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Innovative Applications of O.R.

Game of banks - biform game theoretical framework for ATM network cost sharing

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ABSTRACT

Automated teller machines (ATM) play a major role in the world economy as they enable financial transactions and hence good exchanges and consumption. ATM transaction fees are incurred to cover the cost of running the network and these are often settled among the members including banks and cash machine operators. In this paper, we develop a novel biform game theoretic model for members to optimally invest in the ATM network and to share the cost. This biform game includes both a cooperative game theory mechanism for interchange fee sharing and a non-cooperative counterpart to model the fact that members also wish to maximize their utilities. While the proposed coopetition framework is applicable to general ATM networks, we focus the case study on the UK ATM network thanks to the accessibility of the data in addition to the notable stability issues that the network is currently experiencing as has been widely featured by the mainstream media. On the technical side, we prove the existence of a pure Nash equilibrium, which can be computed efficiently. We also show that, under some settings, the Shapley allocation belongs to the core and hence it is not only fair to all members but also leads to a stable ATM network. In addition, we show that the Shapley value allocation dominates the current mechanism in terms of social welfare. Finally, we provide numerical analysis and managerial insights using real data on the complete UK ATM network.

1. Introduction

ATM networks are fundamental parts of the world payment infrastructures and cash machines are by far the most popular channel for cash withdrawal. Members of the ATM network include banks, which are called card *issuers* as they issue bank cards to their customers. The banks themselves own ATM machines that are available for their customers to use. The ATM network also includes cash machine operators, which are also called independent ATM deployers. While these members do not have customers and hence do not issue bank cards, they build ATMs to enable card holders from other network members to access their cash. These cash operators can open ATM machines in shopping malls, train stations, supermarkets and other locations.

Whenever a card-holder makes a transaction, there is a transaction fee incurred to cover the cost of installing and running the ATM network. In some countries such as the USA and Canada, the transaction fees are paid by the card holders, i.e., the consumers, while in others such as the UK and several other countries in Europe and Asia, these are paid by the card issuers. In the latter case, customers can enjoy using the ATM free of charge without concerning about who the machine owners are. In return, these ATM operators receive interchange fee payments from the card issuers. In both of these cases, it is crucial that network members need to make agreements on who and how much to pay for the transactions.

In this work, we focus the case study on the UK ATM network partly thanks to the publicly available data for demonstration and partly because the issue of how to settle the transaction fees in the UK has been on-going for the past few years and this work might offer a solution. We note, however, that the proposed biform game theoretical framework among network members are applicable to other ATM networks around the world.

1.1. UK ATM landscape

The UK's cash machine network is currently the busiest ATM transaction switch in the world with over 50,000 ATMs (LINK-ATM-Ltd., 2023). Most of the ATMs in the UK are free-to-use and "over 98% of all ATM cash withdrawals by UK cardholders in the UK are made free of charge". The network is managed mostly by LINK ATM Ltd., which was formed around the year 2000 after the gradual merger of several smaller shared ATM

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networks. Currently, LINK has 27 members, which are banks, building societies, and ATM service providers. By joining the network, customers from any member can access to cash across the whole of the UK by using ATMs owned by itself or by other members.

The LINK network is a fundamental part of the UK's payments infrastructure and cash machines are by far the most popular channel for cash withdrawal in the UK, used by millions of consumers every week. According to the LINK latest annual report 2021/2022 (LINK-ATM-Ltd., 2022), there are over 100 million UK issued LINK-enabled cards in circulation. There were 980 m LINK cash withdrawals and 1.5 billion transactions of all types (e.g., including balance inquiries) made by consumers in 2021 — amounting to £79 billion. In another pre-pandemic report (LINK-ATM-Ltd., 2018a), it was shown that the record high daily withdrawal in 2017 was £766 m in value — with 8.7 m cash withdrawals (on 22 December 2017) and, at its busiest, LINK has processed over 20,000 transactions a minute (LINK-ATM-Ltd., 2018a).

The major objective of LINK (and the UK government) with regard to the UK cash payment system is to provide free ATM service to as many people as possible. In part of the effort, LINK has a special Financial Inclusion Programme, which safeguards free-to-use ATMs in remote and deprived areas. Recently, LINK has announced "to launch 'Super Premiums' to protect remote ATMs" (LINK-ATM-Ltd., 2018c).

1.2. Current methods for interchange fee setting and stability issues

The free-to-use cash machines network is a great privilege for the consumers; and the UK government and policy makers have worked to maintain this facility to ensure a healthy economy. The main objective of LINK (and the UK government) with regard to the UK cash payment system is to provide a free ATM service to as many consumers as possible. This, however, means that members of the ATM network need to share the underlying cost of operations. A crucial part of the LINK Scheme operation is to settle the interchange fee between members.

Since commercially negotiated bilateral agreements are unmanageable for this large network (27 members need 351 such agreements), the LINK scheme has come up with a simple *equal per-transaction cost sharing* model (or *equal cost sharing* for brevity). Here, the per-transaction cost is calculated by dividing the total cost of operating the ATM network by the total number of transactions (more details can be found in Appendix A). Once this is calculated, each LINK member will receive (or pay) a settlement amount equal to the product of the per-transaction-cost and the difference between the inflow, i.e., the number of transactions other members used its ATMs, minus the outflow, i.e., the number of transactions its customers use other members' ATMs. With all the transactions going through the LINK network, records of how many transactions coming in and out from each member is available for this calculation. The estimation of the per-transaction cost is carried out independently by KPMG (2006), a network of professional service firms and one of the Big Four auditors in the world. KPMG carries out annual surveys with all LINK members and provides a thorough analysis of all the cost involved from fixed terms such as ATM purchase depreciation and installation, market rent, repairs and maintenance to variable costs including communication, cash purchase, delivery, and management etc. The calculated per-transaction cost will then be recommended for the following year. Minor modifications to the cost allocation methodology are also agreed from time to time. This methodology is subject to competition law, and secured regulatory approval in 2001 by the Payment Systems Regulator (PSR, 2019).

While the equal-cost-sharing scheme is quite simple and easy to use, not all members are happy with it. This disagreement peaked in early 2017 when some members threatened to leave the network as they considered that the fees they were paying (or receiving) were too high (low). This has been captured in most of the mainstream media reporting such as the BBC, The Guardian and other newspapers (BBCNews, 2017; The GuardianNews, 2017a, 2017b; The Telegraph News, 2017). The disagreement among members is summarized in The Telegraph News (2017) as "Free access to cash is under threat as banks and other organizations fall out over who pays to maintain Britain's network of ATMs".

Reaching agreement over fees and changes in methodology continues to pose a challenge. There may be commercial tensions and conflicting views between members, particularly between the large net acquirers and the large net issuers. The BBC described the situation as follows:

> Banks have been discussing for months how to bring down the cost, which has risen to over a billion pounds a year. The banks think the fees paid to the independent operators is too high (BBCNews, 2017).

Following lengthy debates among the stakeholders, in January 2018, LINK announced a phased 20 per cent reduction in interchange rates over four years, from around 25p to 20p (LINK-ATM-Ltd., 2018b, 2018d). However, although reduction might help lower the cost for some members; at the same time, it might lead to a fall in the revenue of other members, particularly ATM providers, and this could lead to the closure of some ATMs. Peter McNamara, chief executive of independent operator Note Machine, warned that "such a squeeze being proposed by LINK would result in more customers being charged" since "the free-to-use machines would not be economical". In that case, Note Machine would consider closing down ATMs. Peter McNamara claims that "We estimate that you could be losing up to a quarter of the free-to-use ATM sites in the UK", BBCNews (2017). Indeed, after the reduction to 24p in 2018, alongside the closure of ATMs, the LINK Scheme has recently announced the postponement of further reduction until further settled due to the disagreement among members and the prospect of closure of part of the ATM network.

Research aims

The ultimate aim of this research is to resolve the stability issues in ATM networks where members have both conflicting interests and at the same time can realize the benefit of collaboration. In particular, we aim to provide not only an alternative method for interchange fee settlement but also analyse how members can optimally invest in the ATM network, what the resulting equilibrium is, and how this performs in terms of social welfare, as well as other questions relating to the stability of the ATM network.

2. Related literature and contributions

2.1. ATM cooperative game

The ATM interchange fee game was proposed by Gow and Thomas in Gow and Thomas (1998), who demonstrate the ATM game with a small example of four banks and show how the cost sharing mechanism suggested by the cooperative game theory framework can be used to settle the interchange fee among banks. The ATM cooperative game is then further extended in Bjørndal, Hamers, and Koster (2004), Hinojosa, Mármol, and Thomas (2005), Parilina and Sedakov (2014) and Rathnam, Kanapaka, and Neelisetti (2015) in which other factors such as additional objectives or the locations of the ATM are considered.

Gow and Thomas (1998) model the ATM network as a cooperative game with m banks or ATM operators. Let $\mathcal{B} = \{1, 2, \ldots, m\}$ be the set of all the members in the network. The members have the option to work by themselves, i.e., to have their customers using only the ATM machines own by themselves, or to form coalitions. Here, we call $\mathcal{B}$ as the grand coalition that is formed by all the members. When a coalition $S \subseteq \mathcal{B}$ is formed, we can view the coalition as a single entity with all the aggregated customers and ATMs owned by its members. The coalition as a whole then incurs a cost for running the sub-network which includes the fixed cost of building and maintaining the ATMs as well as the variable costs incurring from transactions.

With the cost function for each coalition defined, Gow and Thomas (1998) model the ATM network as a cost game $G(\mathcal{B}, c(\cdot))$ where $c : 2^{\mathcal{B}} \rightarrow \mathbb{R}$ maps each coalition to a real value reflecting the total cost incurred by the coalition. The cost game can be transformed into a profit game by different methods such as simply setting the value function as the negative of the cost function or as the total cost saved by the coalitions compared to working individually.

Once the game is defined, different solution concepts such as the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969) are calculated and used as guidance for setting pairwise interchange fees. The Shapley value is one of the main solution concepts in cooperative game theory thanks to its economic intuition and is considered as a fair solution in the sense that the total cost that each player pays is the weighted average of the marginal costs/contributions that the player introduces into the game. Another class of solution concepts includes the imputation, the core (Gillies, 1959), the least core and the nucleolus which all aim for the stability of the game in the sense that they avoid individual players or sub-groups of players to break away from the grand coalition.

While the cooperative game theory model provides an intuitive framework for cost sharing as it directly addresses the key feature of stability and fairness among the members of the ATM network, the main limitation is that it does not consider how the proposed sharing mechanism would affect the members' decisions on how much to invest in the ATM network, which in turns changes the value function of the cooperative game. The fact that some members threatened to closed their ATMs following the recent changes on the per-transaction cost is a perfect example to show that there is an interplay between the sharing mechanism and the level of investment that members put into the game. This naturally leads us to consider a *coopetition model* in the subsequent section *that includes an underlying cooperative sharing mechanism embedded within a non-cooperative utility maximization framework*. The ATM cooperative game to be described will itself extend Gow and Thomas's model in Gow and Thomas (1998) in several ways to describe the reality better. For example, we will consider different types of transaction as well as the benefit of serving customers in order to keep the customers in their businesses. We will therefore model the ATM cooperative game as a profit game which take both the benefit and the cost into account.

2.2. Coopetition and biform game theoretical models

The terms 'coopetition' has long appeared in the literature and the first book on the topic is by Nalebuff, Brandenburger, and Maulana (1996) in which the authors' main aim was to discuss game theory with a practical focus on both the aspects of creating values and capturing values through cooperation and competition, respectively. Bengtsson and Kock (2000) present various different (real) lines of business in which both elements of cooperation and competition are present and describe the complex interaction and relationship among the firms. In a recent article in The CEO Magazine (Wayne, 2022), the author describes modern applications of coopetition such as the collaboration between rivals Apple and Samsung on iPhone screens, between Pfizer and BioNTech on developing vaccines against COVID-19, or between Apple, Amazon, Google and more than 170 other companies on Connected Home over IP. Within the Management Sciences field, Bengtsson and Kock (2000) argue firms can benefit from both cooperation and competition simultaneously while Guo and Wu (2018) and Meca, García-Jurado, and Borm (2003) study models in which competing firms work together to reduce inventory costs through capacity sharing.

The coopetition framework has been studied in different settings; among which the framework that we consider matches mostly with that in Bo and Nguyen (2019), Brandenburger and Stuart (2007), Feess and Thun (2014) and van Beek, Malmberg, Borm, Quant, and Schouten (2023). Brandenburger and Stuart (2007) introduce the biforms game in which cooperative games are embedded in a strategic noncooperative game; that is, the inner cooperative payoff sharing is part of the outer utility function of the strategic game. The biform game framework has since then be applied to different contexts including the surplus division in supply chains on investment incentives (Feess & Thun, 2014). Most recently, van Beek et al. (2023) study the biform linear production process in which firms are competing to make strategic decision on which resource bundle to choose among two potential locations before cooperating in a standard linear production cooperative game. Independently, Bo and Nguyen (2019) study the linear production coopetition game in which firms make strategic decision on how much resources to invest in given that their resources will be aggregated in a cooperative linear production game and the reward are shared among the players. Bo and Nguyen (2019) also compare which (cooperative) sharing scheme would lead to the optimal social welfare. The focus of this paper is to study the properties of the Shapley allocation in the ATM biform game in comparison with the current equal-cost-sharing model.

The title of the paper starts with "Game of Banks" as is inspired by the recent television series "Game of Thrones" which features many games within games being played between the main characters and in which both cooperation and competition are present. The field of Game Theory, as formally introduced by John von Neumann and Oskar Morgenstern in their classic book "Theory of Games and Economic Behaviour" (Von Neumann & Morgenstern, 2007), started out by covering a balance of materials from both the two main sub-fields of Cooperative and Non-cooperative Game Theory. Since then, the two sub-fields have diverged with the non-cooperative game theory literature have seemed to attract more attention in the past decades. In the "Game of Banks", however, we argue that both elements of competition and cooperation need to be considered in order to resolve the on-going conflict and to keep the free-to-use ATM network alive for the consumers.

2.3. Contributions

Our contributions include the following:

(1) *Modelling and novel application*: We develop a novel ATM biform game that includes a cooperative framework for profit sharing embedded within a non-cooperative framework to model the fact that banks and card service providers also care about how to optimally invest on their ATM network to maximize their utilities. This work enriches both the cooperative game theory and non-cooperative game theory literature by including an on-going real-life novel application involving the whole UK network of banks and cash-machine operators.

