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Measuring productivity in networks: A game-theoretic approach

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ABSTRACT

Measuring individual productivity (or equivalently distributing the overall productivity) in a network structure of workers displaying peer effects has been a subject of ongoing interest in many areas ranging from academia to industry. In this paper, we propose a novel approach based on cooperative game theory that takes into account the peer effects of worker productivity represented by a complete bipartite network of interactions. More specifically, we construct a series of cooperative games where the characteristic function of each coalition of workers is equal to the sum of each worker intrinsic productivity as well as the productivity of other workers within a distance discounted by an attenuation factor. We show that these (truncated) games are balanced and converge to a balanced game when the distance of influence grows large. We then provide an explicit formula for the Shapley value and propose an alternative coalitionally stable distribution of productivity which is computationally much more tractable than the Shapley value. Lastly, we characterize this alternative distribution based on three sensible properties of a logistic network. This analysis enhances our understanding of game-theoretic analysis within logistics networks, offering valuable insights into the peer effects' impact when assessing the overall productivity and its distribution among workers.

1. Introduction

Game theory and network productivity are two fields that have been applied to the study of logistics networks. In general, game theory is a branch of mathematics that studies strategic decision making in various interactions, while network productivity is concerned with the efficiency and effectiveness of networks. In the context of logistics, these fields have been used to study how decisions made by individual actors within a supply chain can affect the overall efficiency and productivity of the network. Potential advances in this area could include the development of new mathematical models and/or algorithms to analyze logistics networks, the application of game theory and network productivity principles to real-world logistics problems, or the integration of these fields with other areas of logistics research.

This paper focuses on analyzing the measurement of worker productivity in a logistics network represented by a complete bipartite network. Such a network structure is particularly interesting from the perspective of cooperative game theory, as all its induced sub-networks maintain the same network structure. This network structure effectively simulates various productive and logistical relationships, such as the interconnection between goods suppliers and consumers, where collaboration between both groups is crucial for efficient provision of goods. Another real-world example of a complete bipartite

network could be a food supply system connecting producers with retailers, with producers forming one team and retailers forming the other. Additionally, the concept is applicable to the internal structure of companies, where work teams are divided into two fully connected groups. In this scenario, the network's efficiency depends not only on the individual productivity of workers within each team but also on the connectivity and collaboration between the two teams.

Measuring the productivity of workers in a network is crucial as it enables the identification (and reward) of the most effective employees in their roles. This knowledge empowers managers to concentrate their resources and training initiatives on those individuals who require performance improvement. Additionally, productivity measurement assists in detecting bottlenecks within the network and areas where efficiency enhancements can be made. Such insights aid managers in making informed decisions regarding network re-organization or job reassignments to enhance overall efficiency.

The network productivity can be thought of as a public good for several reasons. Firstly, network productivity is essential for the efficient functioning and provision of common goods across various contexts such as the environment, health, and logistics. Given that common goods are accessible to the majority of society, network productivity is crucial to ensuring the availability and effective distribution of

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these goods. Secondly, network productivity is based on the interconnection and collaboration among different actors, including both public institutions and private companies. In the context of providing common goods, actors must work together and share resources to achieve optimal results. Network productivity plays a fundamental role in optimizing interactions and collaboration among these actors, contributing to the efficient provision of common goods.

Moreover, an increase in network productivity can generate positive externalities that benefit society as a whole. For example, higher productivity in logistics can lead to more efficient delivery of common goods, such as medical supplies during a health crisis. These positive externalities have a beneficial impact on society by improving quality of life and promoting economic and social development. Lastly, effective provision of common goods often requires collaboration between the public and private sectors. Network productivity is a critical component in facilitating cooperation and synergy between these actors, enabling them to coordinate efforts, share resources, and optimize the provision of common goods, especially in crisis or emergency situations. Finally, each agent intrinsic productivity can be viewed as public good that provide different (based on network position), non-rival benefits to all members of society.

All the above examples share a common theme: measuring the productivity of agents within a network can help identify opportunities to target interventions in the network. The paper [1] fills a gap in the current literature on communication networks by presenting a unique tutorial on the application of cooperative game theory. It comprehensively covers the theory and technical aspects, and provides practical examples drawn from game theory and communication applications. Within [2], a cooperative game theory-driven method is proposed, specifically focusing on community detection in social networks. Individuals are viewed as players, and communities are seen as coalitions formed by players. The authors use a utility function to measure preference and propose an algorithm to identify a coalition profile with maximal utility values. Experimental results demonstrate the effectiveness of the approach. In [3], the authors investigate a cooperative differential game model applied to networks, where players have the ability to cut connections with their neighboring nodes. This enables the evaluation of a characteristic function that measures the value of coalitions based on cooperation. The authors prove the convexity of the game, ensuring the Shapley value belongs to the core.

