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ORIGINAL RESEARCH

Construction of copulas for bivariate failure rates

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Abstract
This paper aims to develop a method to construct an asymmetric copula, based on which a closed form of the cumulative bivariate failure rate can be obtained. The construction method differs from existing ones. This new method can facilitate the derivation of some results such as the estimation of the expected number of occurrences for a system whose failure process is modelled by a bivariate stochastic process or the expected cost in optimisation of maintenance policies.

Keywords Dependence · Copula · Asymmetric copula · Bivariate failure rate

1 Introduction

1.1 Motivation

In reliability engineering, the concept of the failure rate plays a key role. For the one-dimensional (1-D) scenarios, the calculation of the failure rate (or cumulative failure rate) is straightforward and the failure rate normally has a simple expression. For the two-dimensional (2-D) scenarios, however, the expression of the bivariate failure rate can be too complicated to be used in applications such as estimation of the amount of warranty claims. It is noted that the bivariate reliability and therefore bivariate failure rates are widely used in many real applications. For example, in the automotive industry, vehicles are sold with 2-D warranty, which defines that vehicle users can claim warranty within a time limit and a cumulative usage limit. Researchers and practitioners need closed-forms of bivariate failure rates in development of reliability estimation, warranty policy optimisation, or budget planning (Wu, 2012; Wang & Zhang, 2011; Dai et al., 2021; Shang et al., 2022).

As discussed in Wu (2014, 2024), when considering two dimensions (i.e., age and cumulative usage) in warranty claims analysis, a better approach for modelling the two dimensions is to use an asymmetric copula as *items with large age implies large cumulative usage whereas items with large cumulative usage do not mean large age,* which is due to the fact that some items may not be used often.

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The expression of an asymmetric copula can become more complicated than that of a symmetric copula. As a result, the cumulative bivariate failure rate of an asymmetric copula may not have a closed form, which can hinder its applications. This necessitates an exploration of the way to select bivariate copulas and to construct asymmetric copulas with which the corresponding bivariate failure rates can have closed forms. This paper serves for this purpose.

1.2 Prior work

In an attempt to show that there does not exist an absolutely continuous bivariate exponential distribution with constant bivariate failure rate and with the marginals being dependently distributed, Basu (1971) proposed a definition of bivariate failure rates. Johnson and Kotz (1975) defined the bivariate failure rate as a vector, which is also referred to as the hazard gradient. Navarro (2008) obtained a general procedure to characterize bivariate absolutely continuous distributions by using the bivariate failure (hazard) rate function. Barbiero (2022a) proposed two methods for deriving a bivariate discrete probability distribution from a continuous one by retaining some specific features of the original stochastic model. Barbiero (2022b) investigated (1) pseudo-random simulation, (2) attainable Pearson's correlations, (3) stress-strength reliability parameter, and (4) parameter estimation for a bivariate discrete probability distribution, considering the associated failure rate function. Kayid (2022) derived two characterizations of the weak bivariate failure rate order over the bivariate Laplace transform order of two-dimensional residual lifetimes.

1.3 Challenges and our proposed methods

The existing work has the following three drawbacks:

- Suppose there is a bivariate joint distribution $F(x, y; \boldsymbol{\theta})(= P(X < x, Y < y))$ and the marginal distributions of X and Y are $F_X(x; \boldsymbol{\theta}_1)$ and $F_Y(y; \boldsymbol{\theta}_2)$, respectively. The relationship between $P(X < x, Y < y)$ and $P(X \geq x, Y \geq y)(= \bar{F}(x, y))$ is $\bar{F}(x, y) = 1 - F_X(x; \boldsymbol{\theta}_1) - F_Y(y; \boldsymbol{\theta}_2) + F(x, y; \boldsymbol{\theta})$. Basu (1971) defined the bivariate failure rate as $r(x, y) = \frac{f(x,y;\boldsymbol{\theta})}{\bar{F}(x,y)}$, where $f(x, y; \boldsymbol{\theta})$ is the bivariate density function. Nevertheless, some papers mistakenly derive bivariate failure rates based on the relationship $\bar{F}(x, y; \boldsymbol{\theta}) = 1 - F(x, y; \boldsymbol{\theta})$ (see Jack et al. (2009), for example).
- Most research assumes a simple expression of a bivariate failure rate (see Navarro (2008), for example). Nevertheless, in many other applications, an associated bivariate failure rate function is derived from a bivariate joint probability distribution, which is estimated based on the time-to-occurrence data. As such, assuming a simple expression of a bivariate failure rate may result in a different bivariate joint probability distribution as the one derived from the real observations.
- Many failure rate functions such as the power law function are used as the failure intensity functions, with which maintenance policies are optimised. There is often a need to obtain a closed form of the expression of the cumulative bivariate failure intensity functions.

A challenge in modelling 2-D failures is that the expression of a bivariate failure rate can be very complex, which may confine its applications. This paper therefore proposes a method to construct asymmetric copulas, which can facilitate the use of the bivariate failure rate in applications.

1.4 Overview

The structure of the remainder of this paper is as follows. Section 2 introduces the challenges that are identified and proposes a method to overcome the challenges. Section 3 constructs an asymmetric copula for a special case: the Gumbel Barnett copula. Section 4 discusses another approach to constructing an asymmetric copula and discusses a limitation of the proposed approaches. Section 5 concludes this paper.

2 Bivariate failure rates

This section introduces the definitions of the bivariate failure rate, proposed by Basu (1971); Johnson and Kotz (1975), respectively.