(2) *Theoretical results*: We show attractive properties of the game such as the ATM coopetition game is a special potential game and hence it has attractive properties like the existence of the pure NE as well as the convergence of best-response algorithms. We also show the additivity and convexity of the game in some settings. This leads to the stability (in addition to fairness) of the Shapley allocation. In addition, the Shapley allocation is shown to dominate the current equal (per-transaction) cost sharing scheme in terms of social welfare.
(3) *Numerical scheme and proof technique*: We develop numerical schemes for finding the Nash equilibrium and the Shapley value. The best-response problem for each member involves optimizing potential functions which are sums of an exponentially large number of terms (for the case of the ATM coopetition game with 27 members, the objective functions are the sum of over 130 million terms). While the technique make use of a known randomization technique, the convergence analysis is rather complicated for the case of the ATM coopetition game given its complex value functions. The underlying technique of exploiting the Cramer device and a generalized version of the multivariate Centre Limit Theorem would be of interests on their own right as they open up the door for efficient Shapley value estimation in other large cooperative games with complex characteristic functions.
(4) *Managerial insights*: We analyse various business reports to collect real data to demonstrate the case study on the complete UK ATM network with 27 members (existing literature only considered toy examples with a small number of banks and only on the cooperative aspect). We also consider a more complete setting with practical consideration such as the different types of transactions. We provide managerial insights on how different members with different characteristics strategically behave in the game and in reality. We answer questions such as why some members are threatening to leave and would the current method resolve it? How fair is the current equal-per-transaction cost sharing model compared to the proposed Shapley value allocation? Which coalitions would have the incentive to stay and leave the network? How the two cost sharing schemes ultimately affect the social welfare?

3. Modelling the bank ATM network as a biform game

3.1. Problem settings and coalitional values

In the ATM game proposed by Gow and Thomas in Gow and Thomas (1998), it was assumed that the number of ATMs owned by the members are fixed. As motivated in Section 1.2, the level of investment in the ATM network is among the key decisions considered by banks and ATM operators. We will incorporate this into a higher-level framework which contains both the (cooperative) sharing of the cost among members and the (competitive) strategic play on the ATM network investment.

The decision of the members are how many ATMs to build and whether they should join a network (or a sub-network) or to work individually. When a group of members join a (sub-)network, customers of any bank member can use any ATMs owned by any other members in the network free of charge. This means members must also need to agree among themselves a mechanism to share their investment and operation cost.

Let t_i denote the number of ATMs that member ith is going to build. Once a coalition $\mathcal{S} \subseteq \mathcal{B}$ is formed, we can view its members as an entity that serves their aggregated customers by using their available ATMs. There are two main types of ATM transactions which are cash transactions and non-cash transactions (including balance inquiries and incomplete cash transactions). The latter type of transaction is considered in the literature but does affect the cost structure of the game and hence we will include both in this work. Let N_i^1 and N_i^2 denote the number of cash transactions and balance inquiry transactions that customers from bank i wish to take per year. Here, we concern only with ATM transactions and the settlement among members but not other card transactions that might take place, say when paying in a shopping mall. For the rest of this paper, we use superscript $k = 1$ to denote cash transactions and $k = 2$ to denote balance inquiry transaction. Let us denote $t(\mathcal{S}) := \sum_{i\in\mathcal{S}} t_i$ and $N^k(\mathcal{S}) := \sum_{i\in\mathcal{S}} N_i^k$ as the total number of ATMs that network $\mathcal{S}$ has and the total number of transactions of type $k \in \{1, 2\}$ from their customers, respectively. Let $\alpha(\tau)$ be the proportion of customers using ATMs given that the UK is covered by τ number of ATMs available. The larger is τ, the more transactions are taken place. We call $\alpha(\cdot)$ the ATM *utilization function*.

If a network of banks $\mathcal{S}$ collaborates under their decision $(t_i)_{i\in\mathcal{S}}$, then for each type kth of transactions, the total number of transactions served by the ATM network is approximated as $\alpha(t(\mathcal{S}))N^k(\mathcal{S})$. The remaining $(1 - \alpha(t(\mathcal{S})))N^k(\mathcal{S})$ transactions will not take placed through ATMs either because customers stay in the network but choose other non-ATM means such as going directly to a bank branch, or to leave the network. Let $(\beta, 1 - \beta)$ be the corresponding proportions of customers staying and leaving the network.

For $i, j \in \mathcal{S}$, let us denote $n_{ij}^k(\mathcal{S}, \boldsymbol{t})$ as the number of transactions of type kth by customers of member ith on ATMs owned by member jth. We assume that the ATMs and the customers are spread out proportionally and hence

$$n_{ij}^k(\mathcal{S}, \boldsymbol{t}) = \frac{t_j}{t(\mathcal{S})} N_i^k \alpha(t(\mathcal{S})), \tag{1}$$

that is, among those $N_i^k\alpha(t(\mathcal{S}))$ transactions being served, each select an ATM out of the set of $t(\mathcal{S})$ ATMs available with equal chances. While the above proportional reallocation of transactions is an approximation of reality, this is probably the best we can assume unless we go into the micro level of individual ATM locations. The same assumption has been used in the literature such as in Gow and Thomas (1998).

Let c^k be the cost per transaction of type kth from a customer whose card is issued by a bank on using ATMs owned by another member. The cost per transaction for those using their own bank's ATM is $(c^k - \Delta^k)$ for some $\Delta^k \geq 0$ to show that these transactions are cheaper operationally (because the customers are already in the same internal system and there is no inter-bank external communication needed). Let κ_1 be the per-transaction *benefit* of serving and keeping customers in the network. Let κ_2^k be the per-transaction *cost* for serving transactions of type k by using other means such as going directly to a local bank branch. Let κ_3 be the cost of not serving a transaction (and hence potentially having the customer leaving the network). Here, we assume the cost of interest rate and management fee etc. have been incorporated into these parameters. For the optimization of the ATM operations, we refer readers to Ágoston, Benedek, and Gilányi (2016), who study how banks and cash management firms collaborate on the fees setting to achieve Pareto optimal on their cost-revenue tradeoff.

Let us denote $v(\mathcal{S}, \boldsymbol{t})$ as the total profit of all members in subset $\mathcal{S}$ if the coalition $\mathcal{S}$ is formed, we have

$$\begin{aligned} v(\mathcal{S}, \boldsymbol{t}) = \sum_{i\in\mathcal{S}} \Bigg\{ \sum_{k\in\{1,2\}} \Bigg(& N_i^k \alpha(t(\mathcal{S}))\kappa_1 - N_i^k(1 - \alpha(t(\mathcal{S})))\beta\kappa_2^k - N_i^k(1 - \alpha(t(\mathcal{S})))(1 - \beta)\kappa_3 \\ & - \sum_{j\in\mathcal{S}} (c^k - \Delta^k 1_{i=j}) n_{ji}^k(\mathcal{S}, \boldsymbol{t}) \Bigg) - h_i(t_i) \Bigg\}, \end{aligned} \tag{2}$$

where, for each $k = \{1, 2\}$, the first part $N_i^k \alpha(t(S))\kappa_1$ is the total value generated by serving $N_i^k \alpha(t(S))$ transactions (and hence effectively keeping customers happy), the second part $N_i^k(1-\alpha(t(S)))\kappa_2^k$ is the total cost incurred by serving customers by other means, the third part is the cost of having customers leaving the network, the fourth part is the total variable cost incurred through using ATMs provided by i and by all customers of all members in the coalition S, while the last part $h_i(t_i)$ is the total fixed cost of building and maintaining the ATMs incurred to member ith; that is the cost that are still incurred even without any transaction.

The coalitional value can be further simplified as

$$v(S, \boldsymbol{t}) = \sum_{k\in\{1,2\}} \left(\delta_1^k N^k(S)\alpha(t(S)) - \delta_2^k N^k(S) + \frac{\Delta^k}{t(S)} \alpha(t(S)) \sum_{i\in S} t_i N_i^k \right) - \sum_{i\in S} h_i(t_i), \tag{3}$$

where $\delta_1^k = (\kappa_1 + \beta\kappa_2^k + (1-\beta)\kappa_3 - c^k)$, $\delta_2^k = (\beta\kappa_2^k + (1-\beta)\kappa_3)$.

The value function as defined above is more complete than that in the literature such as Gow and Thomas (1998) in several aspects. First, we include the both types of cash and non-cash transactions to reflect the fact that these have significantly different costs. Second, we include the first three terms to reflect the benefit of keeping customers and the cost of having them leaving the network. We define the cooperative game as a profit-sharing game instead of a cost-sharing game as this naturally leads to the question of how banks can make the optimal choice on their ATM investment to maximize their 'utilities'. Finally, we assume that the proportion of transactions being served $\alpha(t(S))$ is dependent on the size of the ATM network as opposed to that in Gow and Thomas (1998) where the parameter α was fixed (due to the number of ATMs in their model were fixed).

3.2. Bank ATM coopetition game

For each $i \in \mathcal{B}$, let $\mathcal{F}_i := [0, \bar{t}_i]$ be the feasible range for the number of ATMs that bank ith consider having and let $\mathcal{F} = \times_{i\in\mathcal{B}} \mathcal{F}_i$ as the joint feasible space. For each choice of $\boldsymbol{t} \in \mathcal{F}$, we have a cooperative game $G(N, v(\cdot, \boldsymbol{t}))$ which lead to a profit allocation $u_i(\boldsymbol{t})$, which could be calculated based on an allocation rule such as the current equal-cost sharing scheme or some cooperative game theory solution concepts such as the Shapley value or the Nucleolus.

We call $\langle \mathcal{B}, \mathcal{F}, (u_i)_{i\in\mathcal{B}} \rangle$ an ATM coopetition game. Suppose each member aims to find the optimal t_i to maximize the utility function $u_i(t_i, \boldsymbol{t}_{-i})$. The Nash equilibrium t^{NE} is defined as

$$t_i^{NE} = \underset{0\le t_i\le \bar{t}_i}{\arg\max}\ u_i(t_i, \boldsymbol{t}_{-i}^{NE}),\ \forall i \in [1, \dots, n].$$

We will study the ATM coopetition game and compare the Shapley allocation sharing scheme with the equal cost sharing scheme. We consider the Shapley allocation in our coopetition framework as this is one of the most prominent sharing schemes, which has been used by Gow and Thomas (1998) in the ATM cooperative games as well as in various other recent cooperative game settings such as the liability games (Csóka, Illés, & Solymosi, 2022) and the fishing devices allocation games (Bergantiño, Groba, & Sartal, 2023). We will show that the choice of the Shapley allocation leads to attractive analytical properties.

Given the characteristic function $v(\cdot, \boldsymbol{t})$ as defined in (3), the Shapley value for player i is

$$\phi_i = \sum_{S\subset\mathcal{B}\setminus\{i\}} \lambda_{|S|}(v(S\cup\{i\}, \boldsymbol{t}) - v(S, \boldsymbol{t})), \tag{4}$$

which is the weighted average of the marginal contribution $(v(S \cup \{i\}, \boldsymbol{t}) - v(S, \boldsymbol{t}))$ that player ith contributes to each coalition $S \in \mathcal{B}$. Here, $\lambda_{|S|} := \frac{|S|!(n-|S|-1)!}{n!}$ is the probability of having player ith joining right after coalition S in a random permutation of players (and hence player ith is contributing to the coalition). Under the Shapley allocation rule, the utility function of the coopetition game is defined with $u_i(t_i, \boldsymbol{t}_{-i}) := \phi_i(\boldsymbol{t})$.

4. Properties of ATM coopetition game

4.1. Pure NE existence

We first show the pure Nash equilibrium existence. Bo and Nguyen (2019) show that resource-sharing coopetition games with separable costs are potential games, which was introduced in Monderer and Shapley (1996). Similar results that link between Shapley allocation and potential games have also appeared in Norde, Özen, and Slikker (2016), Qin (1996) and Ui (2000). Utilizing these, we can establish Nash equilibrium existence result as is formally stated in the following theorem.

Theorem 1. *There exists a pure Nash equilibrium in the ATM coopetition game.*

Proof. Let us first rewrite the utility function. We have

$$\begin{aligned} u_i(t_i, \boldsymbol{t}_{-i}) &= \sum_{S\subseteq\mathcal{B}\setminus\{i\}} \lambda_{|S|}\left(v(S\cup\{i\}, \boldsymbol{t}) - v(S, \boldsymbol{t})\right) \\ &= \sum_{S\subseteq\mathcal{B}\setminus\{i\}} \lambda_{|S|} v(S\cup\{i\}, \boldsymbol{t}) - \sum_{S\subseteq\mathcal{B}\setminus\{i\}} \lambda_{|S|} v(S, \boldsymbol{t}) \qquad (5) \\ &= \underbrace{\sum_{S'\subseteq\mathcal{B}} \lambda_{|S'|-1} v(S', \boldsymbol{t})}_{\Phi(\boldsymbol{t})} - \underbrace{\left[\sum_{S'\subseteq\mathcal{B}\setminus\{i\}} \lambda_{|S'|-1} v(S', \boldsymbol{t}) + \sum_{S\subseteq\mathcal{B}\setminus\{i\}} \lambda_{|S|} v(S, \boldsymbol{t})\right]}_{g_i(\boldsymbol{t}_{-i})}, \qquad (6) \end{aligned}$$

where index S' in the first two terms of (6) is equivalent to $S \cup \{i\}$ in (5); we just change the index name to avoid confusion with the third term. In other words, for each player ith, $u_i(t_i, \boldsymbol{t}_{-i})$ can be rewritten as the difference between a common function $\phi(\boldsymbol{t})$, which does not depend on index ith, and a remainder $g_i(\boldsymbol{t}_{-i})$, which depend only on $\boldsymbol{t}_{-i}$ but not t_i.

Thus,

$$\arg\max_{t_i} \ u_i(t_i, \boldsymbol{t}_{-i}) = \arg\max_{t_i} \ \Phi(\boldsymbol{t}), \quad \forall i \in \mathcal{B}. \tag{7}$$

The best-response problem for each player ith is therefore equivalent to the maximization problem on a common function $\Phi(\boldsymbol{t})$ on component t_i. Since the function $\Phi(\boldsymbol{t})$ is continuous and the domain $\mathcal{F}$ is compact, there exists a global optimal solution $\boldsymbol{t}^*$. Since at $\boldsymbol{t}^*$, no player ith can improve by unilaterally changing t_i, we have $\boldsymbol{t}^*$ as a pure Nash equilibrium. ■

4.2. Stability of ATM coopetition games

We now study the stability of the ATM cooperative game in detail. To that end, we need to impose some structure on the ATM utilization function $\alpha(\cdot)$. In the literature, such as in Gow and Thomas (1998), Hinojosa et al. (2005) and Parilina and Sedakov (2014), $\alpha(\tau)$ has been assumed to take a constant value since the number of ATMs in these models is fixed. In this section, we extend this to model the fact that the proportion of transactions served by the ATM network is a non-decreasing function of the number of ATMs. Besides, for each member, the cost of serving transactions from customers from other members are usually higher than or equal to that from its own customers. It is, therefore, reasonable to assume the following.

Assumption 1. The ATM utilization function $\alpha(\tau)$ is a non-decreasing function on τ, and $\Delta^k \geq 0, \ k = 1, 2$.

We have the following analytical properties of the game.

Theorem 2. *Under Assumption 1, for each fixed decision t, the super/sub-additivity of the corresponding ATM cooperative game $G(\mathcal{B}, v(\cdot, t))$ can be summarized in the following table:*

	$\Delta^k = 0, \forall k = 1, 2$	$\Delta^k > 0, \forall k = 1, 2$
$\alpha(\tau)$ constant on τ	Additive	Sub-additive
$\frac{\alpha(\tau)}{\tau}$ constant on τ	Super-additive	Super-additive
$\frac{\alpha(\tau)}{\tau}$ non-decreasing on τ	Super-additive	Super-additive
General cases with $\alpha(\tau)$ increasing on τ	Super-additive	Depending on convexity of $\alpha(\tau)$ and relative magnitudes of Δ, δ_1

Remark. When the cooperative game is additive, there is no gain or loss for banks and other cash machine operators to collaborate. Members should (respectively, should not) collaborate if the game is super-additive (respectively, sub-additive).