In this paper, we explore a cooperative game framework that considers the influence of peer effects on worker productivity in complete bipartite networks. The investigation of peer effects has recently undergone expansion within networks (refer to [4] for a recent survey). Our analysis focuses on a series of cooperative games where each worker's characteristic function incorporates their own productivity and the productivity of nearby workers within a specified distance. The interconnections are weighted using an attenuation factor, highlighting the impact of neighboring workers on an individual's overall productivity. We show that these games are balanced and converge to a balanced game when the distance of influence grows large provided that the attenuation factor is below a certain threshold.

We propose three different approaches to distributing productivity among workers. The first approach is the status quo granting each work his individual productivity, which accounts for peer effects. The second approach utilizes the Shapley value to share the overall productivity, while the third approach, called the Link Ratio Productivity Distribution (LRP distribution), takes into account the network's structure and the connectivity of the workers. We characterize the LRP distribution and analyze its impact on the efficiency of the logistics network. Our study emphasizes the significance of measuring productivity of workers in a logistics network represented by a complete bipartite network and explores how to distribute the overall productivity to individual according to their contributions. This analysis contributes to enhancing our understanding of game-theoretic networks within logistics systems,

offering insights into the peer effects' impact when assessing the overall productivity and its distribution among workers.

The utilization of cooperative games based on network elements to establish objective criteria for benefit/cost sharing among network members is a well-established topic in the literature. In [5], authors examine different solution concepts in cooperative game theory using a graph-based game, demonstrating the computational complexity of core computation and the potential undecidability of the existence of von Neumann–Morgenstern solutions. The proposed approach in the study by [6] introduces allocation rules for network games that consider possible changes in the network structure made by players. These rules allocate value based on alternative network structures, providing a comprehensive analysis of the dynamics within network games. The research conducted by [7] analyzes reward games in network structures, investigating link monotonic allocation schemes and characterizing conditions for link monotonicity in the Myerson and position allocation schemes. In the work by [8], the average tree solution is presented as a unique solution for cooperative games with communication structures depicted by undirected graphs. The study demonstrates that the game possesses a non-empty core, and under the concept of link-convexity (a weaker condition than convexity), the average tree solution resides within the core. This research provides valuable insights into the solvability and stability of cooperative games within communication networks. The authors in [9] propose algorithms that detect and eliminate the most influential node in order to weaken leadership positions. They employ a greedy approach based on modifying the network's structure. To measure a node's leadership, they utilize the Shapley value and develop algorithms for overthrowing leaders. For further information, we recommend consulting the surveys by [10,11].

The structure of the paper is as follows. It begins with a preliminary section introducing cooperative game theory and networks. Section 3 describes finite attenuation network games (FAN games) and examines their main properties. In Section 4, the focus is on establishing a necessary and sufficient condition for FAN games to converge to a new class of cooperative games: attenuation network games (AN games), which are shown to be totally balanced and convex. A coalitionally stable productivity sharing distribution based on network-generated productivity is also presented, along with an explicit form of the Shapley value in relation to the network structure. Section 5 explores an alternative productivity distribution that considers network structure and worker connectivity, providing an easier calculation method than the Shapley value. The concept of difference games, obtained by subtracting consecutive FAN games, is introduced, and the analysis demonstrates how productivity increases with distance. A series of distributions for the difference games is proposed, converging to an overall productivity distribution for AN games known as the link ratio productivity distribution (LRP distribution). The coalitional stability of LRP is established, and it is characterized based on three desirable properties for a realistic and functional network. Finally, Section 6 discusses implications and suggests potential avenues for future research in the field, catering to both academics and practitioners.

2. Preliminaries

To ensure clarity, we have incorporated in this section the fundamental principles of cooperative game theory and graph theory that are essential for comprehending and validating the findings presented in the paper.

A cooperative (profit) TU-game is a pair (N, v) where $N = \{1, 2, \dots, n\}$ is a finite set of players. The set of all coalitions S in N is represented by $\mathcal{P}(N)$, and the characteristic function $v : \mathcal{P}(N) \rightarrow \mathbb{R}$ is defined such that $v(\emptyset) = 0$. The value $v(S)$ denotes the maximum profit obtainable by coalition $S \subseteq N$, where N is commonly referred to as the grand coalition. The profit vector or allocation is denoted as $x \in \mathbb{R}^{|N|}$, where $|N|$ refers to the cardinality of the grand coalition. We also denote $s = |S|$ for simplicity.

