone, and then calculate analogous expressions for $y_{n+1}^*$ and $y_{n-1}^*$. Hence we find that $u_n = y_n^*$ is another solution of the Somos-5 QRT map (2.32), with the same coefficients $\tilde{\alpha}, \tilde{\beta}$, and as a direct consequence the sequence $(\sigma_n)$ also satisfies (2.33). □

Both the preceding observation and its proof were based on recognizing the specific ratio of tau functions $\tau_n$ for $y_n$ appearing in (2.19). However, a more systematic approach to deriving and proving Somos-type relations, which does not require any *a priori* information, is to regard them as linear relations between degree 2 products of tau functions. We will illustrate this approach by considering the tau functions $\chi_m, \theta_m, \xi_m, \eta_m$ generated by iterating the bilinear system (2.27).

Let us suppose that the sequence $(\xi_m)$ generated by iteration of (2.27) satisfies a Somos-$k$ recurrence relation for some $k$. The simplest non-trivial case to try is $k = 4$. In that case, we write down two adjacent iterations of a Somos-4 recurrence for $\xi_m$, in the form

$$\begin{aligned} \xi_{m+4}\xi_m &= \alpha\,\xi_{m+3}\xi_{m+1} + \beta\,\xi_{m+2}^2, \\ \xi_{m+5}\xi_{m+1} &= \alpha\,\xi_{m+4}\xi_{m+2} + \beta\,\xi_{m+3}^2. \end{aligned} \tag{2.35}$$

A direct way to check whether such a relation is valid is to write down one more iteration of the recurrence, and verify that a corresponding $3 \times 3$ determinant vanishes; and this method can be extended to check a Somos-$k$ relation of arbitrary order $k$ (cf. the proof of theorem 3.4 in the next section). In any case, the equations (2.35) provide a linear system for the coefficients $\alpha, \beta$, with solution

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \xi_{m+3}\xi_{m+1} & \xi_{m+2}^2 \\ \xi_{m+4}\xi_{m+2} & \xi_{m+3}^2 \end{pmatrix}^{-1} \begin{pmatrix} \xi_{m+4}\xi_m \\ \xi_{m+5}\xi_{m+1} \end{pmatrix},$$

so the Somos-4 relation is valid if and only if the components of the above vector are constant (that is, independent of the index $m$). In the particular case at hand, we have initial data given by the cluster $\mathbf{x}^\dagger = (\chi_0, \theta_0, \xi_0, \eta_0, \theta_1, \xi_1)$, and to determine all the terms appearing in (2.35) with $m = 0$ we need to perform 4 iterations of the bilinear system (2.27). Then from solving the pair of linear equations for the coefficients, we find that $\alpha = (c-1)^2 d$, which only depends on the parameters $c, d$, so is clearly constant, while $\beta = \beta(c, d, \mathbf{x}^\dagger)$ is a rather complicated rational function (in fact, a Laurent polynomial) of $c, d$ and the initial data $\mathbf{x}^\dagger$. Nevertheless, under the action of the cluster map $\tilde{\psi}$ it can be verified directly that $\tilde{\psi}^*(\beta) = \beta$, so that this coefficient is constant along each orbit. This then suggests that $\beta$ can be rewritten as a polynomial function of $c, d$ and $K_1$, the first integral for the map (2.17), which indeed turns out to be the case. We can apply exactly the same method to seek Somos-type relations for the other tau functions $\chi_m, \theta_m, \eta_m$, and find that they all satisfy the same Somos-4 recurrence. The final result is summarized as follows.

**Theorem 2.4.** *Along each orbit of the cluster map $\tilde{\psi}$, given by iteration of the bilinear system (2.27), the sequence of tau functions $(\xi_n)$ satisfies the Somos-4 relation*

$$\xi_{m+4}\xi_m = \alpha\,\xi_{m+3}\xi_{m+1} + \beta\,\xi_{m+2}^2, \tag{2.36}$$

*with constant coefficients given by*

$$\alpha = (c-1)^2\,d, \qquad \beta = cK_1^2 + (c+1)\,d^2K_1 + d^4 + (c-1)^2\,d^2,$$

*where $K_1$ is the corresponding value of the first integral (2.14) for (2.17), obtained in terms of the initial tau functions by pulling it back via the map $\bar{\pi}$, for $u_1 = y_0$, $u_2 = w_0$ given by (2.26) with $m = 0$. Similarly, on each orbit of $\tilde{\psi}$, the sequences $(\chi_m), (\theta_m), (\eta_m)$ all satisfy the Somos-4 relation (2.36) with the same coefficients $\alpha, \beta$.*

### 2.6. Tau functions on the $\mathbb{Z}^2$ lattice

The preceding results show that the action of the commuting integrable birational maps (2.17) and (2.13) in the plane lifts to a pair of commuting cluster maps $\tilde{\psi}$ and $\tilde{\varphi}$, which act on seeds in the same cluster algebra of rank 6 (with 2 additional frozen variables). Thus far we have used the two letters $m, n$ to index iterations of the two different maps and/or their lifts, so now it makes sense to combine them into a pair $(m,n) \in \mathbb{Z}^2$, and write

$$\hat{\psi}^m\hat{\varphi}^n(\mathbf{u}) = (y_{m,n}, w_{m,n}), \qquad (m,n) \in \mathbb{Z}^2,$$

as well as introducing a tau function $T_{m,n}$ on the $\mathbb{Z}^2$ lattice, such that

$$y_{m,n} = \frac{T_{m,n-1}T_{m,n+2}}{T_{m,n}T_{m,n+1}}, \qquad w_{m,n} = \frac{T_{m+1,n+1}T_{m,n}}{T_{m+1,n}T_{m,n+1}}. \tag{2.37}$$

Then the initial seed in the associated cluster algebra is $(\tilde{\mathbf{x}}, \tilde{B})$, with the extended exchange matrix $\tilde{B}$ as in (2.22), and the initial cluster being specified by

$$\tilde{\mathbf{x}} = (\tilde{x}_j)_{1\leqslant j\leqslant 8} = (T_{0,-1}, T_{0,0}, T_{0,1}, T_{0,2}, T_{1,0}, T_{1,1}, c, d). \tag{2.38}$$

Under the combined actions of the cluster maps $\tilde{\psi}$ and $\tilde{\varphi}$, which are equivalent to iterating the bilinear equations given by the systems (2.27) and (2.20), respectively, a generic set of initial values results in a complete set of tau functions defined at each point in the lattice, that is $(T_{m,n})_{(m,n)\in\mathbb{Z}^2}$. More precisely, due to the Laurent phenomenon, if each of the initial $T_{i,j}$ appearing in (2.38) is non-zero, then all other $T_{m,n}$ are obtained by evaluating suitable Laurent polynomials at these initial values. In fact, these functions on the lattice are completely characterized as the solution of a system of four bilinear lattice equations.

**Theorem 2.5.** *The tau functions on $\mathbb{Z}^2$, associated via (2.37) with combined iteration of the commuting integrable maps (2.17) and (2.13), satisfy the following system of bilinear equations:*

$$\begin{aligned} T_{m+1,n+2}T_{m,n-1} &= d\,T_{m+1,n+1}T_{m,n} + c\,T_{m+1,n}T_{m,n+1}, \\ T_{m,n+2}T_{m+1,n-1} &= d\,T_{m,n+1}T_{m+1,n} + T_{m+1,n+1}T_{m,n}, \\ T_{m+2,n+1}T_{m,n} &= c\,T_{m+1,n+1}T_{m+1,n} + T_{m+1,n+2}T_{m+1,n-1}, \\ T_{m+2,n}T_{m,n+1} &= T_{m+1,n+1}T_{m+1,n} + T_{m+1,n+2}T_{m+1,n-1}. \end{aligned} \tag{2.39}$$

*Conversely, any solution of this system of bilinear lattice equations produces a simultaneous solution $(y_{m,n}, w_{m,n})$ of the pair of iterated maps (2.17) and (2.13).*

**Proof.** Essentially this is just a matter of rewriting the results of theorem 2.2 with appropriate indices in $\mathbb{Z}^2$, and similarly for the preceding statements about the bilinear system for the tau functions of the map (2.13), and checking that they are compatible. First of all, for the tau functions of the QRT map (2.17), we can identify the non-frozen part of any cluster via

$$(T_{m,n-1}, T_{m,n}, T_{m,n+1}, T_{m,n+2}, T_{m+1,n}, T_{m+1,n+1}) \equiv (\chi_m, \theta_m, \xi_m, \eta_m, \theta_{m+1}, \xi_{m+1}),$$

where the equivalence means that, going from left to right, we simply suppress the dependence on the second index $n$. The action of a single iteration of the map $\tilde{\psi}$ on any such cluster, corresponding to the shift $m \to m+1$, is obtained by solving each of the four equations (2.39) in turn, to obtain the transformation

$$\begin{aligned} \tilde{\psi}: \quad & (T_{m,n-1}, T_{m,n}, T_{m,n+1}, T_{m,n+2}, T_{m+1,n}, T_{m+1,n+1}) \\ & \mapsto (T_{m+1,n-1}, T_{m+1,n}, T_{m+1,n+1}, T_{m+1,n+2}, T_{m+2,n}, T_{m+2,n+1}), \end{aligned}$$

which is equivalent to one iteration of the four bilinear equations (2.27), or to the composition of the four successive cluster mutations (2.28) together with a permutation. Similarly, for the tau functions of the map (2.13), we can write the 'forgetful' equivalence

$$(T_{m,n-1}, T_{m,n}, T_{m,n+1}, T_{m,n+2}, T_{m+1,n}, T_{m+1,n+1}) \equiv (\tau_{n-1}, \tau_n, \tau_{n+1}, \tau_{n+2}, \sigma_n, \sigma_{n+1}),$$

where the dependence on the first index $m$ is suppressed upon moving from left to right. Then an iteration of the map $\tilde{\varphi}$ acting on such a cluster corresponds to the shift $n \to n+1$, which is achieved by solving the first equation in (2.39) to find $T_{m+1,n+2}$, then using the second equation in the shifted form

$$T_{m,n+3}T_{m+1,n} = d\, T_{m,n+2}T_{m+1,n+1} + T_{m+1,n+2}T_{m,n+1}$$

to find $T_{m,n+3}$, so that overall one has

$$\begin{aligned} \tilde{\varphi}: \quad & (T_{m,n-1}, T_{m,n}, T_{m,n+1}, T_{m,n+2}, T_{m+1,n}, T_{m+1,n+1}) \\ & \mapsto (T_{m,n}, T_{m,n+1}, T_{m,n+2}, T_{m,n+3}, T_{m+1,n+1}, T_{m+1,n+2}), \end{aligned}$$

and clearly this is equivalent to performing one iteration of the pair of bilinear equations (2.20), or to the composition of the two cluster mutations (2.23) together with a permutation. The converse statement follows immediately, because whenever the double sequence $(y_{m,n}, w_{m,n})_{(m,n)\in\mathbb{Z}^2}$ is specified in terms of a solution of the system (2.39) by the formulae (2.37), the bilinear equations imply that $\hat{\psi}\big((y_{m,n}, w_{m,n})\big) = (y_{m+1,n}, w_{m+1,n})$ and $\hat{\varphi}\big((y_{m,n}, w_{m,n})\big) = (y_{m,n+1}, w_{m,n+1})$ hold for all $m, n$. □

**Remark 2.6.** By theorems 2.3 and 2.4, whenever $T_{m,n}$ is a solution of the bilinear lattice system (2.39), it also satisfies a Somos-5 relation in $n$, and a Somos-4 relation in $m$. Moreover, the coefficients of both of these Somos relations are constant (that is, independent of both $m$ and $n$).

### 2.7. Tropical dynamics and degree growth

Following [9], it is constructive to consider the structure of the Laurent polynomials in a cluster algebra in terms of the so-called d-vectors, which permit the degree growth of the cluster variables to be determined from the tropical analogues of the exchange relations. In this case, we can write each cluster of tau functions related by iteration of the lattice system (2.39) as

$$\tilde{\mathbf{x}}_{m,n} := (T_{m,n-1}, T_{m,n}, T_{m,n+1}, T_{m,n+2}, T_{m+1,n}, T_{m+1,n+1}) = (\tilde{x}_j(m,n))_{1\leqslant j\leqslant 6}, \tag{2.40}$$

where, in terms of the initial cluster $\tilde{\mathbf{x}}$ given by (2.38), the Laurent property means that we may write

$$\tilde{x}_j(m,n) = \frac{\mathrm{N}_j(m,n;\tilde{\mathbf{x}})}{\tilde{\mathbf{x}}^{\mathbf{d}_j(m,n)}}, \qquad j = 1,\ldots,6, \tag{2.41}$$

with the numerators $\mathrm{N}_j(m,n;\tilde{\mathbf{x}}) \in \mathbb{Z}[\tilde{\mathbf{x}}]$ being polynomials in the variables of the initial cluster $\tilde{\mathbf{x}}$ which are not divisible by any of $\tilde{x}_1,\ldots,\tilde{x}_6$, and the denominators being monomials whose exponents are encoded in the integer d-vectors $\mathbf{d}_j(m,n) \in \mathbb{Z}^6$. Then it is convenient to combine the six d-vectors in each cluster into a $6\times 6$ matrix $D_{m,n}$, that is

$$D_{m,n} := (\mathbf{d}_1(m,n)\ \mathbf{d}_2(m,n)\ \mathbf{d}_3(m,n)\ \mathbf{d}_4(m,n)\ \mathbf{d}_5(m,n)\ \mathbf{d}_6(m,n)). \tag{2.42}$$

While the degrees of the denominators grow, so that for large enough $m,n$ all the d-vectors belong to $\mathbb{Z}^6_{>0}$, the initial conditions require that

$$\mathrm{N}_j(0,0;\tilde{\mathbf{x}}) = 1 \quad \text{for} \quad j = 1,\ldots,6$$

and

$$D_{0,0} = -I,$$

where $I$ denotes the $6\times 6$ identity matrix. Due to homogeneity of the cluster variables (equivalently, the fact that the tau functions satisfy bilinear equations, so they all have the same weight), the d-vectors encode everything about the degrees of the Laurent polynomials: indeed, homogeneity requires that the total degree of each numerator (regarding $c,d$ as fixed constants) is

$$\deg_{\tilde{\mathbf{x}}}(\mathrm{N}_j(m,n;\tilde{\mathbf{x}})) = \mathbf{e}^T\mathbf{d}_j(m,n) + 1, \qquad \mathbf{e} = (1,1,1,1,1,1)^T,$$

that is, one more than the total degree of the monomial denominators.

Using standard arguments from [9], it can be shown that under the action of cluster mutation, the components of the d-vectors satisfy the $(\max,+)$ tropical version of the exchange relations. To be precise, the action of the cluster map $\tilde{\varphi} = \rho^{-1}\tilde{\mu}_5\tilde{\mu}_1$ on an initial cluster of d-vectors (a tropical seed) takes the form

$$\tilde{\varphi}: \quad (\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3,\mathbf{d}_4,\mathbf{d}_5,\mathbf{d}_6) \mapsto \left(\mathbf{d}_2,\mathbf{d}_3,\mathbf{d}_4,\mathbf{d}_5',\mathbf{d}_6,\mathbf{d}_1'\right),$$

which is composed of the combination of the $(\max,+)$ analogues of the mutations $\tilde{\mu}_1,\tilde{\mu}_5$, namely

$$\begin{aligned} \tilde{\mu}_1: &\quad \mathbf{d}_1' + \mathbf{d}_1 = \max(\mathbf{d}_2+\mathbf{d}_6, \mathbf{d}_3+\mathbf{d}_5), \\ \tilde{\mu}_5: &\quad \mathbf{d}_5' + \mathbf{d}_5 = \max\left(\mathbf{d}_4+\mathbf{d}_6, \mathbf{d}_1'+\mathbf{d}_3\right), \end{aligned} \tag{2.43}$$

followed by the inverse of the cyclic permutation $\rho = (123456)$. (Note that the terms corresponding to the frozen variables $\tilde{x}_7, \tilde{x}_8$ are absent from (2.43), since the components of the d-vectors only measure degrees of the non-frozen variables appearing in the monomial denominators of cluster variables.) The action of $\tilde{\varphi}$ on a tropical seed corresponds to the shift $n \to n+1$ which transforms (2.42) to a new matrix $D_{m,n+1}$. Similarly, the shift $m \to m+1$ which transforms (2.42) to a new d-vector matrix $D_{m+1,n}$, is achieved via the action of $\tilde{\psi} = \bar{\rho}^{-1}\tilde{\mu}_3\tilde{\mu}_2\tilde{\mu}_4\tilde{\mu}_1$ on a tropical seed, taking the form

$$\tilde{\psi}: \quad (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4, \mathbf{d}_5, \mathbf{d}_6) \mapsto \left(\mathbf{d}_4', \mathbf{d}_5, \mathbf{d}_6, \mathbf{d}_1', \mathbf{d}_3', \mathbf{d}_2'\right),$$

which is the composition of the $(\max, +)$ versions of four mutations, given by

$$\begin{aligned} \tilde{\mu}_1: &\quad \mathbf{d}_1' + \mathbf{d}_1 = \max\left(\mathbf{d}_2 + \mathbf{d}_6, \mathbf{d}_3 + \mathbf{d}_5\right), \\ \tilde{\mu}_4: &\quad \mathbf{d}_4' + \mathbf{d}_4 = \max\left(\mathbf{d}_2 + \mathbf{d}_6, \mathbf{d}_3 + \mathbf{d}_5\right), \\ \tilde{\mu}_2: &\quad \mathbf{d}_2' + \mathbf{d}_2 = \max\left(\mathbf{d}_5 + \mathbf{d}_6, \mathbf{d}_1' + \mathbf{d}_4'\right), \\ \tilde{\mu}_3: &\quad \mathbf{d}_3' + \mathbf{d}_3 = \max\left(\mathbf{d}_5 + \mathbf{d}_6, \mathbf{d}_1' + \mathbf{d}_4'\right), \end{aligned} \tag{2.44}$$

followed by the inverse of the cyclic permutation $\bar{\rho} = (14)(2536)$. (Note that, due to the absence of frozen variables, in (2.44) the right-hand sides of $\tilde{\mu}_1$ and $\tilde{\mu}_4$ are identical, and the same is true for $\tilde{\mu}_2$ and $\tilde{\mu}_3$.)

It was noted in the last subsection that we can determine all the seeds obtained via iteration of $\tilde{\varphi}$ and $\tilde{\psi}$ from a single cluster variable indexed by $(m,n) \in \mathbb{Z}^2$, that is

$$T_{m,n} = \frac{\mathrm{N}(m,n;\tilde{\mathbf{x}})}{\tilde{\mathbf{x}}^{\mathbf{d}(m,n)}},$$

where we identify $\mathrm{N}_2(m,n;\tilde{\mathbf{x}}) = \mathrm{N}(m,n;\tilde{\mathbf{x}})$ and $\mathbf{d}_2(m,n) = \mathbf{d}(m,n)$ for all $m,n$. This simplifies the analysis of the tropical dynamics considerably, as we see from (2.40) that the d-vector matrix of any cluster is specified by the single $\mathbb{Z}^2$-indexed d-vector $\mathbf{d}(m,n)$, according to

$$D_{m,n} = \left(\mathbf{d}(m,n-1)\ \mathbf{d}(m,n)\ \mathbf{d}(m,n+1)\ \mathbf{d}(m,n+2)\ \mathbf{d}(m+1,n)\ \mathbf{d}(m+1,n+1)\right).$$

Then, by theorem 2.5, it follows that all the components of $\mathbf{d}(m,n)$, and hence all components of the d-vector matrix, satisfy the same $(\max, +)$ difference equations on the lattice, which immediately leads to the following result.

**Proposition 2.7.** *The matrix of d-vectors satisfies the tropical analogue of the system (2.39), that is*

$$\begin{aligned} D_{m+1,n+2} + D_{m,n-1} &= \max\left(D_{m+1,n+1} + D_{m,n}, D_{m+1,n} + D_{m,n+1}\right), \\ D_{m,n+2} + D_{m+1,n-1} &= \max\left(D_{m,n+1} + D_{m+1,n}, D_{m+1,n+1} + D_{m,n}\right), \\ D_{m+2,n+1} + D_{m,n} &= \max\left(D_{m+1,n+1} + D_{m+1,n}, D_{m+1,n+2} + D_{m+1,n-1}\right), \\ D_{m+2,n} + D_{m,n+1} &= \max\left(D_{m+1,n+1} + D_{m+1,n}, D_{m+1,n+2} + D_{m+1,n-1}\right). \end{aligned} \tag{2.45}$$

Manipulation of the above tropical equations reveals that in both the $m$ and $n$ directions, the d-vectors of the lattice system satisfy linear difference equations, which allow the degree growth to be calculated exactly. A key step in deriving the linear relations satisfied by the d-vectors is the consideration of the tropical analogues of the symplectic coordinates $(y,w)$, which are defined by the $(\max, +)$ versions of the formulae (2.37), namely

$$\begin{aligned}\mathbf{Y}_{m,n} &= \mathbf{d}(m,n-1) - \mathbf{d}(m,n) - \mathbf{d}(m,n+1) + \mathbf{d}(m,n+2),\\ \mathbf{W}_{m,n} &= \mathbf{d}(m+1,n+1) - \mathbf{d}(m+1,n) - \mathbf{d}(m,n+1) + \mathbf{d}(m,n).\end{aligned} \quad (2.46)$$

Each component of the pair of vectors $(\mathbf{Y}_{m,n}, \mathbf{W}_{m,n})$ satisfies the same set of coupled difference equations corresponding to iterations of the shifts $m \to m+1$ and $n \to n+1$, and the dynamics of these two tropical maps turns out to be completely periodic in both lattice directions, with periods that are inherited from the original undeformed dynamical systems (with $c = d = 1$).

**Lemma 2.8.** *Each component of the vectors (2.46) satisfies the tropical analogue of the map (2.13), namely*

$$\hat{\varphi}_{\rm trop} : (Y_{m,n}, W_{m,n}) \mapsto (Y_{m,n+1}, W_{m,n+1}), \quad (2.47)$$

*where*

$$Y_{m,n+1} + Y_{m,n} = \Big[W_{m,n} + [Y_{m,n}]_+\Big]_+, \quad W_{m,n+1} + W_{m,n} = [W_{m,n}]_+ - Y_{m,n},$$

*as well as the tropical analogue of (2.17), given by*

$$\hat{\psi}_{trop} : (Y_{m,n}, W_{m,n}) \mapsto (|W_{m,n}| - Y_{m,n}, -W_{m,n}). \quad (2.48)$$

*For arbitrary initial data $(Y_{0,0}, W_{0,0}) = (Y, W) \in \mathbb{R}^2$, the orbit of $\hat{\varphi}_{trop}$ is periodic with period 3, and the orbit of $\hat{\psi}_{trop}$ is periodic with period 2.*

**Proof.** The calculation of the components of the maps $\hat{\varphi}_{trop}$ and $\hat{\psi}_{trop}$ is achieved directly by taking the appropriate combinations of d-vectors as in (2.46), and transforming them under the actions of $\tilde{\varphi} = \rho^{-1}\tilde{\mu}_5\tilde{\mu}_1$ and $\tilde{\psi} = \bar{\rho}^{-1}\tilde{\mu}_3\tilde{\mu}_2\tilde{\mu}_4\tilde{\mu}_1$, respectively, according to the formulae in (2.43) and (2.44). (As in the definition of matrix mutation (1.5), in order to write the tropical maps concisely we have found it convenient to use the notation $[r]_+ = \max(r, 0)$ for real numbers $r$.) Proving that any real orbit of the map (2.47) has period 3 can be checked directly via a tedious case-by-case analysis, by considering the action on pairs of values $(Y, W) \in \mathbb{R}^2$ lying in different sectors of the plane; we leave the details as an exercise for the reader. It can also be proved by adapting known results about dynamics on tropical elliptic curves [30]. For the map (2.48) the analysis is more straightforward: the second component gives the relation

$$(\mathcal{S} + 1)\, W_{m,n} = W_{m+1,n} + W_{m,n} = 0,$$

where $\mathcal{S}$ denotes the shift operator corresponding to $m \to m+1$, and hence

$$W_{m,n} = (-1)^m\, W_{0,n}$$

for all $m, n$, which oscillates with period 2 in $m$. Together with the first component of (2.48) this also implies that

$$Y_{m+1,n} + Y_{m,n} = |W_{m,n}| = |W_{0,n}| \implies \left(\mathcal{S}^2 - 1\right) Y_{m,n} = (\mathcal{S} - 1)\, |W_{0,n}| = 0,$$

so $Y_{m+2,n} = Y_{m,n}$ as required. □

**Remark 2.9.** The preceding lemma is actually much stronger than what is needed to calculate the exact growth of the d-vectors appearing in the matrices (2.42) that are generated by tropical mutations applied to the specific initial seed $D_{0,0} = -I$, which is all that is required to calculate the degree growth of clusters generated by the lattice system (2.39). Indeed, all we really require is that a particular set of initial conditions for the maps $\hat{\varphi}_{trop}$ and $\hat{\psi}_{trop}$ should have periodic orbits of length 3 and 2, respectively. The exact periodicity of these particular orbits follows directly from the known Zamolodchikov periods of the original undeformed maps with $c = d = 1$, since these specific initial conditions correspond precisely to the d-vectors obtained from seeds of the $A_3$ cluster algebra, as generated by the cluster mutations (2.2).

We can now determine a system of linear difference equations satisfied by the d-vectors.

**Lemma 2.10.** *Let $\mathcal{S}, \mathcal{T}$ denote the shift operators corresponding to $m \to m+1$ and $n \to n+1$, respectively. Then any solution of the tropical lattice system (2.45) satisfies a linear ordinary difference equation of order 4 in the m direction, namely*

$$\left(\mathcal{S}^4 - 2\mathcal{S}^3 + 2\mathcal{S} - 1\right) D_{m,n} = 0 \tag{2.49}$$

*and a linear ordinary difference equation of order 6 in the n direction, that is*

$$\left(\mathcal{T}^6 - \mathcal{T}^5 - \mathcal{T}^4 + \mathcal{T}^2 + \mathcal{T} - 1\right) D_{m,n} = 0, \tag{2.50}$$

*together with the mixed linear relations*

$$(\mathcal{S} - 1)\left(\mathcal{T}^3 - 1\right) D_{m,n} = 0, \qquad \left(\mathcal{S}^2 - 1\right)(\mathcal{T} - 1)\, D_{m,n} = 0. \tag{2.51}$$

**Proof.** We begin by noting that, from the definitions (2.46), each component of $\mathbf{Y}_{m,n}$ and $\mathbf{W}_{m,n}$ can be rewritten in the form

$$Y_{m,n} = \left(\mathcal{T}^3 - \mathcal{T}^2 - \mathcal{T} + 1\right) d_{m,n-1}, \qquad W_{m,n} = (\mathcal{S} - 1)(\mathcal{T} - 1)\, d_{m,n},$$

respectively, where the scalar $d_{m,n}$ represents any component of a d-vector. The ordinary difference equation of order 6 in the $n$ direction follows immediately from the period 3 behavior of the map $\hat{\varphi}_{\mathrm{trop}}$ noted above, as we have

$$0 = \left(\mathcal{T}^3 - 1\right) Y_{m,n} = \left(\mathcal{T}^3 - 1\right)\left(\mathcal{T}^3 - \mathcal{T}^2 - \mathcal{T} + 1\right) d_{m,n-1},$$

and since this holds for each element of $D_{m,n}$, this produces the relation (2.50). Similarly, considering shifts in $m$, from the proof of the previous lemma we have

$$(\mathcal{S} + 1)\, W_{m,n} = 0 \implies \left(\mathcal{S}^2 - 1\right)(\mathcal{T} - 1)\, d_{m,n} = 0,$$

which immediately yields the second mixed relation in (2.51). On the other hand, the first linear relation in (2.51) is a direct consequence of the lattice system for $D_{m,n}$: it follows by subtracting the second equation in (2.45) from the first, since the right-hand sides of these two equations are identical. The most involved part of the proof is to derive the ordinary difference equation of order 4 in the $m$ direction. To show this, we introduce the 4th order difference operator that appears, which is

$$\mathcal{L} := (\mathcal{S} - 1)^3 (\mathcal{S} + 1) = \mathcal{S}^4 - 2\mathcal{S}^3 + 2\mathcal{S} - 1.$$

Now for convenience we revert to using the previous notation from (2.42) for the first four components in a cluster of d-vectors, writing $\mathbf{d}(m,n-1), \mathbf{d}(m,n), \mathbf{d}(m,n+1), \mathbf{d}(m,n+2)$ as $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4$, and note that, in this notation, the first relation in (2.51) is equivalent to

$$(\mathcal{S} - 1)(\mathbf{d}_4 - \mathbf{d}_1) = 0 \implies \mathcal{L}(\mathbf{d}_4 - \mathbf{d}_1) = 0,$$

while the right-hand sides of the third and fourth equations in (2.45) are the same, so that subtracting them implies the relation

$$\left(\mathcal{S}^2-1\right)\left(\mathbf{d}_3-\mathbf{d}_2\right)=0 \implies \mathcal{L}\left(\mathbf{d}_3-\mathbf{d}_2\right)=0,$$

and the definition of $\mathbf{Y}_{m,n}$ in (2.46) and the two-periodicity in $m$, as in lemma 2.8, yields the relation

$$\left(\mathcal{S}^2-1\right)\left(\mathbf{d}_4-\mathbf{d}_3-\mathbf{d}_2+\mathbf{d}_1\right)=0 \implies \mathcal{L}\left(\mathbf{d}_4-\mathbf{d}_3-\mathbf{d}_2+\mathbf{d}_1\right)=0.$$

Hence we see that three independent linear combinations of the quantities $\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3,\mathbf{d}_4$ are annihilated by the operator $\mathcal{L}$, so it suffices to obtain one more independent combination lying in the kernel. If we add the third and fourth equations in (2.45), and subtract twice the first entry in the max from both sides, followed by using $\mathbf{Y}_{m+2,n}=\mathbf{Y}_{m,n}$ once again, then we find

$$\left(\mathcal{S}-1\right)^2\left(\mathbf{d}_2+\mathbf{d}_3\right)=2\left[\mathbf{Y}_{m+1,n}\right]_+ \implies \mathcal{L}\left(\mathbf{d}_2+\mathbf{d}_3\right)=0,$$

which is the desired fourth linearly independent relation. Hence each component of the quadruple $\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3,\mathbf{d}_4$ lies in the kernel of $\mathcal{L}$, and the required result (2.49) follows. □

These linear relations mean that it is a fairly straightforward matter to obtain the explicit solution for the double sequence of d-vector matrices $D_{m,n}$ subject to specifying the initial cluster of d-vectors via $D_{0,0}=-I$.