The intuition behind the results in Theorem 2 is as follows. If $\Delta^k = 0, \forall k = 1, 2$, as shown in the second column of the table, then there is no loss in collaboration since the cost of using any ATM is still the same from any coalition member's point of view. If $\alpha(\tau)$ is a constant as in the case of the second row, there is no help in having additional ATMs and hence there is no benefit of forming coalitions. Putting these together, we obtain all the results stated in the second column and the second row of the table. The remaining details of the proof of Theorem 2 are provided in Appendix B, which exploits the form of the coalitional value function in Formulation (3).

In the remainder of this section, we consider the case when $\alpha(\tau) = \frac{\tau}{T}$ and derive some analytical insights on the game. Here, T is assumed to be a threshold number on the number of ATMs above which adding additional ATM does not help increase the number of transactions being served; that is, the model is appropriate for cases when the ATM network is not yet saturated. In Section 5, we extend this to the case when $\alpha(\cdot)$ is a concave linear fractional function. For concreteness, we formally state the assumption and refer to it when appropriate in the subsequent results.

Assumption 2. The ATM utilization function $\alpha(\tau)$ is a linear function on τ; that is, $\alpha(\tau) = \frac{\tau}{T}$ for some given T.

From Theorem 2, we know that the ATM cooperative game is super-additive under Assumption 2. This means the grand coalition should be formed. We next show additional properties of the game and the Shapley value.

Theorem 3. *Under Assumption 2, the following results hold:*

(a) The characteristic function is super-modular. As a consequence, the core exists. In addition, the Shapley value belongs to the core.
(b) The Shapley value has the following closed form:

$$\phi_i^{SH} = \sum_{k=1}^{2} \left(\sum_{j \in \mathcal{B}} \frac{\delta_1^k}{2T} \left(N_i^k t_j + N_j^k t_i \right) - \delta_2^k N_i^k + \frac{\Delta^k}{T} N_i^k t_i \right) - h_i(t_i), \tag{8}$$

We provide detailed proof of the theorem in Appendix C. Here, we note some implications on the desirable properties of the Shapley allocation:

(i) *Fairness and stability* are the two main criteria in judging a solution concept in cooperative game theory. The result in Theorem 3 shows that, not only does the Shapley value have the attractive fairness properties, as it is designed for, but also the appealing stability property that the core has. These are very desirable in general applications and in particular, are of crucial importance in this ATM game given that members are concerned about staying or breaking away from the network.
(ii) Having the simple *closed-form solution* for the Shapley value is appealing not only because of its mathematical beauty but also its simplicity which means it is easier to compute and to communicate the proposed sharing scheme with practitioners.
(iii) The formulation of the Shapley value suggests that each member ith should pay for their fixed cost of running their own ATMs, i.e., the term $h_i(t_i)$, but not those incurred by others. The current equal-cost-sharing scheme, however, divides the total cost (of approximately £900 m), i.e., containing the term $\sum_{i \in \mathcal{B}} h_i(t_i)$, by the number of transactions, which means that inefficiency from one member might affect profitability of others.

(iv) The Shapley value contains a linear term on t and the cost term $h_i(t_i)$. If the cost function $h_i(t)$ is convex on t_i, as is usually assumed in the economics literature, then $\phi_i^{SH}(t)$ is a concave function on t_i and we can solve a convex optimization problem to find the optimal t_i for each member. Otherwise, the problem can still be efficiently solvable as we are optimizing over one-dimensional variable t_i, which takes integer values and hence a brute-force search can be used as the last resource.

4.3. Dominance of Shapley allocation over current cost sharing scheme

Under equal cost sharing, each member in the network will pay a per-transaction cost of $\pi^1 = c^1 + g^1$ for cash withdrawal and $\pi^2 = c^2 + g^2$ for other non-cash transactions; that is, g^1, g^2 are the premiums above the (real) operational costs c^1, c^2 that members have to pay. The details for the calculations of these terms are provided in Appendix A. Following the disagreement among members (BBCNews, 2017; The GuardianNews, 2017a, 2017b; The Telegraph News, 2017), LINK has recently implemented a strategy to make adjustment to the estimates of the per-transaction cost π_1 from 25p to 24p per cash-withdrawal transaction and the plan is to have this further reduced in the next five years (LINK-ATM-Ltd., 2018d). We will show that choosing any pair of (π^1, π^2), or equivalently (g^1, g^2), would still result in a lower social welfare compared to using Shapley value allocation.

Assumption 3. For each $k = 1, 2$, $c^k \le \pi^k \le \min(\kappa_1^k, \kappa_2^k, \kappa_3^k)$.

The first part of Assumption 3, i.e. $c^k \le \pi^k$ essentially states that the cost of operating the ATMs is smaller than the price that the ATM operators charge others (so that the operators can make a profit). The second part states that the benefit of serving customers is higher than the price paid for the ATMs usage since otherwise members would not receive enough benefit to cover the cost of building and running the ATMs.

Theorem 4. *Under Assumptions 2 and 3 and under convex cost function $h_i(\cdot)$, $\forall i \in \mathcal{B}$, the Shapley allocation dominates the equal-cost-sharing scheme in terms of social welfare.*

The proof of the theorem is provided in Appendix D. The intuition behind this theorem is that, since the Shapley allocation is designed so as each member chooses its level of investment to maximize the weighted average of its marginal contributions to the game, the member's objective function is better 'aligned' to the centralized social welfare objective function compared to that of the equal-cost-sharing scheme. As a result, the Nash equilibrium under Shapley allocation is shown to be closer to the social optimal solution compared to that of the equal-cost-sharing scheme.

5. Computation of Nash equilibrium and Shapley allocation for general cases

In Section 4.2, we provide close-formed solution for the Shapley allocation and the pure Nash Equilibrium for affine function $\alpha(\cdot)$. In this section, we extend the model to the linear fractional form and provide numerical approach for finding solutions of the ATM coopetition game. We first present a general framework for finding pure NE in Section 5.1. A crucial part in this framework is to solve the best response problem for each player, which was shown to be equivalent to the maximization of the potential function. The numerical method for optimizing that is presented in Section 5.2 where we follow the Owen's multi-linear extension form (Owen, 1972) to approximate the potential function and its first two partial derivatives. Finally, we present the methods for solving large ATM coopetition games and for finding the Shapley allocation in Appendices F and G, respectively.

Assumption 4. Function $\alpha(\tau)$ has a linear fractional form $\alpha(\tau) = \frac{a\tau}{1+a\tau}$ for some given parameter a.

Assumption 4 is quite reasonable as it satisfies the following. First, it is an increasing function on τ with $\alpha(0) = 0$ and $\lim_{\tau\to\infty} \alpha(\tau) = 1$. Second, the function is concave which is a reasonable assumption in economics, which essentially states that the marginal value of having additional ATMs is a non-increasing function on τ. Finally, the choice of a parametrized model with the fine-tuning parameter a allows us to control the curvature of the function while fixing the two ends $\alpha(0) = 0$ and $\lim_{\tau\to\infty} \alpha(\tau) = 1$.

We note that function $\alpha(\tau)$ can be approximated to the linear function $a\tau$ when the denominator $(1 + a\tau)$ is close to 1. This means the linear fractional model is close to the linear model stated in Assumption 2 when τ is relatively small; that is when the ATM network is not yet saturated with many ATMs. In the numerical experiments, we document how α was estimated.

It is worth to recap that the result on the coopetition game belonging to the class of potential game does not depend on the form of $\alpha(\cdot)$ and hence the existence of a pure Nash equilibrium is still applicable.

5.1. General framework for finding pure Nash equilibrium

In this section, we show that we can compute a pure Nash equilibrium through Algorithm 1, which is an iterative algorithm, where at each iteration, we find and update the best response for each player while fixing the strategies of others the same as they were from the previous iteration.

The following result shows that, starting from any initial strategy $\mathbf{t}^0$, the iterative algorithm converges to a pure Nash equilibrium.

Theorem 5. *The iterative best-response algorithm applying to any coopetition game converges to an ϵ-Pure Nash equilibrium for any $\epsilon > 0$.*

Proof. The proof comes directly from the fact that the coopetition game is a potential game. During the algorithm, if it is possible for at least one player to deviate and increase the profit by at least ϵ, the potential function increases by at least ϵ. Nevertheless, since the potential function is bounded above, this process must be finite and hence the algorithm should stop at an ϵ-pure Nash equilibrium. ■

The main repetitive step in Algorithm 1 is to solve the best response problem. We describe the numerical scheme for this in the next section.

Algorithm 1 - Iterative best-response algorithm for finding a pure Nash equilibrium

Input: Coopetition game $\langle \mathcal{B}, \mathcal{F}, (u_i)_{i\in\mathcal{B}} \rangle$, precision parameter ϵ;
Initialize: Set $k = 0$ and choose a random initial strategy $\boldsymbol{t}^k$;
repeat
  for $i = 1 : n$ **do**
    Set $\boldsymbol{t}_i^{k+1} \in \arg\max_{0\le t_i\le \bar{t}_i} u_i(t_i, \boldsymbol{t}_{-i}^k)$.
  end for
  Set $k = k + 1$;
until $||\boldsymbol{t}_i^k - \boldsymbol{t}_i^{k-1}|| < \epsilon$;
Stop, output $\boldsymbol{t}_i^t$ as a pure Nash equilibrium;

5.2. Solving Best Response Problems (BRP)

The best-response problem for each player ith is equivalent to finding the optimal t_i that maximizes the potential function. While the potential function has a close form, its computation is not easy for large games, i.e., with the number of players above 20, since a brute-force method would require computing the values of 2^n coalitions and then taking the weighted average. Since we need to repeatedly evaluate the potential function for different cooperative games $G(\mathcal{B}, v(\cdot, \boldsymbol{t}))$ in order to find the Nash equilibrium, we wish to find a computationally efficient method.

The potential function contains an exponentially large number of terms in the same way that the Shapley value does and is therefore generally difficult to compute and analyse. We utilize the probabilistic view from Owen's multi-linear extension form (Owen, 1972) to approximate the potential function and its first two derivatives. For each given $x \in [0, 1]$ and for each $j = 1, \dots, n$, let X_j be a Bernoulli random variable which receives value 1 with probability x and value 0 with probability $(1 - x)$. We can view each outcome $X = (X_1, \dots, X_n)$ as an indicator vector, which corresponds to a coalition $\mathcal{S}$ that is formed by those players j with $X_j = 1$. We use X and $\mathcal{S}$ interchangeably in this section. For example, $v(X, \boldsymbol{t})$ is equivalent to $v(\mathcal{S}, \boldsymbol{t})$ and $v(X \mid X_i = 1, \boldsymbol{t})$ is equivalent to $v(\mathcal{S} \cup \{i\}, \boldsymbol{t})$.

We have the following alternative formulations for the potential function and its derivatives. Here, we write X instead of $X(x)$, i.e., X as a function of x, to simplify the notations.

Theorem 6. *If the characteristic function $v(\mathcal{S}, t)$ is twice differentiable for all $\mathcal{S} \subseteq \mathcal{B}$, then we have the following alternative formulations for the potential function and its first two partial derivatives for each $i \in \mathcal{B}$:*

$$\Phi(\boldsymbol{t}) = \int_{0^+}^1 \frac{1}{x}\,\mathrm{E}\left[v(X, \boldsymbol{t})\right] dx - \sum_{i\in\mathcal{B}} h_i(t_i). \tag{9}$$

$$\frac{\partial \Phi_i(\boldsymbol{t})}{\partial t_i} = \int_0^1 \mathrm{E}\left[\frac{\partial v(X \mid X_i = 1, \boldsymbol{t})}{\partial t_i}\right] dx - \frac{\partial h_i(t_i)}{\partial t_i}. \tag{10}$$

$$\frac{\partial^2 \Phi_i(\boldsymbol{t})}{\partial t_i^2} = \int_0^1 \mathrm{E}\left[\frac{\partial^2 v(X \mid X_i = 1, \boldsymbol{t})}{\partial t_i^2}\right] dx - \frac{\partial^2 h_i(t_i)}{\partial t_i^2}, \tag{11}$$

where the expectations over $(X \mid X_i = 1)$ means the randomness is over $(X_j)_{j\in\mathcal{B}\setminus\{i\}}$ while X_i is fixed at 1.

Proof. We have

$$\lambda_k = \frac{k!(m-k-1)!}{m!} = \int_0^1 x^k (1-x)^{m-k-1} dx;$$

Thus,

$$\begin{aligned}
\Phi(\boldsymbol{t}) &= \sum_{\mathcal{S}\subseteq\mathcal{B}} \lambda_{|\mathcal{S}|-1} v(\mathcal{S}, \boldsymbol{t}) - \sum_{i\in\mathcal{B}} h_i(t_i) \\
&= \sum_{\mathcal{S}\subseteq\mathcal{B}} \left(\int_0^1 x^{|\mathcal{S}|-1}(1-x)^{m-|\mathcal{S}|} dx\right) \sum_{\mathcal{S}\subseteq\mathcal{B}\setminus\{i\}} \lambda_{|\mathcal{S}|-1} v(\mathcal{S}, \boldsymbol{t}) - \sum_{i\in\mathcal{B}} h_i(t_i) \\
&= \int_0^1 \frac{1}{x}\left\{\sum_{\mathcal{S}\subseteq\mathcal{B}} x^{|\mathcal{S}|}(1-x)^{m-|\mathcal{S}|} v(\mathcal{S}, \boldsymbol{t})\right\} dx - \sum_{i\in\mathcal{B}} h_i(t_i).
\end{aligned}$$

The probability of having coalition $\mathcal{S}$ formed is $x^{|\mathcal{S}|}(1-x)^{m-|\mathcal{S}|}$. Thus, the term within the brackets can be viewed as the expectation of $v(X, \boldsymbol{t}) \equiv v(\mathcal{S}, \boldsymbol{t})$ given that each $X_j \sim B(x)$, which denotes a Bernoulli random variable with success rate x. This leads to Eq. (9).