A TU-game (N, v) is considered monotone if larger coalitions receive more significant benefits, which is expressed as $v(S) \leq v(T)$ for all coalitions $S \subseteq T \subseteq N$. Additionally, the game is said to be superadditive if the benefit obtained by the combination of any two disjoint coalitions is at least as much as the sum of their individual benefits. Specifically, $v(S \cup T) \geq v(S) + v(T)$ holds for all disjoint coalitions $S, T \subseteq N$. It is noteworthy that in superadditive games, it is reasonable for the grand coalition to form. This is because the benefit acquired by the grand coalition is at least as great as the sum of the benefits of any other coalition and its complement, i.e., $v(N) \geq v(S) + v(N \setminus S)$, for all $S \subseteq N$.

The set of all vectors that efficiently allocate the benefits of the grand coalition and are coalitionally stable is referred to as the core of the game (N, v) , which is denoted as $Core(N, v)$. More specifically, no group of players in the grand coalition has an incentive to leave, and each coalition is guaranteed to receive at least the profit allocated by the characteristic function:

$$Core(N, v) = \left\{ x \in \mathbb{R}^{|N|} : \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \forall S \subseteq N \right\}.$$

A TU-game is balanced only when the core is nonempty, as detailed in [12,13]. If the core of every subgame is nonempty, the game (N, v) is considered to be a totally balanced game (see [14]). A game (N, v) is regarded as convex if for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subseteq N$ with $i \in S$, then $v(S) - v(S \setminus \{i\}) \leq v(T) - v(T \setminus \{i\})$. It is widely acknowledged that convex games are superadditive, and superadditive games are totally balanced. Shapley establishes in [15] that the core of convex games is large enough.

A single-valued solution φ is an application that assigns to each TU game (N, v) an allocation of $v(N)$, the profit obtained by the grand coalition. Formally, φ is defined as follows: $\varphi : G^N \rightarrow \mathbb{R}^{|N|}$, where G^N is the set of all TU-games with player set N , and $\varphi_i(v)$ represents the profit assigned to player $i \in N$ in the game $v \in G^N$. Hence, $\varphi(v) = (\varphi_i(v))_{i \in N}$ is a profit vector or allocation of $v(N)$. For a comprehensive understanding of cooperative game theory, we recommend referring to [16].

The Shapley value, first introduced in [17], is a widely recognized single-valued solution in cooperative game theory. The Shapley value of convex games always belongs to the core and it is the baricenter of the core (see [15]). Moreover, it is a linear operator on the set of all TU games. For a profit game (N, v) , ϕ is defined as $\phi(N, v) = (\phi_i(N, v))_{i \in N}$, where for each $i \in N$

$$\phi_i(N, v) = \sum_{S \subseteq N: i \in S} \frac{(s-1)!(n-s)!}{n!} \cdot [v(S) - v(S \setminus \{i\})].$$

We consider a network \mathbf{g} of $N = \{1, 2, \dots, n\}$ players represented by an adjacency matrix $\mathbf{G}(N)$; where $g_{ij} = 1$ indicates a link between players i and j , and $g_{ij} = 0$ otherwise. Since the adjacency matrix $\mathbf{G}(N)$ is symmetric and non-negative it follows that its eigenvalues are real and the maximum eigenvalue $\lambda_{\max}(N)$ is positive and dominates in magnitude all other eigenvalues.

A complete bipartite network is a network $\mathbf{g} = (K, M, E)$ of $N = \{1, 2, \dots, n\}$ nodes such that the set N can be divided into two disjoint sets $K, M \subseteq N$, satisfying that $N = K \cup M$ and $g_{ij} = 0$ if i and j belong to the same set (K or M) and $g_{ij} = 1$ otherwise. E is the set of edges. For any coalition of nodes $S \subseteq N$, let $\mathbf{g}(S)$ denote the subnetwork induced by S , with adjacency matrix $\mathbf{G}(S)$, and $\lambda_{\max}(S)$ is its maximum eigenvalue. For any coalition $S \subseteq N$ we can rewrite it as $S = K(S) \cup M(S)$ with $K(S) := S \cap K \subseteq K$ and $M(S) := S \cap M \subseteq M$ disjoint sets, and $E(S)$ the set of edges of coalition S . We denote $|K(S)|$ by k_S and $|M(S)|$ by m_S for simplicity.