**Theorem 2.11.** *The matrix of d-vectors for the tau function solutions of the system of bilinear lattice equations (2.39), generated by the initial cluster of d-vectors*

$$D_{0,0}=\left(\mathbf{d}\left(0,-1\right)\ \mathbf{d}\left(0,0\right)\ \mathbf{d}\left(0,1\right)\ \mathbf{d}\left(0,2\right)\ \mathbf{d}\left(1,0\right)\ \mathbf{d}\left(1,1\right)\right)=-I,$$

*is given by the exact formula*

$$\begin{aligned}D_{m,n}=&\left(\tfrac{m^2}{4}+\tfrac{n^2}{12}\right)\mathbf{e}\mathbf{e}^T+\tfrac{m}{2}\left(\hat{\mathbf{e}}\mathbf{e}^T-\mathbf{e}\hat{\mathbf{e}}^T\right)+\tfrac{n}{6}\left(\mathbf{e}\mathbf{f}^T-\mathbf{f}\mathbf{e}^T\right)+\mathrm{E}-\tfrac{(-1)^m}{24}\left(\mathbf{e}-2\hat{\mathbf{e}}\right)\left(\mathbf{e}^T-2\hat{\mathbf{e}}^T\right)\\&-\tfrac{(-1)^n}{8}\mathbf{g}\mathbf{g}^T+\left(\mathrm{F}+(-1)^m\mathrm{G}\right)e^{2\pi\mathrm{i}n/3}+\left(\mathrm{F}^*+(-1)^m\mathrm{G}^*\right)e^{2\pi\mathrm{i}n/3}\end{aligned} \tag{2.52}$$

*for* $(m,n)\in\mathbb{Z}^2$*, where star denotes the complex conjugate, and*

$$\mathrm{E}=-\frac{1}{36}\begin{pmatrix}14&11&2&-13&2&-7\\11&14&11&2&5&2\\2&11&14&11&2&5\\-13&2&11&14&-7&2\\2&5&2&-7&14&11\\-7&2&5&2&11&14\end{pmatrix},\qquad \mathrm{F}=-\frac{1}{18}\mathbf{h}\mathbf{h}^\dagger,\qquad \mathrm{G}=-\frac{1}{6}\mathbf{k}\mathbf{k}^\dagger,$$

*with the constant vectors*

$$\begin{aligned}&\mathbf{e}^T=(1,1,1,1,1,1),\quad \hat{\mathbf{e}}^T=(0,0,0,0,1,1),\quad \mathbf{f}^T=(0,1,2,3,1,2),\quad \mathbf{g}^T=(1,-1,1,-1,-1,1),\\&\mathbf{h}^T=\left(1,e^{-2\pi\mathrm{i}/3},e^{2\pi\mathrm{i}/3},1,e^{-2\pi\mathrm{i}/3},e^{2\pi\mathrm{i}/3}\right),\quad \mathbf{k}^T=\left(1,e^{-2\pi\mathrm{i}/3},e^{2\pi\mathrm{i}/3},1,e^{2\pi\mathrm{i}/3},e^{-\pi\mathrm{i}/3}\right)\end{aligned}$$

*(and the dagger means Hermitian conjugate).*

**Proof.** This is mostly a straightforward exercise in solving linear difference equations with matrix coefficients. The action of the maps $\tilde{\varphi}$ and $\tilde{\psi}$ on an initial cluster of d-vectors given by the matrix $D_{0,0} = -I$ can be used to produce a complete set of matrices on a six-point stencil on $\mathbb{Z}^2$, namely $D_{0,-1}, D_{0,0}, D_{0,1}, D_{0,2}, D_{1,0}, D_{1,1}$. Fixing the values of $D_{m,n}$ on these 6 points completely specifies an initial value problem for (2.45), the matrix version of the tropical lattice equations. The linear ordinary difference equation (2.50) has the characteristic roots $1, 1, 1, -1, e^{2\pi \mathrm{i}/3}, e^{-2\pi \mathrm{i}/3}$, so in order to find the explicit formula for $D_{m,n}$, we can begin by writing down a general solution of this linear equation in the form

$$D_{m,n} = \mathrm{A}_0\, n^2 + \mathrm{B}_0\, n + \mathrm{C}_0 + \mathrm{D}_0\, (-1)^n + \mathrm{E}_0\, e^{2\pi \mathrm{i} n/3} + \mathrm{E}_0^*\, e^{-2\pi \mathrm{i} n/3}, \qquad (2.53)$$

where *a priori* the coefficient matrices $\mathrm{A}_0, \mathrm{B}_0$, etc are all functions of $m$. Then, upon applying the first mixed relation in (2.51) to the above formula, we find

$$(\mathcal{S} - 1)\left(\mathrm{A}_0\,(6n+9) + 3\mathrm{B}_0 - 2\mathrm{D}_0\,(-1)^n\right) = 0$$

for all $n$, which implies that the coefficients $\mathrm{A}_0, \mathrm{B}_0$ and $\mathrm{D}_0$ are all constants (independent of $m$). Similarly, applying the second mixed relation in (2.51) to the general formula (2.53) for $D_{m,n}$ then implies that

$$\left(\mathcal{S}^2 - 1\right)\mathrm{E}_0 = 0 = \left(\mathcal{S}^2 - 1\right)\mathrm{E}_0^*,$$

so these last two coefficients must both be period 2 functions of $m$, and we may write

$$\mathrm{E}_0 = \mathrm{F} + (-1)^m\,\mathrm{G}$$

for some constant matrices F, G, and $\mathrm{E}_0^*$ is given by the same formula with F, G replaced by their complex conjugates (since the d-vectors are all real). This only leaves the $m$-dependence of the coefficient $\mathrm{C}_0$ undetermined, but then the pure linear relation in $m$, namely (2.49), requires that all the coefficients in (2.53) must lie in the kernel of the operator $\mathcal{L}$, so in particular $\mathcal{L}\,\mathrm{C}_0 = 0$, hence $\mathrm{C}_0$ must have the general form

$$\mathrm{C}_0 = \mathrm{C}_1\, m^2 + \mathrm{C}_2\, m + \mathrm{C}_3 + \mathrm{C}_4\,(-1)^m.$$

It remains to generate a sufficient number of matrices $D_{m,n}$ (for small $m,n$) via the system (2.45), in order to fix the exact values of the constant coefficients $\mathrm{A}_0, \mathrm{B}_0, \mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{D}_0$, as well as F, G and their complex conjugates, which reduces the problem to solving systems of linear equations with computer algebra. □

**Remark 2.12.** The fact that the coefficients of the quadratic terms $m^2$ and $n^2$ are non-zero, with no $mn$ terms in the formula (2.52), can be used to show that the maps $\hat{\psi}$ and $\hat{\varphi}$ correspond to independent translations along each fiber of the pencil of curves (2.15). This means that the Mordell-Weil group of the associated rational elliptic surface has minimal rank 2 (for generic parameters $c, d$). More detailed arguments that show why the rank should be exactly 2 are relegated to the first appendix.

## 3. Integrable deformation of the $\mathrm{C}_2$ cluster map

In this section, we consider deformations of the periodic cluster map constructed from the cluster algebra of type $\mathrm{C}_2$.

### 3.1. *Deformed* $C_2$ *map*

The Cartan matrix for the $C_2$ root system is

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},$$

and this is the companion to the exchange matrix $B = (b_{ij})$ given by

$$B = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \tag{3.1}$$

which is obtained from $C$ by removing the diagonal terms and adjusting the signs of the off-diagonal terms appropriately (with the requirement that if $b_{ij} \neq 0$, then $b_{ji}$ should have the opposite sign). The latter matrix is skew-symmetrizable, since for $D = \mathrm{diag}(1,2)$ we have that

$$\Omega = BD = (\omega_{ij})$$

is the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}. \tag{3.2}$$

(Note that, up to switching $x_1 \leftrightarrow x_2$, this example is equivalent to the case of $B_2$, which has the transpose of the Cartan matrix for $C_2$.)

Starting from an initial cluster $\mathbf{x} = (x_1, x_2)$, we consider a pair of deformed mutations, of the form

$$\begin{array}{lll} \mu_1: & (x_1,x_2) \mapsto (x_1',x_2), & x_1'x_1 = a_1x_2^2 + b_1, \\ \mu_2: & (x_1',x_2) \mapsto (x_1',x_2'), & x_2'x_2 = a_2x_1' + b_2. \end{array} \tag{3.3}$$

One can confirm that, after applying the corresponding pair of matrix mutations, namely $\mu_1$ followed by $\mu_2$, according to the rule (1.5), the exchange matrix (3.1) is mutation periodic under this composition of mutations, that is to say

$$\mu_2\mu_1(B) = B.$$

So, similarly to the situation for the skew form (2.3) constructed from the exchange matrix of type $A_3$, by a minor variation on theorem 1.3 in [20], adjusting the presymplectic structure to the skew-symmetrizable setting (see [21], for instance), the map $\varphi = \mu_2\mu_1$ composed from the pair of deformed cluster mutations (3.3) preserves the log-canonical two-form

$$\omega = \sum_{i<j} \omega_{ij} \mathrm{d}\log x_i \wedge \mathrm{d}\log x_j = \frac{2}{x_1x_2}\mathrm{d}x_1 \wedge \mathrm{d}x_2. \tag{3.4}$$

The latter is the skew form built from the coefficients of the matrix $\Omega$ in (3.2), obtained from skew-symmetrization of (3.1). In other words, $\varphi^*(\omega) = \omega$, and since the two-form (3.4) is non-degenerate in this case, the deformed $C_2$ cluster map $\varphi$ is symplectic, for arbitrary values of the parameters $a_i, b_i$. However, note that when these parameters take generic values, the composition of transformations in (3.3) is not a cluster map, because it does not generate Laurent polynomials in $x_1, x_2$.

The original undeformed mutations obtained from the exchange matrix (3.1), which generate the cluster algebra of type $C_2$, are recovered by setting all the parameters $a_i, b_i$ in (3.3) to 1. Since the Coxeter number of $C_2$ is 4, Zamolodchikov periodicity implies that the undeformed cluster map $\varphi = \mu_2\mu_1$ has period $3 = \frac{1}{2}(4+2)$, so that

$$\varphi = \mu_2\mu_1 \quad (\text{with } a_1 = b_1 = a_2 = b_2 = 1) \implies \varphi^3(\mathbf{x}) = \mathbf{x}.$$

Therefore, due to the periodicity, for any function $f: \mathbb{C}^2 \to \mathbb{C}$, the associated symmetric function given by the product over an orbit, that is

$$K_f(\mathbf{x}) = \prod_{j=0}^{2} (\varphi^*)^j (f)(\mathbf{x}) = \prod_{j=0}^{2} f\Big((\varphi^*)^j(\mathbf{x})\Big),$$

is invariant under the cluster map $\varphi$. Here we consider

$$K = \prod_{j=0}^{2} (\varphi^*)^j (x_2) = x_2 + \frac{2}{x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{1}{x_1x_2}. \tag{3.5}$$

Before proceeding further with the general deformed case, for arbitrary non-zero parameters $a_i, b_i$, we can apply rescaling $x_i \to \lambda_i x_i$ to each cluster variable, with a suitable choice of parameters $(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$, in order to arrange it so that $a_1 = 1 = a_2$, which simplifies the calculations. With the remaining parameters $b_1, b_2$ fixed, the iteration of the deformed map $\varphi$ is given by a system of recurrences

$$\begin{aligned} x_{1,n+1}x_{1,n} &= x_{2,n}^2 + b_1, \\ x_{2,n+1}x_{2,n} &= x_{1,n+1} + b_2. \end{aligned} \tag{3.6}$$

An invariant function for this deformed map can be constructed by the same procedure as was used in [20], and described for type $A_3$ above, whereby we modify each Laurent monomial in (3.5) by inserting arbitrary coefficients $\kappa_i$ in front of each monomial, similarly to (2.12), so that (after fixing the leading coefficient to be 1) we have

$$\tilde{K} = x_2 + \frac{\kappa_1}{x_2} + \frac{\kappa_2 x_1}{x_2} + \frac{\kappa_3 x_2}{x_1} + \frac{\kappa_4}{x_1x_2}. \tag{3.7}$$

Next, we proceed to impose the condition of invariance on this Laurent polynomial, that is $\varphi^*(\tilde{K}) = \tilde{K}$, which puts constraints on the coefficients $\kappa_i$ and $b_i$. This gives rise to a necessary and sufficient condition for the deformed map (3.6) to be Liouville integrable, leading to the following result.

**Theorem 3.1.** *The necessary and sufficient condition for a rational function of the form (3.7) to be a first integral for the map defined by (3.6) is that*

$$b_1 = b_2 = \beta, \tag{3.8}$$

*in which case $\tilde{K}$ is given by*

$$\tilde{K} = x_2 + \frac{1+\beta}{x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{\beta}{x_1x_2}. \tag{3.9}$$

*Hence the deformed symplectic map $\varphi$ given by*

$$\begin{array}{rcccl}\varphi & : & \mathbb{C}^2 & \rightarrow & \mathbb{C}^2 \\ & & \mathbf{x} = (x_1, x_2) & \mapsto & \mathbf{x}' = (x_1', x_2') = \left( \frac{(x_2)^2+\beta}{x_1}, \frac{x_1'+\beta}{x_2} \right) .\end{array} \tag{3.10}$$

*is Liouville integrable whenever the condition (3.8) holds.*

**Proof.** The proof follows from an explicit calculation, which is best achieved using a computer algebra package such as MAPLE: assuming that a first integral of the form (3.7) exists, the equation $\varphi^*(\tilde{K}) = \tilde{K}$ can be rewritten as an identity between two polynomials in $x_1, x_2$, and then comparing coefficients at each degree yields a set of linear equations in the coefficients $\kappa_i$; this linear system has a solution if and only if (3.8) holds. □

**Remark 3.2.** The level sets of the first integral (3.9) are biquadratic curves, hence we can construct a QRT map given by the composition of two involutions, of the same form as in (2.16). However, comparison with the formula (3.10) shows that in this case, on the biquadratic pencil

$$x_1 (x_2)^2 + (1+\beta) x_1 + (x_1)^2 + (x_2)^2 + \beta = \tilde{\kappa} x_1 x_2$$

corresponding to the level sets $\tilde{K} = \tilde{\kappa}$, the transformation $\mu_1$ is the horizontal switch, and $\mu_2$ is the vertical switch (see [4]), hence the map $\varphi = \mu_2 \cdot \mu_1$ coincides with the QRT map.

We have seen that, when the parameters satisfy the constraints (3.8), the symplectic map given by (3.6) is Liouville integrable. However, as already mentioned above, the general deformed cluster map $\varphi$ is not itself a cluster map, and this continues to be true for the constrained version (3.10), since for $\beta \neq 1$ the sequence of pairs of coordinates $\tilde{\varphi}^n(\mathbf{x})$ generated by the latter map do not belong to the ring of Laurent polynomials in $x_1, x_2$. In an attempt to resolve this issue, we must go a step further, and try to apply Laurentification, analogously to what was carried out in [20] for deformed maps of type A in low dimension (and described for deformed $A_3$ in the previous section).

### *3.2. Laurentification of deformed type* $C_2$ *map*

As we saw for the case of the deformed $A_3$ map in the previous section, the word Laurentification refers to a transformation that lifts a given birational map to a map acting on a new set of coordinates in higher dimensions, where the lifted map possesses the Laurent property. Several methods have been used to achieve Laurentification, such as the recursive factorization approach taken in [14] (see also [37, 38] for some related results and observations). Here we consider another method, introduced in [18], which involves the singularity pattern of the iterates of the deformed map $\varphi$. Instead of performing a general analysis of singularities, we apply an empirical version of $p$-adic analysis, which is done by inspecting the prime factorization of the terms given by the iteration. To see the procedure, we consider the rational orbit of the map (3.10) obtained by setting the value of the initial cluster to be $(x_1, x_2) = (x_{1,0}, x_{2,0}) = (1, 1)$, with parameters $b_1 = 2 = b_2$, and find the prime factorizations of the numerators and denominators of successive terms, as in the table below:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $x_{1,n}$ | 3 | $3^2$ | $\frac{19}{5^2}$ | $\frac{569}{11^2}$ | $\frac{17^2 \cdot 107}{3^2 \cdot 23^2}$ | $\frac{139 \cdot 3299}{811^2}$ | $\frac{457737691}{8089^2}$ | $\frac{3 \cdot 457 \cdot 81689827}{7^2 \cdot 23039^2}$ |
| $x_{2,n}$ | 5 | $\frac{11}{5}$ | $\frac{3 \cdot 23}{5 \cdot 11}$ | $\frac{5 \cdot 811}{3 \cdot 11 \cdot 23}$ | $\frac{11 \cdot 8089}{3 \cdot 23 \cdot 811}$ | $\frac{3 \cdot 7 \cdot 23 \cdot 23039}{811 \cdot 8089}$ | $\frac{13 \cdot 173 \cdot 811 \cdot 3793}{7 \cdot 23039 \cdot 8089}$ | $\frac{5^3 \cdot 41 \cdot 39461 \cdot 8089}{7 \cdot 13 \cdot 173 \cdot 23039 \cdot 3793}$ |

We can see that for each of the primes $p = 5, 11, 23, 811, 8089$ (for instance), the $p$-adic norms of $x_{1,n}$ and $x_{2,n}$ exhibit the patterns

$$\begin{array}{llllll} |x_{1,n}|_p: & 1, & 1, & p^2, & 1, & 1 \\ |x_{2,n}|_p: & p^{-1}, & p, & p, & p^{-1} & 1. \end{array} \tag{3.11}$$

Furthermore, other primes, such as $p = 19, 569, 107, 139, 3299$, appear successively as factors in the numerator of $x_{1,n}$, but not in $x_{2,n}$. This suggests that there should be the following singularity patterns:

$$\begin{array}{ll} \text{Pattern 1:} & \ldots, (R,0), (R,\infty), \left(\infty^2,\infty\right), (R,0), \ldots \\ \text{Pattern 2:} & \ldots, (0,R), \ldots, \end{array}$$

where $R$ denotes a regular (non-zero) finite value. Then we introduce two tau-functions $\tau_n$, $\sigma_n$, associated with pattern 1 and pattern 2, respectively, in order to arrange it so that, for isolated values of $n$, at $n = n_p$ say, we have $\tau_{n_p} \equiv 0 \,(\bmod p)$ for the first set of primes, and $\sigma_{n_p} \equiv 0 \,(\bmod p)$ for the second set. Then, to recover the two different singularity patterns, we define a monomial rational map $\pi : \mathbb{C}^5 \to \mathbb{C}^2$, which is specified by the following transformation of dependent variables:

$$\pi: \qquad x_{1,n} = \frac{\sigma_n}{\tau_{n+1}^2}, \quad x_{2,n} = \frac{\tau_n \tau_{n+3}}{\tau_{n+1}\tau_{n+2}}. \tag{3.12}$$

When the two expressions (3.12) are substituted directly into the components (3.6) of $\varphi$, with the parameters constrained so that $b_1 = \beta = b_2$, one obtains the system of recurrence relations

$$\begin{aligned} \sigma_n \sigma_{n+1} &= \beta \tau_{n+1}^2 \tau_{n+2}^2 + \tau_n^2 \tau_{n+3}^2, \\ \tau_n \tau_{n+4} &= \beta \tau_{n+2}^2 + \sigma_{n+1}. \end{aligned} \tag{3.13}$$

If we iterate the latter pair of equations with initial values $(\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3) = (1,1,1,1,1)$ and $\beta = 2$, then we obtain a pair of integer sequences, with the first few terms presented in the following table:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\sigma_n$ | 3 | 9 | 19 | 569 | 30923 | 458561 | 457737691 | 111996752817 |
| $\tau_{n+4}$ | 5 | 11 | 69 | 811 | 8089 | 161273 | 8530457 | 202237625 |

Observe that the primes appearing separately as isolated factors in each of these integer sequences are the same ones that were identified as factors of the numerators and denominators in the previous table.

The system of recurrence relations (3.13) can be interpreted as iteration of a birational map $\psi : \mathbb{C}^5 \to \mathbb{C}^5$ which is intertwined with $\varphi$ via $\pi$, that is

$$\psi: \quad (\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3) \mapsto (\sigma_1, \tau_1, \tau_2, \tau_3, \tau_4), \qquad \varphi \cdot \pi = \pi \cdot \psi.$$

Then we would like to identify the initial data for the map $\psi$ as an initial cluster in a seed for a cluster algebra of rank 5, so that $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5) = (\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3)$. To verify that the Laurent property holds when the deformed map $\varphi$ is lifted to the map $\psi$ on the space of tau functions, we need to find a cluster algebra structure defined by an initial seed $(\tilde{\mathbf{x}}, \tilde{B})$, for a suitable exchange matrix $\tilde{B} \in \mathrm{Mat}_5(\mathbb{Z})$. We will then proceed to show that this extends to a seed $(\hat{\mathbf{x}}, \hat{B})$, where the initial cluster $\hat{\mathbf{x}} = (\tilde{\mathbf{x}}, \beta)$ includes the parameter $\beta$ as a frozen variable, and $\hat{B}$ is an extended $6 \times 5$ exchange matrix (with an additional row to incorporate the frozen variable).

To start with, we calculate the pullback of the symplectic form (3.4) by the rational map $\pi$, to obtain the presymplectic form

$$\tilde{\omega} = \pi^* \omega = \sum_{i<j} \tilde{\omega}_{ij} \frac{\mathrm{d}\tilde{x}_i \wedge \mathrm{d}\tilde{x}_j}{\tilde{x}_i \tilde{x}_j},$$

which gives rise to a new skew-symmetric matrix,

$$\tilde{\Omega} = (\tilde{\omega}_{ij}) = \begin{pmatrix} 0 & -2 & 2 & 2 & -2 \\ 2 & 0 & -4 & 0 & 0 \\ -2 & 4 & 0 & -4 & 4 \\ -2 & 0 & 4 & 0 & 0 \\ 2 & 0 & -4 & 0 & 0 \end{pmatrix}. \tag{3.14}$$

Similar to the matrix in (3.2), the $\tilde{\Omega}$ can be expressed as a product $\tilde{\Omega} = \tilde{B}\tilde{D}$ of skew-symmetrizable matrix $\tilde{B}$ and diagonal matrix $\tilde{D}$. By post-multiplying by the diagonal matrix $\tilde{D}^{-1} = \mathrm{diag}(1, {}^1\!/\!_2, {}^1\!/\!_2, {}^1\!/\!_2, {}^1\!/\!_2)$, this gives the $5 \times 5$ exchange matrix

$$\tilde{B} = \tilde{\Omega}\tilde{D}^{-1} = \left(\tilde{B}_{ij}\right) = \begin{pmatrix} 0 & -1 & 1 & 1 & -1 \\ 2 & 0 & -2 & 0 & 0 \\ -2 & 2 & 0 & -2 & 2 \\ -2 & 0 & 2 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

Now observe that if we apply the composition of mutations $\tilde{\mu}_2\tilde{\mu}_1$ defined by the latter exchange matrix, applying the mutation $\tilde{\mu}_1$ associated with index 1, followed by the mutation $\tilde{\mu}_2$ associated with index 2, then the initial cluster $\tilde{\mathbf{x}} = (\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3)$ gets transformed to $\tilde{\mu}_2\tilde{\mu}_1(\tilde{\mathbf{x}}) = (\tilde{x}_1', \tilde{x}_2', \tilde{x}_3, \tilde{x}_4, \tilde{x}_5) = (\sigma_1, \tau_4, \tau_1, \tau_2, \tau_3)$, where the new cluster variables $\sigma_1, \tau_4$ are obtained from a single iteration of each of the recurrences in (3.13), setting $n = 0$ and $\beta = 1$ therein. To generate the general sequence of mutations for tau functions that corresponds to (3.13) with arbitrary $\beta$, it is necessary to extend the initial cluster to $\hat{\mathbf{x}} = (\tilde{\mathbf{x}}, \beta)$ by inserting the frozen variable $\beta$, and then a further calculation shows that we can define the extended exchange matrix

$$\hat{B} = \begin{pmatrix} 0 & -1 & 1 & 1 & -1 \\ 2 & 0 & -2 & 0 & 0 \\ -2 & 2 & 0 & -2 & 2 \\ -2 & 0 & 2 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.16}$$

which is obtained by inserting an extra row at the bottom of (3.15). The form of the recurrence system (3.13) also requires that we permute the cluster variables after applying the two mutations $\tilde{\mu}_1$ and $\tilde{\mu}_2$.

**Theorem 3.3.** *Let $\rho$ be the permutation (2345). Then $\psi = \rho^{-1}\tilde{\mu}_2\tilde{\mu}_1$ is a cluster map that fixes the extended exchange matrix $\hat{B}$. Iteration of $\psi$ generates two sequences of tau functions $(\sigma_n)$, $(\tau_n)$ satisfying the system (3.13). The tau functions are elements of $\mathbb{Z}_{>0}[\beta, \sigma_0^{\pm 1}, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}]$.*

**Proof.** Consider the cluster algebra with initial cluster $\hat{\mathbf{x}} = (\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \beta)$ and extended exchange matrix $\hat{B}$. One can see that, by applying cluster mutation to $(\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \beta) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6)$ in direction 1, mutation $\tilde{\mu}_1$ gives the exchange relation

$$\sigma_1\sigma_0 = \beta\tau_1^2\tau_2^2 + \tau_0^2\tau_3^2,$$

producing the new cluster $\tilde{\mu}_1(\hat{\mathbf{x}}) = (\sigma_1, \tau_0, \tau_1, \tau_2, \tau_3, \beta)$ and the mutated exchange matrix $\hat{B}_1 = \tilde{\mu}_1(\hat{B})$ given by

$$\hat{B}_1 = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ -2 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & -2 & 0 \\ 2 & -2 & 2 & 0 & -2 \\ -2 & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Following this up with a mutation in direction 2, applying $\tilde{\mu}_2$ gives the new cluster variable $\tau_4$ defined by the following relation:

$$\tau_4\tau_0 = \beta\tau_2^2 + \sigma_1.$$

The new cluster is then $\tilde{\mu}_2\tilde{\mu}_1(\hat{\mathbf{x}}) = (\sigma_1, \tau_4, \tau_1, \tau_2, \tau_3, \beta)$. Therefore applying the composition of mutations $\tilde{\mu}_2$ and $\tilde{\mu}_1$ generates this pair of exchange relations, which corresponds to a single iteration of the map $\psi$, but requires an additional cyclic permutation of the middle 4 variables to obtain $\psi(\hat{\mathbf{x}}) = (\sigma_1, \tau_1, \tau_2, \tau_3, \tau_4, \beta)$. Furthermore, we see that the combination of two matrix mutations is equivalent to a permutation of order 4 acting on the corresponding 4 non-frozen labels, namely $\rho = (2345)$, i.e.

$$\tilde{\mu}_2\tilde{\mu}_1\left(\hat{B}\right) = P_1\hat{B}P_2 = \rho\left(\hat{B}\right)$$

where $P_1$ and $P_2$ are the row and column permutation matrices

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

Thus we have shown that the extended exchange matrix $\hat{B}$ given by (3.16) is cluster mutation periodic in the generalized sense defined in [29], so that $\psi(\hat{B}) = \hat{B}$, where the cluster map $\psi = \rho^{-1}\tilde{\mu}_2\tilde{\mu}_1$ generates two sequences of tau functions satisfying the coupled system (3.13). Hence, by the Laurent phenomenon in the cluster algebra, it follows that iteration of the map $\psi$ on the space of tau functions produces Laurent polynomials that are elements of $\mathbb{Z}_{>0}[\beta, \sigma_0^{\pm 1}, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}]$ (where each monomial that appears has a positive integer coefficient, due to positivity [13, 25]). □

### *3.3. Connection with Somos-5 and a special Somos-7 recurrence*

Analogously to the observation made for the case of $A_3$ in the previous section, we note that the formula for $x_{2,n}$ in (3.12) corresponds to the substitution used for Somos-5 in [15]. This allows us to establish a connection between the system (3.13) and a suitable Somos-5 recurrence relation.