The first two partial derivatives of the potential function can be derived in a similar way:

$$\begin{aligned}
\frac{\partial \Phi_i(t_i, t_{-i})}{\partial t_i} &= \sum_{\mathcal{S}\subseteq\mathcal{B}\setminus\{i\}} \lambda_{|\mathcal{S}|} \frac{\partial v(\mathcal{S}\cup\{i\}, \boldsymbol{t})}{\partial t_i} - \frac{\partial h_i(t_i)}{\partial t_i} \\
&= \sum_{\mathcal{S}\subseteq\mathcal{B}\setminus\{i\}} \left(\int_0^1 x^{|\mathcal{S}|}(1-x)^{n-|\mathcal{S}|-1} dx\right) \frac{\partial v(\mathcal{S}\cup\{i\}, t)}{\partial t_i} - \frac{\partial h_i(t_i)}{\partial t_i} \\
&= \int_0^1 \left\{\sum_{\mathcal{S}\subseteq\mathcal{B}\setminus\{i\}} x^{|\mathcal{S}|}(1-x)^{n-|\mathcal{S}|-1} \frac{\partial v(\mathcal{S}\cup\{i\}, \boldsymbol{t})}{\partial t_i}\right\} dx - \frac{\partial h_i(t_i)}{\partial t_i}.
\end{aligned}$$

The term within the integral can be viewed as the expectation of $\frac{\partial v(S\cup\{i\},t)}{\partial t_i}$. This leads to Eqs. (10) and (11). Here, we note a difference between Eqs. (9) and (10) on the term $\frac{1}{x}$. ■

The result of Theorem 6 closely follows the multi-linear extension form for finding the Shapley value in Owen (1972). The similarity should not be of much surprise as the potential function is closely related to that for the Shapley value (see Formulation (6)). The slight change is on the factor $\frac{1}{x}$ which comes from the fact that the potential function includes summation over all coalitions while the Shapley value includes summation over all coalitions containing a specific player. The difficulty lies in evaluating the expectations over random variable X as a function of x. We will derive these for the ATM coopetition game next.

For each $j \in \mathcal{B}$, let us denote $q_j^k = N_j^k t_j$ and $\boldsymbol{q} = (q_j)_{j\in\mathcal{B}}$. We then have the following result.

Theorem 7. *Under Assumption 4, we have*

$$\boldsymbol{\Phi}(t) = \int_{0^+}^{1} \frac{1}{x}\,\mathrm{E}\left[X^T\vartheta_1 - \frac{X^T\vartheta_2}{(1+aX^Tt)}\right] dx - \sum_{i\in\mathcal{B}} h_i(t_i) \tag{12}$$

$$\frac{\partial \boldsymbol{\Phi}_i(t_i,t_{-i})}{\partial t_i} = \int_{0}^{1} \mathrm{E}\left[\frac{X_{+i}^T\vartheta_3}{(1+aX_{+i}^Tt)^2} + \frac{\gamma_{1i}}{(1+aX_{+i}^Tt)}\right] dx - \frac{\partial h_i(t_i)}{\partial t_i}. \tag{13}$$

$$\frac{\partial^2 \boldsymbol{\Phi}_i(t_i,t_{-i})}{\partial t_i^2} = \int_{0}^{1} \mathrm{E}\left[\frac{X_{+i}^T\vartheta_4}{(1+aX_{+i}^Tt)^3} + \frac{\gamma_{2i}}{(1+aX_{+i}^Tt)^2}\right] dx - \frac{\partial h_i(t_i)}{\partial t_i}, \tag{14}$$

where $\vartheta_1 = \sum_{k\in\{1,2\}}(\delta_1^k - \delta_2^k)N^k$, $\vartheta_2 = \sum_{k\in\{1,2\}} -\delta_1^k N^k - a\Delta^k q$, $\vartheta_3 = \sum_{k\in\{1,2\}}(a\delta_1^k N^k - a^2\Delta^k q)$, $\gamma_{1i} = \sum_{k\in\{1,2\}}\Delta^k a N_i^k$, $\vartheta_4 = \sum_{k\in\{1,2\}} -2a^2\delta_1^k N^k + 2a^3\Delta^k q$, *and* $\gamma_{2i} = -2a^2\Delta^k N_i^k$. *The expectation in Eq. (12) is taken over* $X = (X_j)_{j\in\mathcal{B}}$ *while the expectation in Eqs. (13)–(14) are taken over* $X_{+i} \equiv (X \mid X_i = 1)$.

The proof details are provided in Appendix E. We can utilize result from this theorem to develop a numerical scheme for evaluating the potential function and its derivatives. We take random samples X as sequence of Bernoulli random variables with a success rate of x to approximate the expectations in Eqs. (12)–(14). We can then use standard numerical methods for one dimensional integration such as the Simpson's rule to compute $\boldsymbol{\Phi}$ and its derivatives.

5.3. Solving large ATM coopetition games

For very large games, sampling directly on X in the previous section does not work well as we will need a larger sample size. We show an alternative approach by dimensionality reduction and asymptotic analysis. The result in this section only works when the game is large enough (i.e., with more than 20 players when Centre Limit Theorems start to work). Conversely, the approximation technique to be developed is mostly needed for large games as we can use the exact method for smaller sized games anyway.

Let us partition the domain of τ into pieces by $0 = T_0 < T_1 < \cdots < T_P$, where T_P is the upper bound on the number of ATMs. For example, we can set $T_P = \sum_{i\in\mathcal{B}} \bar{t}_i$. We approximate the potential function and its derivatives, as derived in (12)–(14), as

$$\tilde{\boldsymbol{\Phi}}(t) = \int_{0^+}^{1} \frac{1}{x}\,\mathrm{E}\left[\sum_{p=1}^{P} 1_{[T_p\le X^Tt<T_{p+1}]}\left\{X^T\vartheta_1 - \frac{X^T\vartheta_2}{(1+a\bar{T}_p)}\right\}\right] dx - \sum_{i\in\mathcal{B}} h_i(t_i). \tag{15}$$

$$\frac{\partial \tilde{\boldsymbol{\Phi}}_i(t_i,t_{-i})}{\partial t_i} = \int_{0}^{1} \mathrm{E}\left[\sum_{p=1}^{P} 1_{[T_p\le X_{+i}^Tt<T_{p+1}]}\left\{\frac{X_{+i}^T\vartheta_3}{(1+a\bar{T}_p)^2} + \frac{\gamma_{1i}}{(1+a\bar{T}_p)}\right\}\right] dx - \frac{\partial h_i(t_i)}{\partial t_i}. \tag{16}$$

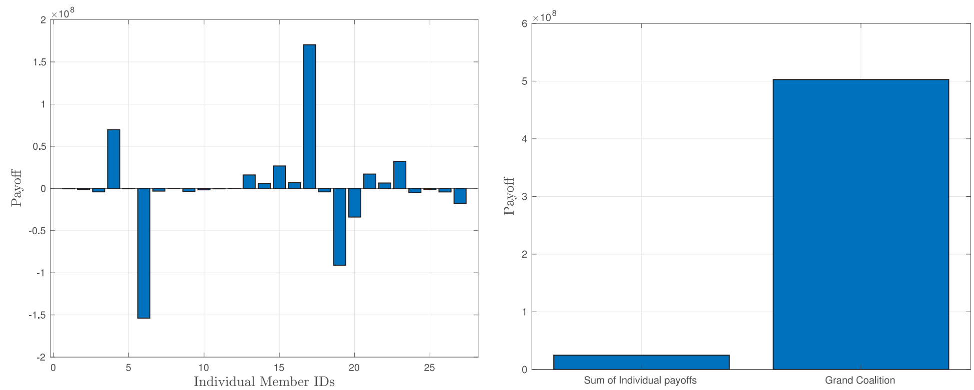
$$\frac{\partial^2 \tilde{\boldsymbol{\Phi}}_i(t_i,t_{-i})}{\partial t_i^2} = \int_{0}^{1} \mathrm{E}\left[\left\{\sum_{p=1}^{P} 1_{[T_p\le X_{+i}^Tt<T_{p+1}]}\frac{X_{+i}^T\vartheta_4}{(1+a\bar{T}_p)^3} + \frac{\gamma_{2i}}{(1+a\bar{T}_p)^2}\right\}\right] dx - \frac{\partial h_i(t_i)}{\partial t_i}. \tag{17}$$

Here, the approximation is done such that the term X^Tt in all the denominators of Eqs. (12)–(14) is replaced by $\bar{T}_p$, where index p is chosen such that $T_p \le X^Tt < T_{p+1}$ in Eq. (15) and $T_p \le X_+^Tt < T_{p+1}$ in Eqs. (16)–(17). The summation over all p and the indicator function $1_{[T_p\le X^Tt<T_{p+1}]}$ is used to ensure that we will be approximating X^Tt by the right piece among P pieces. In other words, we have approximated X^Tt in each of the ranges $[T_p, T_{p+1})$ by its average value. Here, we note that the approximation accuracy will grow with the increase on the number of pieces P. For $P = \sum_{i\in\mathcal{B}} \bar{t}_i$, we essentially recover the original potential function and its derivatives. We will show that the computational complexity grows linearly with the number of pieces, which is desirable as we can control and balance between the approximation accuracy and computational complexity accordingly.

We first note that the summations over p in each of the three Eqs. (15)–(17) can be taken out of the expectations. For each fixed p, the expectation in Eq. (15) is of the form $\mathrm{E}\left[1_{[T_p\le Y\le T_{p+1}]}Z\right]$, where $Y = X^Tt$ and Z is a linear function of X.

We can view Y and Z as the weighted sums of the IID sequence $X = (X_j)_{j\in\mathcal{B}}$ and utilize extensions of the Centre Limit Theorem (CLT), such as the Lyapunov version (Billingsley, 2008), and show that both Y, Z approach normal random variables as m approaches infinity under mild conditions on the weights (for example, there is no weight that is strictly dominating others).

Exploiting CLT for dimensionality reduction was, indeed, introduced by Owen (1972) with examples on weighted voting games. These examples, however, only involve simple characteristic functions that can be transformed into uni-variate cases. In the ATM game, we can see that the characteristic function is of bi-variate form on (Y, Z) and hence we have to use some forms of multivariate CLT to derive the joint distribution. More details on this are presented in Appendix F.



(a) When members work individually (b) Individual sum vs grand coalition

Fig. 1. Benefit of collaboration and stability analysis.

6. Managerial insights

6.1. Experimental setting and data collection

In the experimental results, we test the coopetition model using real data on the complete LINK network. LINK is quite transparent on their statistics and many of these as well as annual/monthly reports and announcements are made publicly available on its website. The latest data about the current members of the LINK network as well as the number of ATMs that each member owns for the year of 2022 are obtained from the LINK website (LINK-ATM-Ltd., 2023). In addition, we use the KPMG's report (KPMG, 2006) to understand the cost structure of the ATM network, which breaks this down into different components including ATM purchase depreciation, installation, maintenance, communication, cash purchase, delivery, and management, among others. This provides us with an estimate of the variable cost per cash transaction (in British Pounds) as $c^1 = 16p$ while the variable cost per non-cash transaction is estimated to be $c^2 = 5p$. We also set $\Delta^k = 10\%c^k$ as the discount for transactions using card issuer own ATMs. We use the general linear fractional form $\alpha(\tau) = \frac{a\tau}{1+a\tau}$ for the ATM utilization function and use algorithms developed in Section 5 for computing the Nash equilibrium and the Shapley value. We chose with $a = 10^{-3}$ as that would correspond to 98%ATM utilization when all members work together in a system of $\tau = 50{,}000$ ATM machines. The remaining parameters on κ and d are estimated such that this is consistent with other published results. For example, it has been mentioned in several media in 2017 that the total cost per year is around £900 m (The GuardianNews, 2017b). Tested data from Gow and Thomas (1998) are also used as references for us to provide estimates on the remaining parameters. We also use the fact that the current interchange fee for cash transaction of 24p is making some members unhappy while the interchange fee of 25p raised concerns to some other members. The information reveals the acceptable ranges that we should set the parameters. Putting altogether, we arrive at the estimate of $d = 10{,}000$, and $\kappa_1^k = 5c^k$, $\kappa_2 = 2c^1$ and $\kappa_3^k = 2c^k$. The total number of transactions N_i^k are recorded by LINK for the purposed of final settlement among members. The LINK website reports annual statistics such as the total number of transactions of each type to the public. For the analysis of the Nash equilibrium, we set the upper bounds $\bar{t}$ to be 10 times the current number of ATMs that members own. It is noted that these data are not easy to estimate as it involves analysing data from all members and to provide a complete cost analysis. In fact, KPMG was hired by LINK to do this task and even then, many assumptions are made during the process. We also provide sensitivity analysis in Appendix H for cases when the numbers of transactions or the numbers of ATMs are changed from the baseline values described above. We note that most of the analysis and the theoretical results in this paper still go through and the methodology for sharing by using cooperative game theory is still applicable if the data changes.

6.2. Benefit of collaboration

Fig. 1(a) shows the payoffs that the 27 members can receives if working alone. The 27 bars show the payoffs that the 27 members can receives by working individually. It should be noted that, some members even have negative payoff if they work alone. For a card issuer, this is mainly because the ATMs that the member has is not enough to serve their customers and this will drive customers away. For an ATM service provider, a negative cost comes from the fact that the member still has to pay for the fixed cost of maintaining the ATMs even if there was no transaction.

Fig. 1(b) show the difference between the sum of the individual payoffs; that is, the sum of the 27 bars in Fig. 1(a) compared to the payoff if they work together to form the grand coalition. We can see that the total payoff from the grand coalition is higher than the sum of payoffs that individual players can make, and this shows the benefit of collaboration. In this case, it is beneficial for the members to collaborate. The underlying reason is that the benefit of being able to serve their customers (and keeping them happy) is still higher than the extra inter-transaction cost Δ^k per transaction for both types of cash and non-cash transactions $k = 1, 2$.

6.3. Stability comparison between shapley allocation and equal-cost sharing on current ATM network

Fig. 2 shows the percentages of coalitions of different sizes that might break away from the grand coalition under equal cost sharing and Shapley allocation. For those coalitions of size less than 11 or more than 21, we are able to enumerate all the possible coalitions. However, for those coalitions of size between 12–20, we draw 10 000 random coalitions and record the percentage of breaking away coalitions among these.

The horizontal axis shows the size of the coalition which ranges from 1 to 26. Here, a size of 1 means that there is only 1 member in the coalitions; that is, the individual members form their own coalitions. A size of 26 means all members but one work together. The vertical axis

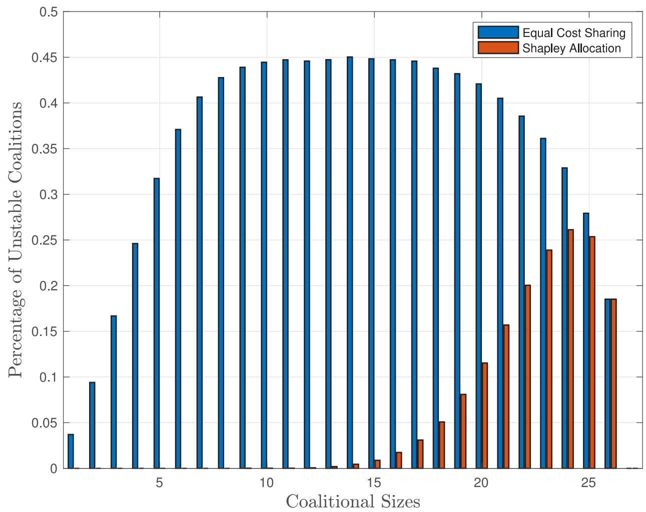
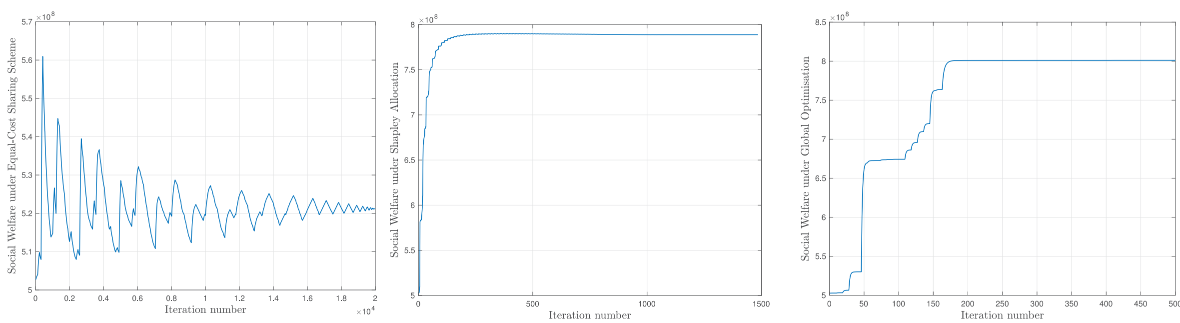


Fig. 2. Percentages of stability.