3. Finite attenuation network games

In order to facilitate the reader's understanding, we consider a real context of application of our study. We focus on a firm where $N = \{1, 2, \dots, n\} = K \cup M$ is the total set of workers and K, M two

different groups of fully connected workers. Formally, we consider a complete bipartite network $\mathbf{g} = (K, M, E)$. For any subset/team of workers $S \subseteq N$, we know that the induced network is a complete bipartite network $\mathbf{g}(S) = (K(S), M(S), E(S))$. Consider $t \geq 0$ as a natural number and $\delta \geq 0$ as a real number. We define the matrix

$$M^t(\mathbf{g}(S), \delta) = \sum_{u=0}^t \delta^u \mathbf{G}^u(S)$$

Note that each entry $m_{ij}^t(\mathbf{g}(S), \delta) = \sum_{u=0}^t \delta^u g_{ij}^u(S)$ counts the number of walks of at most distance t in $\mathbf{g}(S)$ that start in i and end at j weighted by δ^u . In interpretation, the non-negative parameter δ is an attenuation factor that scales down the relative weight of longer walks. Hence, $M^0(\mathbf{g}(S), \delta) = \mathbf{I}_{|S| \times |S|}$ because of $\mathbf{G}^0(S)$ is the identity matrix.

Given a team S , each worker $i \in S$ has an intrinsic productivity of 1 and an actual productivity $p_i^S(\delta, t)$ that benefits from the productivity of the other workers in the team at a distance of at most t (finite attenuation) in $\mathbf{g}(S)$, at a rate of δ . That is:

$$p_i^S(\delta, t) := \sum_{j \in S} m_{ij}^t(\mathbf{g}(S), \delta)$$

Note that $p_i^S(\delta, 0) = 1$ and $p_i^S(\delta, t)$ for $t > 1$ is a measure of the productivity of the worker i in team S that taking into account a peer effects of workers in the team.

Now, given a network $\mathbf{g} = (K, M, E)$ we define the corresponding finite distance attenuation network game (henceforth FAN game) as (N, v_δ^t) with $N = K \cup M$ and $t, \delta \geq 0$, where $v_\delta^t(S) := \sum_{i \in S} p_i^S(\delta, t)$ for all coalition $S \subseteq N$. Note that the characteristic function v_δ^t represents the aggregate productivity of the worker team S up to distance at most t weighted by δ .

The following proposition shows that we can explicitly compute the characteristic function of the FAN games.

Proposition 3.1. *Let $\mathbf{g} = (K, M, E)$ be a complete bipartite network and (N, v_δ^t) the corresponding FAN game. For each coalition $S \subseteq N$ it holds:*

$$v_\delta^t(S) = \begin{cases} |S| & \text{if } t = 0, \\ |S| + (|S| \delta + 2) \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u-1}, & \text{if } t \text{ is even.} \\ |S| + (|S| \delta + 2) \sum_{u=1}^{\frac{t-1}{2}} (k_S^u m_S^u \delta^{2u-1}) + 2k_S^{\frac{t+1}{2}} m_S^{\frac{t+1}{2}} \delta^t, & \text{if } t \text{ is odd.} \end{cases}$$

The reader may notice that $v_\delta^t(S) > 0$ for all $S \subseteq N$ and $t, \delta \geq 0$. The increase in productivity with respect to the increase in distance can be seen more clearly if we relate FAN games at different distances:

$$\begin{aligned} v_\delta^0(S) &= |S| \\ v_\delta^1(S) &= v_\delta^0(S) + 2k_S m_S \delta \\ v_\delta^2(S) &= v_\delta^1(S) + (k_S^2 m_S + k_S m_S^2) \delta^2 \\ v_\delta^3(S) &= v_\delta^2(S) + 2k_S^2 m_S^2 \delta^3 \\ &\vdots \\ v_\delta^t(S) &= \begin{cases} v_\delta^{t-1}(S) + |S| k_S^{\frac{t}{2}} m_S^{\frac{t}{2}} \delta^t, & \text{if } t \text{ is even,} \\ v_\delta^{t-1}(S) + 2k_S^{\frac{t+1}{2}} m_S^{\frac{t+1}{2}} \delta^t, & \text{if } t \text{ is odd.} \end{cases} \end{aligned}$$

This can be interpreted as follows: when we go from distance 0 to 1, each worker (of $K(S)$ or $M(S)$) receives part of the productivity of the workers of the opposite group, hence the aggregate productivity increase of the team is $2k_S m_S \delta = \sqrt{k_S m_S} \cdot 2 \cdot (\sqrt{k_S m_S} \delta)$. When the distance increases to 2, in addition to the above productivity ($v_\delta^1(S)$), each worker also has access to the productivity of his own group for each worker of the opposite group, and so the increase of the team is now $(k_S^2 m_S + k_S m_S^2) \delta^2 = \frac{k_S + m_S}{2} \cdot 2 \cdot (\sqrt{k_S m_S} \delta)^2$. However if we

Table 1
FAN games for $t = 0, 1, 2, 3, 10$ for [Example 3.3](#).