2.1 Definitions of bivariate failure rates

Johnson and Kotz (1975) defined the bivariate failure rate as a vector, as shown below

$$r_{JK}(x, y; \boldsymbol{\theta}) = (r_1(x, y; \boldsymbol{\theta}), r_2(x, y; \boldsymbol{\theta})), \tag{1}$$

which is also called a *hazard gradient*, where

$$r_1(x, y; \boldsymbol{\theta}) = -\frac{\partial}{\partial x} \log \bar{F}(x, y), \tag{2}$$

and

$$r_2(x, y; \boldsymbol{\theta}) = -\frac{\partial}{\partial y} \log \bar{F}(x, y). \tag{3}$$

$r_1(x, y; \boldsymbol{\theta})$ and $r_2(x, y; \boldsymbol{\theta})$ defined above are referred to as JK's bivariate failure rate in this paper.

It is noted that for a bivariate probability distribution, the relationship between $\bar{F}(x, y)$ and $F(x, y; \boldsymbol{\theta})$ is not the same as that in the univariate scenario, that is, $P(X > x, Y > y) = \bar{F}(x, y) \neq 1 - F(x, y; \boldsymbol{\theta})$, instead, it is shown below

$$\bar{F}(x, y) = 1 - F_1(x) - F_2(y) + F(x, y; \boldsymbol{\theta}). \tag{4}$$

Given a bivariate cumulative distribution function $F(x, y; \boldsymbol{\theta})$ and its associated density function $f(x, y; \boldsymbol{\theta})$, Basu (1971) defined the corresponding bivariate hazard rate of $F(x, y; \boldsymbol{\theta})$ as

$$r(x, y; \boldsymbol{\theta}) = \lim_{\substack{\Delta_1 \to 0 \\ \Delta_2 \to 0}} \frac{P\{x \leq X < x + \Delta_1, y \leq Y < y + \Delta_2 | x \leq X, y \leq Y\}}{\Delta_1 \Delta_2}, \tag{5}$$

or equivalently:

$$r(x, y; \boldsymbol{\theta}) = \frac{f(x, y; \boldsymbol{\theta})}{\bar{F}(x, y)}, \tag{6}$$

where $F(x, y; \boldsymbol{\theta})$ is the joint distribution function of random variables X and Y, and $f(x, y; \boldsymbol{\theta})$ is the joint probability density function of X and Y. $\boldsymbol{\theta}$ is the set of parameters in the copula and the bivariate function $F(x, y; \boldsymbol{\theta})$.

$r(x, y; \boldsymbol{\theta})$ defined in Eq. (6) is referred to as Basu's bivariate failure rate in this paper.

2.2 Copula-based bivariate failure rates

A copula is a tool for defining joint probability distributions and is defined as following.

Definition 1 Denote [0, 1] by $\mathbf{I}$. A bivariate copula is a function $C(\omega_1, \omega_2; \boldsymbol{\theta})$: $\mathbf{I}^2 \rightarrow \mathbf{I}$ with the following properties:

(a) **(Grounded)** $C(\omega_1, \omega_2; \boldsymbol{\theta}) = 0$, if $\omega_k = 0$ where $k = 1, 2$.
(b) **(Consistence with margins)** $C(\omega_1, 1; \boldsymbol{\theta}) = \omega_1$ and $C(1, \omega_2; \boldsymbol{\theta}) = \omega_2$.
(c) **(Rectangle inequality)** $C(u_1, u_2) - C(u_1, \omega_2) - C(\omega_1, u_2) + C(\omega_1, \omega_2) \geq 0$ for all $0 \leq \omega_1 \leq u_1 \leq 1$ and $0 \leq \omega_2 \leq u_2 \leq 1$.

Assume that $\frac{\partial^2 C(\omega_1,\omega_2;\boldsymbol{\theta})}{\partial\omega_1\partial\omega_2}$ exists, then Condition (c) can also be expressed as following:

(c') $\frac{\partial^2 C(\omega_1,\omega_2;\boldsymbol{\theta})}{\partial\omega_1\partial\omega_2} > 0$.

Hence, for a given copula $C(\omega_1, \omega_2; \boldsymbol{\theta})$, one can obtain its associated failure rate function as follows.

Denote $c(\omega_1, \omega_2; \boldsymbol{\theta}) = \frac{\partial^2 C(\omega_1,\omega_2;\theta)}{\partial\omega_1\partial\omega_2}$, then

$$\begin{aligned} c(\omega_1, \omega_2; \boldsymbol{\theta}) &= \frac{\partial^2 F(F_1^{-1}(\omega_1), F_2^{-1}(\omega_2))}{\partial\omega_1\partial\omega_2} \\ &= \frac{f(F_1^{-1}(\omega_1), F_2^{-1}(\omega_2))}{f_1(F_1^{-1}(\omega_1)) f_2(F_2^{-1}(\omega_2))} \\ &= \frac{f(x, y; \boldsymbol{\theta})}{f_1(x) f_2(y)}, \end{aligned} \tag{7}$$

where $x = F_1^{-1}(\omega_1)$, $y = F_2^{-1}(\omega_2)$, $f(x, y; \boldsymbol{\theta}) = \partial^2 F(x, y; \boldsymbol{\theta})/\partial x \partial y$, $f_1(x) = \int_0^\infty f(x, y; \boldsymbol{\theta}) dy$, and $f_2(y) = \int_0^\infty f(x, y; \boldsymbol{\theta}) dx$.