(a) Equal Weight Sharing (b) Shapley Allocation (c) Centralised Social Welfare Maximisation

Fig. 3. Social welfare comparison.

shows the percentages of coalitions that wants to break away among all coalitions of the specified sizes. Here, a percentage of 42% under equal-cost sharing for coalitions of size 20 means that, among all coalitions of that size (there are 888 030 such coalitions), 42% of these are unhappy with the equal cost sharing scheme. The corresponding percentage of unhappy coalitions under the Shapley allocation is 12%.

We can see that, under the current sharing scheme, many coalitions have the incentive to break away while that is smaller under the Shapley allocation. This shows that the Shapley allocation is a more attractive approach for sharing compared to the current equal weight sharing scheme in the stability criteria in addition to the fact that the Shapley sharing scheme is fair from the economic point of view in that members receives payoff that is proportional to the marginal contributions that the member has on the network.

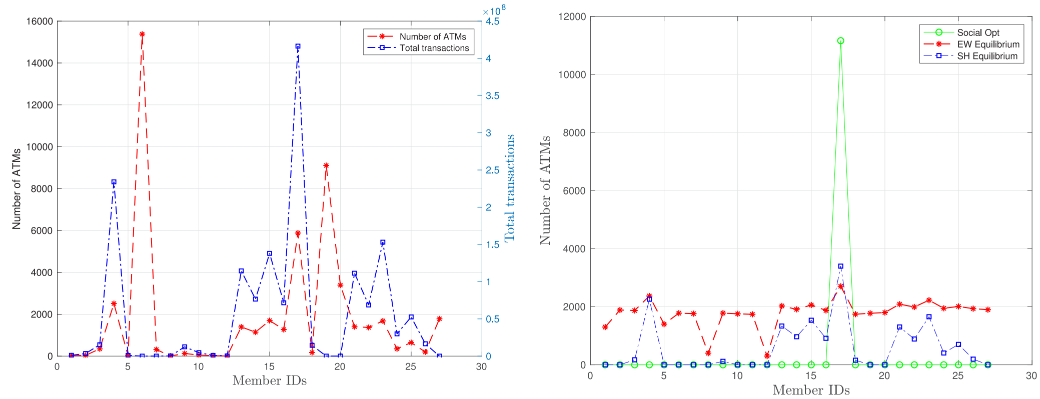
6.4. Social welfare

Fig. 3 shows how the social welfare evolves through iterations when each individual member maximizes its utility under equal-cost sharing and Shapley allocation, and when all members work together to maximize the centralized social welfare. In the case of the equal-cost-sharing scheme, we can see in Fig. 3a that the social utility (measured by the value of the grand coalition) might go up and down and eventually converge to a stable value, even though the utility of the individual member might increase in each round.

The price of anarchy (Koutsoupias & Papadimitriou, 1999) can be used as a measure of how good the system is designed, i.e., the *mechanism design*, to enable members to act rationally while still achieving high social welfare. As seen in Fig. 3, the price of anarchy for the equal-cost-sharing scheme is quite high. For the Shapley allocation case, the price of anarchy is also present when the individual member maximizes its utility. Nevertheless, by comparing Fig. 3b and c, we can see a lesser degree of system loss due to competition. Indeed, at Nash equilibrium, the social welfare under the Shapley allocation is around $7.85e8$ while that of the centralized optimization is around $8e8$; both of these translate to the price of anarchy of around 2% which is quite good. Under the equal-cost-sharing scheme, almost all members try to increase their number of ATMs, which degenerates the overall system. The Nash equilibrium under the Shapley allocation, however, is quite close to the system optimal solution and this matches with the implications of the results in Theorem 3 as well as explaining why it performs well on the social welfare.

6.5. Nash equilibria

Fig. 4a shows the current ATM network in terms of the number of ATMs and the total number of transactions for each of the 27 members. Fig. 4b shows the Nash equilibria for both equal weight sharing (Fig. 4a) and Shapley allocation (Fig. 4b) in comparison to the system optimal solution,



(a) Current Network (b) Social Optimal vs ECS vs Shapley Allocation

Fig. 4. Nash equilibria.

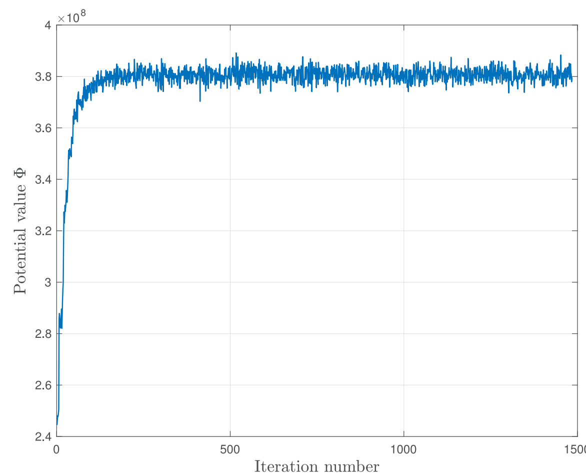


Fig. 5. Convergence of potential function in finding Nash equilibrium under Shapley allocation.

i.e., when all members collaborate to maximize the total grand coalition payoff. Under the system optimal solution, i.e., the grand coalition's decision-making problem, we solve a single optimization problem on how many ATMs that each member should optimally builds. Although all members can use all ATMs built by others in the same network, the members still have to pay for the variable transaction costs. The optimal solution is therefore to have the card-issuing member with the greatest number of transactions to build all the ATMs needed (instead of spreading that out to multiple members) to minimize the cross-member transaction cost premium Δ as the fixed cost of building the ATMs is the same no matter who own these ATMs when they work together. In this case, member number 17 has the greatest number of transactions to be fulfilled and hence was chosen to build all the ATMs. The Nash Equilibrium under the equal-cost sharing scheme spread out the number of ATMs to be built rather evenly with bank members (i.e., card issuers) build more ATMs that their current level while card service providers build less ATMs than they currently own. Finally, combining information from both Fig. 4a and b, we can see that the Nash equilibrium under the Shapley allocation have members building their ATMs mostly proportionally to the numbers of transactions that the members have to fulfilled.

6.6. Algorithm convergence

Fig. 5 shows how the potential function evolves on reaching the Nash equilibrium under the Shapley allocation. We can see that the potential function has an increasing trend throughout the iterations and this matches with the theoretical convergence result on maximizing the single potential function. We can see some fluctuation in the upward trend and this is partly due to the fact that the sampling approach in estimating the potential function has some noises and partly because of non-optimal step size in the Newton-based approach. It is possible to resolve these issues by choosing optimal step size through line search and to impose higher accuracy estimation (or even computing the true values of the potential function and its derivative if computational time is not an issue). Nevertheless, for this application when we only wish to understand the system behaviour, we leave these as they are.

6.7. Limitation and future research

In this work, we provide a coopetition framework that includes a cost-sharing mechanism and an analysis on how members can act strategically on their ATM infrastructure investment. While the general messages on the benefit of collaborations, the convergence of the algorithm, and the dominance of the Shapley sharing mechanism compared to the current equal-cost sharing method are proven theoretically, some of the numerical analyses are dependent on the specific parameters that we use. Although we have tried our best to estimate these from a wide-range of sources and while we have done some sensitivity analysis, we should be cautious on the specific numerical values shown on the results as these are only applicable to the stated assumptions and the choices of the parameters, such as κ, d, s. For example, the equilibrium on the numbers of ATMs that each member should build in the equilibrium are highly dependent on these parameters. In addition, even if these parameters reflect the realities, by the definition of the Nash equilibrium, the number of ATMs shown are only optimal once all other members also choose their strategies according to what suggested in the results. Nevertheless, the model can still provide members with a powerful way to find their optimal response given the predicted behaviour of other members. Finally, although we have tried to make the model as close to reality as possible, there are other factors that can affect the decision such as the government subsidizes to certain areas (e.g., rural areas) and to card-service providers to enable the free-payment network, and other operational aspects of ATMs' cash management as discussed in Ágoston et al. (2016).

7. Conclusion

In conclusion, this work introduces a novel coopetition game that includes a cooperative framework for banks and ATM operators to share their cost of the ATM network and a competitive game theory framework in which members maximize utilities. We obtain desirable results such as showing the pure Nash equilibrium existence and present an iterative-best response algorithm for finding such a solution. The proposed Shapley allocation has attractive properties such as they are not only fair but also stable and it dominates the current sharing scheme in terms of the social welfare. We collect the most updated data and provide a case study with the current complete LINK network of banks and ATM service operators. Managerial insights were drawn where we demonstrate the benefit of collaboration. We explained why current sharing scheme can lead to instability where sub-coalitions has the incentive to break away and how the Shapley allocation alleviates this issue. We also measure the social welfare and demonstrate that the Shapley allocation attains closer system optimality compared to the equal-cost sharing scheme. As the game is considerably large, we develop new techniques for finding pure Nash equilibrium and the Shapley value allocation.

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Appendix A. Equal cost sharing scheme

Under equal cost sharing, each bank in the network will pay a per-transaction cost for both the fixed cost and the variable transaction costs. For the fixed cost, the per-transaction cost is:

$$\pi^0 = \frac{\sum_{i\in\mathcal{B}} h_i(t_i)}{(N^1(\mathcal{B}) + N^2(\mathcal{B}))\alpha(t(\mathcal{B}))}.$$

For cash withdrawal transaction, the per-transaction cost is:

$$\begin{aligned}
\pi^1 &= \pi^0 + \frac{C^1(\mathcal{B},\boldsymbol{t})}{N^1(\mathcal{B})\alpha(t(\mathcal{B}))} = \pi^0 + \frac{\sum_{i,j\in\mathcal{B}}(c^1 - \Delta^1 1_{i=j}) n^1_{ij}(\mathcal{B},\boldsymbol{t})}{N^1(\mathcal{B})\alpha(t(\mathcal{B}))}\\
&= \pi^0 + \frac{c^1 N^1 \frac{t(\mathcal{B})}{T} - \Delta^1 \sum_{i\in\mathcal{B}} N^1_i t_i}{N^1(\mathcal{B})t(\mathcal{B})}\\
&= c^1 + \underbrace{\pi^0 - \frac{\Delta^1 \sum_{i\in\mathcal{B}} N^1_i t_i}{N^1(\mathcal{B})t(\mathcal{B})}}_{g^1(\boldsymbol{t})},
\end{aligned}$$

where $C^1(\mathcal{B},\boldsymbol{t})$ denotes the total variable cost of the network on cash-transactions.

The per-transaction cost for non-cash transactions is defined as

$$\begin{aligned}
\pi^2 &= \pi^0 + \frac{C^2(\mathcal{B},\boldsymbol{t})}{N^2(\mathcal{B})\alpha(t(\mathcal{B}))} = \frac{\sum_{i,j\in\mathcal{B}}(c^2 - \Delta^2 1_{i=j}) n^2_{ij}(\mathcal{B},\boldsymbol{t})}{N^2(\mathcal{B})\alpha(t(\mathcal{B}))}\\
&= \pi^0 + \frac{c^2 N^2 t(\mathcal{B}) - \Delta^2 \sum_{i\in\mathcal{B}} N^2_i t_i}{N^2(\mathcal{B})t(\mathcal{B})}\\
&= c^2 + \underbrace{\pi^0 - \frac{\Delta^2 \sum_{i\in\mathcal{B}} N^2_i t_i}{N^2(\mathcal{B})t(\mathcal{B})}}_{g^2(\boldsymbol{t})},
\end{aligned}$$

where $C^2(\mathcal{B},\boldsymbol{t})$ denotes the total cost of the network on non-cash transactions.

Appendix B. Proof of Theorem 2 - Benefit of collaboration

Proof. For any disjoint coalitions $\mathcal{S}_a, \mathcal{S}_b \in \mathcal{B}$, from Formulation (3), we have

$$v(\mathcal{S}_a \cup \mathcal{S}_b, \boldsymbol{t}) - v(\mathcal{S}_a, \boldsymbol{t}) - v(\mathcal{S}_b, \boldsymbol{t}) = \sum_{k\in\{1,2\}} \left(\delta^k_1 N^k(\mathcal{S}^a)\right)\left[\alpha(t(\mathcal{S}^a \cup \mathcal{S}^b)) - \alpha(t(\mathcal{S}^a))\right]$$

$$
\begin{aligned}
&+ \sum_{k\in\{1,2\}} \left(\delta_1^k N^k(\mathcal{S}^b)\right)\left[\alpha(t(\mathcal{S}^a\cup\mathcal{S}^b)) - \alpha(t(\mathcal{S}^b))\right]\\
&+ \Delta^k \sum_{i\in\mathcal{S}^a} t_i N_i^k \left[\frac{\alpha(t(\mathcal{S}^a\cup\mathcal{S}^b))}{t(\mathcal{S}^a\cup\mathcal{S}^b)} - \frac{\alpha(t(\mathcal{S}^a))}{t(\mathcal{S}^a)}\right]\\
&+ \Delta^k \sum_{i\in\mathcal{S}^b} t_i N_i^k \left[\frac{\alpha(t(\mathcal{S}^a\cup\mathcal{S}^b))}{t(\mathcal{S}^a\cup\mathcal{S}^b)} - \frac{\alpha(t(\mathcal{S}^b))}{t(\mathcal{S}^b)}\right],
\end{aligned}
$$

by cancelling our all the remaining terms given that $\mathcal{S}^a$ and $\mathcal{S}^b$ are disjoint.

If $\Delta = 0$, the last two terms are zero and hence, given that the first term is always non-negative, banks should collaborate. For $\Delta > 0$, the sign of the second term depends on function $\alpha(\cdot)$. We consider the following cases.