S	$v_\delta^t(S)$	$v_\delta^0(S)$	$v_\delta^1(S)$	$v_\delta^2(S)$	$v_\delta^3(S)$	$v_\delta^{10}(S)$
$\{i\}$	1	1	1	1	1	1
$\{2, 3\}$	2	2	2	2	2	2
$\{1, i\}$	$\begin{cases} 2, & \text{if } t = 0, \\ 2 + 6 \sum_{u=1}^{\frac{t}{2}} \left(\frac{1}{4}\right)^u, & \text{if } t \text{ is even} \\ 2 + \left(\frac{1}{2}\right)^{t-1} + 6 \sum_{u=1}^{\frac{t-1}{2}} \left(\frac{1}{4}\right)^u, & \text{if } t \text{ is odd.} \end{cases}$	2	3	3.5	3.75	3.998
N	$\begin{cases} 3, & \text{if } t = 0, \\ 3 + 7 \sum_{u=1}^{\frac{t}{2}} \left(\frac{1}{2}\right)^u, & \text{if } t \text{ is even,} \\ 3 + \left(\frac{1}{2}\right)^{\frac{t-3}{2}} + 7 \sum_{u=1}^{\frac{t-1}{2}} \left(\frac{1}{2}\right)^u, & \text{if } t \text{ is odd.} \end{cases}$	3	5	6.5	7.5	9.78125

increase the distance to 3 each worker receives, in addition to the above productivity ($v_\delta^2(S)$), the productivity of the other group ($K(S)$ or $M(S)$) for each path of distance 2 that may exist, and now the increase of the team is $2k_S^2 m_S^2 \delta^3 = \sqrt{k_S m_S} \cdot 2 \cdot \left(\sqrt{k_S m_S} \delta\right)^3$ and so on.

Our next objective is to analyze the properties of FAN games. It is easy to see that when the team of workers increases, we add more productivity to the team, hence FAN games are monotonic. The natural question that arises is whether the snowball effect in productivity whereby the returns of joining a coalition of workers increases as the coalition grows large occurs in our game (i.e., FAN games are convex). The following theorem provides affirmative answer.

Theorem 3.2. *Every FAN game is convex.*

The fact that any FAN game is convex has two important consequences. FAN games are totally balanced and the Shapley value always belongs to the core of these games. Next, we illustrate how to calculate different FAN games by changing the distance range t , through the analysis of a logistic network with several distribution centers.

Example 3.3. We consider the analysis of a logistic network involving three distribution centers: 1, 2, and 3. Distribution centers 2 and 3 do not have a direct relationship in terms of collaboration or resource exchange in this specific logistic network. Each distribution center can operate independently, and its productivity can be influenced by internal factors such as operational efficiency and service quality. However, distribution center 1 is connected to both distribution center 2 and 3. This indicates that its productivity can be influenced by the collaboration and advancements of both distribution centers. There can be information exchange, service provision, or resource sharing between distribution center 1 and distribution centers 2 and 3, which benefits the overall productivity.

Additionally, we consider the flow of innovations among the distribution centers measured as a distance. This distance reflects the number of steps it takes for innovations to reach a particular distribution center after being evaluated and filtered by others. If the distance is one, each distribution center has direct access to the innovations of the other centers. For example, distribution center 1 can access the results of 2 and 3. If the distance is two, in addition to the aforementioned access, distribution center 1 will also be able to access its own innovations after they have been evaluated by distribution centers 2 and 3.

In this situation, we assume an attenuation factor of $\frac{1}{2}$ meaning that the productivity of each distribution center is halved with each iteration. This factor represents the diminishing impact of previously shared innovations as they propagate through the network.

Formally, we define a complete bipartite network with $K = \{1\}$, $M = \{2, 3\}$ and $\delta = \frac{1}{2}$. The following table shows the corresponding FAN game with $\delta = \frac{1}{2}$ and $t \in \{0, 1, 2, 3, 10\}$ as shown in [Table 1](#).

Table 2
Productivity in N for $t = 0, 1, 2, 3, 10$ for [Example 3.3](#).

Worker	$p_i^N(\frac{1}{2}, 0)$	$p_i^N(\frac{1}{2}, 1)$	$p_i^N(\frac{1}{2}, 2)$	$p_i^N(\frac{1}{2}, 3)$	$p_i^N(\frac{1}{2}, 10)$
1	1	2	2.5	3	3.90625
2	1	1.5	2	2.25	2.9375
3	1	1.5	2	2.25	2.9375

[Table 2](#) shows the productivity of each center i in the overall network for the above flows of innovations (distances).