Basu's bivariate failure rate function can be defined with the copula as follows:

$$\begin{aligned} r(x, y; \boldsymbol{\theta}) =& \frac{f(x, y; \boldsymbol{\theta})}{\bar{F}(x, y)} \\ =& \frac{c(\omega_1, \omega_2; \boldsymbol{\theta}) f_1(F_1^{-1}(\omega_1)) f_2(F_2^{-1}(\omega_2))}{1 - \omega_1 - \omega_2 + C(\omega_1, \omega_2; \boldsymbol{\theta})}. \end{aligned} \tag{8}$$

For most copulas, the expression $r(x, y; \boldsymbol{\theta})$ is very complicated. When $r(x, y; \boldsymbol{\theta})$ is used for further derivations, for instance, if $\Lambda(x_0, y_0) = \int_0^{x_0} \int_0^{y_0} r(x, y; \boldsymbol{\theta}) dx dy$ needs to be calculated, a closed-form expression of $\Lambda(x_0, y_0)$ may not be obtained. An intuitive idea is to choose a copula with the following form,

$$C(\omega_1, \omega_2; \boldsymbol{\theta}) = \omega_1 + \omega_2 - 1 + g(\omega_1, \omega_2; \boldsymbol{\theta}), \tag{9}$$

$C(\omega_1, \omega_2; \boldsymbol{\theta})$ in Eq. (9) is the bivariate survival copula of $g(\omega_1, \omega_2; \boldsymbol{\theta})$. With Eq. (9), one hopes to eliminate the part $1 - \omega_1 - \omega_2$ in the denominator of $r(x, y; \boldsymbol{\theta})$ so that the expression of $r(x, y; \boldsymbol{\theta})$ can be simplified.

With Definition 1, we can obtain Lemma 1.

Lemma 1 *$g(\omega_1, \omega_2; \boldsymbol{\theta})$ should satisfy the following conditions*

- **Condition A:** $g(\omega_1, 0; \boldsymbol{\theta}) = 1 - \omega_1$ *and* $g(0, \omega_2; \boldsymbol{\theta}) = 1 - \omega_2$,
- **Condition B:** $g(\omega_1, 1; \boldsymbol{\theta}) = g(1, \omega_2; \boldsymbol{\theta}) = 0$,

- **Condition C:** $\frac{\partial^2 g(\omega_1,\omega_2;\boldsymbol{\theta})}{\partial\omega_1\partial\omega_2} > 0$.

In addition to the three conditions in Lemma 1, we also hope that the bivariate failure rate has a closed form. Nevertheless, in the literature, there does not exist a widely accepted and rigorous definition of a "closed form", although some definitions are given (see Chow (1999); Borwein and Crandall (2013), for example). In our case, we hope that $\Lambda(x_0, y_0) = \int_0^{x_0}\int_0^{y_0} r(x, y; \boldsymbol{\theta})dxdy$ (with $x_0, y_0 \in (0, +\infty)$), has a closed-form. To this end, we impose that $g(\omega_1, \omega_2; \boldsymbol{\theta})$ should also satisfy the following condition:

- **Condition D:** $\int_0^{F_1(x_0)}\int_0^{F_2(y_0)} \frac{f_1(F_1^{-1}(\omega_1))f_2F_2^{-1}(\omega_2))\frac{\partial^2 g(\omega_1,\omega_2;\boldsymbol{\theta})}{\partial\omega_1\partial\omega_2}}{g(\omega_1, \omega_2; \boldsymbol{\theta})}d\omega_1 d\omega_2$ has a closed-form expression.

In some practical applications, some probability distributions are not symmetric. For example, the relationship between age and accumulated usage should be modelled by an asymmetric copula (Wu, 2014).

We say a copula lacks permutation symmetry (Joe (2014), page 65) if

$$C(\omega_1, \omega_2; \boldsymbol{\theta}) \neq C(\omega_2, \omega_1; \boldsymbol{\theta}).$$

Unfortunately, using existing methods for constructing asymmetric copulas (see Liebscher (2008); Wu (2014); Mukherjee et al. (2018)) do not provide a better method, with which an elegant expression of $r(x, y; \boldsymbol{\theta})$ can be obtained. We therefore propose a new method to construct asymmetric copulas, as shown in the following section.

It is noted that the bivariate failure rate $r(x, y)$, in general, does not necessarily determine its associated $\bar{F}(x, y)$ uniquely (Yang & Nachlas, 2001; Finkelstein, 2003), but it determines $\bar{F}(x, y)$ under some reasonable assumptions (Navarro, 2008). As such, one normally determines the bivariate joint distribution, based on which the failure rate can be provided. This requires one to develop new approaches to simplifying the expression of the failure rate, which motivates the development of this paper.

Example 1 Assume $C(\omega_1, \omega_2; \boldsymbol{\theta})$ is the Clayton copula, which is a commonly used copula in the reliability engineering. Let $\omega_1 = 1 - e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}}$ and $\omega_2 = 1 - e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}}$, which suggests that the marginal distribution is the Weibull distribution. Then, $C(\omega_1, \omega_2; \boldsymbol{\theta}) = \left[\omega_1^{-\theta} + \omega_2^{-\theta} - 1\right]^{-1/\theta}$. If we plug $C(\omega_1, \omega_2; \boldsymbol{\theta})$ into Eq. (8), then the denominator of Eq. (8) becomes

$$\begin{aligned} &1 - \omega_1 - \omega_2 + C(\omega_1, \omega_2; \boldsymbol{\theta}) \\ &\quad = e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}} + e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}} - 1 + \left\{\left[1 - e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}}\right]^{-\theta} + \left[1 - e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}}\right]^{-\theta} - 1\right\}^{-\frac{1}{\theta}} \end{aligned} \tag{10}$$

and

$$\begin{aligned} c(\omega_1, \omega_2; \boldsymbol{\theta}) =&(\theta + 1)(\omega_1\omega_2)^{-(\theta+1)}(\omega_1^{-\theta} + \omega_2^{-\theta} - 1)^{\frac{2\theta+1}{\theta}} \\ =&(\theta + 1)\left[\left(1 - e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}}\right)\left(1 - e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}}\right)\right]^{\theta+1} \times \end{aligned}$$

$$\left[\left(1-e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}}\right)^{-\theta}+\left(1-e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}}\right)^{-\theta}-1\right]^{\frac{2\theta+1}{\theta}}. \tag{11}$$