**Theorem 3.4.** *The sequence of tau functions $(\tau_n)$ generated by iteration of (3.13) satisfies a Somos-5 relation with coefficients that are constant along each orbit, given by*

$$\tau_n\tau_{n+5} = \zeta\,\tau_{n+1}\tau_{n+4} + \theta\,\tau_{n+2}\tau_{n+3} \tag{3.18}$$

*where the coefficients are*

$$\zeta = 1-\beta, \quad \theta = \beta\tilde{K}, \tag{3.19}$$

*with $\tilde{K}$ being the value of the first integral (3.9). Hence $u_n = x_{2,n}$ satisfies the Somos-5 QRT map (2.32) with coefficients $\tilde{\alpha} = \zeta$, $\tilde{\beta} = \theta$ as in (3.19) along any orbit of the deformed $C_2$ map (3.10).*

**Proof.** The first three iterations of the Somos-5 sequence can be represented in matrix form as

$$\underbrace{\begin{pmatrix} \tau_0\tau_5 & \tau_1\tau_4 & \tau_2\tau_3 \\ \tau_1\tau_6 & \tau_2\tau_5 & \tau_3\tau_4 \\ \tau_2\tau_7 & \tau_3\tau_6 & \tau_4\tau_5 \end{pmatrix}}_{M} \begin{pmatrix} 1 \\ -\zeta \\ -\theta \end{pmatrix} = 0. \tag{3.20}$$

As the vector $\mathbf{v} = (1,-\eta,-\theta)^T$ is non-zero, $\det(M) = 0$ is a necessary condition for the tau functions $\tau_n$ obtained from (3.13) to satisfy (3.18). With the help of MAPLE software, we can easily confirm that the relation holds. The coefficients $\zeta$ and $\theta$ can be found by computing the kernel of $M$, which turns out to be independent under shifting the indices of each tau function $(n \to n+1)$: to be precise, $\zeta = 1-\beta$ is just a constant (independent of tau functions), while

$$\theta = \frac{\beta\left(\left(\beta\tau_1^2+\sigma_0\right)\tau_2^2+\tau_0^2\tau_3^2\right)\left(\tau_1^2+\sigma_0\right)}{\sigma_0\tau_0\tau_1\tau_2\tau_3},$$

but this is just $\beta$ times the first integral (3.9) lifted to the space of tau functions. Hence the vector $\mathbf{v}$ is constant along each orbit, and remains in the kernel of the matrix $M$ when the replacement $\tau_n \to \tau_{n+1}$ is made for each tau function appearing therein. □

We have seen that, subject to the condition $b_1 = \beta = b_2$, the variable $u_n = x_{2,n}$ satisfying one half of the system (3.6), also satisfies the Somos-5 QRT map (2.32) with appropriate coefficients $\tilde{\alpha},\tilde{\beta}$. This suggests that each invariant curve for the deformed map $\tilde{\varphi}$, given by a level set of (3.9), is birationally equivalent to a corresponding elliptic curve associated with a level set of the Somos-5 QRT map. According to [19], each such curve is also isomorphic to a curve that corresponds to a level set of the Lyness map

$$w_{n+1}w_{n-1} = \tilde{\zeta}w_n + \tilde{\theta} \tag{3.21}$$

(for suitable $\tilde{\zeta}$ and $\tilde{\theta}$), which is the integrable deformation of the periodic map of type $A_2$. Applying the results from [19], it can be shown that the iterates $x_{2,n}$ of the deformed $C_2$ map $\tilde{\varphi}$ are also associated with the Lyness map, via the transformation

$$w_n = x_{2,n} + \frac{\theta}{\zeta} = \frac{1}{\zeta}\frac{\tau_{n-1}\tau_{n+4}}{\tau_{n+1}\tau_{n+2}}.$$

The above substitution is consistent with the fact that $\tau_n$ satisfies the bilinear recurrence

$$\tau_{n+7}\tau_n = \tilde{\zeta}\tau_{n+6}\tau_{n+1} + \tilde{\theta}\tau_{n+3}\tau_{n+4}. \tag{3.22}$$

which a special type of Somos-7 recurrence, namely the same as (1.3) associated with the Lyness map (1.2). This is another type of Somos sequence generated by a sequence of mutations in a cluster algebra of rank 7 (for further detail see [10, 11]). The same Somos-7 relation will also be seen to appear in the next section.

**Remark 3.5.** Another way to see that the existence of the special Somos-7 relation (3.22) follows from theorem 3.4, is to apply a result from [15] (see also [33]), which says that every Somos-5 sequence also satisfies a Somos-$k$ relation of odd order, for each odd integer $k \geqslant 7$: so every Somos-5 is also a Somos-7. The converse is not quite true, however: every Somos-7 does satisfy a relation of Somos-5 type, but generically it has one coefficient that is periodic with period 3, rather than having two constant coefficients $\zeta, \theta$ as in (3.18). This result is proved in appendix B, along with a number of results about the Somos-7 recurrence (3.22) that have not been collected elsewhere.

### *3.4. Tropicalization and degree growth for deformed $C_2$ map*

Given an initial cluster $\hat{\mathbf{x}} = (\sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \beta)$ for the deformed $C_2$ cluster map, as in theorem 3.3, we can associate a tropical cluster of d-vectors in $\mathbb{Z}^5$, encapsulated in the matrix

$$(\mathbf{d}_0\ \mathbf{e}_0\ \mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) = -I, \tag{3.23}$$

where $I$ denotes the $5 \times 5$ identity matrix. The Laurent property implies that the sequences of tau functions $\sigma_n, \tau_n$ generated by the system (3.13) take the form

$$\sigma_n = \frac{\mathrm{N}_n^{(1)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{d}_n}}, \qquad \tau_n = \frac{\mathrm{N}_n^{(2)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{e}_n}},$$

with the numerators $\mathrm{N}_n^{(1)}, \mathrm{N}_n^{(2)}$ being polynomials in $\mathbb{Z}[\hat{\mathbf{x}}]$ that are not divisible by any of the initial cluster variables, while the basic results on d-vectors in [9] imply that the sequences $\mathbf{d}_n, \mathbf{e}_n$ appearing as exponents in the denominators satisfy the $(\max, +)$ version of this system, given by

$$\begin{aligned} \mathbf{d}_{n+1} + \mathbf{d}_n &= 2\max(\mathbf{e}_{n+2} + \mathbf{e}_{n+1}, \mathbf{e}_{n+3} + \mathbf{e}_n), \\ \mathbf{e}_{n+4} + \mathbf{e}_n &= \max(2\mathbf{e}_{n+2}, \mathbf{d}_{n+1}). \end{aligned} \tag{3.24}$$

(Note that, as for the tropical version of deformed $A_3$ in the previous section, we do not count degrees with respect to frozen variables, so there are no terms corresponding to the parameter $\beta$ in the system.)

We now consider the tropical analogue of the map $\pi$ defined in (3.12), which leads us to introduce the quantities

$$\mathbf{X}_{1,n} = \mathbf{d}_n - 2\mathbf{e}_{n+1}, \qquad \mathbf{X}_{2,n} = \mathbf{e}_{n+3} - \mathbf{e}_{n+2} - \mathbf{e}_{n+1} + \mathbf{e}_n \tag{3.25}$$

The dynamics of these combinations of the d-vectors holds the key to their degree growth.

**Lemma 3.6.** *Whenever* $\mathbf{d}_n, \mathbf{e}_n$ *satisfy the system (3.24), each component of the quantities (3.25) is a solution of the tropical (or ultradiscrete) QRT map* $\varphi_{trop}$ *defined by*

$$\begin{aligned} X_{1,n+1} + X_{1,n} &= 2\,[X_{2,n}]_+\,, \\ X_{2,n+1} + X_{2,n} &= [X_{1,n+1}]_+\,. \end{aligned} \tag{3.26}$$

*Given arbitrary initial data* $\left(X_{1,0}, X_{2,0}\right) \in \mathbb{R}^2$*, the orbit of the latter map is periodic with period 3.*

**Proof.** The map $\varphi_{\text{trop}}$ presented in (3.26) is the $(\max,+)$ analogue of the QRT map (3.10), which follows immediately from the system (3.24) by rearranging and rewriting it in terms of the components of the quantities defined in (3.25). The fact that every orbit is period 3 can be checked by a direct case-by-case analysis for initial data in different sectors of the plane. It can also be deduced from Nobe's general results on periods of ultradiscrete QRT maps [30]. □

**Remark 3.7.** For the main case of interest, the initial tropical cluster in (3.23) produces the pair of vectors of initial values

$$\mathbf{X}_{1,0} = (-1,0,2,0,0)^T, \qquad \mathbf{X}_{2,0} = (0,-1,1,1,-1)^T$$

for (3.26), and this initial pair produces the subsequent pairs of terms

$$\mathbf{X}_{1,1} = (1,0,0,2,0)^T, \qquad \mathbf{X}_{2,1} = (1,1,-1,1,1)^T \qquad \text{and} \qquad \mathbf{X}_{1,2} = (1,2,0,0,2)^T, \\ \mathbf{X}_{2,2} = (0,1,1,-1,1)^T$$

under iteration, with the iterates repeating thereafter. The components of these vectors are all built from two particular orbits of the scalar map (3.26), namely

$$(-1,0) \to (1,1) \to (1,0)\,, \qquad (0,-1) \to (0,1) \to (2,1)\,, \tag{3.27}$$

which correspond precisely to the sequence of pairs of d-vectors arising from the Zamolodchikov periodicity of the orbit of the original undeformed $C_2$ map, defined by (3.3) with all parameters $a_i, b_i$ set to 1. Indeed, these two orbits can be read off from the denominators in the corresponding sequence of $C_2$ cluster variables, viz.

$$(x_1,x_2) \longrightarrow \left(\frac{x_2^2+1}{x_1}, \frac{x_2^2+x_1+1}{x_1x_2}\right) \longrightarrow \left(\frac{x_2^2+(x_1+1)^2}{x_1x_2^2}, \frac{x_1+1}{x_2}\right).$$

For finite type cluster algebras, it is known that there is a direct correspondence between cluster variables and a specific subset of the roots in the associated root system, whereby the coefficients of linear combinations of simple roots determine the d-vectors (see e.g. theorem 4.10 in [5]). In this case, the pairs of d-vectors corresponding to the exponents of $x_1, x_2$ in the monomials above are given by the sequence of $2\times 2$ matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

whose first and second rows, respectively, yield the two orbits (3.27) above, while the columns correspond to the sequence of pairs of roots $(-\alpha_1,-\alpha_2)\to(\alpha_1,\alpha_1+\alpha_2)\to(\alpha_1+2\alpha_2,\alpha_2)$ in the $C_2$ root system. To determine the degree growth of the tau functions generated by (3.13), we only need to make use of the fact that these particular sequences have period 3, rather than the more general result of lemma 3.6.

**Theorem 3.8.** *All sequences of d-vectors* $\mathbf{d}_n,\mathbf{e}_n$ *that are solutions of the* $(\max,+)$ *system (3.24) lie in the kernel of the 6th order linear difference operator*

$$\hat{\mathcal{L}}=\left(\mathcal{T}^3-1\right)\left(\mathcal{T}^2-1\right)\left(\mathcal{T}-1\right).$$

*Hence the particular d-vectors corresponding to the sequences of tau functions* $\sigma_n,\tau_n$ *produced by (3.13) give the leading order degree growth*

$$\mathbf{d}_n=\frac{n^2}{6}\left(1,1,1,1,1\right)^T+O(n),\qquad \mathbf{e}_n=\frac{n^2}{12}\left(1,1,1,1,1\right)^T+O(n).\tag{3.28}$$

**Proof.** The second relation in (3.25) can be written as $(\mathcal{T}^2-1)(\mathcal{T}-1)\,\mathbf{e}_n=\mathbf{X}_{2,n}$, and then by lemma 3.6 we have

$$\hat{\mathcal{L}}\,\mathbf{e}_n=\left(\mathcal{T}^3-1\right)\left(\mathcal{T}^2-1\right)\left(\mathcal{T}-1\right)\mathbf{e}_n=\left(\mathcal{T}^3-1\right)\mathbf{X}_{2,n}=0,$$

which shows that $\mathbf{e}_n$ is annihilated by the linear operator $\hat{\mathcal{L}}$ (which is the same operator that appeared in (2.50) in the deformed $A_3$ case). Then the first relation in (3.25) implies that

$$\hat{\mathcal{L}}\,\mathbf{d}_n=\hat{\mathcal{L}}\left(\mathbf{Y}_{1,n}-\mathbf{e}_{n+1}\right)=\left(\mathcal{T}^2-1\right)\left(\mathcal{T}-1\right)\left(\mathbf{Y}_{1,n+3}-\mathbf{Y}_{1,n}\right)-\hat{\mathcal{L}}\,\mathbf{e}_{n+1}=0,$$

as required, where we used the fact that $\mathbf{Y}_{1,n}$ has period 3, from the same lemma. Now for the particular tropical seed (3.23), which corresponds to the d-vectors of the initial set of tau functions in (3.13), we can use $(\max,+)$ relations (3.24) to calculate more terms in the sequence of d-vectors. We can also use the corresponding 3-periodic sequence of solutions of the tropical QRT map, as presented in remark 3.7, to generate additional terms of the sequence $(\mathbf{e}_n)$ from the formula

$$\mathbf{e}_{n+3}=\mathbf{e}_{n+2}+\mathbf{e}_{n+1}-\mathbf{e}_n+\mathbf{X}_{2,n},$$

which produces

$$\mathbf{e}_4=\left(1,1,0,0,0\right)^T,\qquad \mathbf{e}_5=\left(1,2,1,0,0\right)^T,$$

so that $\mathbf{e}_j$, $0\leqslant j\leqslant 5$ provide enough initial values to generate the 6 vector constants that specify the explicit solution of the linear difference equation $\hat{\mathcal{L}}\,\mathbf{e}_n=0$ in terms of its characteristic roots. In fact, for this choice of initial values it is straightforward to show by induction that the components of $\mathbf{e}_n$ take the form

$$\mathbf{e}_n=\left(\tilde{e}_n,e_{n+3},e_{n+2},e_{n+1},e_n\right),$$

so these vectors are completely specified by the following two scalar recurrence sequences:

$$\begin{array}{ll}(\tilde{e}_n): & 0,0,0,0,1,1,2,3,4,5,7,8,10,12,14,16,19,21,\ldots;\\ (e_n): & 0,0,0,-1,0,0,0,1,2,2,4,5,6,8,10,11,14,16,\ldots.\end{array}$$

The sequences $\tilde{e}_n, e_n$ determine, respectively, the exponents of $\sigma_0$, and the exponents of the other $\tau_j$ for $j = 0,1,2,3$, that appear in the denominators of $\tau_n$. We omit the full details of the explicit formula for the sequence of vectors $\mathbf{e}_n$, and merely note that we can determine the leading order behavior $\mathbf{e}_n = \mathbf{a}\,n^2(1+o(n))$ by calculating the constant vector

$$\left(\mathcal{T}^3-1\right)\left(\mathcal{T}^2-1\right)\mathbf{e}_n = 12\mathbf{a} = (1,1,1,1,1)^T,$$

which fixes the value of $\mathbf{a}$. Similarly, a simple inductive argument can be used to show that $\mathbf{d}_n$ has components of the form

$$\mathbf{d}_n = \left(\tilde{d}_n, d_{n+3}, d_{n+2}, d_{n+1}, d_n\right),$$

specified by the scalars $\tilde{d}_n, d_n$. The latter two scalar sequences determine, respectively, the exponents of $\sigma_0$, and the exponents of the other $\tau_j$ for $j = 0,1,2,3$, that appear in the denominators of $\sigma_n$; their initial terms are as follows:

$$\begin{array}{ll} \left(\tilde{d}_n\right): & -1,1,1,1,3,5,5,9,11,13,17,21,23,29,33,37,43,49,\ldots; \\ \left(d_n\right): & 0,0,0,0,0,2,2,4,6,8,10,14,16,20,24,28,32,38,\ldots. \end{array}$$

We omit the complete details of the explicit formula for $\mathbf{d}_n$, but rather make use of the first relation in (3.25) once again, to see that the leading order behavior of the sequence is found from

$$\mathbf{d}_n = 2\mathbf{e}_{n+1} + \mathbf{X}_{1,n} = 2\mathbf{a}\,n^2\left(1+o(n)\right),$$

with the same constant vector $\mathbf{a}$, since $\mathbf{X}_{1,n}$ has period 3. Together, this yields the required expression for the leading order quadratic growth of $\mathbf{d}_n, \mathbf{e}_n$ as in (3.28), with an $O(n)$ correction in each case. □

## 4. Integrable deformations of the $\mathrm{B}_3$ cluster map

In this section, we consider the deformation of a 3D periodic cluster map which arises from mutations in the cluster algebra of type $\mathrm{B}_3$. The original cluster map in three dimensions has period $4 = \frac{1}{2}(6+2)$, which is $\frac{1}{2}\times(\text{Coxeter number}+2)$, and as before our aim is to construct parameter-families of deformations of this map that result in aperiodic dynamics that is Liouville integrable. However, in contrast to all the examples previously considered here and in [20], for the $\mathrm{B}_3$ case we find that there is more than one distinct family of deformations that is integrable (in fact, precisely two distinct 1-parameter families, up to equivalence via scaling transformations).

### 4.1. Deformed map $\mathrm{B}_3$

For the $\mathrm{B}_3$ root system, the Cartan matrix is

$$C = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

which is the companion of the skew-symmetrizable exchange matrix

$$B = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

Skew-symmetrizability of $B$ is seen from the fact that $\Omega = BD = (\omega_{ij})$ is skew-symmetric, where $D = \mathrm{diag}(2,1,1)$ and

$$\Omega = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We now consider the sequence of deformed mutations

$$\begin{array}{lll} \mu_1: & (x_1,x_2,x_3) \mapsto (x_1',x_2,x_3), & x_1'x_1 = b_1 + a_1x_2, \\ \mu_2: & (x_1',x_2,x_3) \mapsto (x_1',x_2',x_3), & x_2'x_2 = b_2 + a_2\,(x_1')^2\,x_3, \\ \mu_3: & (x_1',x_2',x_3) \mapsto (x_1',x_2',x_3'), & x_3'x_3 = b_3 + a_3x_2', \end{array} \tag{4.1}$$

where $a_j, b_j$ are arbitrary parameters. With a generic choice of these parameters, the Laurent property no longer holds for these mutations, so the map $\varphi = \mu_3\mu_2\mu_1$ does not have the Laurent property; moreover, it is no longer completely periodic with period 4. However, similarly to the $C_2$ case, by a minor variation on theorem 1.3 in [20], generalizing it to the skew-symmetrizable case, it is not hard to see that the deformed map $\varphi$ given by the composition of transformations (4.1) preserves the same presymplectic form $\omega = \sum_{i<j}\omega_{ij}\mathrm{d}\log x_i \wedge \mathrm{d}\log x_j$ as in the undeformed case.

Before considering the deformed case (4.1) further, there are two ways to simplify the calculations. Firstly, assuming the case of generic parameter values $a_ib_i \neq 0$ for all $i$, we apply the scaling action of the three-dimensional algebraic torus $(\mathbb{C}^*)^3$, given by $x_i \to \lambda_i x_i$, $\lambda_i \neq 0$, and use this to remove three parameters, so that we may set

$$a_i \to 1, \qquad i = 1,2,3,$$

without loss of generality, but keep $b_i$ arbitrary for $i = 1,2,3$. Having simplified the space of parameters, the map $\varphi$ is equivalent to the iteration of the system of recurrences

$$\begin{aligned} x_{1,n+1}x_{1,n} &= x_{2,n} + b_1, \\ x_{2,n+1}x_{2,n} &= x_{1,n+1}^2x_{3,n} + b_2, \\ x_{3,n+1}x_{3,n} &= x_{2,n+1} + b_3. \end{aligned} \tag{4.2}$$

Secondly, because we are in an odd-dimensional situation where necessarily $\det(\Omega) = 0$ and $\omega$ is degenerate, we can use

$$\ker\Omega = <(1,0,2)^T>, \qquad \mathrm{im}\,\Omega = (\ker\Omega)^\perp = <(0,1,0)^T, (-2,0,1)^T>$$

to generate the one-parameter scaling group $(x_1,x_2,x_3) \to (\lambda x_1, x_2, \lambda^2 x_3)$, $\lambda \in \mathbb{C}^*$ (obtained from the null vector field $x_1\partial_{x_1} + 2x_3\partial_{x_3}$ by exponentiation), and the projection $\pi: \mathbb{C}^3 \to \mathbb{C}^2$ onto its monomial invariants,

$$\pi: \qquad y_1 = x_2, \qquad y_2 = \frac{x_3}{x_1^2}.$$

On the $(y_1, y_2)$-plane, $\varphi$ induces the reduced map $\hat{\varphi}$, such that $\pi \cdot \varphi = \hat{\varphi} \cdot \pi$, where

$$\hat{\varphi}: \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1^{-1}\left( (y_1+b_1)^2 y_2 + b_2 \right) \\ y_1^{-1}\left( 1 + \frac{b_3 y_1 + b_2}{y_2 (y_1+b_1)^2} \right) \end{pmatrix}. \tag{4.3}$$

The reduced map is symplectic, that is to say $\hat{\varphi}^*(\hat{\omega}) = \hat{\omega}$, where the non-degenerate two-form preserved by $\hat{\varphi}$ is

$$\hat{\omega} = \mathrm{d}\log y_1 \wedge \mathrm{d}\log y_2, \qquad \pi^* \hat{\omega} = \omega. \tag{4.4}$$

In the original case where all parameters are 1, the reduced map (4.3) with $b_1 = b_2 = b_3 = 1$ has period 4, because $x_{2,n+4} = x_{2,n}$ and $x_{3,n+4}/x_{1,n+4}^2 = x_{3,n}/x_{1,n}^2$ for all $n$. In that case we can construct two functionally independent first integrals in the plane, $K^{(i)}, K^{(ii)}$ say. Here we will just focus on one of these, namely

$$\begin{aligned} K^{(i)} &:= \textstyle\sum_{i=0}^{3} (\hat{\varphi}^*)^i (y_1) \\ &= y_1 y_2 + y_1 + 3y_2 + 3\tfrac{y_2}{y_1} + \tfrac{y_2}{y_1^2} + \tfrac{5}{y_1} + \tfrac{1}{y_2} + \tfrac{2}{y_1^2} + \tfrac{2}{y_1 y_2} + \tfrac{1}{y_1^2 y_2}, \end{aligned} \tag{4.5}$$

which satisfies $\hat{\varphi}^*(K^{(i)}) = K^{(i)}$ when $b_1 = b_2 = b_3 = 1$.

Next, we modify $K^{(i)}$ by inserting constant coefficients in front of each of the Laurent monomials in $y_1, y_2$ that appear, fixing the coefficient of the first term to be 1 without loss of generality, to obtain

$$\hat{K} = y_1 y_2 + c_1 y_1 + c_2 y_2 + c_3 \frac{y_2}{y_1} + c_4 \frac{y_2}{y_1^2} + \frac{c_5}{y_1} + \frac{c_6}{y_2} + \frac{c_7}{y_1^2} + \frac{c_8}{y_1 y_2} + \frac{c_9}{y_1^2 y_2}. \tag{4.6}$$

If we assume that these modified first integrals are preserved by the deformed map $\hat{\varphi}$ given by (4.3), then this puts a finite number of constraints on the coefficients $c_i$ and the parameters $b_i$, which leads to finding necessary and sufficient conditions for the deformed symplectic map to be Liouville integrable. Thus we obtain the following result.

**Theorem 4.1.** *For the deformed symplectic map (4.3) to admit a first integral of the form (4.6), it is necessary and sufficient that the parameters $b_i$ should satisfy either*

$$b_1 = b_2, \quad b_3 = 1, \tag{4.7}$$

*or*

$$b_2 = b_3 = b_1^2. \tag{4.8}$$

*If we fix $b_1 = \beta$, then in the case that the constraint (4.7) holds, the first integral takes the form*

$$\hat{K}_1 = y_1 y_2 + y_1 + (2\beta+1) y_2 + \beta(\beta+2) \frac{y_2}{y_1} + \beta^2 \frac{y_2}{y_1^2} + \frac{3\beta+2}{y_1} + \frac{1}{y_2} + \frac{2\beta}{y_1^2} + \frac{2}{y_1 y_2} + \frac{1}{y_1^2 y_2}, \tag{4.9}$$

*while in the case that (4.8) holds, the first integral is*

$$\hat{K}_2 = y_1y_2 + y_1 + (2\beta+1)\,y_2 + \beta\,(\beta+2)\,\frac{y_2}{y_1} + \beta^2\frac{y_2}{y_1^2} + \frac{2\beta^2+2\beta+1}{y_1} + \frac{1}{y_2} + \frac{2\beta^2}{y_1^2} + \frac{\beta^2+1}{y_1y_2} + \frac{\beta^2}{y_1^2y_2}. \tag{4.10}$$

*Hence the map $\hat{\varphi}$ given by (4.3) is Liouville integrable whenever either condition (4.7) or (4.8) holds.*

Thus we arrive at two 1-parameter families of integrable maps of the plane associated with the deformation of the $\mathrm{B}_3$ cluster map, namely

$$\hat{\varphi}_1: \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1^{-1}\left((y_1+\beta)^2 y_2 + \beta\right) \\ y_1^{-1}\left(1 + \frac{1}{y_2(y_1+\beta)}\right) \end{pmatrix}, \tag{4.11}$$

which has the first integral $\hat{K}_1$ given by (4.9), and

$$\hat{\varphi}_2: \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1^{-1}\left((y_1+\beta)^2 y_2 + \beta^2\right) \\ y_1^{-1}\left(1 + \frac{\beta^2(y_1+1)}{y_2(y_1+\beta)^2}\right) \end{pmatrix}, \tag{4.12}$$

with the first integral $\hat{K}_2$, as in (4.10). So each map has an invariant pencil of genus 1 curves of degree 5 and bidegree $(3,2)$; that is too high for a QRT map, where the bidegree is $(2,2)$ [4]. Clearly the maps coincide for $\beta = 1$ when the map is completely periodic with period 4 (corresponding to a pencil of elliptic curves with 4-torsion). However, it seems that the maps cannot be birationally conjugate to one another for other values of $\beta$; one way to see this is to look at the j-invariants of the curves in each pencil, which are rational functions of $\beta$ and the value of the invariant $\hat{K}_j = \kappa$ (for $j = 1,2$, respectively): the factorizations of the two different j-invariants have polynomial factors in their denominators that appear with quite different degrees, and this could be used to show that there is no automorphism of $\mathbb{C}(\beta,\kappa)$ which transforms one elliptic fibration into the other; or perhaps there is a geometrical way to see this more easily. It is possible to see that the two maps cannot be conjugate to one another more directly, by considering the fixed points: for generic $\beta$, in the affine plane $\mathbb{C}^2$ the map (4.11) has three fixed points outside the line $y_1 = 0$ where it is singular, whereas the map (4.12) only has one fixed point outside this line.

### *4.2. The deformed map $\hat{\varphi}_1$ for* $\mathrm{B}_3$

Let us consider the deformed map (4.11), which can be rewritten as the pair of recurrence relations

$$\begin{aligned} y_{1,n+1}y_{1,n} &= (y_{1,n}+\beta)^2\,y_{2,n} + \beta, \\ y_{2,n+1}y_{2,n}y_{1,n}\,(y_{1,n}+\beta) &= (y_{1,n}+\beta)\,y_{2,n} + 1. \end{aligned} \tag{4.13}$$

Following the same process as in the previous section, we study the singularity structures of the deformed map (4.11) by observing the $p$-adic properties of iterates defined over $\mathbb{Q}$. Then we find that there are two singularity patterns for $y_{1,n}$ and $y_{2,n}$, namely

$$\begin{aligned}&\text{Pattern 1}: (y_{1,n},y_{2,n}) = \ldots, \left(R,\infty^1\right), \left(\infty^1,R\right), \left(\infty^1,0^1\right), \left(R,0^2\right), \left(R,\infty^2\right), \ldots\\ &\text{Pattern 2}: (y_{1,n},y_{2,n}) = \ldots, (0,R), \ldots.\end{aligned} \tag{4.14}$$

This indicates that the $y_{1,n}$ and $y_{2,n}$ can be written in terms of tau functions $\tau_n$ and $\eta_n$ as

$$y_{1,n} = \frac{\eta_n}{\tau_{n+2}\tau_{n+3}}, \quad y_{2,n} = \rho_n \frac{\tau_{n+1}^2\tau_{n+2}}{\tau_n^2\tau_{n+4}}, \tag{4.15}$$

where the quantity $\rho_n$ is an additional prefactor. However, substituting these expressions directly into (4.13) gives rise to relations between tau functions that are not in the form of cluster exchange relations. So in addition to $p$-adic analysis, we proceed to consider the singularity patterns more closely via explicit analysis with the introduction of a small quantity $\epsilon$.

A discrete dynamical system defined by a birational map can have two types of singularities: the points in phase space at which the map is undefined, and the points where the Jacobian of the map vanishes. From (4.11), one can see that the deformed map $\hat{\varphi}_1$ possesses a singularity at $y_1 = -\beta$. Performing singularity analysis by setting $y_{1,n} = -\beta + \epsilon$, we find the confined singularity pattern

$$\begin{pmatrix}-\beta\\ C\end{pmatrix} \to \begin{pmatrix}-1\\ \infty^1\end{pmatrix} \to \begin{pmatrix}\infty^1\\ -1\end{pmatrix} \to \begin{pmatrix}\infty^1\\ 0^1\end{pmatrix} \to \begin{pmatrix}-1\\ 0^2\end{pmatrix} \to \begin{pmatrix}-\beta\\ \infty^2\end{pmatrix} \to \begin{pmatrix}C'\\ -1/\beta\end{pmatrix}, \tag{4.16}$$

where $C, C'$ are regular values, and when $\epsilon \to 0$ the subsequent terms are not indeterminate (they are generic, regular values). By comparing (4.16) with (4.14), it is clear that (4.16) corresponds to Pattern 1, but with more detail revealed. The detailed form of the singularity pattern suggests another way to relate $y_{1,n}$ to the tau function $\tau_n$, after shifting by the parameter $\beta$, expressing it as

$$y_{1,n} = -\beta + \vartheta_n \frac{\tau_{n+5}\tau_n}{\tau_{n+3}\tau_{n+2}}, \tag{4.17}$$

where $\vartheta_n$ is another prefactor. Defining a new variable $w_n = y_1 + \beta$ leads to a system of three recurrence relations, expressed in terms of $w_n, y_{1,n}$ and $y_{2,n}$. Furthermore, subtracting the first relation in (4.13) from $w_n$ times the second and removing a common factor of $y_{1,n}$ results in simplifying the recurrence for $y_{2,n}$, yielding the three equations

$$\begin{aligned} w_n &= y_{1,n} + \beta,\\ y_{1,n+1}y_{1,n} &= w_n^2 y_{2,n} + \beta,\\ y_{2,n+1}y_{2,n}w_n^2 &= y_{1,n+1} + 1.\end{aligned} \tag{4.18}$$

The way that the tau function $\tau_n$ appears in (4.17) suggests that there is a close connection between the iteration of $w_n$ in (4.18) and the Lyness map.