We may notice that the larger t the higher individual and aggregate productivities. Moreover, productivities seem to converge to a certain value as the flow of innovation t increases, i.e., $p^N(\frac{1}{2}, t) \approx (4, 3, 3)$ for t enough large. In conclusion, we can say that distribution center 1 has a higher final productivity than the others.

A question that may arise naturally is whether FAN games converge to a particular game when t increases. In the following section we determine necessary and sufficient conditions on attenuation factor δ for FAN games to converge (when t goes to infinity).

4. Converging FAN games to Attenuation Network games

In this section we investigate what happens when each worker in a team benefits from the productivity of the others at any distance, that is, what happens to FAN games when the distance goes to infinity. We are interested in study under what conditions FAN games converge to a well-defined TU-game.

Consider a complete bipartite network $\mathbf{g} = (K, M, E)$ and $\Lambda(g, \delta) := \{(N, v_\delta^t) / t \in \mathbb{N}\}$ the family of all possible FAN games with an attenuation factor $\delta \geq 0$. It is easy to check that $\lambda_{\max}(S) = \sqrt{k_S m_S}$, for all $S \subseteq N$.

The first theorem provides a necessary and sufficient condition for the family of FAN games to converge. Before showing it we need the following technical lemma.

Lemma 4.1. *Let \mathbf{g} be a complete bipartite network and $\Lambda(g, \delta)$ the corresponding family of FAN games with δ . Then, $\{v_\delta^t(S)\}_{t \in \mathbb{N}}$ converges to a real value $v_\delta(S)$, for each coalition $S \subseteq N$, if and only if $\delta \in \left[0, \frac{1}{\lambda_{\max}(S)}\right]$.*

Note that this technical condition sets a different condition for the convergence of the productivity of each team based on the same attenuation factor. The following result provides a unique condition in terms of the network's overall productivity.

Theorem 4.2. *Let \mathbf{g} be a complete bipartite network and $\Lambda(g, \delta)$ the corresponding family of FAN games with δ . Then, $\{v_\delta^t\}_{t \in \mathbb{N}}$ converges to a finite TU game v_δ if and only if $\delta \in \left[0, \frac{1}{\lambda_{\max}(N)}\right]$.*

Table 3
Convergence of the FAN-games for [Example 4.3](#).

S	$\{i\}$	$\{2, 3\}$	$\{1, i\}$	N
$v_\delta^0(S)$	1	2	2	3
$v_\delta^1(S)$	1	2	3	5
$v_\delta^2(S)$	1	2	3.5	6.5
$v_\delta^3(S)$	1	2	3.75	7.5
$v_\delta^{10}(S)$	1	2	3.998	9.78125
\vdots	\vdots	\vdots	\vdots	\vdots
$v_\delta(S)$	1	2	4	10

Given g a complete bipartite network and the associated family of FAN games $A(g, \delta)$ with $\delta \in \left[0, \frac{1}{\lambda_{\max}(N)}\right]$, we can define an attenuation network game (N, v_δ) as the limit of $\{v_\delta^t\}_{t \in \mathbb{N}}$. Notice this game is well defined because of the above theorem. Henceforth, we will refer to (N, v_δ) as a AN game. Moreover, by [Lemma 4.1](#), we have an explicit formula for AN games, that is, for any $S \subseteq N$,

$$v_\delta(S) = \frac{k_S + m_S + 2k_S m_S \delta}{1 - k_S m_S \delta^2}.$$

The following example illustrates AN games and the distribution of the individual productivity in the grand coalition.

Example 4.3. Consider the [Example 3.3](#) with $K = \{1\}$, $M = \{2, 3\}$ and $\delta = \frac{1}{2}$. Notice that $\lambda_{\max}(N) = \sqrt{2}$ and $\delta = 1/2 \in \left[0, \frac{1}{\sqrt{2}}\right]$.

[Table 3](#) shows that the limit of the family of FAN games is a TU game with finite values

Moreover, the limit of the individual productivity for the grand coalition, $\lim_{t \rightarrow \infty} p^N(\frac{1}{2}, t) = (4, 3, 3) =: p^N(\frac{1}{2})$ is a stable (in the sense of the core) distribution of the total productivity ($v_\delta(N) = 10$).

Next proposition shows that $p^N(\delta) := \lim_{t \rightarrow \infty} p^N(\delta, t)$, is always a core allocation for (N, v_δ) . Hence, AN games are totally balanced, because of every subgame of an AN game is also an AN game.