- If we have $\alpha(\tau)$ to be a constant independent on τ and $\Delta = 0$, then both terms are equal to zero. The game is additive and this means there is no gain or loss from collaboration.
- If we have $\alpha(\tau)$ to be a constant independent on τ and $\Delta > 0$, then the first term is equal to zero while the second term is negative. This means the game is sub-additive and the members should not collaborate to avoid the extra charge of Δ when using ATMs other than each own.
- If we have $\alpha(\tau)$ is a linearly increasing function on τ, then the first term is positive while the second term is equal to zero. This means the game is super-additive and banks should collaborate to form the grand coalition.
- If $\frac{\alpha(\tau)}{\tau}$ is a non-decreasing function on τ, then the first term is positive while the second term is non-negative. This also means the game is super-additive and the grand coalition is formed.
- For others cases, the sign of the second term depend on the monotonicity of $\frac{\alpha(\tau)}{\tau}$, or in other words the convexity of $\alpha(\tau)$. For example, when $\alpha(\tau)$ is a piece-wise linear concave function, the second term is negative. The relative magnitudes between the two terms will decide the sign of the quantity $v(\mathcal{S}_a\cup\mathcal{S}_b, \boldsymbol{t}) - v(\mathcal{S}_a, \boldsymbol{t}) - v(\mathcal{S}_b, \boldsymbol{t})$ which will decide whether member should collaborate. Here, we also note the roles that different constant factors such as δ_1^k and Δ^k, $k = 1, 2$, play when banks consider forming coalitions. ■

Appendix C. Proof of Theorem 3 - Convexity and Shapley value closed form

Proof. For each $i, j \in \mathcal{B}$, let us define $\lambda_i = \sum_{k=1}^2 \delta_2^k N_i^k$, $\omega_i = \sum_{k=1}^2 \frac{\Delta^k}{T} N_i^k t_i$, $\sigma_{i,j} = \sum_{k=1}^2 \frac{\delta_1^k}{2T}\left(N_i^k t_j + N_j^k t_i\right)$ and $\delta_3^k = \frac{\Delta^k}{T}$. Let $\Lambda = (\lambda_i)_{i\in\mathcal{B}}$, $\omega = (\omega_i)_{i\in\mathcal{B}}$ and $\Sigma = (\sigma_{ij})_{i,j\in\mathcal{B}}$. Let us also define $s_i = \sum_{j\in\mathcal{B}} \sigma_{ij}$.

For each coalition $\mathcal{S} \subset \mathcal{B}$, let $e(\mathcal{S}) \in \{0,1\}^m$ be the corresponding indicator vector with those components equal to one are precisely members of $\mathcal{S}$. Under the assumption that $\alpha(t(\mathcal{S})) = \frac{t(\mathcal{S})}{T}$, we can rewrite (3) as follows

$$
\begin{aligned}
v(\mathcal{S}, \boldsymbol{t}) &= \sum_{k\in\{1,2\}} \left(\delta_1^k N^k(\mathcal{S})\alpha(t(\mathcal{S})) - \delta_2^k N^k(\mathcal{S}) + \frac{\Delta^k}{t(\mathcal{S})}\alpha(t(\mathcal{S}))\sum_{i\in\mathcal{S}} t_i N_i^k\right) - \sum_{i\in\mathcal{S}} h_i(t_i)\\
&= \sum_{k\in\{1,2\}} \left(\frac{\delta_1^k}{T} e(\mathcal{S})(N^k t^T)e(\mathcal{S}) - \delta_2^k e(\mathcal{S})N^k + \frac{\Delta^k}{T}\sum_{i\in\mathcal{S}} t_i N_i^k\right) - \sum_{i\in\mathcal{S}} h_i(t_i)\\
&= e(\mathcal{S})^T \Sigma e(\mathcal{S}) - e(\mathcal{S})^T \Lambda + e(\mathcal{S})^T \omega - \sum_{i\in\mathcal{B}} h_i(t_i).
\end{aligned}
$$

(a) For any i and any $\mathcal{S} \not\ni i$, we have

$$
v(\mathcal{S}\cup i, \boldsymbol{t}) - v(\mathcal{S}, \boldsymbol{t}) = \sigma_{ii} + \sum_{j\in\mathcal{S}} \sigma_{ij} - \lambda_i + \omega_i - h_i(t_i), \tag{C.1}
$$

where σ_{ij} is the (i,j) element of Σ, and where λ_i, ω_i are the ith element of Λ, ω.
From Eq. (C.1), for any pair of coalitions $\mathcal{S}^a, \mathcal{S}^b$ with $\mathcal{S}^a \subsetneq \mathcal{S}^b$, with $i \notin \mathcal{S}^b$, we have

$$
\left(v(\mathcal{S}^b\cup i, \boldsymbol{t}) - v(\mathcal{S}^b, \boldsymbol{t})\right) - \left(v(\mathcal{S}^a\cup i, \boldsymbol{t}) - v(\mathcal{S}^a, \boldsymbol{t})\right) = \left(\sum_{j\in\mathcal{S}^b}\sigma_{ij} - \sum_{j\in\mathcal{S}^a}\sigma_{ij}\right) = \sum_{j\in\mathcal{S}^b\setminus\mathcal{S}^a}\sigma_{ij} > 0.
$$

Thus, the game is convex. Therefore, the core exists and the Shapley value belongs to the core.

(b) The Shapley value for player i is

$$
\begin{aligned}
\phi_i^{SH} &= \sum_{\mathcal{S}\subset\mathcal{B}\setminus} p_{|\mathcal{S}|}(v(\mathcal{S}\cup i, \boldsymbol{t}) - v(\mathcal{S}, \boldsymbol{t}))\\
&= \sum_{k=0}^{n-1} p_k \sum_{|\mathcal{S}|=k, i\notin\mathcal{S}} (v(\mathcal{S}\cup i, \boldsymbol{t}) - v(\mathcal{S}, \boldsymbol{t}))\\
&= \sigma_{ii} - \lambda_i + \omega_i - h_i(t_i) + \sum_{k=0}^{n-1} p_k \sum_{|\mathcal{S}|=k, i\notin\mathcal{S}} \sum_{j\in\mathcal{S}} \sigma_{ij}.
\end{aligned}
$$

For $k = 0$, we have $\sum_{|\mathcal{S}|=k, i\notin\mathcal{S}} \sum_{j\in\mathcal{S}} \sigma_{ij} = 0$. For $k \geq 1$, we have

$$
\sum_{|\mathcal{S}|=k, i\notin\mathcal{S}} \sum_{j\in\mathcal{S}} \sigma_{ij} = \binom{n-1}{k-1}(s_i - \sigma_{ii}),
$$

where s_i is the sum of column ith of Σ. Thus,

$$
\phi_i^{SH} = \sigma_{ii} - \lambda_i + \omega_i - h_i(t_i) + \sum_{k=0}^{n-1} p_k \binom{n-1}{k-1}(s_i - \sigma_{ii})
$$

$$= \sigma_{ii} - \lambda_i + \omega_i - h_i(t_i) + (s_i - \sigma_{ii})$$
$$= s_i - \lambda_i + \omega_i - h_i(t_i).$$

We can then rewrite the Shapley value as

$$\phi_i = \sum_{k=1}^{2}\left(\sum_{j\in\mathcal{B}} \frac{\delta_1^k}{2T}\left(N_i^k t_j + N_j^k t_i\right) - \delta_2^k N_i^k + \delta_3^k N_i^k t_i\right) - h_i(t_i). \quad \blacksquare$$

Appendix D. Proof of Theorem 4 - Shapley allocation dominates equal cost sharing

Proof. Let us first write down the social welfare function

$$\begin{aligned}
v(\mathcal{B},\boldsymbol{t}) &= \sum_{k\in\{1,2\}}\left(\delta_1^k N^k(\mathcal{B})\alpha(t(\mathcal{B})) - \delta_2^k N^k(\mathcal{B}) + \frac{\Delta^k}{t(\mathcal{B})}\alpha(t(\mathcal{B}))\sum_{i\in\mathcal{B}} t_i N_i^k\right) - \sum_{i\in\mathcal{B}} h_i(t_i)\\
&= \sum_{k\in\{1,2\}}\sum_{i\in\mathcal{B}}\left(\frac{\delta_1^k}{T}N^k t_i - \delta_2^k N_i^k + \delta_3^k t_i N_i^k\right) - \sum_{i\in\mathcal{B}} h_i(t_i)\\
&= \sum_{i\in\mathcal{B}} \xi_i(t_i),
\end{aligned}$$

where $\xi_i(t_i) = \sum_{k\in\{1,2\}}\left(\frac{\delta_1^k}{T}N^k t_i - \delta_2^k N_i^k + \delta_3^k t_i N_i^k\right) - h_i(t_i)$.

Let $u_i^{SH}(\boldsymbol{t})$ and $u_i^{ES}(\boldsymbol{t})$ be the utility of player ith under Shapley allocation and equal-cost sharing respectively. Let t^{SH} and t^{ES} be the corresponding Nash equilibriums under Shapley allocation and equal-cost sharing while t^{SO} be the social optimal solution.

Let us denote $v_i(\mathcal{S},\boldsymbol{t})$ as the benefit that member ith receives from joining coalition $\mathcal{S}$ (and before the settlement among members)

$$\begin{aligned}
v_i(\mathcal{S},\boldsymbol{t}) &= \sum_{k\in\{1,2\}}\Bigg(N_i^k\alpha(t(\mathcal{S}))\kappa_1 - N_i^k(1-\alpha(t(\mathcal{S})))\beta\kappa_2^k - N_i^k(1-\alpha(t(\mathcal{S})))(1-\beta)\kappa_3\\
&\quad - \sum_{j\in\mathcal{S}}(c^k - \Delta^k 1_{i=j})n_{ji}^k(\mathcal{S},\boldsymbol{t})\Bigg) - h_i(t_i),\\
&= \sum_{k\in\{1,2\}}\left(N_i^k\alpha(t(\mathcal{S}))(\delta_1^k + c^k) - N_i^k\delta_2^k - \sum_{j\in\mathcal{S}}(c^k - \Delta^k 1_{i=j})n_{ji}^k(\mathcal{S},\boldsymbol{t})\right) - h_i(t_i)\\
&= \sum_{k\in\{1,2\}}\left(N_i^k\alpha(t(\mathcal{S}))(\delta_1^k + c^k) - N_i^k\delta_2^k - c^k N^k(\mathcal{S})\frac{t_i}{t(\mathcal{S})}\alpha(t(\mathcal{S})) + \frac{\Delta^k t_i}{t(\mathcal{S})}N_i^k\alpha(t(\mathcal{S}))\right) - h_i(t_i),
\end{aligned}$$

which forms individual parts of the coalitional value in Formulations (2) and (3). Here, we utilize the formulation for $n_{ij}^k(\mathcal{S},\boldsymbol{t})$ from (1) for the derivation.

The utility function for player ith under equal cost-sharing π is

$$\begin{aligned}
u_i^{ES}(\boldsymbol{t}) &= v_i(\mathcal{B},\boldsymbol{t}) + \sum_{k\in\{1,2\}}\pi^k\left(\sum_{j\in\mathcal{B}} n_{ji}^k(\mathcal{B},\boldsymbol{t}) - \sum_{j\in\mathcal{B}} n_{ij}^k(\mathcal{B},\boldsymbol{t})\right) - h_i(t_i)\\
&= \sum_{k\in\{1,2\}}\left(N_i^k\alpha(t(\mathcal{B}))(\delta_1^k + c^k) - N_i^k\delta_2^k - c^k N^k(\mathcal{B})\frac{t_i}{t(\mathcal{B})}\alpha(t(\mathcal{B})) + \frac{\Delta^k t_i}{t(\mathcal{B})}N_i^k\alpha(t(\mathcal{B}))\right) - h_i(t_i)\\
&\quad + \sum_{k\in\{1,2\}}(c^k + g^k)\left(\sum_{j\in\mathcal{B}} n_{ji}^k(\mathcal{B},\boldsymbol{t}) - \sum_{j\in\mathcal{B}} n_{ij}^k(\mathcal{B},\boldsymbol{t})\right)\\
&= \sum_{k\in\{1,2\}}\left(N_i^k\alpha(t(\mathcal{B}))(\delta_1^k + c^k) - N_i^k\delta_2^k - c^k N^k(\mathcal{B})\frac{t_i}{t(\mathcal{B})}\alpha(t(\mathcal{B})) + \frac{\Delta^k t_i}{t(\mathcal{B})}N_i^k\alpha(t(\mathcal{B}))\right) - h_i(t_i)\\
&\quad + \sum_{k\in\{1,2\}}(c^k + g^k)\alpha(t(\mathcal{B}))\left(\frac{t_i}{t(\mathcal{B})}N^k(\mathcal{B}) - N_i^k\right)\\
&= \sum_{k\in\{1,2\}}\left[N_i^k\alpha(t(\mathcal{B}))\delta_1^k - N_i^k\delta_2^k + \frac{\Delta^k t_i}{t(\mathcal{B})}N_i^k\alpha(t(\mathcal{B})) + g^k\alpha(t(\mathcal{B}))\left(\frac{t_i}{t(\mathcal{B})}N^k(\mathcal{B}) - N_i^k\right)\right] - h_i(t_i).
\end{aligned}$$

Under Assumption 2, this can be further simplified as

$$u_i^{ES}(\boldsymbol{t}) = \sum_{k\in\{1,2\}}\left(\delta_1^k N_i^k\frac{t(\mathcal{B})}{T} - \delta_2^k N_i^k + \delta_3^k N_i^k t_i + g^k\frac{N^k t_i - N_i^k t(\mathcal{B})}{T}\right) - h_i(t_i).$$

For equal cost per transaction, t_i^{ES} solves:

$$0 = \frac{\partial u_i^{ES}(\boldsymbol{t})}{\partial t_i}\Big|_{t_i^{ES}} = \sum_{k=1}^{2}\frac{\delta_1^k}{T}N_i^k + \delta_3^k N_i^k + g^k\frac{N^k - N_i^k}{T} - \frac{\partial h_i(t_i)}{\partial t_i}\Big|_{t_i^{ES}}, \forall\, i\in\mathcal{B}. \tag{D.1}$$

For Shapley value, we first use the formulation for the Shapley allocation in (8) to derive the utility function for member i as

$$u_i^{SH}(\boldsymbol{t}) = \phi_i - h_i(t_i) = \sum_{k=1}^{2}\left(\sum_{j\in\mathcal{B}}\frac{\delta_1^k}{2T}\left(N_i^k t_j + N_j^k t_i\right) - \delta_2^k N_i^k + \delta_3^k N_i^k t_i\right) - h_i(t_i).$$

The NE under the Shapley allocation solves:

$$0 = \frac{\partial u_i^{SH}(\boldsymbol{t})}{\partial t_i}|_{t_i^{SH}} = \sum_{k=1}^{2} \frac{\delta_1^k}{T}(N_i^k + \frac{1}{2}(N^k - N_i^k)) + \delta_3^k N_i^k - \frac{\partial h_i(t_i)}{\partial t_i}|_{t_i^{SH}}, \forall\ i \in \mathcal{B}. \tag{D.2}$$

For social optimal, NE solves:

$$0 = \frac{\partial v(\mathcal{B},\boldsymbol{t})}{\partial t_i}|_{t_i^{SO}} = \sum_{k=1}^{2} \frac{\delta_1^k}{T} N^k + \delta_3^k N_i^k - \frac{\partial h_i(t_i)}{\partial t_i}|_{t_i^{SO}}, \forall\ i \in \mathcal{B}. \tag{D.3}$$

Under Assumption 3, we can show that

$$g^k = \pi^k - c^k \le \frac{1}{2}(\kappa_1^k + \beta\kappa_2^k + (1-\beta)\kappa_3^k - c^k) = \frac{\delta_1^k}{2},$$

which leads to

$$\sum_{k=1}^{2} \frac{\delta_1^k}{T} N_i^k + \delta_3^k N_i^k + g^k \frac{N^k - N_i^k}{T} \le \sum_{k=1}^{2} \frac{\delta_1^k}{T}(N_i^k + \frac{1}{2}(N^k - N_i^k)) + \delta_3^k N_i^k \le \sum_{k=1}^{2} \frac{\delta_1^k}{T}^k N^k + \delta_3^k N_i^k$$

This together with Eqs. (D.1)–(D.3) lead to,

$$\frac{\partial h_i(t_i)}{\partial t_i}|_{t_i^{ES}} \le \frac{\partial h_i(t_i)}{\partial t_i}|_{t_i^{SH}} \le \frac{\partial h_i(t_i)}{\partial t_i}|_{t_i^{SO}}.$$

As $h_i(t_i)$ is non-decreasing on t_i, t_i^{SH} is in the range between that for social optimal and equal cost sharing. Since the function $h_i(t_i)$ is convex, $\xi_i(t_i)$ is also concave. Thus, $\xi_i(t_i)$ is monotonically non-decreasing between $[0, t_i^{SO}]$ and hence

$$\xi_i(t_i^{ES}) \le \xi_i(t_i^{SH}) \le \xi_i(t_i^{SO}).$$

This leads to the dominance of the Shapley allocation over equal cost sharing allocation in terms of social welfare. ■

Remark. Note that the above result also holds if, for each $i \in \mathcal{B}$, function $\xi_i(t_i)$ is unimodal.