Then the Basu failure rate is given by

$$r(x,y;\boldsymbol{\theta}) = \frac{\frac{1}{\alpha_1\alpha_2}\left(\frac{x}{\alpha_1}\right)^{\beta_1-1}\left(\frac{y}{\alpha_2}\right)^{\beta_2-1}e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}-\left(\frac{y}{\alpha_2}\right)^{\beta_2}}\left[\left(1-e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}}\right)^{-\theta}+\left(1-e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}}\right)^{-\theta}-1\right]^{\frac{2\theta+1}{\theta}}}{e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}}+e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}}-1+\left\{\left[1-e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}}\right]^{-\theta}+\left[1-e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}}\right]^{-\theta}-1\right\}^{-\frac{1}{\theta}}}. \tag{12}$$

The above denominator cannot be eliminated in the bivariate failure rate after it is plugged into Eq. (8). The numerator is more complex than the denominator. As a result, if the failure rate $r(x, y; \boldsymbol{\theta})$ is used as the failure intensity function of a bivariate stochastic process, then its cumulative failure intensity, which is calculated by $\Gamma(x_0, y_0) = \int_0^{x_0}\int_0^{y_0} r(x, y; \boldsymbol{\theta})dxdy$, does not have a closed-form.

Proposition 1 *If* $C(\omega_1, \omega_2; \boldsymbol{\theta}) \neq C(\omega_2, \omega_1; \boldsymbol{\theta})$ *for* $\omega_1 \neq \omega_2$, *then* $r(x, y; \boldsymbol{\theta}) \neq r(x, y; \boldsymbol{\theta})$ *for* $\omega_1 \neq \omega_2$.

Proof It is noted that

$$\begin{aligned} C(\omega_1,\omega_2;\boldsymbol{\theta}) =& P(X<x, Y<y) \\ =& P(F_1(X)<F_1(x), F_2(Y)<F_2(y)) \\ =& P(F_1(X)<\omega_1, F_2(Y)<\omega_2), \end{aligned} \tag{13}$$

we therefore obtain

$$\begin{aligned} r(x,y;\boldsymbol{\theta}) =& \lim_{\substack{\Delta_1\to 0\\ \Delta_2\to 0}} \frac{P\{x\le X<x+\Delta_1, y\le Y<y+\Delta_2|X\ge x, Y\le y\}}{\Delta_1\Delta_2} \\ \neq& \lim_{\substack{\Delta_1\to 0\\ \Delta_2\to 0}} \frac{P\{y\le Y<y+\Delta_2, x\le X<x+\Delta_1|X\ge x, Y\ge y\}}{\Delta_1\Delta_2} \\ =& r(u,x;\boldsymbol{\theta}). \end{aligned} \tag{14}$$

This completes the proof. □

Proposition 1 suggests that the bivariate failure rate is asymmetric if its associated copula is asymmetric. This is useful even when one constructs an asymmetric failure intensity function for repairable systems.

3 The case of the Gumbel-Barnett copula

Intuitively, one may suggest the Gumbel-Barnett copula due to Gumbel (1960) and Barnett (1980), which is defined by

$$C(\omega_1, \omega_2; \boldsymbol{\theta}) = \omega_1 + \omega_2 - 1 + (1-\omega_1)(1-\omega_2)\exp[-\phi \log(1-\omega_1)\log(1-\omega_2)], \tag{15}$$

where $\phi \in (0, 1]$. Based on the copula defined in Eq. (15), one can obtain

$$\begin{aligned} r(x, y; \boldsymbol{\theta}) =& \frac{f(x, y; \boldsymbol{\theta})}{\bar{F}(x, y)} \\ =& f_1(F_1^{-1}(\omega_1)) f_2(F_2^{-1}(\omega_2)) \times \\ & \left[1 - \phi \log(1-\omega_2) - \phi \log(1-\omega_1) + (1 - \phi \log(1-\omega_1)\log(1-\omega_2))\,\phi\right]. \end{aligned} \tag{16}$$

The expression in Eq. (16) is fairly elegant.

3.1 A method for constructing asymmetric copulas

To construct a new copula that is asymmetric and meets the conditions listed in Lemma 1, we need to ensure that $g(\omega_1, \omega_2; \boldsymbol{\theta}) \neq g(\omega_2, \omega_1; \boldsymbol{\theta})$.

Define a bivariate function as following

$$\begin{aligned} \widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta}) = &\, \omega_1 + \omega_2 - 1 + (1-\omega_1)(1-\omega_2) \\ & \exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}]. \end{aligned} \tag{17}$$

When $\theta_1 = \theta_2 = 1$, the model in Eq. (17) reduces to the Gumbel-Barnett copula.

Proposition 2 $\widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta})$ *defined in* Eq. (17) *is a copula.*

Proof Apparently, it is easy to prove that $\widetilde{C}(0, \omega_2) = \widetilde{C}(\omega_1, 0) = 0$, $\widetilde{C}(1, \omega_2) = \omega_2$ and $\widetilde{C}(\omega_1, 1) = \omega_1$.