Proposition 4.4. Let g be a complete bipartite network and (N, v_δ) be the corresponding AN game. Then, $p^N(\delta) \in \text{Core}(N, v_\delta)$.

Next theorem proves that AN games are convex. Before introducing it, let us demonstrate the following technical lemma, which shows the marginal productivity of a worker to a team.

Lemma 4.5. Let g be a complete bipartite network and (N, v_δ) be the corresponding AN game. Then, for any $i \in S \subseteq N$,

$$v_\delta(S) - v_\delta(S \setminus \{i\}) = \begin{cases} \frac{(1+m_S \delta)^2}{(1-k_S m_S \delta^2)(1-k_S m_S \delta^2 + m_S \delta^2)}, & \text{if } i \in K(S), \\ \frac{(1+k_S \delta)^2}{(1-k_S m_S \delta^2)(1-k_S m_S \delta^2 + k_S \delta^2)}, & \text{if } i \in M(S). \end{cases}$$

The following theorem shows that the marginal productivity of a worker to a team is greater the larger the team is.¹

Theorem 4.6. Every AN game is convex.

As mentioned above, the Shapley value, $\phi(v_\delta)$, always belongs to the core of the AN game (N, v_δ) . Next theorem provides an explicit formula for the Shapley value of AN games.

Theorem 4.7. Let g a complete bipartite network and (N, v_δ) the corresponding AN game. Then, for all $i \in K$

¹ It is worth noting that convexity of AN games can be also shown to follow, via a limit argument, from the convexity of the FAN games.

Table 4
Productivity in N vs. Shapley value for [Example 4.8](#).

Worker	$p^N(\frac{1}{2})$	$\phi(v_{\frac{1}{2}})$
1	4	4
2	3	3
3	3	3

Table 5
Productivity in N vs. Shapley value for [Example 4.9](#).

Worker	$p^N(\frac{1}{3})$	$\phi(v_{\frac{1}{3}})$
1	3	3.14
2	2	1.95
3	2	1.95
4	2	1.95

$$\phi_i(v_\delta) = \sum_{k=1}^{|K|} \sum_{m=0}^{|M|} \Pi_M^K(k, m) \cdot \frac{(1+m\delta)^2}{(1-km\delta^2)(1-km\delta^2+m\delta^2)}$$

and for all $i \in M$

$$\phi_i(v_\delta) = \sum_{k=0}^{|K|} \sum_{m=1}^{|M|} \Pi_K^M(m, k) \cdot \frac{(1+k\delta)^2}{(1-km\delta^2)(1-km\delta^2+k\delta^2)}$$

$$\text{where } \Pi_Y^X(i, j) = \binom{|Y|}{j} \cdot \binom{|X|-1}{i-1} \cdot \frac{(i+j-1)!(|X|+|Y|-i-j)!}{(|X|+|Y|)!}$$

The reader may notice that once we obtain the Shapley value for a worker $i \in K$, $\phi_i(v_\delta)$, it is easy to calculate it for workers $j \in M$. Indeed, $\phi_j(v_\delta) = \frac{v_\delta(N) - |K| \cdot \phi_i(v_\delta)}{|M|}$ for all $j \in M$ and $i \in K$.

Recall that, as we already discussed earlier, the overall productivity can be considered a public good. The Shapley value acts then as an individual measure for productivity. Additionally, the Shapley value can be interpreted as an individual's contribution to the public good, demonstrating a voluntary willingness to contribute to the sustainability of that shared productivity. Next example illustrate the Shapley value for AN games.

Example 4.8. Consider again [Example 3.3](#) with $K = \{1\}$, $M = \{2, 3\}$ and $\delta = \frac{1}{2}$. [Table 4](#) compares the Shapley value with individual productivity for the grand coalition.

In this example, both productivity distributions coincides but this is not the case in general. After an extended interaction among the centers, center 1 contributes to the network with a productivity level of 4, while the rest of the centers contribute with a level of 3 each.

The following example shows that Shapley value can be close to the individual productivity for the grand coalition.

Example 4.9. Consider the logistic network given in [Example 3.3](#) expanded with a new distribution center 4, that is, $K = \{1\}$, $M = \{2, 3, 4\}$, but now $\delta = \frac{1}{3}$. Notice that $\lambda_{\max}(N) = \sqrt{3}$ and $\delta = 1/3 \in \left[0, \frac{1}{\sqrt{3}}\right]$.

[Table 5](#) compares the Shapley value with individual productivity for the grand coalition.

The reader may notice that center 1 has an individual productivity level of 3, but contributes to the network with a productivity level of 3.14. On the other hand, the rest of the centers have an individual productivity level of 2, while their contribution to the network is lower (1.95).