Appendix E. Proof of Theorem 7 - Solving best response problem

Proof. We can rewrite the characteristic function in terms of X as follows.

$$\begin{aligned}
v(S,t) &= \sum_{k\in\{1,2\}} \left(\delta_1^k N^k(S)\alpha(t(S)) - \delta_2^k N^k(S) + \frac{\Delta^k}{t(S)}\alpha(t(S)) \sum_{j\in S} t_j N_j^k \right) - \sum_{i\in S} h_i(t_i) \\
&= \sum_{k\in\{1,2\}} \left(\delta_1^k N^k(S)\frac{at(S)}{1+at(S)} - \delta_2^k N^k(S) + \Delta^k \frac{a}{1+at(S)} \sum_{j\in S} t_j N_j^k \right) - \sum_{i\in S} h_i(t_i) \\
&= \sum_{k\in\{1,2\}} \delta_1^k N^k(S) - \delta_2^k N^k(S) - \frac{\delta_1^k N^k(S) - a\Delta^k \sum_{j\in S} t_j N_j^k}{1+at(S)} - \sum_{i\in S} h_i(t_i) \\
&= \left(X^T \vartheta_1 - \frac{X^T\vartheta_2}{(1+aX^T t)} \right) - \sum_{i\in S} h_i(t_i).
\end{aligned} \tag{E.1}$$

We also have

$$\begin{aligned}
\frac{\partial v(S,t)}{\partial t_i} &= \sum_{k\in\{1,2\}} \left(\delta_1^k N^k(S)\frac{a}{(1+at(S))^2} + \Delta^k a \frac{(1+at(S))N_i^k - a\sum_{j\in S} t_j N_j^k}{(1+at(S))^2} \right) - \frac{\partial h_i(t_i)}{\partial t_i} \\
&= \sum_{k\in\{1,2\}} \left(\frac{a\delta_1^k N^k(S) - a^2\Delta^k \sum_{i\in S} t_i N_i^k}{(1+at(S))^2} + \Delta^k a \frac{N_i^k}{(1+at(S))} \right) - \frac{\partial h_i(t_i)}{\partial t_i} \\
&= \sum_{k\in\{1,2\}} \left(\frac{X^T(a\delta_1^k N^k - a^2\Delta^k q)}{(1+aX^T t)^2} + \Delta^k a \frac{N_i^k}{(1+aX^T t)} \right) - \frac{\partial h_i(t_i)}{\partial t_i} \\
&= \left(\frac{X^T\vartheta_3}{(1+aX^T t)^2} + \frac{\gamma_{1i}}{(1+aX^T t)} \right) - \frac{\partial h_i(t_i)}{\partial t_i}.
\end{aligned} \tag{E.2}$$

Finally, we have

$$\begin{aligned}
\frac{\partial^2 v(S,t)}{\partial t_i^2} &= \sum_{k\in\{1,2\}} \Bigg(-\delta_1^k N^k(S)\frac{2a^2}{(1+at(S))^3} - \Delta^k N_i^k \frac{a^2}{(1+at(S))^2} \\
&\qquad -\Delta^k a^2 \frac{N_i^k(1+at(S)) - 2a\sum_{i\in S} t_i N_i^k}{(1+at(S))^3} \Bigg) - \frac{\partial h_i^2(t_i)}{\partial t_i^2} \\
&= \sum_{k\in\{1,2\}} \left(-\delta_1^k N^k(S)\frac{2a^2}{(1+at(S))^3} - \Delta^k N_i^k \frac{2a^2}{(1+at(S))^2} + 2\Delta^k a^3 \frac{\sum_{i\in S} t_i N_i^k}{(1+at(S))^3} \right) - \frac{\partial h_i^2(t_i)}{\partial t_i^2} \\
&= \sum_{k\in\{1,2\}} \left(\frac{-2a^2\delta_1^k N^k(S) + 2a^3\Delta^k \sum_{i\in S} t_i N_i^k}{(1+at(S))^3} \right) - \left(\frac{2a^2\Delta^k N_i^k}{(1+at(S))^2} \right) - \frac{\partial h_i^2(t_i)}{\partial t_i^2}
\end{aligned}$$

$$= \left(\frac{X^T \vartheta_4}{(1 + aX^T t)^3} + \frac{\gamma_{2i}}{(1 + aX^T t)^2} \right) - \frac{\partial h_i^2(t_i)}{\partial t_i^2}. \tag{E.3}$$

Eqs. (E.1)–(E.3) together with results from Theorem 6 lead to the results stated in the theorem. ■

Appendix F. Solving large ATM coopetition game

While the literature on generalized univariate CTL is well established and there are existing results on multivariate CTL too, the latter requires IID assumption which does not hold in our case. We will instead utilize the Cramer-Wold's device (Cramér & Wold, 1936) and the Lyapunov version of univariate CTL (Billingsley, 2008) for this purpose.

For a sequence of positive parameters $\theta_1, \theta_2 \dots$, let $W_j = \theta_j X_j$ be a random variable which receives value θ_j w.p. x and 0 w.p. $(1-x)$. Here, we assume that $0 < x < 1$ as for $x = 0$ or $x = 1$, we have the corresponding equivalent deterministic cases, which are more straightforward to compute the expectations.

We have $W_1, W_2 \dots$ is a sequence of independent random variables with finite expected values $\mu_j = x\theta_j$ and $\varsigma_j^2 = x(1-x)\theta_j^2$. Let $s_n^2 = \sum_{j=1}^n \varsigma_j^2 = x(1-x)\sum_{j=1}^n \theta_j^2$. If there exists some $\eta > 0$ such that the following Lyapunov's condition holds,

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\eta}} \sum_{i=1}^n \mathrm{E}\left[|W_i - \mu_i|^{2+\eta}\right] = 0,$$

then $\sum_{j=1}^n W_j$ converges in distribution to a normal distribution $N(x\sum_{j=1}^n \theta_j, x(1-x)\sum_{j=1}^n \theta_j^2)$.

We can show that the Lyapunov's condition with $\eta = 2$ on the sequence W is equivalent to

$$\frac{1}{s_n^{2+\eta}} \sum_{i=1}^n \mathrm{E}\left[|W_i - \mu_i|^{2+\eta}\right] = \frac{1}{s_n^{2+\eta}} \sum_{i=1}^n \left(x(1-x)^4\theta_j^4 + (1-x)x^4\theta_j^4 \right) = \frac{1 - 3x + 3x^2}{x(1-x)} \frac{\sum_{i=1}^n \theta_j^4}{\left(\sum_{j=1}^n \theta_j^2\right)^2}.$$

Thus, the Lyapunov's condition with $\eta = 2$ is equivalent to

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \theta_j^4}{(\sum_{j=1}^n \theta_j^2)^2} = 0. \tag{F.1}$$

This condition is satisfied if there is no single element of vector θ that dominates the rest as Owen has referred to in his example on estimating the Shapley value of a majority voting game. This can also be applied to more generic situations such as when there is a fixed proportion of largest elements that are bounded. For example, if we know there exist $\underline{\theta}, \bar{\theta}$ such that a portion of the largest elements of θ are bounded, i.e., $0 \le \underline{\theta} \le |\theta_j| \le \bar{\theta}, \forall j \in \mathcal{B}_1$ and $|\theta_i| \le \underline{\theta}$, $\forall i \in \mathcal{B}\backslash\mathcal{B}_1$, and that the size of $\mathcal{B}_1$ goes to infinity, we then have

$$0 \le \lim_{n\to\infty} \frac{\sum_{i=1}^n \theta_j^4}{(\sum_{j=1}^n \theta_j^2)^2} \le 2 \lim_{n\to\infty} \frac{\sum_{j\in\mathcal{B}_1} \theta_j^4}{(\sum_{j\in\mathcal{B}_1} \theta_j^2)^2} \le 2 \lim_{|\mathcal{B}_1|\to\infty} \frac{|\mathcal{B}_1|\bar{\theta}^4}{|\mathcal{B}_1|^2\underline{\theta}^4} = \frac{2\bar{\theta}^4}{\underline{\theta}^4} \lim_{|\mathcal{B}_1|\to\infty} \frac{1}{|\mathcal{B}_1|} = 0.$$

The following assumption states the Lyapunov's condition on weighted versions of t and ϑ_j, $j = 1, \dots, 4$.

Assumption 5. For each $\varrho \in \mathbb{R}$, vector $\theta := \varrho t + (1-\varrho)\vartheta_j$ satisfies the Lyapunov's condition (F.1) for each $j = 1, \dots, 4$.

Remark. Assumption 5 requires that vectors t and ϑ_j are not perfectly correlated and there are no small portions of their elements that dominate the rest as shown earlier.

Remark. It should be noted that Assumption 5 is required for the perfect estimation of the potential function and its derivatives using multivariate normal distributions. For practical purposes where an approximation is acceptable, the assumption does not need to hold perfectly.

Eqs. (15)–(17) have the same form so we will show the result for the first derivative of the potential function only. Results for the potential function and the second derivative can be derived in the same manner.

Theorem 8. *Under Assumption 5, as $n \to \infty$, we have*

$$\frac{\partial\tilde{\Phi}_i(t_i, t_{-i})}{\partial t_i} \to \int_0^1 \sum_{p=1}^P \left\{ \mu_y(\Psi_y(T_{p+1}) - \Psi_y(T_p)) - \sigma_{yz}(\psi_y(T_{p+1}) - \psi_y(T_p)) \right\} dx - \frac{\partial h_i(t_i)}{\partial t_i},$$

where

$$\begin{bmatrix} \mu_y \\ \mu_z \end{bmatrix} = \begin{bmatrix} x \sum_{j\in\mathcal{B}\backslash\{i\}} t_j \\ x\left(\dfrac{\sum_{j\in\mathcal{B}} \vartheta_{3j}}{(1 + a\bar{T}_p)^2} + \dfrac{\gamma_{1i}}{(1 + a\bar{T}_p)} \right) \end{bmatrix}, \quad \begin{bmatrix} \sigma_y^2 & \sigma_{yz} \\ \sigma_{yz} & \sigma_z^2 \end{bmatrix} = x(1-x) \begin{bmatrix} \sum_{j\in\mathcal{B}\backslash\{i\}} t_j^2 & \dfrac{\sum_{j\in\mathcal{B}\backslash\{i\}} t_j\vartheta_{3j}}{(1 + a\bar{T}_p)^2} \\ \dfrac{\sum_{j\in\mathcal{B}\backslash\{i\}} t_j\vartheta_{3j}}{(1 + a\bar{T}_p)^2} & \dfrac{\sum_{j\in\mathcal{B}\backslash\{i\}} \vartheta_{3j}^2}{(1 + a\bar{T}_p)^4} \end{bmatrix},$$

and where $\psi_y(y_0) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y_0-\mu_y)^2}{2\sigma_y^2}}$ and $\Psi_y(y_0) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y_0-\mu_y)^2}{2\sigma_y^2}}$ are the density function and cumulative distribution function of normal random variable Y.

Proof. We can take the summation out of the Expectation in Eq. (16) and rewrite it as

$$\frac{\partial \tilde{\Phi}_i(t_i,t_{-i})}{\partial t_i} = \int_0^1 \sum_{p=1}^{P} 1_{[T_p \le X_{+i}^T t < T_{p+1}]} \mathrm{E}\left[\left\{ \frac{X_{+i}^T \vartheta_3}{(1+a\bar{T}_p)^2} + \frac{\gamma_{1i}}{(1+a\bar{T}_p)} \right\}\right] dx - \frac{\partial h_i(t_i)}{\partial t_i} \tag{F.2}$$

$$= \int_0^1 \sum_{p=1}^{P} 1_{[T_p \le X_{-i}^T t + t_i < T_{p+1}]} \mathrm{E}\left[\left\{ \frac{X_{-i}^T \vartheta_3 + \vartheta_{3i}}{(1+a\bar{T}_p)^2} + \frac{\gamma_{1i}}{(1+a\bar{T}_p)} \right\}\right] dx - \frac{\partial h_i(t_i)}{\partial t_i} \tag{F.3}$$

$$= \int_0^1 \sum_{p=1}^{P} 1_{[T_p - t_i \le X_{-i}^T t < T_{p+1} - t_i]} \mathrm{E}\left[\left\{ X_{-i}^T \vartheta_5 + \gamma_{3i} \right\}\right] dx - \frac{\partial h_i(t_i)}{\partial t_i}, \tag{F.4}$$

where $X_{-i} = X \mid X_i = 0$, and $\vartheta_5 = \frac{\vartheta_3}{(1+a\bar{T}_p)^2}$ and $\gamma_{3i} = \frac{\vartheta_{3i}}{(1+a\bar{T}_p)^2} \frac{\gamma_{1i}}{(1+a\bar{T}_p)}$

Let us define $Y := X_{-i}^T t = \sum_{j\in\mathcal{B}\setminus\{i\}} t_j X_j$ and $Z := X_{-i}^T \vartheta_5 + \gamma_{3i}$. Since vectors t and ϑ_5 satisfy the Lyapunov's condition, we have both Y and Z converge to normal distributions as n approaches infinity.

For any $\varrho \in \mathbb{R}$, we have $\varrho Y + (1-\varrho) Z$ is also a weighted average of IID random variables X_{-i} with the new set of weights satisfying the Lyapunov's condition by Assumption 5. Applying the Cramer-Wold's device, as $n \to \infty$, we have (Y, Z) approaches $N\left(\begin{bmatrix}\mu_y\\ \mu_z\end{bmatrix}, \begin{bmatrix}\sigma_y^2 & \sigma_{yz}\\ \sigma_{yz} & \sigma_z^2\end{bmatrix}\right)$, where the two moments are defined in the theorem.