Now as long as we can prove that $\widetilde{c}(\omega_1, \omega_2; \boldsymbol{\theta}) = \frac{\partial^2 \widetilde{C}(\omega_1,\omega_2;\boldsymbol{\theta})}{\partial\omega_1\partial\omega_2} > 0$, we prove that $\widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta})$ is d-decreasing in ω_1 and ω_2.

$$\begin{aligned} \frac{\partial \widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta})}{\partial \omega_1} =& 1 + \left[\omega_2 - 1 - \phi\theta_1(1-\omega_2)(-\log(1-\omega_1))^{\theta_1 - 1}(-\log(1-\omega_2))^{\theta_2}\right] \\ & \exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}] \end{aligned} \tag{18}$$

Using the inequality $\frac{x}{1+x} \leq \log(1+x) \leq x$ for all $x > -1$ (Love, 1980), and considering Eq. (18) and, we have

$$\begin{aligned} \widetilde{c}(\omega_1, \omega_2; \boldsymbol{\theta}) =& \frac{\partial^2 \widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta})}{\partial\omega_1\partial\omega_2} \\ =& \big[1 + \phi\theta_1(-\log(1-\omega_1))^{\theta_1-1}(-\log(1-\omega_2))^{\theta_2} - \phi\theta_1\theta_2(-\log(1-\omega_1))^{\theta_1-1}(-\log(1-\omega_2))^{\theta_2-1} \\ & + \phi\theta_2\left[1 + \phi\theta_1(-\log(1-\omega_1))^{\theta_1-1}(-\log(1-\omega_2))^{\theta_2}\right]\left[(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2-1}\right]\big] \\ & \exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}] \\ =& \big[(-\log(1-\omega_1))^{1-\theta_1}(-\log(1-\omega_2))^{1-\theta_2} - \phi\theta_1\log(1-\omega_2) - \phi\theta_1\theta_2 - \phi\theta_2\log(1-\omega_1) \end{aligned}$$

$$+\phi^2\theta_1\theta_2(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}\Big]\Big[(-\log(1-\omega_1))^{\theta_1-1}(-\log(1-\omega_2))^{\theta_2-1}\Big]$$
$$\exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}]$$
$$\geq\Bigg[\omega_1^{1-\theta_1}\omega_2^{1-\theta_2}+\phi\theta_1\frac{\omega_2}{1-\omega_2}-\phi\theta_1\theta_2+\phi\theta_2\frac{\omega_1}{1-\omega_1}$$
$$+\phi^2\theta_1\theta_2\omega_1^{\theta_1}\omega_2^{\theta_2}\Bigg](-\log(1-\omega_1))^{\theta_1-1}(-\log(1-\omega_2))^{\theta_2-1}$$
$$\exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}] \tag{19}$$
$$\geq\Bigg[\phi\theta_1\frac{\omega_2}{1-\omega_2}-\phi\theta_1\theta_2+\phi\theta_2\frac{\omega_1}{1-\omega_1}+2\phi\sqrt{\theta_1\theta_2}\Bigg]$$
$$(-\log(1-\omega_1))^{\theta_1-1}(-\log(1-\omega_2))^{\theta_2-1}$$
$$\exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}] \tag{20}$$
$$\geq\Bigg[\phi\theta_1\frac{\omega_2}{1-\omega_2}+\phi\theta_2\frac{\omega_1}{1-\omega_1}\Bigg](-\log(1-\omega_1))^{\theta_1-1}(-\log(1-\omega_2))^{\theta_2-1}$$
$$\exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}] \tag{21}$$
$$\geq 0. \tag{22}$$

The above inequality (19) was derived by using $\frac{x}{1+x} \leq \log(1+x) \leq x$; inequality (20) used the inequality $a+b \geq 2\sqrt{ab}$ for $a, b > 0$; inequality (21) was obtained because $2\phi\sqrt{\theta_1\theta_2} > \phi\theta_1\theta_2$, and inequality (22) is obvious.

This completes the proof. □

For the copula defined in Eq. (17), if $\theta_1 \neq \theta_2$, it is easy to prove the following proposition.

Proposition 3 $\widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta}) \neq \widetilde{C}(\omega_2, \omega_1; \boldsymbol{\theta})$ *for* $\theta_1 \neq \theta_2$.

Proposition 3 can easily be established by using Eq. (17).

Proposition 3 shows that the proposed copula $\widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta})$ is an asymmetric copula.

From the definition of the failure rate shown in Eq. (8), we can obtain

$$r(x, y; \boldsymbol{\theta}) = \frac{f(x, y; \boldsymbol{\theta})}{\bar{F}(x, y)}$$
$$= \frac{\widetilde{c}(\omega_1, \omega_2; \boldsymbol{\theta}) f_1(F_1^{-1}(\omega_1)) f_2(F_2^{-1}(\omega_2))}{1-\omega_1-\omega_2+\widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta})}$$
$$= \frac{f_1(F_1^{-1}(\omega_1)) f_2(F_2^{-1}(\omega_2))}{(1-\omega_1)(1-\omega_2)}$$
$$\Big[(-\log(1-\omega_1))^{1-\theta_1}(-\log(1-\omega_2))^{1-\theta_2}-\phi\theta_1\log(1-\omega_2)-\phi\theta_1\theta_2-\phi\theta_2\log(1-\omega_1)$$
$$+\phi^2\theta_1\theta_2(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}\Big]\Big[(-\log(1-\omega_1))^{\theta_1-1}(-\log(1-\omega_2))^{\theta_2-1}\Big]. \tag{23}$$

Example 2 Suppose $\omega_1 = 1 - e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}}$ and $\omega_2 = 1 - e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}}$. Then $f_1(F_1^{-1}(\omega_1)) = \frac{1}{\alpha_1}\left(\frac{x}{\alpha_1}\right)^{\beta_1-1} e^{-\left(\frac{x}{\alpha_1}\right)^{\beta_1}}$, $f_2(F_2^{-1}(\omega_2)) = \frac{1}{\alpha_2}\left(\frac{y}{\alpha_2}\right)^{\beta_2-1} e^{-\left(\frac{y}{\alpha_2}\right)^{\beta_2}}$, $(-\log(1-\omega_1))^{\theta_1} = \left(\frac{x}{\alpha_1}\right)^{\theta_1\beta_1}$ and $(-\log(1-\omega_2))^{\theta_2} = \left(\frac{y}{\alpha_2}\right)^{\theta_2\beta_2}$. Then the failure rate is given by