**Theorem 4.2.** *Under the iteration of (4.18), the quantity $w_n$ satisfies the Lyness recurrence*

$$w_{n+1}w_{n-1} = \tilde{\alpha} w_n + \tilde{\beta}, \tag{4.19}$$

*where the coefficients along each orbit of the map $\hat{\varphi}_1$ are $\tilde{\alpha} = 1 - \beta$ and $\tilde{\beta} = \beta \hat{K}_1 + 2\beta^2 + \beta + 1$.*

**Proof.** By following the same approach as used in the proof of theorem 3.4, after setting the prefactor $\vartheta_n \to 1$ in (4.17), one can show that $\tau_n$ satisfies a special Somos-7 relation of the same form as (3.22), namely

$$\tau_{n+7}\tau_n = \tilde{\alpha}\tau_{n+6}\tau_{n+1} + \tilde{\beta}\tau_{n+3}\tau_{n+4}. \tag{4.20}$$

where $\tilde{\alpha} = 1 - \beta$, and the coefficient $\tilde{\beta}$ is given as above in terms of $\beta$ and the conserved quantity $\hat{K}_1$. Then from (4.17) and the first relation in (4.18), it is clear that $w_n$ is given in terms of the tau function $\tau_n$ by

$$w_n = \frac{\tau_{n+5}\tau_n}{\tau_{n+3}\tau_{n+2}}, \tag{4.21}$$

and from this it is an immediate consequence that $w_n$ satisfies the Lyness recurrence (4.19) with these coefficients, which are constant along each orbit. $\square$

As noted in the introduction, the special Somos-7 recurrence (4.20) corresponds to a cluster algebra of rank 7, where $(\tau_n)$ is a sequence of cluster variables and the coefficients $\tilde{\alpha}, \tilde{\beta}$ are regarded as frozen variables; and in that setting, the associated exchange matrix has rank 2 (for further details, see [11]). In the discussion of the deformed $C_2$ map, we already noted that there is a close connection between this special Somos-7 relation and Somos-5. This leads to a related result for the map defined by (4.13).

**Theorem 4.3.** *Under iteration of (4.13), the quantity $v_n = y_{1,n} + 1$ satisfies the Somos-5 QRT map, in the form*

$$v_{n+1}v_nv_{n-1} = \hat{\alpha}v_n + \hat{\beta} \tag{4.22}$$

*where the coefficients along each orbit of the map $\hat{\varphi}_1$ are given by $\hat{\alpha} = \tilde{K}_1 + \beta + 3$ and $\hat{\beta} = (\beta - 1)\hat{\alpha}$.*

**Proof.** This result, including the above formulae for the coefficients $\hat{\alpha}, \hat{\beta}$, is a consequence of theorem 1 in [19], which states that each invariant curve of the Lyness map is birationally equivalent to an invariant curve corresponding to the Somos-5 QRT map, and hence there is a direct correspondence between the orbits of the two maps, whenever the parameters of the maps related to each other in a specific way. See also proposition B.2 in the second appendix below. $\square$

Recall from the discussion around theorem 3.4 that a substitution of the form

$$v_n = \frac{\hat{\tau}_{n+4}\hat{\tau}_{n+1}}{\hat{\tau}_{n+3}\hat{\tau}_{n+2}} \tag{4.23}$$

relates (4.22) directly to the Somos-5 recurrence, that is

$$\hat{\tau}_{n+5}\hat{\tau}_n = \hat{\alpha}\hat{\tau}_{n+1}\hat{\tau}_{n+4} + \hat{\beta}\hat{\tau}_{n+2}\hat{\tau}_{n+3}. \tag{4.24}$$

Now the substitution (4.23) and the definition of the quantity $v_n$ implies that

$$y_{1,n} = -1 + \frac{\hat{\tau}_{n+4}\hat{\tau}_{n+1}}{\hat{\tau}_{n+3}\hat{\tau}_{n+2}},$$

but in general this is not compatible with the substitution (4.21) that relates $w_n$ to a solution of (4.20), in the sense that the tau functions $\tau_n$ and $\hat{\tau}_n$ need not be the same, but instead they are related by a gauge factor that depends on $n$. Rather, the most general way to relate $v_n$ to $\tau_n$ is to write

$$v_n = y_{1,n} + 1 = \xi_n \frac{\tau_{n+4}\tau_{n+1}}{\tau_{n+3}\tau_{n+2}}, \tag{4.25}$$

with another prefactor $\xi_n$ that depends on $n$. It will turn out that, with an appropriate choice of gauge, this quantity is periodic with period 3. (See theorem 4.4 below, and appendix B.)

Observe that, with the extra variable $v$ added to $y_1, y_2$ and $w$, the map defined by (4.13) is equivalent to iteration of a system of four equations, namely

$$\begin{aligned} w_n &= y_{1,n} + \beta, \\ y_{1,n+1}y_{1,n} &= w_n^2 y_{2,n} + \beta, \\ v_n &= y_{1,n} + 1, \\ y_{2,n+1}y_{2,n}w_n^2 &= v_{n+1}, \end{aligned} \tag{4.26}$$

and upon substituting for $y_1, y_2$ from (4.15), for $w$ from (4.17), and for $v$ from (4.25) the most general set of relations between the tau functions is found to be the following:

$$\begin{aligned} &\vartheta_n \tau_{n+5}\tau_n = \beta\,\tau_{n+3}\tau_{n+2} + \eta_n, \\ &\eta_{n+1}\eta_n = \rho_n (\vartheta_n)^2\, \tau_{n+5}^2\tau_{n+1}^2 + \beta\,\tau_{n+4}\tau_{n+3}^2\tau_{n+2}, \\ &\xi_n\tau_{n+4}\tau_{n+1} = \tau_{n+3}\tau_{n+2} + \eta_n, \\ &\rho_{n+1}\rho_n (\vartheta_n)^2 = \xi_{n+1}. \end{aligned} \tag{4.27}$$

**Theorem 4.4.** *There is a choice of gauge which fixes $\vartheta_n \to 1$ in the system (4.27), and implies that $\xi_{n+3} = \xi_n$ and $\rho_{n+6} = \rho_n$ for all n, with*

$$\prod_{i=0}^{5} \rho_i = \prod_{j=1}^{3} \xi_j = \hat{K}_1 + \beta + 3. \tag{4.28}$$

*In that case, the system corresponds to a lift of the deformed* $\mathrm{B}_3$ *map* $\hat{\varphi}_1$ *to a birational map on an extended space of tau functions, that is*

$$\Phi:\ (\tau_0,\tau_1,\tau_2,\tau_3,\tau_4,\eta_0,\rho_0,\beta) \mapsto (\tau_1,\tau_2,\tau_3,\tau_4,\tau_5,\eta_1,\rho_1,\beta), \tag{4.29}$$

*where the sequences $(\tau_n)$, $(\eta_n)$ possess the Laurent property, but the periodic coefficients $\rho_n$ do not.*

**Proof.** By definition, in the context of the tau function formulae (4.15), a gauge transformation is any transformation of the tau functions which leaves the variables $y_{1,n}, y_{2,n}$ invariant. If we make the replacement $\tau_n \to g_n\tau_n$, where the dependence of $g_n$ on $n$ is arbitrary, then clearly replacing $\eta_n \to g_{n+2}g_{n+3}\eta_n$ leaves $y_{1,n}$ the same, while replacing $\rho_n \to g_n^2 g_{n+4} g_{n+1}^{-2} g_{n+2}^{-1}\rho_n$ leaves $y_{2,n}$ unchanged. Now in (4.17), regardless of what non-zero prefactor $\vartheta_n$ appears to begin with, we can always make the replacement $\vartheta_n \to g_{n+2}g_{n+3}g_n^{-1}g_{n+5}^{-1}\vartheta_n = 1$; to be precise, this is achieved by specifying any solution of a linear difference equation of order 5 for $\log g_n$. With that choice of gauge, the variable $w_n$ is given in terms of $\tau_n$ by (4.21), and the sequence $(\tau_n)$ satisfies the special Somos-7 recurrence (4.20), as in the proof of theorem 4.2. However, as already mentioned above, in general the prefactor $\xi_n$ appearing in (4.25) cannot be simultaneously fixed to be 1 (rather, fixing $\xi_n \to 1$, so that $\tau_n$ satisfies the Somos-5 relation (4.24), is a *different* gauge choice). Thus, in the 'Somos-7 gauge', where $\theta_n = 1$, the system of recurrences (4.27) becomes

$$\begin{aligned} \tau_{n+5}\tau_n &= \beta\,\tau_{n+3}\tau_{n+2} + \eta_n, \\ \eta_{n+1}\eta_n &= \rho_n\,\tau_{n+5}^2\tau_{n+1}^2 + \beta\,\tau_{n+4}\tau_{n+3}^2\tau_{n+2}, \\ \xi_{n+1}\,\tau_{n+5}\tau_{n+2} &= \tau_{n+4}\tau_{n+3} + \eta_{n+1}, \\ \rho_{n+1}\rho_n &= \xi_{n+1} \end{aligned} \tag{4.30}$$

(having shifted $n \to n+1$ in the third relation). In the above, an extended 'cluster' of initial data, including the fixed parameter ('frozen variable') $\beta$, is given by $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \eta_0, \rho_0, \beta)$, and via (4.15) this fixes initial data $\mathbf{y}_0 = (y_{1,0}, y_{2,0})$ for the map $\hat{\varphi}_1$. Now iterating each of the equations (4.30) one by one, in order, starting from $n = 0$, produces in turn $\tau_5, \eta_1, \xi_1, \rho_1$, giving the image of the lifted map $\Phi$ as in (4.29). Notice that the intermediate step of finding $\xi_1$ can be skipped: for each $n$, by combining the last two relations, we have

$$\rho_{n+1}\rho_n = \frac{\tau_{n+4}\tau_{n+3} + \eta_{n+1}}{\tau_{n+5}\tau_{n+2}}.$$

Hence the first two relations in (4.30) appear like a pair of cluster exchange relations, with one of them having a coefficient $\rho_n$ that is non-autonomous (dependent on $n$). Upon iteration of the map $\Phi$, we obtain the three sequences $(\tau_n)$, $(\eta_n)$, $(\rho_n)$, which together specify the orbit $\mathbf{y}_n = \hat{\varphi}_1^n(\mathbf{y}_0)$, as well as the sequence $(\xi_n)$ of intermediate values, which appear in the formula (4.25) for the quantities $v_n$. Now consider the ring of Laurent polynomials

$$\mathcal{R} = \mathbb{Z}\left[\beta, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}, \tau_4^{\pm 1}, \eta_0^{\pm 1}, \rho_0^{\pm 1}\right].$$

Direct calculation of three steps of $\Phi$ shows by inspection that $\xi_1, \xi_2 \in \mathcal{R}$, and

$$\xi_3 = \frac{\tau_3\tau_2 + \eta_0}{\tau_4\tau_1} = \xi_0 \in \mathcal{R},$$

hence the sequence $(\xi_n)$ has period 3, or in other words $(\mathcal{T}^3 - 1)\,\xi_n = 0$ (where $\mathcal{T}$ denotes the shift operator that sends $n \to n+1$). Then, upon taking logarithms on both sides of the fourth relation in (4.30), we have

$$\begin{aligned} (\mathcal{T}+1)\,\log\rho_n = \log\xi_{n+1} &\implies \left(\mathcal{T}^3 - 1\right)(\mathcal{T}+1)\,\log\rho_n = 0 \\ &\implies \left(\mathcal{T}^6 - 1\right)\,\log\rho_n = \left(\mathcal{T}^3 - 1\right)\left(\mathcal{T}^3 + 1\right)\,\log\rho_n = 0, \end{aligned}$$

hence the sequence $(\rho_n)$ has period 6, as required. However, while $\rho_0, \rho_1, \rho_5 \in \mathcal{R}$, we find that $\rho_2, \rho_3, \rho_4 \notin \mathcal{R}$: the latter three terms have non-monomial factors appearing in their denominators, so they cannot be cluster variables. A direct calculation shows that the product of three adjacent $\xi_n$ is

$$\xi_1\xi_2\xi_3 = \hat{K}_1 + \beta + 3 \in \mathcal{R},$$

where here $\hat{K}_1$ is used to denote the value of the invariant along an orbit of the lifted map $\Phi$, considered as a function of $\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \eta_0, \rho_0, \beta$; hence $\hat{K}_1 \in \mathcal{R}$, and using the fourth relation in (4.30) once more, we see that the latter product is equal to $\rho_0\rho_1\rho_2\rho_3\rho_4\rho_5$, so (4.28) holds, as required. Next, we claim that $\tau_n, \eta_n \in \mathcal{R}$. To see this, we just need to show that $\tau_n \in \mathcal{R}$ for all $n$, since if this holds then the first relation in (4.30) implies immediately that $\eta_n = \tau_{n+5}\tau_n - \beta\,\tau_{n+3}\tau_{n+2} \in \mathcal{R}$. So we consider the aforementioned fact that, due to the gauge choice, $\tau_n$ satisfies the special Somos-7 recurrence (4.20), which has coefficients $\tilde{\alpha}, \tilde{\beta}$, and there is an associated Lyness invariant quantity (see [19], for instance), which we denote by $\tilde{K}$. Then, by a minor modification of theorem 3.7 in [16] and its proof, it follows that the Somos-7 recurrence has the strong Laurent property, in the sense that $\tau_n \in \tilde{\mathcal{R}}$ for all $n \geqslant 0$, where

$$\tilde{\mathcal{R}} = \mathbb{Z}\left[\tilde{\alpha}, \tilde{\beta}, \tilde{K}, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}, \tau_4, \tau_5, \tau_6\right].$$

(For further details, see theorem B.5 in the second appendix below.) By inspection of the first two iterates of $\Phi$, we can verify directly that $\tau_5, \tau_6 \in \mathcal{R}$, while from theorem 4.2 we have $\tilde{\alpha} = 1 - \beta$, $\tilde{\beta} = \beta\hat{K}_1 + 2\beta^2 + \beta + 1$ and a short explicit calculation with computer algebra shows that $\tilde{K} = \hat{K}_1 + 2\beta + 2$. Since, as already noted, $\hat{K}_1 \in \mathcal{R}$ on an orbit of $\Phi$, it follows that $\tilde{\alpha}, \tilde{\beta}, \tilde{K} \in \mathcal{R}$, hence $\tilde{\mathcal{R}}$ is a subring of $\mathcal{R}$. Thus we see that $\tau_n \in \mathcal{R}$ for $n \geqslant 0$, and an analogous argument extends this to $n < 0$ and completes the proof of the theorem. □

**Remark 4.5.** The first two relations in (4.30) resemble exchange relations in a cluster algebra, but the third and fourth relations (which define $\rho_n$) do not. Thus, the sequence of tau functions cannot be produced by cluster exchange relations with frozen variables alone. Nevertheless, this can be considered an 'almost Laurentification' of the deformed map: the tau functions $\tau_n$ and $\eta_n$ are Laurent polynomials, while the periodic quantities $\rho_n$ only contain a finite number of non-monomial factors in their denominators, so this is an example of the extended Laurent property [28], where only a finite extension of the ring $\mathcal{R}$ is required. The coefficients $\rho_n$ are reminiscent of y-variables in a cluster algebra with coefficients, which can be used to generate non-autonomous difference equations, including those of discrete Painlevé type [17, 32]. We have attempted to construct the relations (4.30) from a suitable Y-system, by pulling back the two-form (4.4) to derive an associated exchange matrix (cf. the formulae (4.42)–(4.44) for the case of the map $\hat{\varphi}_2$ below), but as yet we have not succeeded in doing this in a consistent way.

### 4.3. *Deformed map $\hat{\varphi}_2$ for* $\mathrm{B}_3$

The action of the deformed map $\hat{\varphi}_2$ given by (4.12) is equivalent to iteration of the coupled pair of recurrences

$$\begin{aligned} y_{1,n+1}y_{1,n} &= w_n^2 y_{2,n} + \beta^2, \\ y_{2,n+1}y_{2,n}y_{1,n}w_n^2 &= w_n^2 y_{2,n} + \beta^2\left(y_{1,n}+1\right), \end{aligned} \tag{4.31}$$

where for convenience we made use of the same variable $w_n = y_{1,n} + \beta$ as in the previous discussion of the map $\hat{\varphi}_1$. Subtracting the first relation from the second gives rise to a simplified relation for $y_{2,n+1}$, and thus iterates of the map are generated by the system of four relations

$$\begin{aligned} w_n &= y_{1,n} + \beta, \\ y_{1,n+1}y_{1,n} &= w_n^2 y_{2,n} + \beta^2, \\ v_{n+1} &= y_{1,n+1} + \beta^2, \\ y_{2,n+1}y_{2,n}w_n^2 &= v_{n+1}, \end{aligned} \tag{4.32}$$

where, in contrast to (4.26), there is a different definition for the quantity $v_n = y_{1,n} + \beta^2$. By looking into the prime factorization of some orbits of (4.32) defined over $\mathbb{Q}$, we observe the following confined singularity patterns:

$$\begin{aligned}
&\text{Pattern 1}: (y_{1,n}, y_{2,n}, w_n) = \dots, (R,0,R), (R,\infty,R), (\infty,R,\infty), (\infty,0,\infty), (R,0,R), \dots \\
&\text{Pattern 2}: (y_{1,n}, y_{2,n}, w_n) = \dots, (R,R,0), \left(R,\infty^2,0\right), \dots \\
&\text{Pattern 3}: (y_{1,n}, y_{2,n}, w_n) = \dots, (0,R,R), \dots .
\end{aligned} \tag{4.33}$$

(The corresponding patterns for $v_n$ have been omitted.) We introduce tau functions $\tau_n, \sigma_n$ and $\eta_n$ which correspond to Patterns 1,2 and 3, respectively, such that $y_{1,n}$, $y_{2,n}$ and $w_n = y_{1,n} + \beta$ can be written as

$$y_{1,n} = \frac{\eta_n}{\tau_{n+2}\tau_{n+1}}, \quad y_{2,n} = \frac{\tau_{n+4}\tau_{n+1}\tau_n}{\sigma_n^2 \tau_{n+3}}, \quad w_n = \frac{\sigma_{n+1}\sigma_n}{\tau_{n+2}\tau_{n+1}}, \tag{4.34}$$

and then the fourth equation in (4.32) immediately implies that

$$v_n = y_{1,n} + \beta^2 = \frac{\tau_{n+4}\tau_{n-1}}{\tau_{n+2}\tau_{n+1}}. \tag{4.35}$$

Upon inspecting the structure of a particular singularity further by approaching it in the limit of a small parameter $\epsilon \to 0$, one can see that the singularities of $y_{1,n}$ and $y_{2,n}$ in Pattern 1 correspond to the sequence

$$\begin{pmatrix} C \\ -\frac{\beta^2(C+1)}{C^2+2\beta C+\beta^2} \end{pmatrix} \to \begin{pmatrix} -\beta^2 \\ 0^1 \end{pmatrix} \to \begin{pmatrix} -1 \\ \infty^1 \end{pmatrix} \to \begin{pmatrix} \infty^1 \\ -1 \end{pmatrix} \to \begin{pmatrix} \infty^1 \\ 0^1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0^1 \end{pmatrix} \to \begin{pmatrix} -\beta^2 \\ C' \end{pmatrix} \tag{4.36}$$

with $C, C'$ being regular values, which propagates from the point $(C, -\frac{\beta^2(C+1)}{C^2+2\beta C+\beta^2})$ where the Jacobian of the deformed map $\hat{\varphi}_2$ is zero. Noting that the value $y_1 = -1$ appears in the singularity pattern, we can consider another variable $u_n = y_{1,n} + 1$, and find that

$$u_n = y_{1,n} + 1 = \xi_n \frac{\tau_{n+3}\tau_n}{\tau_{n+2}\tau_{n+1}}, \tag{4.37}$$

where the prefactor $\xi_n$ cannot be removed without a change of gauge, which would modify the form of some of the expressions in (4.35). (As explained in appendix B, the quantity $\xi_n$ is periodic with period 3.)

Notice that the ratios of tau functions in (4.35) and (4.37) are identical to the substitutions associated with the Lyness map and Somos-5 QRT map, respectively. This suggests that the quantities $v_n = y_{1,n} + \beta^2$ and $u_n = y_{1,n} + 1$ should provide solutions of these maps under iteration, as described by the following statement.

**Theorem 4.6.** *The quantities $v_n$ generated under iteration of the system of recurrences (4.32) satisfy the Lyness map which is equivalent to the recurrence*

$$v_{n+1}v_{n-1} = \gamma v_n + \delta, \tag{4.38}$$

*where the coefficients along each orbit of the $\hat{\varphi}_2$ are specified by $\gamma = 1 - \beta^2$ and $\delta = \beta^2 \hat{K}_2 + 2\beta(\beta^3 + 1)$, and the associated sequence of tau functions $(\tau_n)$ related via (4.35) satisfies the Somos-7 recurrence*

$$\tau_{n+7}\tau_n = \gamma\, \tau_{n+6}\tau_{n+1} + \delta\, \tau_{n+4}\tau_{n+3}. \tag{4.39}$$

*The corresponding iterates of* $u_n = y_{1,n} + 1$ *satisfy the Somos-5 QRT map which is given by the recurrence*

$$u_{n+1}u_nu_{n-1} = \hat{\gamma}u_n + \hat{\hat{\delta}}, \tag{4.40}$$

*where* $\hat{\gamma} = \hat{K}_2 + 2\beta + 2$ *and* $\hat{\hat{\delta}} = (\beta^2 - 1)\hat{\gamma}$.

**Proof.** This follows from analogous arguments to those used in proving theorems 4.2 and 4.3. For a more detailed explanation of the connection between the Lyness map (4.38) and the Somos-5 QRT map (4.40), see proposition B.2 in the second appendix below. □

The tau function expressions (4.34) and (4.35) can be substituted directly into (4.32), giving rise to the system of equations

$$\begin{aligned} \sigma_{n+1}\sigma_n &= \beta\tau_{n+2}\tau_{n+1} + \eta_n, \\ \eta_{n+1}\eta_n &= \tau_{n+4}\tau_n\sigma_{n+1}^2 + \beta^2\tau_{n+3}\tau_{n+2}^2\tau_{n+1}, \\ \tau_{n+5}\tau_n &= \beta^2\tau_{n+3}\tau_{n+2} + \eta_{n+1}. \end{aligned} \tag{4.41}$$

Since the three recurrences above are all of the right form for an exchange relation, it appears likely that their iteration can be described by a sequence of cluster mutations in an appropriate cluster algebra. To verify this is the case, we set the initial cluster to be

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7) = (\eta_0, \sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \tau_4),$$

and then determine a new exchange matrix via the pullback of the symplectic form (4.4) by the rational map $\tilde{\pi} : \mathbb{C}^7 \to \mathbb{C}^2$ defined by the equations for $(y_{1,0}, y_{2,0}) \in \mathbb{C}^2$ given by setting $n = 0$ in (4.34). As a result, one finds a presymplectic form on the space of tau functions, written in terms of the cluster variables $\tilde{x}_j$ for $1 \leqslant j \leqslant 7$ as

$$\tilde{\omega} = \tilde{\pi}^*\omega = \sum_{ij} \tilde{\Omega}_{ij} \mathrm{d}\log\tilde{x}_i \wedge \mathrm{d}\log\tilde{x}_j,$$

where the $7 \times 7$ matrix $\tilde{\Omega}$ is given by

$$\tilde{\Omega} = \begin{pmatrix} 0 & -2 & 1 & 1 & 0 & -1 & 1 \\ 2 & 0 & 0 & -2 & -2 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 1 & -1 \\ 0 & 2 & -1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{4.42}$$

Given the skew-symmetric matrix $\Omega$ as above, a skew-symmetrizable exchange matrix $\tilde{B}$ such that $\tilde{B}\tilde{D} = \tilde{\Omega}$ can be determined by post-multiplying with the diagonal matrix $\tilde{D}^{-1} = \mathrm{diag}(1, {}^1\!/_2, 1, 1, 1, 1, 1)$, to obtain

$$\tilde{B} = \tilde{\Omega}\tilde{D}^{-1} = (\tilde{B}_{ij}) = \begin{pmatrix} 0 & -1 & 1 & 1 & 0 & -1 & 1 \\ 2 & 0 & 0 & -2 & -2 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{4.43}$$

The matrix $\tilde{B}$ above generates a coefficient-free cluster algebra, but both the parameter $\beta$ and its square appear in front of some of the terms in (4.41). To incorporate this into the exchange relations, we extend the initial cluster by adding the frozen variable $\tilde{x}_8 = \beta$ and adjoining an extra row with entries $(0,1,0,0,0,0,-2)$ to the exchange matrix $\tilde{B}$. Then we can obtain the following statement, which constitutes the Laurentification of the deformed $\mathrm{B}_3$ map $\hat{\varphi}_2$.

**Theorem 4.7.** *Let* $(\hat{\mathbf{x}},\hat{B})$ *be given as an initial seed which is composed of the extended initial cluster*

$$\hat{\mathbf{x}} = (\tilde{x}_j)_{1\leqslant j\leqslant 8} = (\eta_0,\sigma_0,\tau_0,\tau_1,\tau_2,\tau_3,\tau_4,\beta)$$

*together with the associated extended exchange matrix*

$$\hat{B} = \begin{pmatrix} 0 & -1 & 1 & 1 & 0 & -1 & 1 \\ 2 & 0 & 0 & -2 & -2 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \tag{4.44}$$

*and consider the permutation* $\rho = (34567)$. *Then the iteration of the cluster map* $\psi = \rho^{-1}\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2$ *is equivalent to the system of recurrences (4.41), and for all* $n\in\mathbb{Z}$ *the tau functions* $\eta_n,\sigma_n,\tau_n$ *are elements of the Laurent polynomial ring* $\mathbb{Z}[\beta,\eta_0^{\pm1},\sigma_0^{\pm1},\tau_0^{\pm1},\tau_1^{\pm1},\tau_2^{\pm1},\tau_3^{\pm1},\tau_4^{\pm1}]$, *with positive integer coefficients.*

**Proof.** Let us consider the seed $(\hat{\mathbf{x}}',\hat{B}') = \tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2(\hat{\mathbf{x}},\hat{B}) =$ that arises from applying the sequence of mutations $\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2$ to the given initial seed, where (as usual) we use $\tilde{\mu}_j$ to denote mutations in the cluster algebra associated with Laurentification of the deformed map. The new cluster is $\hat{\mathbf{x}}' = (\tilde{x}_1',\tilde{x}_2',\tilde{x}_3',\tilde{x}_4,\tilde{x}_5,\tilde{x}_6,\tilde{x}_7,\tilde{x}_8)$, where the new cluster variables (with primes) are obtained from the exchange relations

$$\begin{aligned} \tilde{x}_2'\tilde{x}_2 &= \tilde{x}_8\tilde{x}_4\tilde{x}_5 + \tilde{x}_1, \\ \tilde{x}_1'\tilde{x}_1 &= (\tilde{x}_8)^2\tilde{x}_4(\tilde{x}_5)^2\tilde{x}_6 + (\tilde{x}_2{}')^2\tilde{x}_3\tilde{x}_7, \\ \tilde{x}_3'\tilde{x}_3 &= (\tilde{x}_8)^2\tilde{x}_5\tilde{x}_6 + \tilde{x}_1{}', \end{aligned} \tag{4.45}$$

while the mutated exchange matrix $\hat{B}' = \tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2(\hat{B})$ is given by

$$\begin{pmatrix} 0 & -1 & 1 & 1 & 1 & 0 & -1 \\ 2 & 0 & 0 & 0 & -2 & -2 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.46}$$

For the new cluster variables, if we identify $\tilde{x}_1{}' = \eta_1$, $\tilde{x}_2{}' = \sigma_1$, $\tilde{x}_3{}' = \tau_5$ and replace all variables $\tilde{x}_i$ for $4\leqslant i\leqslant 8$ with the corresponding tau functions and frozen variable from the original cluster $\hat{\mathbf{x}}$, then we find that the exchange relations (4.45) are equivalent to the recurrence formulae (4.41) for $n=0$. As for the exchange matrix $\hat{B}'$, we can rewrite it in the following way:

$$\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2\left(\hat{B}\right) = P_1\hat{B}P_2 = \rho\left(\hat{B}\right).$$

In the above, the action of the permutation $\rho = (34567)$ is equivalent to applying the row and column permutation matrices

$$P_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0 \end{pmatrix}. \tag{4.47}$$

Hence we see that the cluster map defined by $\psi = \rho^{-1}\tilde{\mu}_3\tilde{\mu}_1\tilde{\mu}_2$ satisfies $\psi(\hat{B}) = \hat{B}$, and its action on any cluster is equivalent to the shift $n \to n+1$ on the indices of the tau functions. Since they are cluster variables, these tau functions exhibit the Laurent property. Moreover, the cluster variables are Laurent polynomials with positive integer coefficients, due to the positivity property [13, 25]. □

### 4.4. Tropicalization and degree growth for deformed $\mathrm{B}_3$ cluster map

We have seen that deformation of the periodic dynamics in type $\mathrm{B}_3$ yields two integrable symplectic maps, but only the second map $\hat{\varphi}_2$ given by (4.31) has been shown to correspond to mutations in a cluster algebra, as in theorem 4.7. By the Laurent property, we can write the tau functions generated by the system (4.41) in the form

$$\eta_n = \frac{\mathrm{N}_n^{(1)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{d}_n}}, \qquad \sigma_n = \frac{\mathrm{N}_n^{(2)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{e}_n}}, \qquad \tau_n = \frac{\mathrm{N}_n^{(3)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{f}_n}},$$

and find that the corresponding d-vectors $\mathbf{d}_n, \mathbf{e}_n, \mathbf{f}_n \in \mathbb{Z}^7$ satisfy the $(\max,+)$ tropical relations

$$\begin{aligned} \mathbf{e}_{n+1} + \mathbf{e}_n &= \max\left(\mathbf{f}_{n+2} + \mathbf{f}_{n+1}, \mathbf{d}_n\right), \\ \mathbf{d}_{n+1} + \mathbf{d}_n &= \max\left(\mathbf{f}_{n+4} + \mathbf{f}_n + 2\mathbf{e}_{n+1}, \mathbf{f}_{n+3} + 2\mathbf{f}_{n+2} + \mathbf{f}_{n+1}\right), \\ \mathbf{f}_{n+5} + \mathbf{f}_n &= \max\left(\mathbf{f}_{n+3} + \mathbf{f}_{n+2}, \mathbf{d}_{n+1}\right), \end{aligned} \tag{4.48}$$

where, as usual, the tropical relations do not contain analogues of coefficient terms associated with the parameter $\beta$, since the denominators of the tau functions only depend on the non-frozen variables $\eta_0, \sigma_0, \tau_0, \tau_1, \tau_2, \tau_3, \tau_4$ in the initial cluster $\hat{\mathbf{x}}$.