Let $\psi(\cdot)$ and $\Psi(\cdot)$ be the density function and cumulative distribution of the standard bivariate normal distribution, respectively. We need to find $\mathrm{E}\left[1_{T_p - t_i \le Y \le T_{p+1} - t_i} Z\right]$. Note that $Z \mid Y$ is a normal distribution $N(\mu_z + \frac{\sigma_{yz}}{\sigma_y^2}(Y-\mu_y), \sigma_z^2 - \frac{\sigma_{yz}^2}{\sigma_y^2})$. Thus, $\mathrm{E}[Z \mid Y] = \mu_z + \frac{\sigma_{yz}}{\sigma_y^2}(Y - \mu_y)$. As a result, we have

$$\begin{aligned}
\mathrm{E}\left[1_{T_p - t_i \le Y \le T_{p+1} - t_i} Z\right] &= \mathrm{E}\left[1_{T_p - t_i \le Y \le T_{p+1} - t_i} \mathrm{E}[Z \mid Y]\right] \\
&= \mathrm{E}\left[1_{T_p - t_i \le Y \le T_{p+1} - t_i} \left(\mu_z + \frac{\sigma_{yz}}{\sigma_y^2}(Y-\mu_y)\right)\right] \\
&= \mathrm{E}\left[1_{T_p - t_i \le Y \le T_{p+1} - t_i} \left(\mu_z - \frac{\sigma_{yz}}{\sigma_y^2}\mu_y + \frac{\sigma_{yz}}{\sigma_y^2} Y\right)\right] \\
&= \left(\mu_z - \frac{\sigma_{yz}}{\sigma_y^2}\mu_y\right) \mathrm{E}\left[1_{T_p - t_i \le Y \le T_{p+1} - t_i}\right] + \frac{\sigma_{yz}}{\sigma_y^2} \mathrm{E}\left[1_{T_p - t_i \le Y \le T_{p+1} - t_i} Y\right] \\
&= \left(\mu_z - \frac{\sigma_{yz}}{\sigma_y^2}\mu_y\right) (\Psi_y(T_{p+1} - t_i) - \Psi_y(T_p - t_i)) + \frac{\sigma_{yz}}{\sigma_y^2} \int_{T_p - t_i}^{T_{p+1} - t_i} y f \psi_y(y) dy \\
&= \left(\mu_z - \frac{\sigma_{yz}}{\sigma_y^2}\mu_y\right) (\Psi_y(T_{p+1} - t_i) - \Psi_y(T_p - t_i)) + \frac{\sigma_{yz}}{\sigma_y^2} \int_{T_p - t_i}^{T_{p+1} - t_i} y \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}} dy \\
&= \mu_z (\Psi_y(T_{p+1} - t_i) - \Psi_y(T_p - t_i)) - \frac{\sigma_{yz}}{\sigma_y} (\psi_y(T_{p+1} - t_i) - \psi_y(T_p - t_i)) \\
&= \mu_z \left(\Psi\left(\frac{T_{p+1} - t_i - \mu_y}{\sigma_y}\right) - \Psi\left(\frac{T_p - t_i - \mu_y}{\sigma_y}\right)\right) \\
&\quad - \frac{\sigma_{yz}}{\sigma_y} \left(\psi\left(\frac{T_{p+1} - t_i - \mu_y}{\sigma_y}\right) - \psi\left(\frac{T_p - t_i - \mu_y}{\sigma_y}\right)\right).
\end{aligned}$$

This leads to the results stated in the theorem. ■

Remark. The complexity of the algorithm is linear on P and hence it is possible to use a reasonably large P if we desire more accuracy on the estimations of the potential function and its derivatives.

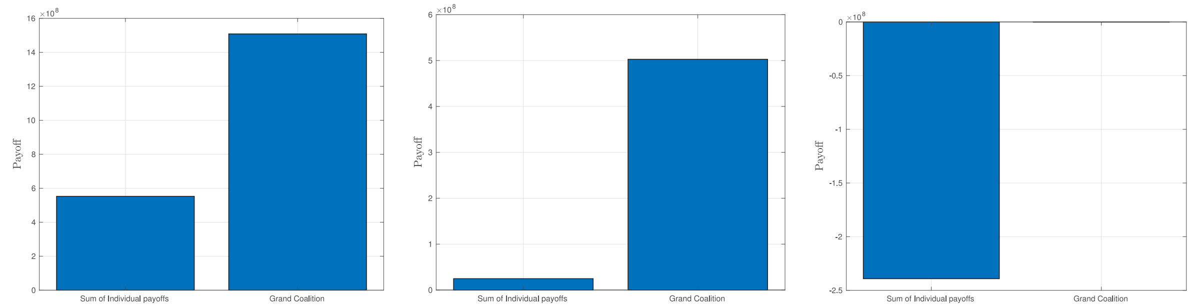
Remark. For other cooperative games, the function $u(x)$ might be of more complex forms and it might not be straightforward to find a closed-form solution. Nevertheless, the process of deriving $u(x)$ as an expectation over some function of multi-variate normal random variable (Y, Z) would be very useful. We can then take random samples (Y, Z) to approximate $u(x)$. While both (Y, Z) and $(X_j)_{j\in\mathcal{B}}$ are random variables, sampling over (Y, Z) is computationally cheaper than to sample from X due to dimensionality reduction. In fact, we use this technique for estimating the Shapley value and the potential function in our numerical demonstration.

Appendix G. Finding Shapley allocation

Similar to the potential function, although the Shapley value has a close form, its computation is not easy for large general games, i.e., with the number of players above 20. We can use the same method developed for solving the BRP as follows.

We have

$$\begin{aligned}
\phi_i(t) &= \sum_{S\subseteq\mathcal{B}\setminus\{i\}} \lambda_{|S|} (v(S\cup\{i\}, t) - v(S, t)) - h_i(t_i) \\
&= \sum_{S\subseteq\mathcal{B}\setminus\{i\}} \int_0^1 x^{|S|}(1-x)^{n-|S|-1} (v(S\cup\{i\}, t) - v(S, t))\, dx - h_i(t_i) \\
&= \int_0^1 \sum_{S\subseteq\mathcal{B}\setminus\{i\}} x^{|S|}(1-x)^{n-|S|-1} (v(S\cup\{i\}, t) - v(S, t))\, dx - h_i(t_i).
\end{aligned}$$



(a) Total transactions are increased twice (b) Base Line (c) Total transactions are halved

Fig. H.6. Varying total number of transactions.

Let X be the indicator vector of $\mathcal{S}$. We will rewrite $v(\mathcal{S}\cup\{i\},\boldsymbol{t})-v(\mathcal{S},\boldsymbol{t})$ as a function of X as follows.

$$\begin{aligned}
v(\mathcal{S}\cup\{i\},\boldsymbol{t})-v(\mathcal{S},\boldsymbol{t}) &= \sum_{k\in\{1,2\}}(\delta_1^k-\delta_2^k)N_i^k-\frac{\delta_1^k N^k(\mathcal{S}\cup\{i\})-a\Delta^k\sum_{j\in\mathcal{S}\cup\{i\}}t_jN_j^k}{1+at(\mathcal{S}\cup\{i\})}\\
&\quad+\frac{\delta_1^kN^k(\mathcal{S})-a\Delta^k\sum_{i\in\mathcal{S}}t_iN_i^k}{1+at(\mathcal{S})}-h_i(t_i)\\
&=\sum_{k\in\{1,2\}}(\delta_1^k-\delta_2^k)N_i^k-\frac{A}{B}-h_i(t_i),
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\delta_1^kN^k(\mathcal{S}\cup\{i\})-a\Delta^k\sum_{j\in\mathcal{S}\cup\{i\}}t_jN_j^k\right)(1+at(\mathcal{S}))-\left(\delta_1^kN^k(\mathcal{S})-a\Delta^k\sum_{i\in\mathcal{S}}t_iN_i^k\right)(1+at(\mathcal{S}\cup\{i\}))\\
&=(1+at(\mathcal{S}))(\delta_1^kN_i^k-a\Delta^kt_iN_i^k)+at_i(\delta_1^kN^k(\mathcal{S})-a\Delta^k\sum_{j\in\mathcal{S}}t_jN_j^k)\\
&=\sum_{k=1}^{2}X^t\boldsymbol{t}\left(a\delta_1^kN_i^k-a^2\Delta^kt_iN_i^k\right)+X^T\left(at_i\delta_1^kN^k-a^2t_i\Delta^k\boldsymbol{q}\right)+\left(\delta_1^kN_i^k-a\Delta^kt_iN_i^k\right),
\end{aligned}$$

and $B=(1+at(\mathcal{S}\cup\{i\}))(1+at(\mathcal{S}))=(1+aX^t\boldsymbol{t}+at_i)(1+aX^t\boldsymbol{t})$.

Thus,

$$\begin{aligned}
\phi_i(\boldsymbol{t}) &= \int_0^1\sum_{\mathcal{S}\subseteq\mathcal{B}\setminus\{i\}}x^{|\mathcal{S}|}(1-x)^{n-|\mathcal{S}|-1}\,(v(\mathcal{S}\cup\{i\},\boldsymbol{t})-v(\mathcal{S},\boldsymbol{t}))\,dx-h_i(t_i)\\
&=(\delta_1^k-\delta_2^k)N_i^k+\int_0^1x^{|\mathcal{S}|}(1-x)^{n-|\mathcal{S}|-1}\sum_{\mathcal{S}\subseteq\mathcal{B}\setminus\{i\}}\left(\frac{\gamma_1X^t\boldsymbol{t}+X^T\boldsymbol{f}+\gamma_2}{(1+aX^t\boldsymbol{t}+at_i)(1+aX^t\boldsymbol{t})}\right)dx-h_i(t_i)\\
&=(\delta_1^k-\delta_2^k)N_i^k+\int_0^1E\left(\frac{\gamma_1X^t\boldsymbol{t}+X^T\boldsymbol{f}+\gamma_2}{(1+aX^t\boldsymbol{t}+at_i)(1+aX^t\boldsymbol{t})}\right)dx-h_i(t_i).
\end{aligned}$$

We can draw sample of X to estimate the expectation. For very large game, we can use the same technique developed in Appendix F and arrive at an approximated Shapley value as

$$\phi_i(\boldsymbol{t})\approx(\delta_1^k-\delta_2^k)N_i^k+\int_0^1\left\{\sum_{p=1}^{P}1_{[T_p\le\bar{T}_p<T_{p+1}]}E\left[\frac{\gamma_1X^t\boldsymbol{t}+X^T\boldsymbol{f}+\gamma_2}{(1+a\bar{T}_p+at_i)(1+a\bar{T}_p)}\right]\right\}dx-h_i(t_i).$$

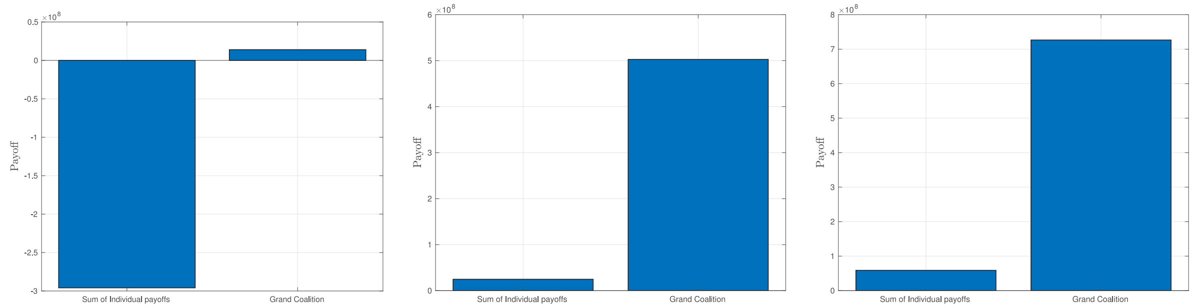
Under similar assumption, we have as $n\to\infty$

$$\phi_i(\boldsymbol{t})\to\int_0^1\sum_{p=1}^{P}\left\{\mu_y(\Phi(T_{p+1})-\Phi(T_p))-\sigma_{yz}(f_z(T_{p+1})-f_z(T_p))\right\}dx-\frac{\partial h_i(t_i)}{\partial t_i},$$

Appendix H. Sensitivity analysis

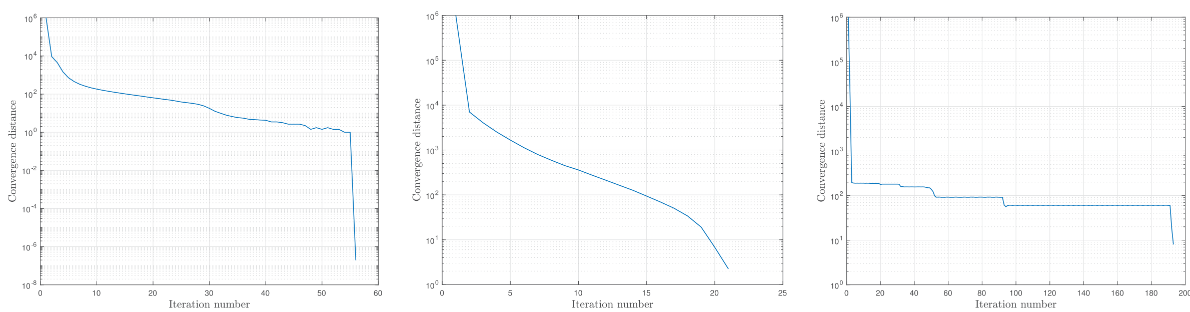
Fig. H.6 shows how the benefit of collaboration might change when the number of transactions is increased to twice as much the current values or to decrease to halves. We can see that, the higher are the total transactions, the higher are the values of the individual coalitions as well as that of the grand coalition. In all cases, we can see that the value of the grand coalition is higher than the sum of the individual payoffs and this shows that it is generally beneficial to collaborate.

Fig. H.7 shows how the benefit of collaboration might change when the numbers of ATMs are increased to twice as much the current values or to decrease to halves while keeping the total number of transactions the same. We can see that, the higher are the total number of ATMs, the smaller are the values of the individual payoffs as well as that of the grand coalition. Similar to the previous example, it is generally beneficial to collaborate as evidenced by the value of the grand coalition to be higher than the sum of the individual payoffs.



(a) Total ATMs are increased twice (b) Base Line (c) Total ATMs are halved

Fig. H.7. Varying total number of ATMs.



(a) Shapley Allocation (b) Equal Weight (c) Social Welfare Optimisation

Fig. I.8. Algorithm Convergence.

Appendix I. Algorithm convergence

Fig. I.8 shows how the Nash equilibrium is reached through iterations by measuring the norms of the differences between consecutive $\boldsymbol{t}^k$. We can see that in all cases, these converges to zero when the solution $\boldsymbol{t}$ no-longer changes. In all cases, the algorithm converges quite fast to Nash equilibrium.

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