$$
\begin{aligned}
r(x, y; \boldsymbol{\theta}) =& \frac{f_1(F_1^{-1}(\omega_1)) f_2(F_2^{-1}(\omega_2))}{(1-\omega_1)(1-\omega_2)} \\
& \left[(-\log(1-\omega_1))^{1-\theta_1}(-\log(1-\omega_2))^{1-\theta_2}\right. \\
& -\phi\theta_1 \log(1-\omega_2) - \phi\theta_1\theta_2 - \phi\theta_2 \log(1-\omega_1) \\
& \left.+\phi^2\theta_1\theta_2(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}\right] \\
& \left[(-\log(1-\omega_1))^{\theta_1-1}(-\log(1-\omega_2))^{\theta_2-1}\right] \\
=& \frac{1}{\alpha_1\alpha_2}\left(\frac{x}{\alpha_1}\right)^{\beta_1\theta_1-1}\left(\frac{y}{\alpha_2}\right)^{\beta_2\theta_2-1}\left[\left(\frac{x}{\alpha_1}\right)^{\beta_1(1-\theta_1)}\left(\frac{y}{\alpha_2}\right)^{\beta_2(1-\theta_2)} + \phi\theta_1\left(\frac{y}{\alpha_2}\right)^{\beta_2}\right. \\
& \left. - \phi\theta_1\theta_2 + \phi\theta_2\left(\frac{x}{\alpha_1}\right)^{\beta_1} + \phi^2\theta_1\theta_2\left(\frac{x}{\alpha_1}\right)^{\theta_1\beta_1}\left(\frac{y}{\alpha_2}\right)^{\theta_2\beta_2}\right]. \quad (24)
\end{aligned}
$$

As can be seen from Eq. (24), the expression of $r(x, y; \boldsymbol{\theta})$ is simpler than the one in Eq. (12) in Example 1 although different copulas are used. Furthermore, the Basu failure rate in Eq. (24) is obtained based on an asymmetric copula.

With Eqs. (4) and (17), the corresponding copula $\bar{C}(\omega_1, \omega_2; \boldsymbol{\theta})$ of $\bar{F}(x, y)$ is given by

$$
\begin{aligned}
\bar{F}(x, y) =& \bar{C}(\omega_1, \omega_2; \boldsymbol{\theta}) \\
=& 1 - \omega_1 - \omega_2 + \widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta}) \\
=& (1-\omega_1)(1-\omega_2)\exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}] \\
=& \exp\left[-\left(\frac{x}{\alpha_1}\right)^{\beta_1} - \left(\frac{y}{\alpha_2}\right)^{\beta_2} - \phi\left(\frac{x}{\alpha_1}\right)^{\beta_1\theta_1}\left(\frac{y}{\alpha_2}\right)^{\beta_2\theta_2}\right]. \quad (25)
\end{aligned}
$$

Hence, JK's bivariate failure rate function can be defined with the copula as follows:

$$
r_1(x, y; \boldsymbol{\theta}) = \left[-\frac{\beta_1}{\alpha_1}\left(\frac{x}{\alpha_1}\right)^{\beta_1-1} - \phi\frac{\beta_1\theta_1}{\alpha_1}\left(\frac{x}{\alpha_1}\right)^{\beta_1\theta_1-1}\left(\frac{y}{\alpha_2}\right)^{\beta_2\theta_2}\right]\bar{C}(\omega_1, \omega_2; \boldsymbol{\theta}), \quad (26)
$$

and

$$
r_2(x, y; \boldsymbol{\theta}) = \left[-\frac{\beta_2}{\alpha_2}\left(\frac{y}{\alpha_2}\right)^{\beta_2-1} - \phi\frac{\beta_2\theta_2}{\alpha_2}\left(\frac{x}{\alpha_1}\right)^{\beta_1\theta_1}\left(\frac{y}{\alpha_2}\right)^{\beta_2\theta_2-1}\right]\bar{C}(\omega_1, \omega_2; \boldsymbol{\theta}). \quad (27)
$$

Copulas are a tool for modelling the dependence between random variables. Both Kendall's tau and Spearman's rho can be expressed by copulas. It seems difficult to obtain Spearman's rho or Kendall's tau of copula $\widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta})$. Nevertheless, we investigate the bounds of Spearman's ρ of $\widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta})$ in the following Proposition.

Proposition 4 *The Spearman's rho, ρ, of $\widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta})$ is*

$$
3e^{-\phi} - 3 \leq \rho \leq -12\frac{e^{4/\phi}}{\phi}Ei[-\frac{4}{\phi}] - 3, \quad (28)
$$

where $Ei(.)$ is given by $Ei(x) = \int_{-\infty}^{x} \frac{e^t}{x} dt$. The bounds are sharp.

Proof Since the relationship between Spearman's rho and a copula is $\rho = 12\int_0^1\int_0^1(\widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta}) - \omega_1\omega_2)d\omega_1 d\omega_2$, we obtain

$$\begin{aligned}\rho =&12\int_0^1\int_0^1(\widetilde{C}(\omega_1,\omega_2;\boldsymbol{\theta})-\omega_1\omega_2)d\omega_1 d\omega_2\\ =&12\int_0^1\int_0^1(\omega_1+\omega_2-1+(1-\omega_1)(1-\omega_2)\exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}]\\ &-\omega_1\omega_2)d\omega_1 d\omega_2\\ =&12\int_0^1\int_0^1((1-\omega_1)(1-\omega_2)\exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}]d\omega_1 d\omega_2-3\end{aligned} \tag{29}$$

Let $-\log(1-\omega_i)=u_i$ for $i=1,2$. That is, $1-\omega_i=e^{-u_i}$. Then,

$$\begin{aligned}\rho =&12\int_0^1\int_0^1 e^{-u_1-u_2-\phi u_1^{\theta_1}u_2^{\theta_2}}d\omega_1 d\omega_2-3\\ =&12\int_0^{+\infty}\int_0^{+\infty}e^{-2u_1-2u_2-\phi u_1^{\theta_1}u_2^{\theta_2}}du_1du_2-3.\end{aligned} \tag{30}$$