To determine the growth of the d-vectors in this case, we introduce the tropical analogues of the substitutions (4.34) and (4.35), namely

$$\begin{aligned} \mathbf{Y}_{1,n} &= \mathbf{d}_n - \mathbf{f}_{n+2} - \mathbf{f}_{n+1}, & \mathbf{Y}_{2,n} &= \mathbf{f}_{n+4} - \mathbf{f}_{n+3} + \mathbf{f}_{n+1} + \mathbf{f}_n - 2\mathbf{e}_n, \\ \mathbf{W}_n &= \mathbf{e}_{n+1} + \mathbf{e}_n - \mathbf{f}_{n+2} - \mathbf{f}_{n+1}, & \mathbf{V}_n &= \mathbf{f}_{n+4} - \mathbf{f}_{n+2} - \mathbf{f}_{n+1} + \mathbf{f}_{n-1}, \end{aligned} \tag{4.49}$$

which turn out to produce periodic quantities under iteration. These substitutions allow us to derive the tropical version of the system (4.32) for the map $\hat{\varphi}_2$.

**Lemma 4.8.** *The combinations of d-vectors defined by (4.49) satisfy the tropical analogue of the deformed* $\mathrm{B}_3$ *map* $\hat{\varphi}_2$*, given by the following system of four equations:*

$$\begin{aligned} \mathbf{W}_n &= [\mathbf{Y}_{1,n}]_+\,, \\ \mathbf{Y}_{1,n+1}+\mathbf{Y}_{1,n} &= [2\mathbf{W}_n+\mathbf{Y}_{2,n}]_+\,, \\ \mathbf{V}_{n+1} &= [\mathbf{Y}_{1,n+1}]_+\,, \\ \mathbf{Y}_{2,n+1}+\mathbf{Y}_{2,n}+2\mathbf{W}_n &= \mathbf{V}_{n+1}. \end{aligned} \tag{4.50}$$

*Given arbitrary initial values* $(Y_{1,0},Y_{2,0})\in\mathbb{R}^2$*, every component of this system is periodic with period 4.*

**Proof.** The $(\max,+)$ equations relating $\mathbf{Y}_{1,n},\mathbf{Y}_{2,n},\mathbf{W}_n$ and $\mathbf{V}_n$ follow directly from the substitutions (4.49) and the tropical analogues of the exchange relations that produce (4.48). Observe that, from the point of view of the dynamics of the main variables $\mathbf{Y}_{1,n},\mathbf{Y}_{2,n}$, the third equation in (4.50) is redundant, since from the first equation it is clear that

$$\mathbf{V}_n=\mathbf{W}_n \tag{4.51}$$

for all $n$, so we could omit the third equation and replace the fourth one by

$$\mathbf{Y}_{2,n+1}+\mathbf{Y}_{2,n}+2\mathbf{W}_n=\mathbf{W}_{n+1}.$$

For the vector system (4.50), a set of initial data consists of a pair of vectors $\mathbf{Y}_{1,0},\mathbf{Y}_{2,0}$, but we can view each component as a piecewise linear map $\hat{\varphi}_{2,trop}:\mathbb{R}^2\to\mathbb{R}^2$, and check directly that any pair arbitrary initial values $(Y_{1,0},Y_{2,0})\in\mathbb{R}^2$ produces a sequence of points in the plane that repeats with period 4; or in other words, $(\hat{\varphi}_{2,\mathrm{trop}})^4=\mathrm{id}$. This can be verified by case-by-case analysis, which we leave to the reader. □

We now show how we can use Somos sequences to simplify the calculation of degree growth for tau functions, by using a tropical analogue of theorem 4.6.

**Lemma 4.9.** *The d-vectors* $\mathbf{f}_n\in\mathbb{Z}^7$ *that specify the denominators of tau functions* $\tau_n$ *generated by the system (4.48) satisfy the tropical Somos-7 relation*

$$\mathbf{f}_{n+7}+\mathbf{f}_n=\max\left(\mathbf{f}_{n+6}+\mathbf{f}_{n+1},\mathbf{c}+\mathbf{f}_{n+4}+\mathbf{f}_{n+3}\right), \tag{4.52}$$

*and the corresponding quantity* $\mathbf{V}_n$ *defined in (4.49) satisfies the ultradiscrete Lyness map*

$$\mathbf{V}_{n+1}+\mathbf{V}_{n-1}=\max\left(\mathbf{V}_n,\mathbf{c}\right), \tag{4.53}$$

*where* $\boldsymbol{c}$ *is the constant vector*

$$\mathbf{c}=(2,2,1,1,1,1,1)^T.$$

**Proof.** Upon substituting the expression $\tau_n=\mathrm{N}_n^{(3)}(\hat{\mathbf{x}})/\hat{\mathbf{x}}^{\mathbf{f}_n}$ into the Somos-7 recurrence (4.39) and comparing denominators on each side, we see that the first term on the right has denominator $\hat{\mathbf{x}}^{\mathbf{f}_{n+6}+\mathbf{f}_{n+1}}$, since the coefficient $\gamma$ in front of this term is a constant that is independent of cluster variables; but the coefficient $\delta$ appearing in front of the second term is linear in the first integral $\hat{K}_2$, as given in (4.10), and pulling this back to a function of the tau functions via the map $\tilde{\pi}:\mathbb{C}^7\to\mathbb{C}^2$ defined by the substitutions for $y_{1,0},y_{2,0}$ in (4.34) gives a Laurent polynomial that takes the form

$$\tilde{\pi}^*(\delta)=\frac{\mathrm{P}(\hat{\mathbf{x}})}{\eta_0^2\sigma_0^2\tau_0\tau_1\tau_2\tau_3\tau_4},$$

for a certain polynomial P, which means that the denominator of the second term on the right-hand side is $\hat{\mathbf{x}}^{\mathbf{c}+\mathbf{f}_{n+4}+\mathbf{f}_{n+3}}$, with the given constant vector $\mathbf{c}$. Hence $\mathbf{f}_n$ satisfies the given $(\max,+)$ version of Somos-7, and it follows immediately that $\mathbf{V}_n$ is a solution of the corresponding analogue of the Lyness map (4.38), which is the relation (4.53) with the same constant $\mathbf{c}$. □

The seed $\hat{\mathbf{x}}=(\eta_0,\sigma_0,\tau_0,\tau_1,\tau_2,\tau_3,\tau_4,\beta)$ gives the initial cluster of d-vectors specified by the matrix

$$(\mathbf{d}_0\ \mathbf{e}_0\ \mathbf{f}_0\ \mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3\ \mathbf{f}_4\ )=-I, \tag{4.54}$$

where $I$ here denotes the $7\times 7$ identity matrix. This provides the initial data

$$\mathbf{Y}_{1,0}=(-1,0,0,1,1,0,0)^T,\qquad \mathbf{Y}_{2,0}=(0,2,-1,-1,0,1,-1)^T$$

for the vector form of the map $\hat{\varphi}_{2,\mathrm{trop}}$, as in (4.50), and gives

$$\mathbf{V}_0=\mathbf{W}_0=[\mathbf{Y}_{1,0}]_+=(0,0,0,1,1,0,0)^T.$$

As for the other root systems considered previously, the precise form of these vectors, and the fact that they produce an orbit with period 4, is related to the Zamolodchikov periodicity of the original $\mathrm{B}_3$ cluster map, given by (4.2) with $b_1=b_2=b_3=1$. After applying a single iteration of the map $\hat{\varphi}_{2,\mathrm{trop}}$ to these initial vectors, we find

$$\mathbf{Y}_{1,1}=[\mathbf{Y}_{1,1}]_+=\mathbf{V}_1=\mathbf{W}_1=(1,2,0,0,1,1,0)^T,\quad \mathbf{Y}_{2,1}=(1,0,1,-1,-1,0,1)^T.$$

Note that, since the sequences $\mathbf{Y}_{1,n}$ and $\mathbf{Y}_{2,n}$ both have period 4, it follows that the associated sequence of quantities $\mathbf{V}_n=[Y_{1,n}]_+$ does as well, and this is the same as $\mathbf{W}_n$, as already noted in (4.51). By making further iterations of $\hat{\varphi}_{2,\mathrm{trop}}$, we record that the next two values of the vector $\mathbf{V}_n$ are

$$\mathbf{V}_2=(2,2,1,0,0,1,1)^T,\qquad \mathbf{V}_3=(1,0,1,1,0,0,1)^T,$$

with all subsequent terms determined from $\mathbf{V}_{n+4}=\mathbf{V}_n$.

Another way to see the periodicity of the quantities $\mathbf{V}_n$ is by noting that (4.53), or rather its scalar version

$$V_{n+1}+V_{n-1}=\max(V_n,c), \tag{4.55}$$

is just one particular example of the family of ultradiscrete QRT maps considered by Nobe. In the specific case of interest here, the first two components of the vectors $\mathbf{V}_0$ and $\mathbf{V}_1$, and the first two components of the vector $\mathbf{c}$, correspond to two different period 4 orbits of the scalar relation (4.55) with parameter value $c=2$, namely

$$(0,1)\to(1,2)\to(2,1)\to(1,0)\quad\text{and}\quad(0,2)\to(2,2)\to(2,0)\to(0,0), \tag{4.56}$$

respectively, while the other five components of these vectors correspond to the same period 4 orbit of (4.55) with parameter value $c = 1$, that is

$$(0,0) \to (0,1) \to (1,1) \to (1,0). \tag{4.57}$$

However, note that, from the result of theorem 4 in [30], choosing different initial data and/or different values of $c$ in (4.55) can produce orbits with a different (arbitrarily large) period.

**Theorem 4.10.** *The d-vectors $\mathbf{d}_n, \mathbf{e}_n, \mathbf{f}_n$ satisfying the $(\max, +)$ system (4.48) all lie in the kernel of the linear difference operator*

$$\tilde{\mathcal{L}} = \left(\mathcal{T}^2 + 1\right)\left(\mathcal{T}^2 - 1\right)^2\left(\mathcal{T}^3 - 1\right).$$

*For the tau functions $\eta_n, \sigma_n, \tau_n$ generated by (4.41), the leading order degree growth of their denominators is given by*

$$\mathbf{d}_n = 2\mathbf{a}\,n^2 + O(n), \quad \mathbf{e}_n = \mathbf{a}\,n^2 + O(n), \quad \mathbf{f}_n = \mathbf{a}\,n^2 + O(n),$$
$$\text{with} \quad \mathbf{a} = \frac{1}{24}(2,2,1,1,1,1,1)^T.$$

**Proof.** By definition, we have

$$\mathbf{V}_n = \left(\mathcal{T}^5 - \mathcal{T}^3 - \mathcal{T}^2 + 1\right)\mathbf{f}_{n-1},$$

but since the sequence $\mathbf{V}_n$ has period 4, this gives

$$\left(\mathcal{T}^4 - 1\right)\mathbf{V}_{n+1} = \left(\mathcal{T}^4 - 1\right)\left(\mathcal{T}^2 - 1\right)\left(\mathcal{T}^3 - 1\right)\mathbf{f}_n = \tilde{\mathcal{L}}\mathbf{f}_n = 0.$$

Then, given that $\mathbf{f}_n$ lies in the kernel of $\tilde{\mathcal{L}}$, from the first formula in (4.49) we find

$$\tilde{\mathcal{L}}\,\mathbf{d}_n = \tilde{\mathcal{L}}\left(\mathbf{f}_{n+2} + \mathbf{f}_{n+1} + \mathbf{Y}_{1,n}\right) = \tilde{\mathcal{L}}\mathbf{f}_{n+2} + \tilde{\mathcal{L}}\mathbf{f}_{n+1} + \left(\mathcal{T}^2 - 1\right)\left(\mathcal{T}^3 - 1\right)\left(\mathbf{Y}_{1,n+4} - \mathbf{Y}_{1,n}\right) = 0,$$

where we used the fact that $\mathbf{Y}_{1,n}$ has period 4. Similarly, the second formula in (4.49) and the fact that $\mathbf{Y}_{2,n}$ has period 4 gives

$$\tilde{\mathcal{L}}\,\mathbf{e}_n = \frac{1}{2}\tilde{\mathcal{L}}\left(\mathbf{f}_{n+4} - \mathbf{f}_{n+3} + \mathbf{f}_{n+1} + \mathbf{f}_n - \mathbf{Y}_{2,n}\right) = 0.$$

Thus the first part of the statement is proved.

For the second part of the statement, note that the value of $\mathbf{V}_0$ calculated above requires that

$$\mathbf{f}_{-1} = (0,0,0,0,0,0,1)^T,$$

and we can use the relation

$$\mathbf{f}_{n+4} = \mathbf{V}_n + \mathbf{f}_{n+2} + \mathbf{f}_{n+1} - \mathbf{f}_{n-1},$$

together with the four-periodicity of $\mathbf{V}_n$, to extend the initial values given in (4.54) and thereby produce $(\mathbf{f}_j)_{-1 \leqslant j \leqslant 7}$, providing the required number of terms to completely solve the initial value problem for the linear difference equation $\tilde{\mathcal{L}}\,\mathbf{f}_n = 0$. In passing, we note that this sequence of d-vectors has the particular form

$$\mathbf{f}_n = \left(f_n^{(1)}, f_n^{(2)}, f_{n+4}^{(3)}, f_{n+3}^{(3)}, f_{n+2}^{(3)}, f_{n+1}^{(3)}, f_n^{(3)}\right),$$

given in terms of the three scalar sequences

$$\begin{aligned} \left(f_n^{(1)}\right): &\quad 0,0,0,0,0,0,1,2,2,3,5,6,7,9,11,13,15,17,20,23,25,28,32,\ldots, \\ \left(f_n^{(2)}\right): &\quad 0,0,0,0,0,0,2,2,2,4,6,6,8,10,12,14,16,18,22,24,26,30,34,\ldots, \\ \left(f_n^{(3)}\right): &\quad 1,0,0,0,0,-1,0,0,0,0,1,1,2,2,3,4,5,5,7,8,9,10,12,\ldots, \end{aligned}$$

which determine the degrees of $\eta_0$, $\sigma_0$ and $\tau_j$ for $0 \leqslant j \leqslant 4$, respectively, that appear in the denominator of the tau functions $\tau_n$. For the leading order quadratic growth of $\mathbf{f}_n$ we find

$$\mathbf{f}_n \sim \mathbf{a}\,n^2 \quad \text{where} \quad (\mathcal{T}+1)\left(\mathcal{T}^4-1\right)\left(\mathcal{T}^3-1\right)\mathbf{f}_n = 48\mathbf{a} = (4,4,2,2,2,2,2)^T,$$

with the value of the constant $\mathbf{a}$ being determined above from the initial values $\mathbf{f}_{-1},\ldots,\mathbf{f}_7$. Then from the first two formulae in (4.49), together with the four-periodicity of $\mathbf{Y}_{1,n}$ and $\mathbf{Y}_{2,n}$, we have

$$\mathbf{d}_n \sim \mathbf{f}_{n+2} + \mathbf{f}_{n+1} \sim 2\mathbf{f}_n \sim 2\mathbf{a}\,n^2, \qquad \mathbf{e}_n \sim \frac{1}{2}\left(\mathbf{f}_{n+4} - \mathbf{f}_{n+3} + \mathbf{f}_{n+1} + \mathbf{f}_n\right) \sim \mathbf{f}_n \sim \mathbf{a}\,n^2,$$

with the same constant $\mathbf{a}$. This completely fixes the quadratic leading order behavior of the d-vectors, and in each case the correction term is $O(n)$. □

**Remark 4.11.** Although they do not appear to be generated by simple cluster mutations, the Laurent polynomials $\tau_n, \eta_n$ produced by iteration of (4.29), associated with the other deformed $\mathrm{B}_3$ map $\hat{\varphi}_1$, are also connected to a Somos-7 recurrence, namely (4.20), so it should be possible to calculate their degrees by adapting the preceding arguments suitably.

## 5. Integrable deformations of the $\mathrm{D}_4$ cluster map

In this section, we consider the deformation of a cluster map in 4D, which is composed of mutations in the cluster algebra of type $\mathrm{D}_4$. We will show that there are two essentially different choices of the deformation parameters that yield a discrete integrable system, each of which lifts to a cluster map in higher dimensions via Laurentification.

### 5.1. *Deformed* $\mathrm{D}_4$ *cluster map*

The Cartan matrix for the $\mathrm{D}_4$ root system is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}. \tag{5.1}$$

The corresponding exchange matrix is

$$B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5.2}$$

The deformed mutations with parameters $a_j, b_j$ for $1 \leqslant j \leqslant 4$ take the form

$$\begin{array}{ll} \mu_1 : (x_1, x_2, x_3, x_4) \mapsto (x_1', x_2, x_3, x_4), & x_1 x_1' = b_1 + a_1 x_2 \\ \mu_2 : (x_1', x_2, x_3, x_4) \mapsto (x_1', x_2', x_3, x_4), & x_2 x_2' = b_2 + a_2 x_3 x_4 x_1' \\ \mu_3 : (x_1', x_2', x_3, x_4) \mapsto (x_1', x_2', x_3', x_4), & x_3 x_3' = b_3 + a_3 x_2' \\ \mu_4 : (x_1', x_2', x_3', x_4) \mapsto (x_1', x_2', x_3', x_4'), & x_4 x_4' = b_4 + a_4 x_2'. \end{array} \tag{5.3}$$

The deformed map $\varphi = \mu_4\mu_3\mu_2\mu_1$ reduces to the original cluster map when we fix the parameters $a_i = 1 = b_i$ for all $i$. The Coxeter number for $D_4$ is 6, and the periodicity for the cluster map is period $4 = \frac{1}{2}(6+2)$, i.e.

$$\varphi \cdot (\mathbf{x}, B) = (\varphi(\mathbf{x}), B) \quad (\text{with } a_j = 1 = b_j) \implies \varphi^4(\mathbf{x}) = \mathbf{x}. \tag{5.4}$$

As usual, we can reduce the number of parameters in the problem by rescaling each of the cluster variables independently, $x_i \to \lambda_i x_i$, and choose the scalings so that the parameters $a_j$ are removed and the sequence of deformed mutations can be rewritten as

$$\begin{aligned} x_{1,n+1} x_{1,n} &= x_{2,n} + b_1, \\ x_{2,n+1} x_{2,n} &= x_{3,n} x_{4,n} x_{1,n+1} + b_2, \\ x_{3,n+1} x_{3,n} &= x_{2,n+1} + b_3, \\ x_{4,n+1} x_{4,n} &= x_{2,n+1} + b_4. \end{aligned} \tag{5.5}$$

Since the exchange matrix is skew-symmetric, by the result of theorem 1.3 in [20] the deformed map preserves the presymplectic form $\omega$ given by

$$\omega = \frac{1}{x_1 x_2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{1}{x_2 x_3} \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \frac{1}{x_2 x_4} \mathrm{d}x_2 \wedge \mathrm{d}x_4. \tag{5.6}$$

Now since $B$ is degenerate and of rank 2, one can reduce the birational map $\varphi$ from 4D to a 2-dimensional symplectic map. The null space and image of $B$ are given by

$$\ker(B) = < (1,0,0,1)^T, (1,0,1,0) >, \qquad \mathrm{im}\,(B) = < (0,1,0,0)^T, (-1,0,1,1)^T > . \tag{5.7}$$

Hence the null distribution of the presymplectic form $\omega$ is spanned by the two commuting vector fields $\mathbf{v}_1 = x_1\partial_{x_1} + x_4\partial_{x_4}$ and $\mathbf{v}_2 = x_1\partial_{x_1} + x_3\partial_{x_3}$. The space of leaves of the null foliation has local coordinates

$$y_1 = x_2, \quad y_2 = \frac{x_3 x_4}{x_1}. \tag{5.8}$$

Then the rational map defined by

$$\begin{array}{rrcl} \pi \quad : & \mathbb{C}^4 & \to & \mathbb{C}^2 \\ & \mathbf{x} = (x_1, x_2, x_3, x_4) & \mapsto & \mathbf{y} = (y_1, y_2) \end{array} \tag{5.9}$$

reduces the cluster map $\varphi$ to the 2D symplectic map

$$\hat{\varphi} \ : \quad \mathbb{C}^2 \ \to \ \mathbb{C}^2$$
$$\mathbf{y} = (y_1, y_2) \ \mapsto \ \left( \frac{(b_1 + y_1) y_2 + b_2}{y_1}, \frac{(b_4 + y_2) y_1 + b_1 y_2 + b_2)((b_3 + y_2) y_1 + b_1 y_2 + b_2)}{y_2 y_1^2 (b_1 + y_1)} \right), \tag{5.10}$$

which is intertwined with $\varphi$ via $\pi$, so that

$$\hat{\varphi} \cdot \pi = \pi \cdot \varphi, \qquad \hat{\varphi}^* (\hat{\omega}) = \hat{\omega},$$

where $\pi^*(\hat{\omega}) = \omega$ is the pullback of the symplectic form

$$\hat{\omega} = \frac{1}{y_1 y_2} \mathrm{d}y_1 \wedge \mathrm{d}y_2 \tag{5.11}$$

under $\pi$. When all of the parameters $b_i = 1$, the reduced map $\hat{\varphi}$ has period 4, and one of the first integrals associated with this map takes the form

$$K = \sum_{i=0}^{3} (\hat{\varphi}^*)^i (y_1) = \frac{(1 + y_1)^3 + \left(2 + 5y_1 + y_1^3\right) y_2 + (1 + y_1)^2 y_2^2}{y_1^2 y_2}. \tag{5.12}$$

By applying the same procedure as in the previous examples, we suppose that there is an analogous first integral that is compatible with the deformed map (5.10), taking the form

$$\tilde{K} = y_1 + \alpha_1 y_2 + \frac{\alpha_2 y_1}{y_2} + \frac{\alpha_3 y_2}{y_1} + \frac{\alpha_4}{y_2} + \frac{\alpha_5}{y_1} + \frac{\alpha_6 y_2}{y_1^2} + \frac{\alpha_7}{y_2 y_1} + \frac{\alpha_8}{y_1^2} + \frac{\alpha_9}{y_2 y_1^2} \tag{5.13}$$

where $\alpha_j$ are undetermined parameters. Then imposing the requirement that $\tilde{K}$ should be preserved, so that $\hat{\varphi}^*(\tilde{K}) = \tilde{K}$, constrains these parameters and leads us to find necessary and sufficient conditions for the map $\hat{\varphi}$ to be integrable, as follows.

**Theorem 5.1.** *For the deformed symplectic map $\hat{\varphi}$ to admit the first integral (5.13), it is necessary and sufficient for the parameters $b_i$ to satisfy one of the following sets of conditions:*

$$(1) \ b_2 = b_4 = b_1 b_3; \tag{5.14a}$$
$$(2) \ b_1 = b_2 = b_3 b_4; \tag{5.14b}$$
$$(3) \ b_2 = b_3 = b_1 b_4. \tag{5.14c}$$

*Hence in each of these cases, the deformed map $\hat{\varphi}$ given by (5.10) is Liouville integrable, preserving the function*

$$\tilde{K} = y_1 + y_2 + \frac{y_1}{y_2} + \frac{(b_1 + 1) y_2}{y_1} + \frac{b_3 + b_4 + 1}{y_2} + \frac{b_1 + b_2 + b_3 + b_4 + 1}{y_1} + \frac{b_1 y_2}{y_1^2} + \frac{b_3 b_4 + b_3 + b_4}{y_1 y_2} + \frac{2b_2}{y_1^2} + \frac{b_3 b_4}{y_1^2 y_2} \tag{5.15}$$

**Remark 5.2.** Observe that the form of the original deformed mutations (5.5) remains invariant under switching $x_3 \leftrightarrow x_4$, $b_3 \leftrightarrow b_4$, and similarly for the form of the reduced map $\hat{\varphi}$ in (5.10) and the first integral (5.15) when these last two parameters are switched. Hence cases (1) and (3) are equivalent to one another, and (1) and (2) are really the only two distinct cases to consider in theorem 5.1.

For the two essentially distinct cases of the reduced map obtained by deformation of the $\mathrm{D}_4$ cluster map, as identified in the preceding theorem, we have two 2-parameter families of integrable maps given by $\hat{\varphi}_1, \hat{\varphi}_2$ respectively, where

$$\hat{\varphi}_1: \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \left( \frac{(b_1+y_1)\,y_2+b_1b_3}{y_1}, \quad \frac{[(b_1+y_1)\,y_2+b_1b_3\,(y_1+1)]\cdot[y_2+b_3]}{y_1^2y_2} \right) \tag{5.16}$$

$$\hat{\varphi}_2: \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \left( \frac{(b_3b_4+y_1)\,y_2+b_3b_4}{y_1}, \quad \frac{[(b_4+y_2)\,y_1+b_3b_4\,(y_2+1)]\cdot[(b_3+y_2)\,y_1+b_3b_4\,(y_2+1)]}{b_4y_1^2y_2\,(b_3b_4+y_1)} \right) \tag{5.17}$$

where the coefficients in each map are fixed in cases (1) and (2), respectively. The corresponding invariant functions $\tilde{K}_1, \tilde{K}_2$ are given by

$$\begin{aligned} \tilde{K}_1 &= y_1+y_2+\frac{y_1}{y_2}+\frac{(b_1+1)\,y_2}{y_1}+\frac{b_3+b_1b_3+1}{y_2}+\frac{b_1+2b_1b_3+b_3+1}{y_1}+\frac{b_1y_2}{y_1^2} \\ &\quad +\frac{b_3\,(b_1b_3+b_1+1)}{y_1y_2}+\frac{2b_1b_3}{y_1^2}+\frac{b_1b_3^2}{y_1^2y_2}, \end{aligned} \tag{5.18}$$

$$\begin{aligned} \tilde{K}_2 &= y_1+y_2+\frac{y_1}{y_2}+\frac{(b_3b_4+1)\,y_2}{y_1}+\frac{b_3+b_4+1}{y_2}+\frac{2b_3b_4+b_3+b_4+1}{y_1}+\frac{b_3b_4y_2}{y_1^2} \\ &\quad +\frac{b_3b_4+b_3+b_4}{y_1y_2}+\frac{2b_3b_4}{y_1^2}+\frac{b_3b_4}{y_1^2y_2}, \end{aligned} \tag{5.19}$$

which are the particular relevant cases of the function (5.15). The level sets of each of the latter functions gives a pencil of plane curves, of which the generic member has genus 1 and hence corresponds to an elliptic curve. It turns out that the two functions above become equivalent to one another when $b_3 = 1$, upon identifying the remaining parameters $b_1$ and $b_4$ in each case, although the associated maps $\hat{\varphi}_1, \hat{\varphi}_2$ remain distinct from one another.

**Remark 5.3.** Since both sets of curves corresponding to $\tilde{K}_1$ and $\tilde{K}_2$ have bidegree (3,2), they do not correspond to QRT maps, which come from curves of bidegree (2,2) (that is, biquadratic curves).

### 5.2. *The deformed map* $\hat{\varphi}_1$ *for* $\mathrm{D}_4$

The iteration of the deformed map $\hat{\varphi}_1$ can be written as the following system of recurrence relations:

$$\begin{aligned} y_{1,n+1}y_{1,n} &= (b_1+y_{1,n})\,y_{2,n}+b_1b_3 \\ y_{2,n+1}y_{2,n}y_{1,n}^2 &= ((b_1+y_{1,n})\,y_{2,n}+b_1b_3\,(y_{1,n}+1))\,(y_{2,n}+b_3)\,. \end{aligned} \tag{5.20}$$

Applying the $p$-adic method as used in previous cases, we choose some rational values for the parameters and initial conditions, and observe the orbit of the map $\hat{\varphi}_1$ as a sequence in $\mathbb{Q}^2$. Then we find three different singularity patterns, given by

$$\begin{aligned} &\text{Pattern 1}: (y_{1,n},y_{2,n}) = \ldots,\left(R,0^1\right),\left(R,\infty^1\right),\left(\infty^1,\infty^1\right),\left(\infty^1,R\right),\left(R,0^1\right),(R,R)\ldots \\ &\text{Pattern 2}: (y_{1,n},y_{2,n}) = \ldots,\left(0^1,R\right),\ldots \\ &\text{Pattern 3}: (y_{1,n},y_{2,n}) = \ldots,\left(R,0^1\right),\left(R,\infty^1\right)\ldots. \end{aligned} \tag{5.21}$$

By associating a tau function with each pattern, so that $\tau_n, r_n, \sigma_n$ correspond to patterns 1,2,3 respectively, we are led to the change of variables

$$y_{1,n} = \frac{r_n}{\tau_{n-1}\tau_n}, \quad y_{2,n} = \frac{\sigma_{n+1}\tau_{n-2}\tau_{n+2}}{\sigma_n\tau_n\tau_{n+1}}. \tag{5.22}$$

If we directly substitute these variables into the recurrences (5.21), then we obtain the relations

$$\begin{aligned} r_{n+1}r_n &= \frac{(b_1\tau_{n-1}\tau_n + r_n)\,\sigma_{n+1}\tau_{n-2}\tau_{n+2} + b_1b_3\tau_{n-1}\tau_n^2\tau_{n+1}\sigma_n}{\sigma_n} \\ \sigma_{n+2}\tau_{n+3} &= \frac{\begin{array}{l}[b_1b_3\sigma_n\tau_n\tau_{n+1} + b_1\sigma_{n+1}\tau_{n-2}\tau_{n+2}] \\ \cdot\left[(b_1\tau_{n-1}\tau_n + r_n)\,\sigma_{n+1}\tau_{n-2}\tau_{n+2} + b_1b_3\sigma_n\tau_n\tau_{n+1}\,(\tau_{n-1}\tau_n + r_n)\right]\end{array}}{b_1r_n^2\tau_{n-2}\sigma_n}. \end{aligned} \tag{5.23}$$

To simplify the above relations and decouple them in such a way that they represent exchange relations, it is helpful to observe the full singularity pattern in 4D, which emerges from applying the sequence of deformed mutations (5.5) subject to the conditions $b_2 = b_4 = \beta = b_1b_3$. Using the $p$-adic method in this case suggests making a transformation on the level of the $x$-variables, defined by

$$x_{1,n} = \rho_n\frac{\sigma_n}{\tau_{n-1}} \quad x_{2,n} = \frac{r_n}{\tau_{n-1}\tau_n} \quad x_{3,n} = \rho_n\frac{\sigma_{n+1}}{\tau_n}, \quad x_{4,n} = \frac{\tau_{n-2}\tau_{n+2}}{\tau_{n-1}\tau_{n+1}}, \tag{5.24}$$

where the extra prefactor $\rho_n$ corresponds to an additional singularity pattern, appearing only on this level. By a short calculation, one can confirm that these new formulae are consistent with the expression for $y_{2,n}$ previously given in (5.22), since we have

$$y_{2,n} = \frac{x_{3,n}x_{4,n}}{x_{1,n}} = \frac{\sigma_{n+1}\tau_{n-1}}{\sigma_n\tau_n}\cdot\frac{\tau_{n-2}\tau_{n+2}}{\tau_{n-1}\tau_{n+1}} = \frac{\sigma_{n+1}\tau_{n-2}\tau_{n+2}}{\sigma_n\tau_n\tau_{n+1}},$$

as required. Thus, with the parameters constrained as in (5.14$a$), the iteration of the deformed map (5.5) is equivalent to a system of four relations, namely

$$\begin{aligned} &\rho_{n+1}\rho_n\sigma_{n+1}\sigma_n = b_1\tau_{n-1}\tau_n + r_n, \\ &r_{n+1}r_n = \rho_{n+1}\rho_n\sigma_{n+1}^2\tau_{n+2}\tau_{n-2} + b_1b_3\tau_{n+1}\tau_n^2\tau_{n-1}, \\ &\rho_{n+1}\rho_n\sigma_{n+2}\sigma_{n+1} = b_3\tau_{n+1}\tau_n + r_{n+1}, \\ &\tau_{n+3}\tau_{n-2} = b_1b_3\tau_n\tau_{n+1} + r_{n+1}. \end{aligned} \tag{5.25}$$

By incorporating the above relations into (5.23), and eliminating $\rho_n$, we are able to decouple them into a total of three recurrences, which all take the form of exchange relations, given by

$$\begin{aligned} &\sigma_{n+2}r_n = b_3\sigma_n\tau_n\tau_{n+1} + \sigma_{n+1}\tau_{n-2}\tau_{n+2} \\ &r_{n+1}\sigma_n = \sigma_{n+1}\tau_{n+2}\tau_{n-2} + b_1\sigma_{n+2}\tau_n\tau_{n-1} \\ &\tau_{n+3}\tau_{n-2} = b_1b_3\tau_n\tau_{n+1} + r_{n+1}. \end{aligned} \tag{5.26}$$

Next, in order to confirm that this gives a cluster map defined on a suitable space of tau functions, we need to build an appropriate exchange matrix which produces (5.26) via a sequence of mutations. Firstly, let us combine the initial tau functions into a cluster in a seed for a coefficient-free cluster algebra, by setting

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8) = (\sigma_0, \sigma_1, r_0, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2),$$

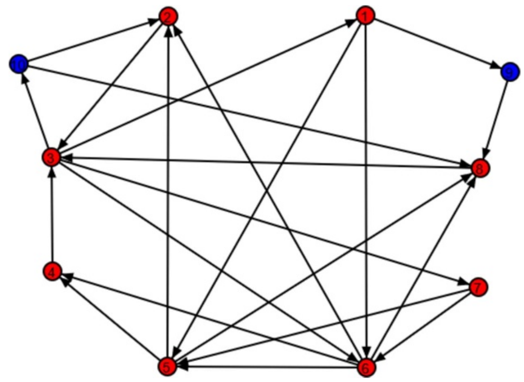

**Figure 2.** Extended quiver associated with the deformed $D_4$ cluster map $\psi_1$.

and let $\tilde{\pi}_1 : \mathbb{C}^8 \to \mathbb{C}^2$ be the rational map defined by (5.22). Then, upon taking the pullback of the symplectic form (5.11) by $\tilde{\pi}_1$, we find

$$\tilde{\omega} = \tilde{\pi}_1^*(\hat{\omega}) = \sum_{1\leqslant i<j\leqslant 8} \tilde{b}_{ij}^{(1)} \mathrm{d}\log\tilde{x}_i \wedge \mathrm{d}\log\tilde{x}_j \tag{5.27}$$

where the coefficients are combined into the matrix $\tilde{B}^{(1)} = (\tilde{b}_{ij}^{(1)})$ given by

$$\tilde{B}^{(1)} = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}. \tag{5.28}$$

We proceed to add two extra rows, associated with the frozen variables $b_1$ and $b_3$, to the bottom of the exchange matrix (5.28), which will result in the construction of the extended exchange matrix $\hat{B}^{(1)}$ shown in (5.29) below. Figure 2 depicts the quiver associated with the full matrix $\hat{B}^{(1)}$.

**Theorem 5.4.** *Given the extended initial cluster*

$$\hat{\mathbf{x}} = (\tilde{x}_j)_{1\leqslant j\leqslant 10} = (\sigma_0, \sigma_1, r_0, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2, b_1, b_3),$$

*and the permutation* $\rho_1 = (123)(45678)$*, the iteration of the cluster map* $\psi_1 = \rho_1^{-1}\tilde{\mu}_4\tilde{\mu}_1\tilde{\mu}_3$ *defined by the extended exchange matrix* $\hat{B}^{(1)}$ *in (5.29) with square submatrix (5.28) is equivalent to the system of recurrences (5.26), which generates elements of* $\mathbb{Z}_{>0}[b_1, b_3, \sigma_0^{\pm1}, \sigma_1^{\pm1}, r_0^{\pm1}, \tau_{-2}^{\pm1}, \tau_{-1}^{\pm1}, \tau_0^{\pm1}, \tau_1^{\pm1}, \tau_2^{\pm1}]$

**Proof.** Let us consider the initial seed $(\hat{\mathbf{x}}, \hat{B}^{(1)})$ containing the extended initial cluster $\hat{\mathbf{x}}$ as above, with the corresponding extended exchange matrix given by

$$\hat{B}^{(1)} = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.29}$$

Applying cluster mutation $\tilde{\mu}_3$ at node 3, followed by $\tilde{\mu}_1$ and $\tilde{\mu}_4$, will give the exchange relations

$$\begin{aligned} \tilde{x}_3'\tilde{x}_3 &= \tilde{x}_{10}\tilde{x}_1\tilde{x}_6\tilde{x}_7 + \tilde{x}_2\tilde{x}_4\tilde{x}_8, \\ \tilde{x}_1'\tilde{x}_1 &= \tilde{x}_2\tilde{x}_4\tilde{x}_8 + \tilde{x}_9\tilde{x}_3{}'\tilde{x}_5\tilde{x}_6, \\ \tilde{x}_4'\tilde{x}_4 &= \tilde{x}_9\tilde{x}_{10}\tilde{x}_6\tilde{x}_7 + \tilde{x}_1{}', \end{aligned} \tag{5.30}$$

which have the same form as the expressions in (5.26). Under this sequence of mutations, the extended exchange matrix $\hat{B}^{(1)}$ satisfies the relation

$$\mu_4\mu_1\mu_3\left(\tilde{B}^{(1)}\right) = \rho_1\left(\tilde{B}^{(1)}\right) = P_1\tilde{B}^{(1)}P_2$$

where the action of the permutation $\rho_1 = (123)(45678)$ on the rows/columns is represented by the matrices

$$P_1 = \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \end{pmatrix}. \tag{5.31}$$

It follows that $\hat{B}^{(1)}$ is preserved by the action of the associated cluster map, that is $\psi_1(\hat{B}^{(1)}) = \hat{B}^{(1)}$, where $\psi_1 = \rho_1^{-1}\mu_4\mu_1\mu_3$, and the combination of the inverse permutation with the exchange relations in (5.30) precisely corresponds to the shift of index $n \to n+1$ acting on the tau functions in each cluster, reproducing the iteration of the system (5.26). Hence this cluster map is a Laurentification of the deformed $\mathrm{D}_4$ map $\hat{\varphi}_1$, generating Laurent polynomials in the initial cluster variables, and positivity for skew-symmetric cluster algebras [25] implies that their coefficients are positive integers. □

**Remark 5.5.** The subquiver in figure 2 consisting of the 8 unfrozen nodes is mutation equivalent to another particular quiver presented by Okubo, which enables a $q$-Painlevé VI equation to be constructed from an appropriate combination of coefficient mutations [32].

### 5.3. The deformed map $\hat{\varphi}_2$ for $\mathrm{D}_4$

Let us now consider iteration of the map $\hat{\varphi}_2$ given by (5.17), which can be written as the recurrence

$$\begin{gathered} y_{1,n+1}y_{1,n} = (b_3b_4 + y_{1,n})\,y_{2,n} + b_3b_4 \\ y_{2,n+1}y_{2,n}y_{1,n}^2b_4\,(b_3b_4 + y_{1,n}) = ((b_4 + y_{2,n})\,y_{1,n} + b_3b_4\,(y_{2,n}+1))\,((b_3 + y_{2,n})\,y_{1,n} + b_3b_4\,(y_{2,n}+1)). \end{gathered} \tag{5.32}$$

Repeating the same procedure as in the previous sections, we find three singularity patterns which arise from orbits of (5.32), namely

$$\begin{aligned} &\text{Pattern 1}: (y_{1,n}, y_{2,n}) = \ldots, (R, \infty^1), (\infty^1, \infty^1), (\infty^1, R), (R, 0^1), (R, \infty^1), (R, R)\ldots \\ &\text{Pattern 2}: (y_{1,n}, y_{2,n}) = \ldots, (0^1, R), \ldots \\ &\text{Pattern 3}: (y_{1,n}, y_{2,n}) = \ldots, (R, 0^1), \ldots. \end{aligned} \tag{5.33}$$

By relating the singularities appearing in each pattern with new variables, we define the following variable transformation, in an attempt to Laurentify the deformed map $\hat{\varphi}_2$:

$$y_{1,n} = \frac{\hat{\eta}_n}{\hat{\tau}_{n-1}\hat{\tau}_n}, \quad y_{2,n} = \frac{\rho_n\hat{\tau}_{n-2}}{\hat{\tau}_{n+1}\hat{\tau}_n\hat{\tau}_{n-3}}. \tag{5.34}$$

By substituting these expressions into (5.32), we find a rather complicated system of equations: in particular, the resulting expression for the product $\rho_{n+1}\rho_n$ cannot be considered as an exchange relation, as it is not immediately given as a binomial expression in the other variables. To resolve this problem, we look at the singularity patterns in the original 4D deformed map (5.5) with $b_1 = b_2 = b_3b_4$, and introduce variable transformations corresponding to these. This suggests that the $x_i$ should be expressed as

$$x_{1,n} = \frac{\hat{\tau}_{n+1}\hat{\tau}_{n-3}}{\hat{\tau}_n\hat{\tau}_{n-2}}, \quad x_{2,n} = \frac{\hat{\eta}_n}{\hat{\tau}_n\hat{\tau}_{n-1}}, \quad x_{3,n} = \frac{\hat{r}_n\xi_n}{\hat{\tau}_n}, \quad x_{4,n} = \frac{\hat{\sigma}_n}{\hat{\tau}_n\xi_n}$$

where $\xi_n$ satisfies $\xi_n\xi_{n+1} = \frac{\hat{\sigma}_n}{\hat{r}_n}$. By directly substituting these variables into (5.5), we find a system of equations written as follows:

$$\begin{aligned} \hat{\tau}_{n+2}\hat{\tau}_{n-3} &= b_3b_4\hat{\tau}_n\hat{\tau}_{n-1} + \hat{\eta}_n, \\ \hat{\eta}_{n+1}\hat{\eta}_n &= \hat{r}_n\hat{\sigma}_n\hat{\tau}_{n-2}\hat{\tau}_{n+2} + b_3b_4\hat{\tau}_{n+1}\hat{\tau}_n^2\hat{\tau}_{n-1}, \\ \hat{r}_{n+1}\hat{\sigma}_n &= b_3\hat{\tau}_n\hat{\tau}_{n+1} + \hat{\eta}_{n+1}, \\ \hat{\sigma}_{n+1}\hat{r}_n &= b_4\hat{\tau}_n\hat{\tau}_{n+1} + \hat{\eta}_{n+1}. \end{aligned} \tag{5.35}$$

Also, by observing the singularity pattern for $y_{1,n}$ explicitly from (5.32), one can see that $y_{1,n}$ should satisfy the relation

$$w_{1,n} := y_{1,n} + b_3b_4 = \frac{\hat{\tau}_{n+2}\hat{\tau}_{n-3}}{\hat{\tau}_n\hat{\tau}_{n-1}}, \tag{5.36}$$

which is in agreement with what is found by combining (5.34) with the first recurrence in (5.35). Furthermore, by setting $\rho_n = \hat{r}_n\hat{\sigma}_n$, the relation for $\rho_{n+1}\rho_n$ obtained from (5.32) follows by taking the product of the last two expressions in (5.35).

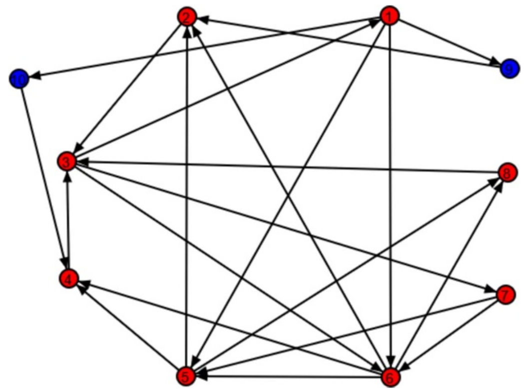

**Figure 3.** Extended quiver associated with the deformed $D_4$ cluster map $\psi_2$.

Using the above, we can consider $(y_{1,0}, y_{2,0}) = (y_1, y_2)$ and define a rational map $\tilde{\pi}_2 : \mathbb{C}^8 \to \mathbb{C}^2$ by

$$\tilde{\pi}_2 : \qquad y_1 = \frac{\hat{\eta}_0}{\hat{\tau}_{-1}\hat{\tau}_0}, \qquad y_2 = \frac{\hat{\sigma}_0 \hat{r}_0 \hat{\tau}_{-2}}{\hat{\tau}_1 \hat{\tau}_0 \hat{\tau}_{-3}}.$$

The exchange matrix describing the cluster dynamics (5.35) is found by pulling back the symplectic form $\hat{\omega}$, as in (5.11), via the rational map $\tilde{\pi}_2$, to obtain the presymplectic form

$$\tilde{\omega} = \tilde{\pi}_2^*(\hat{\omega}) = \sum_{i<j} \frac{\tilde{b}_{ij}^{(2)}}{\tilde{x}_i \tilde{x}_j} \mathrm{d}\tilde{x}_i \wedge \mathrm{d}\tilde{x}_j.$$

Now if we choose to order the coordinates and identify them with variables in a coefficient-free cluster algebra as

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8) = (\hat{\tau}_{-3}, \hat{r}_0, \hat{\eta}_0, \hat{\sigma}_0, \hat{\tau}_{-1}, \hat{\tau}_0, \hat{\tau}_1, \hat{\tau}_{-2}),$$

then we see that the map $\tilde{\pi}_2$ is equivalent to $\tilde{\pi}_1$ defined by (5.22) in case (1) above, so that

$$y_1 = \frac{\tilde{x}_3}{\tilde{x}_5 \tilde{x}_6}, \qquad y_2 = \frac{\tilde{x}_2 \tilde{x}_4 \tilde{x}_8}{\tilde{x}_1 \tilde{x}_6 \tilde{x}_7}$$

and the exchange matrix with entries $\tilde{b}_{ij}^{(2)}$ is identical to the one obtained previously, that is

$$\tilde{B}^{(2)} = \tilde{B}^{(1)} \tag{5.37}$$

as in (5.28).

To obtain an extended version of $\tilde{B}^{(2)}$ that includes $b_3, b_4$ as frozen variables and reproduces (5.35) from a suitable sequence of mutations, we need to construct two extra rows as in (5.38) below. The result of this is represented by the quiver in figure 3, with two frozen nodes.

**Theorem 5.6.** *Given the extended initial cluster*

$$\hat{\mathbf{x}} = (\tilde{x}_j)_{1\leqslant j\leqslant 10} = (\hat{\tau}_{-3}, \hat{r}_0, \hat{\eta}_0, \hat{\sigma}_0, \hat{\tau}_{-1}, \hat{\tau}_0, \hat{\tau}_1, \hat{\tau}_{-2}, b_3, b_4),$$

*and the permutation $\rho_2 = (24)(18567)$, the iteration of the cluster map $\psi_2 = \rho_2^{-1}\tilde{\mu}_2\tilde{\mu}_4\tilde{\mu}_3\tilde{\mu}_1$ defined by the extended exchange matrix $\hat{B}^{(2)}$ in (5.38) with square submatrix (5.28) is equivalent to the system of recurrences (5.35), which generates elements of $\mathbb{Z}_{>0}[b_3, b_4, \hat{r}_0^{\pm 1}, \hat{\eta}_0^{\pm 1}, \hat{\sigma}_0^{\pm 1}, \hat{\tau}_{-3}^{\pm 1}, \hat{\tau}_{-2}^{\pm 1}, \hat{\tau}_{-1}^{\pm 1}, \hat{\tau}_0^{\pm 1}, \hat{\tau}_1^{\pm 1}]$.*

**Proof.** From (5.37) we note that the coefficient-free cluster algebra is identical to that specified by the same $8\times 8$ exchange matrix as was found in case (1) previously, but we need to extend it in such a way that, once $b_3$ and $b_4$ are included as frozen variables, it is compatible with the four relations in (5.35) (whereas in case (1) there were only three relations). In this way, we construct a $10\times 8$ extended exchange matrix $\hat{B}^{(2)}$ from $\tilde{B}^{(2)} = \tilde{B}^{(1)}$, given by

$$\hat{B}^{(2)} = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.38}$$

and we note that the last two rows are different from $\hat{B}^{(1)}$ in (5.29) (as can be seen by comparing the frozen nodes 9 and 10 in figures 2 and 3).

Next, by applying a sequence of mutations starting with mutation $\tilde{\mu}_1$ at node 1 and successively mutating at nodes 3,4 and 2, we find that the nodes are permuted by the given permutation $\rho_2 = (24)(18567)$, so that

$$\tilde{\mu}_2\tilde{\mu}_4\tilde{\mu}_3\tilde{\mu}_1\left(\hat{B}_2\right) = \rho_2\left(\hat{B}_2\right),$$

which is equivalent to the action of a suitable pair of row/column permutation matrices on $\hat{B}_2$. Hence the overall action of $\psi_2 = \rho_2^{-1}\tilde{\mu}_2\tilde{\mu}_4\tilde{\mu}_3\tilde{\mu}_1$ leaves $\hat{B}_2$ invariant, and it is straightforward to check that the corresponding combination of cluster mutations with a permutation is equivalent to one iteration of the relations (5.35). Then as usual, because they are cluster variables, the iterates are elements of the corresponding ring of Laurent polynomials, with positive integer coefficients. □

**Remark 5.7.** Since the subquiver with 8 unfrozen nodes in figure 3 is the same as that in figure 2, it is also mutation equivalent to the quiver associated with the $q$-Painlevé VI equation in [32].

### *5.4. Connection with special Somos-7 relation*

Similarly to what we have seen for the examples of deformations in types A and B, on fixed level sets of these deformed $D_4$ maps we find that the orbits of suitable tau functions satisfy a special Somos-7 relation, which is related to the Lyness map.

**Theorem 5.8.** *For each integrable case of the deformed* $\mathrm{D}_4$ *map, the variable* $w_n = y_{1,n} + \beta$ *satisfies the Lyness map in the form*

$$w_{n+1}w_{n-1} = (1-\beta)\,w_n + \delta, \tag{5.39}$$

*where in case (1) we have*

$$\beta = b_1 b_3, \qquad \delta = \beta \tilde{K}_1 + 2\beta^2 + b_1 + b_3,$$

*on each level set of the invariant function* $\tilde{K}_1$ *given in (5.18), while in case (2) the parameters are specified by*

$$\beta = b_3 b_4, \qquad \delta = \beta \tilde{K}_2 + 2\beta^2 + b_3 + b_4,$$

*with* $\tilde{K}_2$ *as in (5.19). Furthermore, in case (1) we can express* $w_n$ *by the formula*

$$w_n = \frac{\tau_{n+2}\tau_{n-3}}{\tau_n \tau_{n-1}}, \tag{5.40}$$

*where the tau function* $\tau_n$ *satisfies the special Somos-7 relation*

$$\tau_{n+7}\tau_n = (1-\beta)\,\tau_{n+6}\tau_{n+1} + \delta\tau_{n+4}\tau_{n+3}, \tag{5.41}$$

*and for case (2) we have the same expression as (5.40) except that* $\tau_n$ *is replaced by* $\hat{\tau}_n$*, where the latter satisfies the same relation (5.41) but with the modified expression for* $\beta$ *and* $\delta$*, as above. Similarly, in each case the quantity*

$$\hat{w}_n = y_{1,n} + 1$$

*satisfies the Somos-5 QRT map, in the form of the recurrence*

$$\hat{w}_{n+1}\hat{w}_n\hat{w}_{n-1} = \zeta\hat{w}_n + \theta, \tag{5.42}$$

*where, for the appropriate value of* $\beta$ *in each case, the coefficients are given by* $\theta = (\beta-1)\zeta$ *with*

$$\text{case (1)}: \quad \zeta = \tilde{K}_1 + b_1 + b_3 + 2, \qquad \text{case (2)}: \quad \zeta = \tilde{K}_2 + b_3 + b_4 + 2.$$

The proof of the preceding statements is very similar to what was done before for the other examples, so it is omitted. The fact that the quantity $\hat{w}_n$ satisfies the Somos-5 QRT map (5.42) also means that there is a corresponding Somos-5 relation for $\tau_n$ and $\hat{\tau}_n$, but with a periodic coefficient: for further details of this, see appendix B.

### 5.5. Tropicalization and degree growth for deformed $\mathrm{D}_4$ cluster maps

We have shown above that the two distinct integrable deformations of type $\mathrm{D}_4$, given by (5.16) and (5.17), both correspond to cluster maps based on the same coefficient-free cluster algebra, but with a pair of frozen variables adjoined in two different ways in each case. The fact that the underlying coefficient-free cluster algebra is the same means that the tropical dynamics associated with the degree growth of the cluster variables is almost identical for these two cases, so it is convenient to describe them simultaneously and remark briefly on the minor differences between them. Also, because the analysis of the tropical dynamics is very similar to that for the other examples previously considered, we will be more sparing with the details.

In both cases, we denote the extended initial cluster by $\hat{\mathbf{x}}$, as described in theorems 5.4 and 5.6. Then we can write the cluster variables for case (1) in the form

$$\sigma_n = \frac{\mathrm{N}_n^{(1)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{f}_n}}, \qquad r_n = \frac{\mathrm{N}_n^{(2)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{e}_n}}, \qquad \tau_n = \frac{\mathrm{N}_n^{(3)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\mathbf{d}_n}},$$

with three sets of d-vectors associated with the three different types of tau function, while for case (2) we can write the sequences of cluster variables as

$$\hat{\tau}_n = \frac{\hat{\mathrm{N}}_n^{(1)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\hat{\mathbf{d}}_n}}, \qquad \hat{r}_n = \frac{\hat{\mathrm{N}}_n^{(2)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\hat{\mathbf{f}}_n}}, \qquad \hat{\eta}_n = \frac{\hat{\mathrm{N}}_n^{(3)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\hat{\mathbf{e}}_n}}, \qquad \hat{\sigma}_n = \frac{\hat{\mathrm{N}}_n^{(4)}(\hat{\mathbf{x}})}{\hat{\mathbf{x}}^{\hat{\mathbf{g}}_n}},$$

where in the latter case there are four different sets of d-vectors with their corresponding sequences of tau functions. In the first case, the d-vectors satisfy the $(\max,+)$ version of the system (5.26), given by

$$\begin{aligned}
&\mathbf{f}_{n+2}+\mathbf{e}_n = \max\left(\mathbf{f}_n+\mathbf{d}_n+\mathbf{d}_{n+1}, \mathbf{f}_{n+1}+\mathbf{d}_{n-2}+\mathbf{d}_{n+2}\right),\\
&\mathbf{e}_{n+1}+\mathbf{f}_n = \max\left(\mathbf{f}_{n+1}+\mathbf{d}_{n+2}+\mathbf{d}_{n-2}, \mathbf{f}_{n+2}+\mathbf{d}_n+\mathbf{d}_{n-1}\right),\\
&\mathbf{d}_{n+3}+\mathbf{d}_{n-2} = \max\left(\mathbf{d}_n+\mathbf{d}_{n+1}, \mathbf{e}_{n+1}\right),
\end{aligned} \tag{5.43}$$

and in the second case, we find the $(\max,+)$ version of (5.35), namely

$$\begin{aligned}
&\hat{\mathbf{d}}_{n+2}+\hat{\mathbf{d}}_{n-3} = \max\left(\hat{\mathbf{d}}_n+\hat{\mathbf{d}}_{n-1}, \hat{\mathbf{e}}_n\right),\\
&\hat{\mathbf{e}}_{n+1}+\hat{\mathbf{e}}_n = \max\left(\hat{\mathbf{f}}_n+\hat{\mathbf{g}}_n+\hat{\mathbf{d}}_{n-2}+\hat{\mathbf{d}}_{n+2},, \hat{\mathbf{d}}_{n+1}+2\hat{\mathbf{d}}_n+\hat{\mathbf{d}}_{n-1}\right),\\
&\hat{\mathbf{f}}_{n+1}+\hat{\mathbf{g}}_n = \max\left(\hat{\mathbf{d}}_n+\hat{\mathbf{d}}_{n+1}, \hat{\mathbf{e}}_{n+1}\right),\\
&\hat{\mathbf{g}}_{n+1}+\hat{\mathbf{f}}_n = \max\left(\hat{\mathbf{d}}_n+\hat{\mathbf{d}}_{n+1}, \hat{\mathbf{e}}_{n+1}\right).
\end{aligned} \tag{5.44}$$

In both cases, the initial cluster of d-vectors is specified in terms of the $8\times 8$ identity matrix, denoted $I$, with the initial cluster for case (1) being ordered as

$$\left(\mathbf{f}_0\ \mathbf{f}_1\ \mathbf{e}_0\ \mathbf{d}_{-2}\ \mathbf{d}_{-1}\ \mathbf{d}_0\ \mathbf{d}_1\ \mathbf{d}_2\right) = -I, \tag{5.45}$$

whereas in case (2) the ordering of the initial cluster is

$$\left(\hat{\mathbf{d}}_{-3}\ \hat{\mathbf{f}}_0\ \hat{\mathbf{e}}_0\ \hat{\mathbf{g}}_0\ \hat{\mathbf{d}}_{-1}\ \hat{\mathbf{d}}_0\ \hat{\mathbf{d}}_1\ \hat{\mathbf{d}}_{-2}\right) = -I. \tag{5.46}$$

The ordering of the initial d-vectors in the initial clusters (5.45) and (5.46), respectively, corresponds in each case to the ordering of the initial tau functions in the set of unfrozen seed variables $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_8)$. The same ordering determines the appropriate tropical analogues of the symplectic coordinates $y_1, y_2$ that satisfy the maps (5.16) and (5.17), namely

$$\mathbf{Y}_{1,n} = \mathbf{e}_n - \mathbf{d}_{n-1} - \mathbf{d}_n, \qquad \mathbf{Y}_{2,n} = \mathbf{f}_{n+1} - \mathbf{f}_n + \mathbf{d}_{n+2} - \mathbf{d}_{n+1} - \mathbf{d}_n + \mathbf{d}_{n-2}$$

in case (1), and

$$\mathbf{Y}_{1,n} = \hat{\mathbf{e}}_n - \hat{\mathbf{d}}_{n-1} - \hat{\mathbf{d}}_n, \qquad \mathbf{Y}_{2,n} = \hat{\mathbf{f}}_n + \hat{\mathbf{g}}_n - \hat{\mathbf{d}}_{n+1} - \hat{\mathbf{d}}_n + \hat{\mathbf{d}}_{n-2} - \hat{\mathbf{d}}_{n-3}$$

in case (2). We find that the $(\max, +)$ dynamical systems (5.43) and (5.44) lead to the same ultradiscrete map for the vectors $\mathbf{Y}_{1,n}, \mathbf{Y}_{2,n}$ in each case.

**Lemma 5.9.** *The ultradiscrete analogues of (5.20) and (5.32), which are satisfied by the vectors $\mathbf{Y}_{1,n}, \mathbf{Y}_{2,n}$, are the same in each case, being given by the following system of equations:*

$$\begin{aligned} \mathbf{W}_n &= [\mathbf{Y}_{1,n}]_+ , \\ \mathbf{Y}_{1,n+1} + \mathbf{Y}_{1,n} &= [\mathbf{W}_n + \mathbf{Y}_{2,n}]_+ , \\ \mathbf{Y}_{2,n+1} + \mathbf{Y}_{2,n} + 2\mathbf{Y}_{1,n} &= 2\,[\mathbf{Y}_{2,n}]_+ + \mathbf{W}_n. \end{aligned} \tag{5.47}$$

*Given arbitrary initial values $(Y_{1,0}, Y_{2,0}) \in \mathbb{R}^2$, every component of this system is periodic with period 4.*

**Proof.** Since this is very similar to the proof of lemma 4.8, it is omitted. □

The quantity $\mathbf{W}_n$ appearing in the system (5.47) is given as a combination of d-vectors by

$$\mathbf{W}_n = \mathbf{d}_{n+2} - \mathbf{d}_n - \mathbf{d}_{n-1} + \mathbf{d}_{n-3} \tag{5.48}$$

in case (1), associated with denominators of the tau functions $\tau_n$, and by

$$\mathbf{W}_n = \hat{\mathbf{d}}_{n+2} - \hat{\mathbf{d}}_n - \hat{\mathbf{d}}_{n-1} + \hat{\mathbf{d}}_{n-3} \tag{5.49}$$

in case (2), which is a combination of d-vectors associated with denominators of the tau functions $\hat{\tau}_n$. By the results of theorem 5.8, each of these tau functions satisfies a Somos-7 relation of the form (5.41), with a coefficient $\delta$ that can be written in terms of the initial cluster variables as a Laurent polynomial of the form

$$\tilde{\pi}_j^*(\delta) = \frac{\mathrm{P}_j(\hat{\mathbf{x}})}{\tilde{x}_1\tilde{x}_2\tilde{x}_3^2\tilde{x}_4\tilde{x}_5\tilde{x}_6\tilde{x}_7\tilde{x}_8}$$

for $j = 1, 2$, respectively, with a suitable polynomial $\mathrm{P}_j$ in the numerator, and the same form of the denominator in both cases. This immediately yields the following analogue of lemma 4.9.