Since $u_i^{\theta_i}\leq 1$, we obtain

$$\begin{aligned}\rho =&12\int_0^{+\infty}\int_0^{+\infty}e^{-2u_1-2u_2-\phi u_1^{\theta_1}u_2^{\theta_2}}du_1du_2-3\\ \geq&12\int_0^{+\infty}\int_0^{+\infty}e^{-2u_1-2u_2-\phi}du_1du_2-3\\ =&3e^{-\phi}-3,\end{aligned} \tag{31}$$

and with $u_i^{\theta_i}\geq u_i$, we obtain

$$\begin{aligned}\rho =&12\int_0^{+\infty}\int_0^{+\infty}e^{-2u_1-2u_2-\phi u_1^{\theta_1}u_2^{\theta_2}}du_1du_2-3\\ \leq&12\int_0^{+\infty}\int_0^{+\infty}e^{-2u_1-2u_2-\phi u_1u_2}du_1du_2-3\\ =&12\int_0^{+\infty}\frac{e^{-2u_2}}{2+\phi u_2}du_2-3\\ =&12\frac{e^{4/\phi}}{\phi}\int_2^{+\infty}\frac{e^{-\frac{2y}{\phi}}}{y}dy-3\\ =&12\frac{e^{4/\phi}}{\phi}\int_{\frac{4}{\phi}}^{+\infty}\frac{e^{-z}}{z}dz-3\\ =&-12\frac{e^{4/\phi}}{\phi}\mathrm{Ei}[-\frac{4}{\phi}]-3.\end{aligned} \tag{32}$$

Hence, we establish Proposition 4. □

Since Spearman's rho of the Gumbel-Barnett copula is $\rho=-12\frac{e^{4/\phi}}{\phi}\mathrm{Ei}[-\frac{4}{\phi}]-3$, Proposition 4 suggests that Spearman's rho of $\widetilde{C}(\omega_1,\omega_2;\boldsymbol{\theta})$ is smaller than that of the Gumbel-Barnett copula.

Definition 2 If a bivariate copula C is such that

$$\lim_{\omega \to 0+} \frac{1 - C(1-\omega, 1-\omega)}{\omega} = \lambda_U \tag{33}$$

exists, then C has *upper tail dependence* if $\lambda_U \in (0, 1]$ and no upper tail dependence if $\lambda_U = 0$. Similarly, if

$$\lim_{\omega \to 0+} \frac{C(\omega, \omega)}{\omega} = \lambda_L \tag{34}$$

exists, then C has *lower tail dependence* if $\lambda_L \in (0, 1]$ and no upper tail dependence if $\lambda_L = 0$.

It is easy to obtain the following lemma.

Proposition 5 *If a bivariate copula C is such that the tail dependence coefficients of copula* $C(\omega_1, \omega_2; \boldsymbol{\theta})$ *are given by* $\lambda_L = 0$ *and* $\lambda_U = 4$.

Proof

$$\begin{aligned}\lambda_L &= \lim_{\omega \to 0+} \frac{C(\omega, \omega)}{\omega} \\ &= \lim_{\omega \to 0+} \frac{2\omega - 1 + (1-\omega)^2 \exp[-\phi(-\log(1-\omega))^{\theta_1+\theta_2}]}{\omega} \\ &= 0,\end{aligned} \tag{35}$$

$$\begin{aligned}\lambda_U &= \lim_{\omega \to 0+} \frac{1 - C(1-\omega, 1-\omega)}{\omega} \\ &= 2 - \lim_{\omega \to 0+} \frac{1 - 2\omega + \omega^2 \exp[-\phi(-\log\omega)^{\theta_1+\theta_2}]}{\omega} \\ &= 2 - \lim_{\omega \to 0+} (-2 + 2\omega \exp[-\phi(-\log\omega)^{\theta_1+\theta_2}] \\ &+ \omega(-\phi(\theta_1+\theta_2)(-\log\omega)^{\theta_1+\theta_2-1} \exp[-\phi(-\log\omega)^{\theta_1+\theta_2}]) \\ &= 4.\end{aligned} \tag{36}$$

Hence, we establish Proposition 5. □

Example 3 Given a repairable system whose reliability is measured by both its age and cumulative usage intensity, denote its failure intensity by $\lambda(x, y; \boldsymbol{\theta})$. Let $\lambda(x, y; \boldsymbol{\theta}) = r(x, y; \boldsymbol{\theta})$, where $r(x, y)$ is given in Eq. (24). Assume that the repair upon failures between replacements is minimal. That is, a repair restores the system to the status immediately before its failure. Then, the total number of failures in the plane $[0, t] \times [0, u]$ is given by $\int_0^x \int_0^y \lambda(x_0, y_0; \boldsymbol{\theta}) dx_0 dy_0$, from which an explicit expression can be easily obtained.

4 Discussion

Denote $H_1(x) = \int_0^x h_1(u)du$ and $H_2(y) = \int_0^y h_2(u)du$, where $h_1(.)$ is the failure rate associated with the cumulative distribution function (cdf) $F_1(.)$ (i.e., ω_1 in the copula $\widetilde{C}_1(\omega_1, \omega_2; \boldsymbol{\theta})$ and $h_2(.)$ is the failure rate associated with $F_2(.)$. That is, $H_1(x)$ and $H_2(y)$ are the cumulative hazard functions associated with $F_1(.)$ and $F_2(.)$, respectively. Then $-\log(1-\omega_1^{\theta_1}) = H_1(x)$ and $-\log(1-\omega_2^{\theta_2}) = H_2(y)$. We have

$$\begin{aligned}\widetilde{C}(\omega_1, \omega_2; \boldsymbol{\theta}) =& \omega_1 + \omega_2 - 1 + (1-\omega_1)(1-\omega_2)\exp[-\phi(-\log(1-\omega_1))^{\theta_1}(-\log(1-\omega_2))^{\theta_2}] \\ =& F_1(x) + F_2(y) - 1 + F_1(x)F_2(y)\exp[-\phi(H_1(x))^{\theta_1}(H_2(y))^{\theta_2}], \end{aligned} \quad (37)$$

where $\theta_1, \theta_2 \in (0, 1]$.

Another idea to extend the Gumbel-Barnett copula to be an asymmetric form is to define a copula as following

$$\widetilde{C}_1(\omega_1, \omega_2; \boldsymbol{\theta}) = \omega_1 + \omega_2 - 1 + (1-\omega_1)(1-\omega_2)\exp[-\phi\log(1-\omega_1^{\theta_1})\log(1-\omega_2^{\theta_2})]. \quad (38)$$

where $\theta_1 \neq \theta_2$. Nevertheless, the challenge in the copula $\widetilde{C}_1(\omega_1, \omega_2; \boldsymbol{\theta})$ shown in Eq. (38) is that $\log(1-\omega_i^{\theta_i})$ with $i = 1, 2$ may become very complex even for a simple probability distribution like the Weibull distribution.

It is noted that the copula is selected based on some methods such as the maximum likelihood and a performance criterion (or several criteria) such as the Akaike Information Criterion. As such, the construction method of asymmetric copulas proposed in this paper can be used for the scenarios where the performance of the asymmetric Gumbel-Barnett copula is similar to the other best performed copulas.

It is noted that the approach to constructing the asymmetric Gumbel-Barnett copula may not be used for other copulas for forming closed-forms of cumulative failure rates, which is a limitation of this paper.

5 Conclusion

This paper constructed an asymmetric copula from the Gumbel-Barnett copula. With the proposed copula, the expression of the bivariate failure rate becomes elegant, which facilitates its applications in the real world. That is, the main contribution of this paper is its proposal of a method to construct an asymmetric copula for the ease in use of the bivariate failure rate.

References

Barbiero, A. (2022). Discrete analogues of continuous bivariate probability distributions. *Annals of Operations Research, 312*(1), 23–43.

Barbiero, A. (2022). Properties and estimation of a bivariate geometric model with locally constant failure rates. *Annals of Operations Research, 312*(1), 3–22.

Barnett, V. (1980). Some bivariate uniform distributions. *Communications in Statistics Theory and Methods, 9*(4), 453–461.

Basu, A. (1971). Bivariate failure rate. *Journal of the American Statistical Association, 66*(333), 103–104.

Borwein, J. M., Crandall, R. E., et al. (2013). Closed forms: what they are and why we care. *Notices of the AMS, 60*(1), 50–65.

Chow, T. Y. (1999). What is a closed-form number? *The American Mathematical Monthly, 106*(5), 440–448.

Dai, A., Wei, G., Wang, D., He, Z., & He, S. (2021). The opportunity study of pm strategy for second-hand products sold with a two-dimensional warranty. *Reliability Engineering & System Safety, 214*, 107699.

Finkelstein, M. (2003). On one class of bivariate distributions. *Statistics & Probability Letters, 65*(1), 1–6.

Gumbel, E. J. (1960). Bivariate exponential distributions. *Journal of the American Statistical Association, 55*(292), 698–707.

Jack, N., Iskandar, B. P., & Murthy, D. P. (2009). A repair-replace strategy based on usage rate for items sold with a two-dimensional warranty. *Reliability Engineering & System Safety, 94*(2), 611–617.

Joe, H. (2014). *Dependence modeling with copulas*. Boca Raton: CRC Press.

Johnson, N. L., & Kotz, S. (1975). A vector multivariate hazard rate. *Journal of Multivariate Analysis, 5*(1), 53–66.

Kayid, M., et al. (2022). Characterizations of the weak bivariate failure rate order and bivariate ifr aging class. *Journal of Mathematics, 2022*, 2573667.

Liebscher, E. (2008). Construction of asymmetric multivariate copulas. *Journal of Multivariate Analysis, 99*(10), 2234–2250.

Love, E. R. (1980). Some logarithm inequalities. *The Mathematical Gazette, 64*(427), 55–57.

Mukherjee, S., Lee, Y., Kim, J.-M., Jang, J., & Park, J.-S. (2018). Construction of bivariate asymmetric copulas. *Communications for Statistical Applications and Methods, 25*(2), 217–234.

Navarro, J. (2008). Characterizations using the bivariate failure rate function. *Statistics & Probability Letters, 78*(12), 1349–1354.

Shang, L., Liu, B., Cai, Z., & Wu, C. (2022). Random maintenance policies for sustaining the reliability of the product through 2d-warranty. *Applied Mathematical Modelling, 111*, 363–383.

Wang, G. J., & Zhang, Y. L. (2011). A bivariate optimal replacement policy for a cold standby repairable system with preventive repair. *Applied Mathematics and Computation, 218*(7), 3158–3165.

Wu, Q. (2012). Reliability analysis of a cold standby system attacked by shocks. *Applied Mathematics and Computation, 218*(23), 11654–11673.

Wu, S. (2014). Construction of asymmetric copulas and its application in two-dimensional reliability modelling. *European Journal of Operational Research, 238*(2), 476–485.

Wu, S. (2024). A copula-based approach to modelling the failure process of items under two-dimensional warranty and applications. *European Journal of Operational Research, 314*(3), 854–866.

Yang, S.-C., & Nachlas, J. A. (2001). Bivariate reliability and availability modeling. *IEEE Transactions on Reliability, 50*(1), 26–35.