**Lemma 5.10.** *The d-vectors $\mathbf{d}_n \in \mathbb{Z}^8$ that specify the denominators of tau functions $\tau_n$ generated by the system (5.43) satisfy the tropical Somos-7 relation*

$$\mathbf{d}_{n+7} + \mathbf{d}_n = \max(\mathbf{d}_{n+6} + \mathbf{d}_{n+1}, \hat{\mathbf{c}} + \mathbf{d}_{n+4} + \mathbf{d}_{n+3}), \tag{5.50}$$

*with the constant vector*

$$\hat{\mathbf{c}} = (1, 1, 2, 1, 1, 1, 1, 1)^T,$$

*and the d-vectors $\hat{\mathbf{d}}_n \in \mathbb{Z}^8$ that specify the denominators of tau functions $\hat{\tau}_n$ generated by (5.44) satisfy exactly the same relation. Also, the corresponding quantity $\mathbf{W}_n$, defined by either (5.48) or (5.49), satisfies the ultradiscrete Lyness map*

$$\mathbf{W}_{n+1} + \mathbf{W}_{n-1} = \max(\mathbf{W}_n, \hat{\mathbf{c}}), \tag{5.51}$$

*with the same constant $\hat{\mathbf{c}}$.*

Due to the way that the initial seeds for the d-vectors are combined in (5.45) and (5.46), in each case the initial tau functions correspond to the same initial values

$$\mathbf{Y}_{1,0} = (0,0,-1,0,1,1,0,0)^T, \qquad \mathbf{Y}_{2,0} = (1,-1,0,-1,0,1,1,-1)^T$$

for the system (5.47). These initial values produce the period 4 orbit

$$\mathbf{W}_0 = (0,0,0,0,1,1,0,0)^T, \mathbf{W}_1 = (1,0,1,0,0,1,1,0)^T, \mathbf{W}_2 = (1,1,2,1,0,0,1,1)^T,$$
$$\mathbf{W}_3 = (0,1,1,1,1,0,0,1)^T,$$

in which the third component corresponds to an orbit of the scalar map (4.55) with $c = 2$, namely the first orbit listed in (4.56), while every other component corresponds to the orbit (4.57) of the same scalar map with $c = 1$. This allows the d-vectors for the tau functions to be completely determined in both case (1) and case (2). Since the steps of the proof of the following result are very similar to those for theorem 4.10, we leave the details for the reader.

**Theorem 5.11.** *The d-vectors* $\mathbf{d}_n, \mathbf{e}_n, \mathbf{f}_n$ *satisfying the* $(\max, +)$ *system (5.43), as well as the d-vectors* $\hat{\mathbf{d}}_n$, $\hat{\mathbf{e}}_n$, $\hat{\mathbf{f}}_n$, $\hat{\mathbf{g}}_n$ *satisfying (5.44), all lie in the kernel of the linear difference operator*

$$\tilde{\mathcal{L}} = \left(\mathcal{T}^2 + 1\right)\left(\mathcal{T}^2 - 1\right)^2\left(\mathcal{T}^3 - 1\right).$$

*For the tau functions generated by (5.26) and (5.35), the leading order degree growth of their denominators is given by*

$$\mathbf{d}_n = \hat{\mathbf{a}} n^2 + O(n), \quad \mathbf{e}_n = 2\hat{\mathbf{a}} n^2 + O(n), \quad \mathbf{f}_n = \hat{\mathbf{a}} n^2 + O(n)$$

*in case (1), and*

$$\hat{\mathbf{d}}_n = \hat{\mathbf{a}} n^2 + O(n), \quad \hat{\mathbf{e}}_n = 2\hat{\mathbf{a}} n^2 + O(n), \quad \hat{\mathbf{f}}_n = \hat{\mathbf{a}} n^2 + O(n), \quad \hat{\mathbf{g}}_n = \hat{\mathbf{a}} n^2 + O(n)$$

*in case (2), with the same constant vector*

$$\mathbf{a} = \frac{1}{24}(1,1,2,1,1,1,1,1)^T$$

*in each case.*

## 6. Open problems and concluding remarks

The results in this paper constitute a further proof of concept of the idea introduced by one of us with Kouloukas in [20], that periodic dynamical systems arising from Zamolodchikov periodicity of cluster maps associated with finite type simple Lie algebras admit natural deformations to discrete dynamical systems that are completely integrable in the Liouville sense (but no longer completely periodic), and further that this dynamics can be lifted to an enlarged phase space where the Laurent property holds (Laurentification). In all of the low rank cases considered here, over $\mathbb{C}$ the level sets of first integrals are one-dimensional tori, so the spaces of initial conditions correspond to elliptic surfaces. This is just like the well known situation for QRT maps [4], which have been studied for a long time [34]; and indeed, the examples of $A_3$ and $C_2$ treated above are related to particular cases of QRT maps. However, despite the prior results on Laurentification obtained in [14], we do not know of any procedure to endow an arbitrary QRT map with the structure of a cluster algebra. However, if we interpret these examples over $\mathbb{Q}$ (rather than $\mathbb{C}$), then the Zamolodchikov periodicity of the original system can be viewed as providing a family of elliptic curves with a rational torsion point of prescribed order.

Our results on tropical dynamics of d-vectors indicate that Zamolodchikov periodicity plays an essential role in determining the slow (quadratic) growth of degrees of tau functions for these discrete integrable systems. In the first appendix below, we present the derivation of an additional formula (A.6) in the $\mathrm{A}_3$ case, which suggests that the tropical dynamics of d-vectors for cluster variables provides an efficient way to calculate the degrees of maps in projective space. We expect that the same approach can be applied to many other birational maps, shedding light on some of the unexplained observations in [37, 38].

Compared with our previous results, which were all in type A, the examples of $\mathrm{B}_3$ and $\mathrm{D}_4$ considered here have revealed the new feature that the same Zamolodchikov period dynamics can be deformed to an integrable map in more than one distinct way. Nevertheless, there seem to be very close connections between the two cases obtained for $\mathrm{B}_3$; and the close connections between the case (1) and case (2) deformations for $\mathrm{D}_4$ are even more apparent, given that the underlying coefficient-free cluster algebra is the same for both.

The situation for higher rank Lie algebras becomes even more interesting, since with more degrees of freedom one finds that the level sets of first integrals are abelian varieties of dimension greater than 1. In [20] it was shown that the periodic cluster map associated with the $\mathrm{A}_4$ root system has a 2-parameter integrable deformation, which lifts by Laurentification to a cluster map in 11 dimensions, with 2 additional frozen variables. Recently, this construction has been generalized to 2-parameter deformations of $\mathrm{A}_{2N}$ cluster maps for all $N \geqslant 1$, which lift to cluster maps in dimension $4N+3$ associated with a special family of quivers [12]. The complete description of this construction, and analogous results for the odd rank case ($\mathrm{A}_{2N+1}$), will be the subject of future work. In addition to this, we have recently found extensions of the results given here to types B and D in higher rank, which are also under investigation.

## Data availability statement

No new data were created or analysed in this study.

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## Appendix A. Elliptic surface and degree growth for deformed $\mathrm{A}_3$ map

Here we consider the action of the maps $\hat{\varphi}$ and $\hat{\psi}$ on the elliptic surface defined by the first integral (2.15) in the deformed $\mathrm{A}_3$ case. Let $(y, w)$ be a pair of inhomogeneous coordinates for $\mathbb{P}^1 \times \mathbb{P}^1$ and let $Y = \frac{1}{y}$ and $W = \frac{1}{w}$. Then $\mathbb{P}^1 \times \mathbb{P}^1$ is covered by four charts: $(y, w)$, $(Y, w)$, $(y, W)$ and $(Y, W)$. For example, the point $(\infty, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1$ corresponds to $(Y, W) = (0, 0)$. On each chart, $K_1$ is written as

$$K_1 = \frac{(y+dw+c)\,((y+1)\,w+d)}{yw} = \frac{(1+dwY+cY)\,((1+Y)\,w+dY)}{Yw}$$
$$= \frac{(yW+d+cW)\,((y+1)+dW)}{yW} = \frac{(W+dY+cYW)\,((1+Y)+dYW)}{YW}.$$

The pencil defined by $K_1$ has 8 base points on $\mathbb{P}^1\times\mathbb{P}^1$:

(1) $(y,w) = \left(0,-\frac{c}{d}\right)$,
(2) $(y,w) = (0,-d)$,
(3) $(y,w) = (-c,0)$,
(4) $(Y,w) = (0,0)$,
(5) $(Y,w/Y) = (0,-d)$,
(6) $(y,W) = (-1,0)$,
(7) $(Y,W) = (0,0)$,
(8) $(Y,W/Y) = (0,-d)$.

We consider the *generic* situation where the values of the parameters $c,d$ are such that none of these base points coincide, which means that we should assume

$$c\neq 0,1,\qquad d\neq 0,\qquad c\neq d^2. \tag{A.1}$$

Note that the points (5) and (8) are infinitely near (4) and (7), respectively. Blowing up $\mathbb{P}^1\times\mathbb{P}^1$ at these 8 points, we obtain an elliptic surface $\mathcal{X}$ on which $K_1\colon \mathcal{X}\to\mathbb{P}^1$ gives an elliptic fibration (see figures 4 and 5). The commuting maps $\hat{\varphi}$ and $\hat{\psi}$, which were originally defined in (2.10) and (2.17) using the affine chart $(y,w)=(u_1,u_2)$, both extend to automorphisms on $\mathcal{X}$.

Let us use intersection theory to calculate degree growth for $\hat{\varphi}$ and $\hat{\psi}$. As a basis of $\mathrm{Pic}(\mathcal{X})$, it is common to use

$$H_y, H_w, E_1, \ldots, E_8$$

where $H_y$ (resp. $H_w$) is the total transform of the class $\{y=\text{const}\}$ (resp. $\{w=\text{const}\}$) and $E_i$ is the total transform of the exceptional class of the $i$th blowup. For convenience, however, here we use the basis

$$[D_1],\ldots,[D_6],[C_2],[C_3],[C_5],[C_8]$$

(see figure 5). The matrix representations for $\hat{\varphi}$ and $\hat{\psi}$ with respect to this basis are

$$M_{\hat{\varphi}} := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ & & & & & & 0 & 0 & -1 & 0 \\ & & & & & & 0 & 1 & 0 & 0 \\ & & & & & & 0 & 1 & 1 & 1 \\ & & & & & & 1 & -1 & 1 & 0 \end{pmatrix}$$

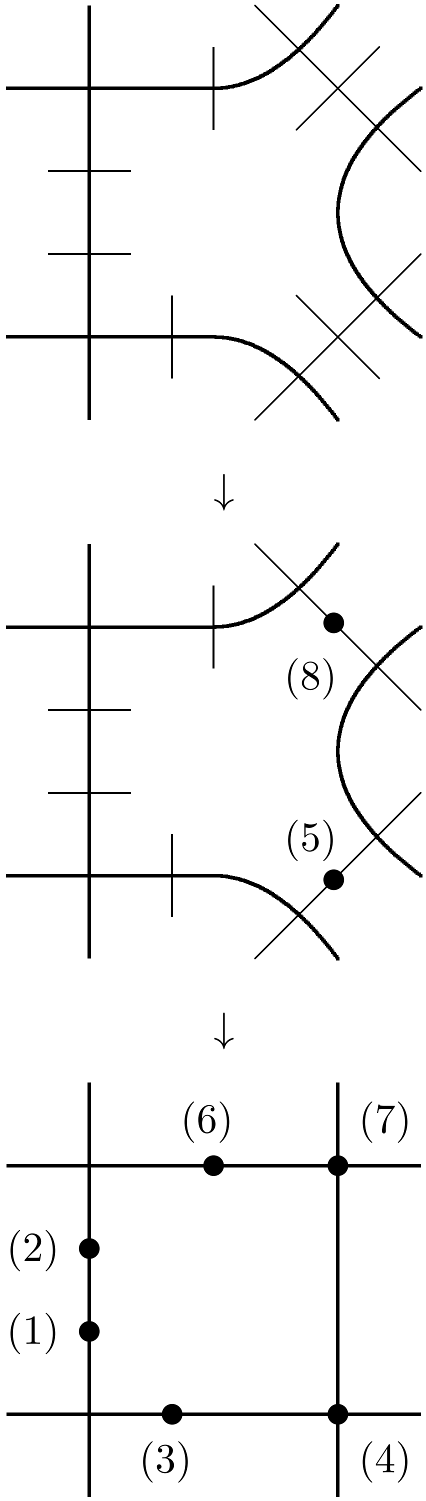


**Figure 4.** Blow-ups needed to obtain an elliptic surface from the pencil defined by $K_1$.

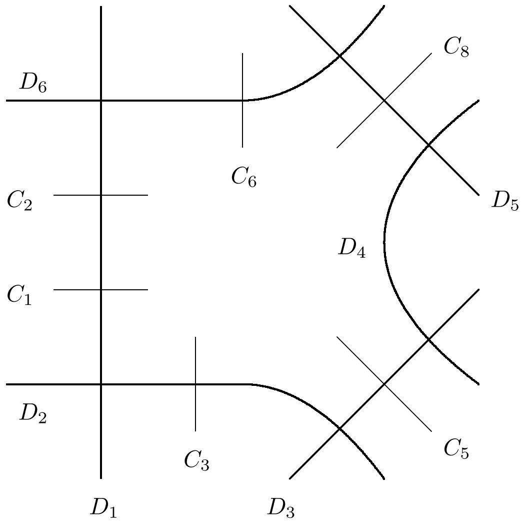


**Figure 5.** The elliptic surface $\mathcal{X}$. The curves $D_1$, $D_4$, $D_2$ and $D_6$ are the strict transforms of the curves $\{y=0\}$, $\{y=\infty\}$, $\{w=0\}$ and $\{w=\infty\}$ in $\mathbb{P}^1\times\mathbb{P}^1$, respectively. The curves $D_1,\ldots,D_6$ have self-intersection $(-2)$ and their intersection pattern forms the Dynkin diagram of type $A_5^{(1)}$ (cf. [31]). The curves $C_1,C_2,C_3,C_5,C_6,C_8$ have self-intersection $(-1)$.

and

$$M_{\hat{\psi}} := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ & & & & & & 1 & 0 & 0 & 0 \\ & & & & & & -1 & 0 & -1 & -1 \\ & & & & & & 0 & 0 & 1 & 0 \\ & & & & & & 1 & 1 & 1 & 2 \end{pmatrix},$$

respectively, where the entries of the empty blocks are all zero. The intersection matrix on $\mathrm{Pic}(\mathcal{X})$ with respect to our basis is

$$A_{\mathcal{X}} := \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

As in figure 6, $\mathcal{X}$ can be blown down to $\mathbb{P}^2$ as well. Since the total transform of the class of lines in $\mathbb{P}^2$ is

$$[D_2] + 2[D_3] + [D_4] + [C_3] + 2[C_5],$$

the degrees with respect to $\mathbb{P}^2$ can be calculated as

$$\deg(\hat{\varphi}^n) = \mathbf{h}^T \cdot A_{\mathcal{X}} M_{\hat{\varphi}}^n \mathbf{h}$$

and

$$\deg\left(\hat{\psi}^m\right) = \mathbf{h}^T \cdot A_{\mathcal{X}} M_{\hat{\psi}}^m \mathbf{h},$$

where

$$\mathbf{h} = (0,1,2,1,0,0,0,1,2,0)^T.$$

A direct calculation shows that

$$\lim_{n\to\infty} \frac{1}{n^2} M_{\hat{\varphi}}^n = \frac{1}{12} \begin{pmatrix} \mathbf{0}_{6,6} & \mathbf{1}_{6,4} \\ \mathbf{0}_{4,6} & \mathbf{0}_{4,4} \end{pmatrix}$$

and

$$\lim_{m\to\infty} \frac{1}{m^2} M_{\hat{\psi}}^m = \frac{1}{3} \begin{pmatrix} \mathbf{0}_{6,6} & \mathbf{1}_{6,4} \\ \mathbf{0}_{4,6} & \mathbf{0}_{4,4} \end{pmatrix},$$

where $\mathbf{1}_{6,4}$ is the matrix of size $6 \times 4$ with all the entries 1. Hence we have

$$\deg(\hat{\varphi}^n) = \frac{3}{4} n^2 + O(n) \tag{A.2}$$

and

$$\deg\left(\hat{\psi}^m\right) = \frac{9}{4} m^2 + O(m). \tag{A.3}$$

The calculation of the degree growth in $\mathbb{P}^2$, as above, can also be derived from the tropical dynamics of the d-vectors, as determined in theorem 2.11. Indeed, the non-frozen variables in a cluster associated with the deformed $A_3$ map $\hat{\varphi}$, and the QRT map $\hat{\psi}$ that commutes with it, determine a point in $\mathbb{P}^2$ according to

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6) \mapsto \left(\tilde{x}_1 \tilde{x}_4 \tilde{x}_5 : (\tilde{x}_2)^2 \tilde{x}_6 : \tilde{x}_2 \tilde{x}_3 \tilde{x}_5\right),$$

so the choice of initial seed

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6) = (X, 1, Z, 1, 1, Y)$$

corresponds to an arbitrary point $\mathbf{X} = (X : Y : Z) \in \mathbb{P}^2$. Thus we can determine the action of $\hat{\varphi}$ and $\hat{\psi}$, and count the degrees of iterates of these maps, via the induced action obtained from the cluster maps $\tilde{\varphi}$ and $\tilde{\psi}$. For instance, by considering $\tilde{\varphi}^n(X, 1, Z, 1, 1, Y)$, we find the sequence

$$\hat{\varphi}^n(\mathbf{X}) = \left(\tau_{n-1}\tau_{n+2}\sigma_n : (\tau_n)^2 \sigma_{n+1} : \tau_n \tau_{n+1} \sigma_n\right)\Big|_{\mathbf{X}} \in \mathbb{P}^2, \tag{A.4}$$

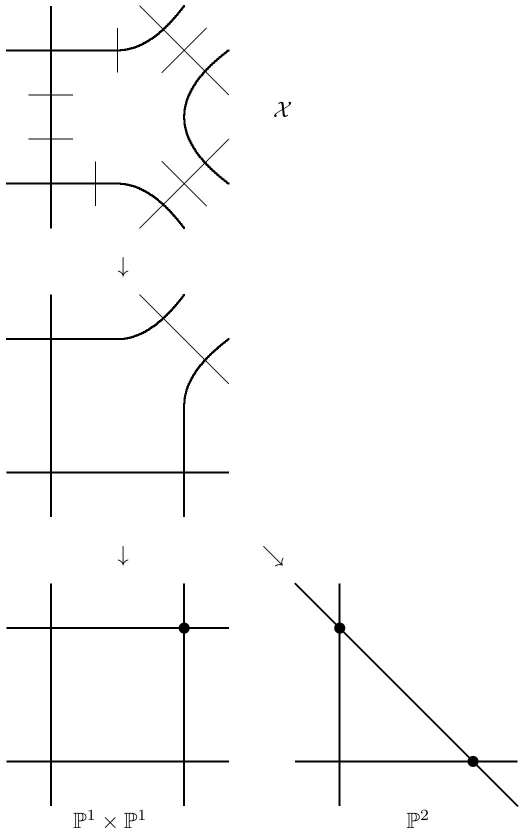


**Figure 6.** The elliptic surface $\mathcal{X}$ can be obtained by blowing up $\mathbb{P}^2$ at 9 points, instead of $\mathbb{P}^1 \times \mathbb{P}^1$ at 8 points. The centres of the blow-ups on $\mathbb{P}^2$ in this figure are $(1:0:0)$ and $(0:1:0)$, where $(y:w:1)$ are inhomogeneous coordinates on $\mathbb{P}^2$.

where the subscript $\mathbf{X}$ on the right-hand side denotes the fact that we substitute the initial values $\tilde{x}_1 = \tau_{-1} = X$, $\tilde{x}_2 = \tau_0 = 1$, $\tilde{x}_3 = \tau_1 = Z$, $\tilde{x}_4 = \tau_2 = 1$, $\tilde{x}_5 = \sigma_0 = 1$, $\tilde{x}_6 = \sigma_1 = Y$ into each Laurent polynomial that appears. More generally, applying a combination of the two maps (associated with shifts in $m,n$ respectively), for any $(m,n) \in \mathbb{Z}^2$ we can compose $m$ steps of $\hat{\psi}$ with $n$ steps of $\hat{\varphi}$, to obtain the $(m,n)$ combined iterate, in the form

$$\hat{\psi}^m \hat{\varphi}^n (\mathbf{X}) = \left( \frac{\mathrm{N}_1\mathrm{N}_4\mathrm{N}_5}{\mathbf{X}^{\mathbf{d}_1+\mathbf{d}_4+\mathbf{d}_5}} : \frac{(\mathrm{N}_2)^2\mathrm{N}_6}{\mathbf{X}^{2\mathbf{d}_2+\mathbf{d}_6}} : \frac{\mathrm{N}_2\mathrm{N}_3\mathrm{N}_5}{\mathbf{X}^{\mathbf{d}_2+\mathbf{d}_3+\mathbf{d}_5}} \right)(m,n;\mathbf{X}), \tag{A.5}$$

where the dependence on $m,n$ and $\mathbf{X}$ indicates the fact that numerator polynomials $\mathrm{N}_j(m,n;\tilde{\mathbf{x}})$ in (2.41) are evaluated at $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4,\tilde{x}_5,\tilde{x}_6) = (X,1,Z,1,1,Y)$, and for the denominators we write

$$\mathbf{X}_j^{\mathbf{d}} = X^{\mathbf{b}_1^T\mathbf{d}_j(m,n)}\, Y^{\mathbf{b}_6^T\mathbf{d}_j(m,n)}\, Z^{\mathbf{b}_3^T\mathbf{d}_j(m,n)},$$

where $\mathbf{b}_k$ denotes the $k$th standard basis vector in $\mathbb{R}^6$. This leads to an exact formula for the degrees of the combined iterates of the two maps in $\mathbb{P}^2$, expressed in terms of the associated d-vectors defined on the $\mathbb{Z}^2$ lattice.

**Theorem A.1.** *For generic complex coefficients $c, d$, the degrees of the combined iterates of the maps $\hat{\psi}$ and $\hat{\varphi}$ in $\mathbb{P}^2$ are given by*

$$\deg\left(\hat{\psi}^m\hat{\varphi}^n\right) = (\mathbf{b}_1+\mathbf{b}_3+\mathbf{b}_6)^T\left(\mathbf{d}(m,n)+\mathbf{d}(m,n+1)+\mathbf{d}(m+1,n)\right)+\tfrac{1}{2}\left(3-(-1)^n\right)$$
$$+\max\left(Y^{(1)}_{m,n}, W^{(1)}_{m,n}, 0\right)+\max\left(Y^{(3)}_{m,n}, W^{(3)}_{m,n}, 0\right)+\max\left(Y^{(6)}_{m,n}, W^{(6)}_{m,n}, 0\right), \tag{A.6}$$

*where $Y^{(k)}_{m,n}, W^{(k)}_{m,n}$ respectively denote the kth component of each of the vectors defined in (2.46), that is*

$$Y^{(k)}_{m,n} = \mathbf{b}_k^T\mathbf{Y}_{m,n}, \qquad W^{(k)}_{m,n} = \mathbf{b}_k^T\mathbf{W}_{m,n}.$$

**Proof.** Each of the entries for the three homogeneous coordinates in (A.5) corresponds to a product of three cluster variables with $X, Y, Z$ and 1 substituted appropriately, hence they are all Laurent polynomials in the homogeneous coordinates $(X:Y:Z)$ of the initial point in $\mathbb{P}^2$. In order to calculate the degrees of the iterates in $\mathbb{P}^2$, we need to remove the denominators, which are monomials in $X, Y, Z$, so that what remains is a coprime triple of homogeneous polynomials in these coordinates; then the degree $\deg(\hat{\psi}^m\hat{\varphi}^n)$ of the combined $(m,n)$ iterate of the maps is the common degree of these three polynomials. We can clear the denominators in (A.5) by multiplying by suitable powers of the projective coordinates $X, Y, Z$. So for instance, to clear the powers of $X$, we must take the maximum of the exponents of $X$ that appear in the denominators of the three homogeneous coordinates in the formula (A.5), and multiply through by

$$X^{\max\left(\mathbf{b}_1^T(\mathbf{d}_1+\mathbf{d}_4+\mathbf{d}_5),\mathbf{b}_1^T(2\mathbf{d}_2+\mathbf{d}_6),\mathbf{b}_1^T(\mathbf{d}_2+\mathbf{d}_3+\mathbf{d}_5)\right)},$$

where each d-vector $\mathbf{d}_j$ is evaluated at $(m,n)\in\mathbb{Z}^6$, corresponding to column $j$ of the matrix $D_{m,n}$ determined in theorem 2.11. If we focus on the third entry in (A.5), then we see that the effect of rescaling by this power of $X$ is to produce an overall prefactor of

$$X^{\max\left(\mathbf{b}_1^T(\mathbf{d}_1-\mathbf{d}_2-\mathbf{d}_3+\mathbf{d}_4),\mathbf{b}_1^T(\mathbf{d}_2-\mathbf{d}_3-\mathbf{d}_5+\mathbf{d}_6),0\right)} = X^{\max\left(Y^{(1)}_{m,n},W^{(1)}_{m,n},0\right)},$$

where the second equality above follows by rewriting each d-vector $\mathbf{d}_j(m,n)$ above as an appropriate shift of a single vector $\mathbf{d}(m,n)$ defined on $\mathbb{Z}^2$, and then noting that $\mathbf{Y}_{m,n}$ and $\mathbf{W}_{m,n}$ are given by the relevant combinations of the latter vector, as in (2.46), with the first component of each, namely $Y^{(1)}_{m,n}$ and $W^{(1)}_{m,n}$, corresponding to the exponent of the homogeneous coordinate $X$. Similarly, by doing an analogous rescaling by powers of the other homogeneous coordinates $Y$ and $Z$, the third homogeneous coordinate in (A.5) becomes a polynomial in $X, Y, Z$ with an overall monomial prefactor of

$$X^{\max\left(Y^{(1)}_{m,n},W^{(1)}_{m,n},0\right)}\, Y^{\max\left(Y^{(6)}_{m,n},W^{(6)}_{m,n},0\right)}\, Z^{\max\left(Y^{(3)}_{m,n},W^{(3)}_{m,n},0\right)}.$$

(The reader should hopefully not be confused by the double meaning of the letter $Y$ in the above formula: the unadorned letter denotes a homogeneous coordinate in $\mathbb{P}^2$, while the same letter appearing in the exponents with indices and a numerical suffix denotes a tropical variable.) By construction, both the first and the second entry have now also become polynomials in $X, Y, Z$, also with monomial prefactors, so that in $\mathbb{P}^2$ we have

$$\hat{\psi}^m\hat{\varphi}^n(\mathbf{X}) = \left(\mathrm{P}_{m,n}(\mathbf{X}) : \mathrm{Q}_{m,n}(\mathbf{X}) : \mathrm{R}_{m,n}(\mathbf{X})\right) \tag{A.7}$$

with $\mathrm{P}_{m,n}, \mathrm{Q}_{m,n}, \mathrm{R}_{m,n} \in \mathbb{C}[X,Y,Z]$; but the monomial prefactors are such that $X$ can never divide all three of these polynomials simultaneously, and the same is true for $Y$ and $Z$. Moreover, it follows from lemma 2.8 that the prefactors oscillate periodically, repeating with period 2 in $m$ and period 3 in $n$.

Apart from the oscillating monomial factors, the main contribution to the degree of the iterates comes from the product of three numerator factors $\mathrm{N}_i\mathrm{N}_j\mathrm{N}_k$ appearing in each homogeneous component of (A.5). There are two essential properties of these polynomials that are inherited from the cluster algebra: firstly, due to the fact that each numerator polynomial $\mathrm{N}_j$ in (2.41) is not divisible by any initial cluster variable $\tilde{x}_i$, it follows that, when the homogeneous coordinates for $\mathbb{P}^2$ are substituted in, the resulting homogeneous polynomial $\mathrm{N}_j(\mathbf{X})$ is not divisible by $X$, $Y$ or $Z$; and secondly, for *generic* values of the coefficients $c,d$, the cluster variables in each cluster are pairwise coprime Laurent polynomials, meaning that for each $(m,n) \in \mathbb{Z}^2$, the polynomials $\mathrm{N}_1(\mathbf{X}), \mathrm{N}_2(\mathbf{X}), \ldots, \mathrm{N}_6(\mathbf{X})$ appearing in (A.5) are themselves pairwise coprime. (It should be noted that the latter assertion ceases to be valid in the non-generic cases (A.1), when the numerator polynomials are reducible and have certain factors in common, which cancel out and cause the degree to drop.) This implies that the degree of the third component $\mathrm{R}_{m,n}(\mathbf{X})$, and hence the degree of the $(m,n)$ combined iterate of the two maps, is equal to the degree of the product of three numerators $\mathrm{N}_2\mathrm{N}_3\mathrm{N}_5$ appearing in (A.5), plus the degree of the oscillating prefactor (A.7). So it remains to calculate the degree of this product of numerators, and show that it is equal to the first line of the formula (A.6).

The products of evaluations of Laurent polynomials appearing in each entry on the right-hand side of (A.5) have a homogeneous degree. To start with, let us fix $m = 0$, and consider the action of $\hat{\varphi}$ (shifting in $n$) on its own, with the sequence of products of three tau functions that appear as components in (A.4). Then for the initial point in $\mathbb{P}^2$ we have the (common) homogeneous degree of each of the projective coordinates, namely

$$\mathrm{hdeg}\left(\tau_{-1}\tau_2\sigma_0, (\tau_0)^2\sigma_1, \tau_0\tau_1\sigma_0\right) = \mathrm{hdeg}(X,Y,Z) = 1,$$

and we can write the pattern of homogeneous degrees in the initial cluster as

$$\mathrm{hdeg}(\tau_{-1},\tau_0,\tau_1,\tau_2,\sigma_0,\sigma_1) = (1,0,1,0,0,1)\,;$$

but after a single iteration of the system (2.20) we find

$$\mathrm{hdeg}\left(\tau_0\tau_3\sigma_1, (\tau_1)^2\sigma_2, \tau_1\tau_2\sigma_1\right) = \mathrm{hdeg}\left(\frac{Y\left(dY(X+Z)+cZ^2\right)}{X}, \frac{Z^2\,(dY+cZ)}{X}, YZ\right) = 2,$$

where from

$$\tau_3 = \frac{dY(X+Z)+cZ^2}{X}, \qquad \sigma_2 = \frac{dY+cZ}{X}$$

we see that the pattern of homogeneous degrees in the new cluster is

$$\mathrm{hdeg}(\tau_0,\tau_1,\tau_2,\tau_3,\sigma_1,\sigma_2) = (0,1,0,1,1,0)\,;$$

and thereafter, with subsequent shifts in $n$, this pattern repeats with period 2. A similar calculation shows that, under each shift $m \to m+1$, the pattern of homogeneous degrees remains the same; so overall it only depends on the parity of $n$. Hence, for all $(m,n)\in\mathbb{Z}^2$, we see that the homogeneous degree of the third component in (A.5) is

$$\mathrm{hdeg}\left(\frac{\mathrm{N}_2\mathrm{N}_3\mathrm{N}_5}{\mathbf{X}^{\mathbf{d}_2+\mathbf{d}_3+\mathbf{d}_5}}(m,n;\mathbf{X})\right)=\deg\left(\mathrm{N}_2\mathrm{N}_3\mathrm{N}_5(m,n;\mathbf{X})\right)$$
$$-\deg\left(\mathbf{X}^{\mathbf{d}_2(m,n)+\mathbf{d}_3(m,n)+\mathbf{d}_5(m,n)}\right)=\frac{1}{2}\left(3-(-1)^n\right);$$

so the rest of the formula (A.6) then follows, by rewriting the sum of d-vectors in the exponent of $\mathbf{X}$ above as $\mathbf{d}(m,n)+\mathbf{d}(m,n+1)+\mathbf{d}(m+1,n)$ and taking the sum of components 1,3 and 6 of the latter vector, which correspond, respectively, to the powers of $X,Z$ and $Y$ in this monomial. □

Observe that the leading order terms in (A.6) are the sum of three components of three shifted copies of the d-vector $\mathbf{d}(m,n)$, and as the growth of each component of this vector is given by the entries in (2.52), overall this contributes a factor of $3\times 3=9$ times the leading order of any of the entries of the matrix $D_{m,n}$, so that

$$\deg\left(\hat{\psi}^m\hat{\varphi}^n\right)=\frac{9}{4}m^2+\frac{3}{4}n^2+O(m)+O(n), \tag{A.8}$$

which provides an independent confirmation of the results (A.2) and (A.3) obtained from intersection theory. We now explain what it tells us about the Mordell-Weil group of the elliptic surface $\mathcal{X}$.

For all but a finite number of values of $\kappa\in\mathbb{P}^1$, the corresponding fiber

$$(y+dw+c)((y+1)w+d)=\kappa yw \tag{A.9}$$

in the pencil defining $\mathcal{X}$ is birationally equivalent to a Weierstrass equation for a smooth cubic curve, that is

$$E_\kappa: \qquad \mathrm{y}^2=\mathrm{x}^3+\mathrm{A}(\kappa)\mathrm{x}+\mathrm{B}(\kappa), \tag{A.10}$$

for certain rational functions $\mathrm{A},\mathrm{B}\in\mathbb{C}(\kappa)$. Hence we can regard $\mathcal{X}$ as an elliptic curve $E/\mathbb{C}(\kappa)$ defined over the function field $\mathbb{C}(\kappa)$, with the Mordell-Weil group being the group of $\mathbb{C}(\kappa)$-rational points. This curve has j-invariant

$$j(E_\kappa)=\frac{\mathrm{f}_4(\kappa)^3}{d^4\kappa^2\,\mathrm{g}_4(\kappa)},$$

for certain degree 4 polynomials of the form

$$\mathrm{f}_4(\kappa)=\kappa^4-4(c+1)\kappa^3+\cdots+\left((c-1)^2+4d^2\right)^2,$$
$$\mathrm{g}_4(\kappa)=c\kappa^4-(c+1)\left(4c-d^2\right)\kappa^3+\cdots+\left(c-d^2\right)\left((c-1)^2+4d^2\right)^2.$$

From the latter one can read off the singular fibers, which correspond to the values of $\kappa$ where the j-invariant has poles: at $0,\infty$ and the four distinct roots of $\mathrm{g}_4$. One can see from $j$ that there are two reducible fibers: the fiber over 0, which has 2 components (double pole in $j$ at $\kappa=0$); and the fiber over $\infty$, which has 6 components (pole of order 6 at $\kappa=\infty$).

Equivalently, we can view the Mordell-Weil group as the set of sections $s : \mathbb{P}^1 \to \mathcal{X}$, which are rational maps satisfying $K_1\big(s(\kappa)\big) = \kappa$ for all $\kappa \in \mathbb{P}^1$. To endow this with a group structure, we must fix a zero section $s_0$ that sends $\kappa$ to $\mathcal{O}_\kappa$, the identity element in each fiber. Then the maps $\hat{\varphi}$ and $\hat{\psi}$ correspond, respectively, to translation by certain points $\mathcal{P}_1$ and $\mathcal{P}_2$ in the group law of each (generic) fiber, which are associated with sections $s_1$ and $s_2$, say. In fact, in terms of the biquadratic model (A.9), viewed as a curve in $\mathbb{P}^1 \times \mathbb{P}^1$, we find that we can take the points $\mathcal{O} = (\infty, \infty)$, $\mathcal{P}_1 = (\infty, 0)$, $\mathcal{P}_2 = (-1, \infty)$ in each fiber: these are just the base points (7),(4),(6) previously identified in the pencil.

The Mordell-Weil group of $\mathcal{X}$, or equivalently, the set of $\mathbb{C}(\kappa)$-rational points on the curve $E\big(\mathbb{C}(\kappa)\big)$, has a canonical height function $\hat{h} : E\big(\mathbb{C}(\kappa)\big) \to \mathbb{R}$, which is a quadratic form having the property that $\hat{h} \geqslant 0$, with $\hat{h}(\mathcal{P}) = 0$ iff $\mathcal{P}$ is a point of finite order (a torsion point), and an associated bilinear pairing

$$< \mathcal{P}, \mathcal{Q} > = \hat{h}(\mathcal{P} + \mathcal{Q}) - \hat{h}(\mathcal{P}) - \hat{h}(\mathcal{Q}).$$

The canonical height is unique, up to an overall choice of scale. In the case at hand, if we fix the normalization to be consistent with the conventions in [35], then we can use the results in Chapter III of the latter book, together with the explicit expression (A.8) for the degree growth of the maps $\hat{\varphi}$ and $\hat{\psi}$, to calculate the canonical heights as follows:

$$\hat{h}(\mathcal{P}_1) = \frac{1}{3} \lim_{n\to\infty} n^{-2} \deg(\hat{\varphi}^n) = \frac{1}{4}, \qquad \hat{h}(\mathcal{P}_2) = \frac{1}{3} \lim_{n\to\infty} n^{-2} \deg\left(\hat{\psi}^n\right) = \frac{3}{4},$$
$$\hat{h}(\mathcal{P}_1 + \mathcal{P}_2) = \frac{1}{3} \lim_{n\to\infty} n^{-2} \deg\left(\hat{\psi}^n \hat{\varphi}^n\right) = 1.$$

Having calculated the canonical heights as above, we then find the Gram matrix with entries $< \mathcal{P}_i, \mathcal{P}_j >$ built from the pairing between these two points, given by

$$\begin{pmatrix} < \mathcal{P}_1, \mathcal{P}_1 > & < \mathcal{P}_1, \mathcal{P}_2 > \\ < \mathcal{P}_1, \mathcal{P}_2 > & < \mathcal{P}_2, \mathcal{P}_2 > \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 3/2 \end{pmatrix}.$$

From this we can conclude that

$$\hat{h}(n_1 \mathcal{P}_1 + n_2 \mathcal{P}_2) = n_1^2 \hat{h}(\mathcal{P}_1) + n_2^2 \hat{h}(\mathcal{P}_2) \neq 0$$

unless $n_1 = n_2 = 0$, which implies that $\mathcal{P}_1$ and $\mathcal{P}_2$ are two independent generators in the Mordell-Weil group, and hence this group has rank at least 2. In fact, according to theorem 3.1 in [31], since the rank of $T$, the root lattice associated with the reducible fibers, is calculated from the multiplicities of these fibers as $\operatorname{rk} T = (2-1) + (6-1) = 6$, the rank of the Mordell-Weil group is

$$\operatorname{rk} E(\mathbb{C}(\kappa)) = 8 - \operatorname{rk} T = 8 - 6 = 2,$$

as expected.

It is worthwhile to note that, if we treat the frozen variables $c, d$ as parameters, then the j-invariant $j(E_\kappa)$ above is defined over $\mathbb{Q}(c, d, \kappa)$, so we can use generic *rational* values $c, d \in \mathbb{Q}$ and rational initial data for the maps to generate interesting examples of elliptic curves defined over $\mathbb{Q}$. One of the simplest cases that produces a curve $E(\mathbb{Q})$ of rank 2 is the choice

$$c = -2, \qquad d = -1, \qquad (y, w) = (1, 1) \implies \kappa = -2.$$

In that case, the curve (A.9) becomes

$$(y-w-2)\,((y+1)\,w-1) = -2yw,$$

which has j-invariant

$$j = -\frac{1771561}{1588},$$

being birationally equivalent to the minimal model

$$y^2 + xy + y = x^3 - 3x + 2,$$

where the latter has Mordell-Weil generators $(x,y) = (1,0)$ and $(-1,2)$[6].

## Appendix B. Three-invariant for the special Somos-7 recurrence

Here we consider the special Somos-7 recurrence relation (1.4) associated with the Lyness map, and present a number of general results about its solutions. We begin by showing that every solution $\tau_n$ also satisfies a relation of Somos-5 type with a coefficient $\xi_n$ that is periodic with period 3, of the form

$$\tau_{n+5}\tau_n = \xi_n\,\tau_{n+4}\tau_{n+1} - a\,\tau_{n+3}\tau_{n+2}. \tag{B.11}$$

This result is connected to several of the examples in this paper, where such Somos-7 relations appear in various places. There is another, analogous result for the Somos-5 recurrence relation itself, namely that every Somos-5 sequence satisfies a relation of Somos-4 type with one coefficient that varies according to the parity of the index $n$ [15]. The corresponding result for the special Somos-7 recurrence can be stated as follows.

**Proposition B.1.** *Suppose that the sequence $(\tau_n)$ satisfies the special Somos-7 relation*

$$\tau_{n+7}\tau_n = a\,\tau_{n+6}\tau_{n+1} + b\,\tau_{n+4}\tau_{n+3}, \tag{B.12}$$

*with constant coefficients $a, b$. Then the quantity*

$$\xi_n = \frac{\tau_{n+5}\tau_n + a\,\tau_{n+3}\tau_{n+2}}{\tau_{n+4}\tau_{n+1}} \tag{B.13}$$

*is periodic with period 3, that is $\xi_{n+3} = \xi_n$ for all n.*

**Proof.** We begin by defining the sequence of ratios

$$f_n = \frac{\tau_{n+2}\tau_n}{\tau_{n+1}^2}. \tag{B.14}$$

In terms of $f_n$, the Somos-7 recurrence is equivalent to the relation

$$f_{n+5}f_{n+4}^2f_{n+3}^3f_{n+2}^3f_{n+1}^2f_n = a f_{n+4}f_{n+3}^2f_{n+2}^2f_{n+1} + b. \tag{B.15}$$

[6] See www.lmfdb.org/EllipticCurve/Q/794/a/1 for more details.

Upon shifting $n \to n+1$ and subtracting (or equivalently, applying the total difference operator to the constant $b$), we find

$$f_{n+6}f_{n+5}^2f_{n+4}^3f_{n+3}^3f_{n+2}^2f_{n+1} - f_{n+5}f_{n+4}^2f_{n+3}^3f_{n+2}^3f_{n+1}^2f_n \\ = a\left(f_{n+5}f_{n+4}^2f_{n+3}^2f_{n+2} - f_{n+4}f_{n+3}^2f_{n+2}^2f_{n+1}\right),$$

and then after dividing by $f_{n+5}f_{n+4}^2f_{n+3}^2f_{n+2}^2f_{n+1}$ and rearranging, this becomes

$$f_{n+6}f_{n+5}f_{n+4}f_{n+3} + \frac{a}{f_{n+5}f_{n+4}} = f_{n+3}f_{n+2}f_{n+1}f_n + \frac{a}{f_{n+2}f_{n+1}}.$$

If we substitute for $f_n$ with the formula (B.14), then the above equality says precisely that

$$\xi_{n+3} = \xi_n,$$

which is the required result. □

The recurrence (B.15) is equivalent to iteration of a map in 5 dimensions, that is

$$\Psi: \quad (f_0, f_1, f_2, f_3, f_4) \mapsto (f_1, f_2, f_3, f_4, f_5),$$

which corresponds to a lift of the Lyness map (1.2) lying between it and Somos-7. The proof of the preceding result shows that, when considered as a function on the 5D phase space, the quantity

$$\xi_0 = f_3f_2f_1f_0 + \frac{a}{f_2f_1},$$

is a three-invariant for the map $\Psi$, in the sense that $(\Psi^*)^3(\xi_0) = \xi_0$, and by viewing Somos-7 as the 7D map defined by (B.12), $\xi_0$ lifts via (B.14) to a three-invariant for this as well.

Note that, since the three quantities $\xi_0, \xi_1, \xi_2$ are functionally independent, the three elementary symmetric functions

$$\xi_0 + \xi_1 + \xi_2, \qquad \xi_0\xi_1 + \xi_1\xi_2 + \xi_2\xi_0, \qquad \xi_0\xi_1\xi_2 \tag{B.16}$$

provide three invariant functions that are also functionally independent; so they constitute three first integrals, both for $\Psi$ and for Somos-7. This is a particular case of the fact that the general Somos-7 relation, with an extra term proportional to $\tau_{n+5}\tau_{n+2}$ included on the right-hand side, has three functionally independent invariants, as was shown by a different method in [11]. It was also remarked there that only one of these three first integrals survives reduction to the plane with coordinates $(u_0, u_1)$ where the Lyness map

$$\hat{\varphi}_L: \qquad \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ (au_1 + b)/u_0 \end{pmatrix}$$

is defined. To be precise, if we define the map $\pi: \mathbb{C}^7 \to \mathbb{C}^2$ by

$$\pi: \qquad u_0 = \frac{\tau_0\tau_5}{\tau_2\tau_3}, \qquad u_1 = \frac{\tau_1\tau_6}{\tau_3\tau_4}, \tag{B.17}$$

then the intermediate lift corresponds to $\tilde{\pi}: \mathbb{C}^5 \to \mathbb{C}^2$, given by

$$u_0 = f_0(f_1f_2)^2f_3, \qquad u_1 = f_1(f_2f_3)^2f_4,$$

and the coordinates $f_j$ make it easy to verify that the only surviving member of the invariant set (B.16) is the product

$$\xi_0\xi_1\xi_2 = a\tilde{K} + b, \tag{B.18}$$

where

$$\tilde{K} = \frac{u_0u_1\,(u_0+u_1) + (u_0+u_1)^2 + \left(a^2+b\right)(u_0+u_1) + ab}{u_0u_1} \tag{B.19}$$

is the form of the first integral for the Lyness map $\hat{\varphi}_L$ that is given in [19]. (As usual, we are abusing notation slightly when we identify the left- and right-hand sides of (B.18): we should really replace $\tilde{K}$ with $\pi^*(\tilde{K})$ or $\tilde{\pi}^*(\tilde{K})$, depending on whether we wish to regard the $\xi_j$ as functions of the quantities $\tau_n$ or $f_n$.)

**Proposition B.2.** *Suppose that the sequence $(u_n)$ satisfies the Lyness map, in the form of the recurrence*

$$u_{n+1}u_{n-1} = au_n + b.$$

*Then the quantity $v_n = u_n + a$ gives a solution of the Somos-5 QRT map, in the form*

$$v_{n+1}v_nv_{n-1} = \hat{\alpha}v_n + \hat{\beta}, \qquad \text{with} \quad \hat{\alpha} = a\tilde{K} + b, \quad \hat{\beta} = -a\hat{\alpha}, \tag{B.20}$$

*where $\tilde{K}$ is the first integral associated with the Lyness map, given by (B.19).*

**Proof.** In terms of a tau function $\tau_n$ satisfying (B.12), we may write

$$u_n = \frac{\tau_{n+5}\tau_n}{\tau_{n+3}\tau_{n+2}} \implies v_n = u_n + a = \frac{\tau_{n+5}\tau_n + a\,\tau_{n+3}\tau_{n+2}}{\tau_{n+3}\tau_{n+2}} = \xi_n\,\frac{\tau_{n+4}\tau_{n+1}}{\tau_{n+3}\tau_{n+2}}$$

from (B.13). Hence, by multiplying out and cancelling the terms that appear in the product of three adjacent $v_n$, we find

$$v_{n+1}v_nv_{n-1} = \xi_{n+1}\xi_n\xi_{n-1}\,\frac{\tau_{n+5}\tau_n}{\tau_{n+3}\tau_{n+2}} = \left(a\tilde{K}+b\right)u_n,$$

where we have used (B.18). If we set $\hat{\alpha} = a\tilde{K} + b$ and replace $u_n$ by $v_n - a$, then we obtain the Somos-5 QRT map in the form (B.20), as required. □

The preceding result shows how the Somos-5 QRT map in theorem 4.3 appears as a consequence of the Lyness map in theorem 4.2, and also explains the connection between the corresponding instances of these maps appearing in theorems 4.6 and 5.8. We now proceed to present some additional results about the solutions of the special Somos-7 recurrence.

**Theorem B.3.** *The general solution of the initial value problem for (B.12), with generic (non-zero) values of the parameters $a, b$ and initial data $\tau_j$, $0 \leqslant j \leqslant 6$, has the form*

$$\tau_n = \exp\left(c_1 + c_2\,n + c_3\,(-1)^n + c_4\,e^{2n\pi\mathrm{i}/3} + c_5\,e^{-2n\pi\mathrm{i}/3}\right)\frac{\sigma(z_0+nz)}{\sigma(z)^{n^2}}, \tag{B.21}$$

*where $\sigma(\cdot) = \sigma(\cdot; g_2, g_3)$ is the Weierstrass sigma function associated with the elliptic curve*

$$E: \qquad y^2 = 4x^3 - g_2x - g_3. \tag{B.22}$$

**Proof.** The proof of this result goes along very similar lines to the corresponding result for Somos-5 in [15], so we will sketch the main details and leave it for the reader to fill in the rest. The first part of the proof requires verifying that the analytic formula does indeed satisfy (B.12), viewed as a difference equation in $n$, while the second part is to show that it gives the general solution, in the sense that for a generic set of coefficients $a, b$ and initial values $\tau_j$, it is possible to find constants $c_j$ and $z_0, z, g_2, g_3$ that solve the corresponding initial value problem. For the first part, note that the set of gauge transformations $\tau_n \to \mathrm{g}_n \tau_n$, which leave invariant both (B.12) and the image $(u_0, u_1)$ of the map $\pi$ in (B.17), consists of an algebraic torus $(\mathbb{C}^*)^5$ defined by the equation $\mathrm{g}_n \mathrm{g}_{n+5} (\mathrm{g}_{n+2} \mathrm{g}_{n+3})^{-1} = 1$, which implies

$$\left(\mathcal{T}^5 - \mathcal{T}^3 - \mathcal{T}^2 + 1\right) \log \mathrm{g}_n = 0,$$

and the solution of the latter equation is precisely the exponential prefactor in (B.21) with arbitrary constants $c_j$, $1 \leqslant j \leqslant 5$. Therefore, to verify the form of the analytic solution, it remains only to check the part involving $\sigma$. This can be done using standard results on Weierstrass functions, and in particular the three-term relation for the sigma function, which shows that (B.21) is a solution of (B.12) if and only if the coefficients are parametrized by $z, g_2, g_3$ according to the formulae

$$a = \frac{\sigma(4z)}{\sigma(2z)\,\sigma(z)^{12}}, \qquad b = -\frac{\sigma(6z)}{\sigma(3z)\,\sigma(2z)\,\sigma(z)^{23}}. \tag{B.23}$$

Now for the second part, begin by observing that the total count of coefficients plus initial values is 9, which is the same as the count of arbitrary constants $c_j$ plus parameters $z_0, z, g_2, g_3$, so it is necessary to account for how these constants/parameters are determined from the initial value problem. Ignoring the non-gauge part of the formula, the four parameters $z_0, z, g_2, g_3$ come from the solution of the initial value problem for the iterates of the Lyness map for given $a, b$ and initial data $(u_0, u_1)$, which has the form

$$(\hat{\varphi}_L)^n (u_0, u_1) = (u_n, u_{n+1}), \qquad \text{with} \qquad u_n = \frac{\sigma(z_0 + nz)\,\sigma(z_0 + (n+5)z)}{\sigma(z_0 + (n+2)z)\,\sigma(z_0 + (n+3)z)\,\sigma(z)^{12}}.$$

Then the solution to the algebraic problem of reconstructing a Weierstrass cubic curve $E$ with invariants $g_2, g_3$ together with two points $P_0 = (\wp(z_0), \wp'(z_0))$, $P = (\wp(z), \wp'(z)) \in E$, starting from an initial point $(u_0, u_1)$ on a fixed level curve of the first integral (B.19) for the Lyness map, is presented explicitly in [19], Theorem 1, up to rescaling the $y$ coordinate by a factor of 2. (Compared with (B.22), the Weierstrass curve in [19] is written as $y^2 = x^3 + Ax + B$.) For generic values of $a, b, \tilde{K}$, the affine biquadratic equation

$$u_0 u_1 (u_0 + u_1) + (u_0 + u_1)^2 + \left(a^2 + b\right)(u_0 + u_1) + ab - \tilde{K} u_0 u_1 = 0$$

defines a smooth curve of genus 1 in $\mathbb{P}^1 \times \mathbb{P}^1$, which is birationally equivalent to a Weierstrass cubic defined by $g_2, g_3$, with non-vanishing discriminant $g_2^3 - 27 g_3^2 \neq 0$. Given the solution to the purely algebraic problem of determining the elliptic curve $E$ and the $(x, y)$ coordinates of the pair of points $P_0, P$, such that the orbit of the Lyness map corresponds to the sequence $P_0 + nP \in E$, it is necessary to determine the associated pair of points on the Jacobian of the curve, that is the values $z_0, z \in \mathrm{Jac}(E) = \mathbb{C}/\Lambda \cong E$, which are found by evaluating the elliptic integrals

$$z_0 = \int_\infty^{P_0} \frac{\mathrm{d}x}{y}, \qquad z = \int_\infty^{P} \frac{\mathrm{d}x}{y},$$

taken modulo the lattice of periods $\Lambda$. Finally, once $z_0, z, g_2, g_3$ have been fixed, the system of 5 linear equations

$$c_1 + c_2 n + c_3 (-1)^n + c_4 e^{2n\pi \mathrm{i}/3} + c_5 e^{-2n\pi \mathrm{i}/3} = \log\left(\frac{\tau_n \sigma(z)^{n^2}}{\sigma(z_0 + nz)}\right), \quad n = 0,1,2,3,4$$

allows the values of the constants $c_j$ appearing in the gauge factor to be determined from the initial values. □

In order to discuss Somos-$k$ relations of higher order that are satisfied by solutions of (B.12), it is convenient to introduce the sequence of division polynomials $(\mathrm{a}_n)$ associated with an elliptic curve (B.22), whose terms correspond to the multiples $nP \in E$ (see [15] and references therein). Given the point $P = (\wp(z), \wp'(z))$ parametrized by $z \in \mathrm{Jac}(E)$, these can be defined analytically by the formula

$$\mathrm{a}_n = \frac{\sigma(nz)}{\sigma(z)^{n^2}}. \tag{B.24}$$

From the definition, the sequence is clearly antisymmetric, in the sense that $\mathrm{a}_n = -\mathrm{a}_{-n}$ for all $n$. This sequence is another solution of the recurrence (B.12), corresponding to the same value of the first integral $\tilde{K}$ as for the solution (B.21), but with different initial conditions; we also refer to $(\mathrm{a}_n)$ as the elliptic divisibility sequence associated with (B.21). In particular, when the parameters and initial conditions are chosen suitably, then $(\mathrm{a}_n)$ consists entirely of integers which satisfy the divisibility property $\mathrm{a}_n | \mathrm{a}_m$ whenever $n|m$: this is the usual meaning of the name elliptic divisibility sequence (EDS).

In general, the terms of the EDS associated with a solution of (B.12) are determined completely as algebraic functions of $a, b, \tilde{K}$. Note that we have $\mathrm{a}_0 = 0, \mathrm{a}_1 = 1$, and the first few terms are specified by

$$\begin{aligned}
&(\mathrm{a}_2)^4 = \tilde{K} + a, \quad (\mathrm{a}_3)^3 = a\tilde{K} + b, \quad \mathrm{a}_4 = \mathrm{a}_2 a, \quad \mathrm{a}_5 = a^2 - b, \\
&\mathrm{a}_6 = -\mathrm{a}_2\mathrm{a}_3 b, \quad \mathrm{a}_7 = -ab\tilde{K} - a^4 + a^2 b - b^2, \\
&\mathrm{a}_8 = -\mathrm{a}_2 a\left(ab\tilde{K} + a^4 - 2a^b + 2b^2\right), \quad \mathrm{a}_9 = -\mathrm{a}_3\left(a^3 b\tilde{K} + a^6 - 2a^4 b + 3a^2 b^2 - b^3\right).
\end{aligned}$$

The terms of the EDS satisfy a Somos-$k$ relation for each $k \geqslant 4$, which means that they can be considered as polynomials in $\mathrm{a}_2, \mathrm{a}_3, a, b, \tilde{K}$ with the appropriate divisibility property, namely

$$\mathrm{a}_n \in \mathcal{R}^* := \mathbb{Z}\left[\mathrm{a}_2, \mathrm{a}_3, a, b, \tilde{K}\right] / \sim, \qquad \text{with} \quad n|m \implies \mathrm{a}_n|\mathrm{a}_m \quad \text{in} \quad \mathcal{R}^*,$$

where $\sim$ denotes the equivalence relation defined by the algebraic identities $(\mathrm{a}_2)^4 = \tilde{K} + a$, $(\mathrm{a}_3)^3 = a\tilde{K} + b$. Now from the analytic definition (B.24) and the formula (B.21), we can use the three-term relation for the sigma function to derive infinitely many higher Somos relations for $\tau_n$, which must be of odd order $k = 2j + 1$, but with the further requirement that $k$ cannot be a multiple of 3, since relations of this order are the only ones that are invariant under the gauge transformations $\mathrm{g}_n$ described in the proof of theorem B.2. In this way, we arrive at infinitely many relations that can be written concisely in the following form:

$$\mathrm{a}_3\mathrm{a}_4\tau_{n+j+1}\tau_{n-j} = \begin{vmatrix} \mathrm{a}_{j+1}\tau_{n+4} & \mathrm{a}_{j-3}\tau_n \\ \mathrm{a}_{j+4}\tau_{n+1} & \mathrm{a}_j\tau_{n-3} \end{vmatrix}. \tag{B.25}$$

However, the above relation can be modified by making use of the Somos-7 recurrence (B.12) to replace the product $\tau_{n+4}\tau_{n-3}$ on the right-hand side of (B.25), and further simplified by using the fact that the terms $\mathrm{a}_j$ of the associated EDS satisfy the same Somos-7 relation. Thus we arrive at

**Corollary B.4.** *For all integers $k \geqslant 7$ that are odd and not a multiple of 3, every solution of (B.12) satisfies a Somos-k relation with constant coefficients, given by*

$$\tau_{n+j+1}\tau_{n-j} = \alpha_j\,\tau_{n+3}\tau_{n-2} + \beta_j\,\tau_{n+1}\tau_n, \tag{B.26}$$

*with $k = 2j+1$ and $j \not\equiv 1 (\bmod 3)$, where*

$$\alpha_j = \frac{\mathrm{a}_{j+1}\mathrm{a}_j}{\mathrm{a}_3\mathrm{a}_2}, \qquad \beta_j = -\frac{\mathrm{a}_{j+3}\mathrm{a}_{j-2}}{\mathrm{a}_3\mathrm{a}_2}. \tag{B.27}$$

We can now conclude this appendix with a strong version of the Laurent property for (B.12), which was an ingredient in the proof of theorem 4.4.

**Theorem B.5.** *The iterates of the special Somos-7 recurrence (B.12) possess the strong Laurent property, in the sense that $\tau_n \in \tilde{\mathcal{R}}$ for all $n \geqslant 0$, where*

$$\tilde{\mathcal{R}} = \mathbb{Z}\left[a, b, \tilde{K}, \tau_0^{\pm 1}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3^{\pm 1}, \tau_4, \tau_5, \tau_6\right].$$

**Proof.** The proof follows very closely the approach used in [16] to show analogous results for Somos-4 and Somos-5. To begin with, it is clear that $\tau_j \in \tilde{\mathcal{R}}$ for $0 \leqslant j \leqslant 10$, taking the 7 initial data and applying (B.12) four times, which requires division by $\tau_0, \tau_1, \tau_2, \tau_3$. Thereafter, the higher relations (Somos-11, Somos-13, etc) can be used to calculate $\tau_j$ for $j \geqslant 11$, setting $n = j, j+1, j+2$ in (B.26) with $j \geqslant 5$ (for $j \not\equiv 1 (\bmod 3)$), so that only divisions by $\tau_0, \tau_1, \tau_2$ are necessary, and all $\tau_n$ appearing on the right-hand side are previously determined elements of $\tilde{\mathcal{R}}$. The result then follows by induction, once it has been shown that the coefficients $\alpha_j, \beta_j \in \mathbb{Z}[a, b, \tilde{K}]$. To see this, it is enough to verify from the recursive relations satisfied by the EDS that each $\mathrm{a}_n$ is equal to a prefactor times an element of $\mathbb{Z}[a, b, \tilde{K}]$, where the prefactor simply repeats the pattern $1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_2, 1, \mathrm{a}_2\mathrm{a}_3$ with period 6, and hence for each $j \not\equiv 1 (\bmod 3)$ the coefficients in (B.27) are of the desired form. □

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