Notice that while $p_i^N(\delta)$ represents the individual productivity of worker i in the network, $\phi_i(v_\delta)$ is interpreted as the average marginal productivity of such a worker i in all the teams. However, despite having an explicit formula for the Shapley value, it is still difficult to

calculate when the number of workers grows. Moreover, we observe that in AN games with few workers the two are close, matching in some cases. Next, we focus on finding an alternative productivity distribution for AN games which takes into account the increase in productivity in the distance of the grand coalition, as well as the degree of connectivity of each worker.

5. Productivity distribution that recognizes workers' connectivity

We first go back to FAN games and study in detail what happens when the distance increases. It is important to measure how much productivity each team generates as the distance t increases. This information will allow us to define an alternative productivity distribution for AN games which, unlike Shapley value, takes into consideration the degree of connectivity of workers.

We start by defining the difference game in t , as the difference between FAN games in t and $t - 1$. Formally, (N, d_δ^t) such that for all coalition $S \subseteq N$, $d_\delta^t(S) := v_\delta^t(S) - v_\delta^{t-1}(S)$.

Next proposition shows and explicit formula for the difference games in t .

Proposition 5.1. *Let g be a complete bipartite network and $A(\delta) := \{(N, v_\delta^t) | t \in \mathbb{N}\}$ the family of FAN games. Then, difference games (N, d_δ^t) with $t \geq 0$, are given by,*

$$d_\delta^t(S) = \frac{1}{2} \left[\left(\sqrt{k_S} + \sqrt{m_S} \right)^2 (\lambda_{\max}(S))^t + \left(\sqrt{k_S} - \sqrt{m_S} \right)^2 (-\lambda_{\max}(S))^t \right] \times \delta^t$$

for all $S \subseteq N$.

Notice that the difference game for a distance t , (N, d_δ^t) , measures the increase in productivity at FAN games per unit of distance. That is, the increase in productivity from (N, v_δ^{t-1}) to (N, v_δ^t) . Moreover, we can rewrite d_δ^t as follows:

$$d_\delta^t(S) = \left[\left(\frac{k_S + m_S}{2} + \sqrt{k_S m_S} \right) (\lambda_{\max}(S))^t + \left(\frac{k_S + m_S}{2} - \sqrt{k_S m_S} \right) (-\lambda_{\max}(S))^t \right] \delta^t$$

Thus, we can distinguish that in even periods the increase in productivity is influenced by the arithmetic mean, $\frac{k_S + m_S}{2}$, while in odd periods it is influenced by the geometric mean, $\sqrt{k_S m_S}$. Given that, $\frac{k_S + m_S}{2} \geq \sqrt{k_S m_S}$ for all team $S \subseteq N$, we can deduce that productivity increases more when we extend the possibility for workers to obtain productivity from odd to even distance than vice versa. This effect is due to the complete bipartite structure of the network as mentioned in the previous section.

Based on this definition we can also rewrite the game v_δ^t in the following way

$$v_\delta^t(S) = |S| + \frac{1}{2} \sum_{u=1}^t \left(\left[\left(\sqrt{k_S} + \sqrt{m_S} \right)^2 (\lambda_{\max}(S))^u + \left(\sqrt{k_S} - \sqrt{m_S} \right)^2 (-\lambda_{\max}(S))^u \right] \delta^u \right)$$

We use now the structure of difference games to define a productivity distribution for AN games. Given $d_\delta^t(N)$ for any $t \geq 1$, we define a productivity distribution $x^t(\delta) = (x_i^t(\delta))_{i \in N}$ such that

$$x_i^t(\delta) = \begin{cases} \frac{d_\delta^t(N)}{|N|} \cdot \frac{|M|}{|K|}, & \text{if } i \in K, \\ \frac{d_\delta^t(N)}{|N|} \cdot \frac{|K|}{|M|}, & \text{if } i \in M. \end{cases}$$

We may notice that we first divide the total productivity among all workers equally, then we weight it by the ratio between the number

Table 6
Difference games for Example 5.3.

S	$\{i\}$	$\{2, 3\}$	$\{1, i\}$	N
$d_\delta^0(S)$	0	0	$\left(\frac{1}{2}\right)^{t-1}$	$\left[\left(\frac{3}{2} + \sqrt{2}\right)(\sqrt{2})^t + \left(\frac{3}{2} - \sqrt{2}\right)(-\sqrt{2})^t\right] \left(\frac{1}{2}\right)^t$
$d_\delta^1(S)$	0	0	1	2
$d_\delta^2(S)$	0	0	0.5	1.5
$d_\delta^3(S)$	0	0	0.25	1
$d_\delta^4(S)$	0	0	0.125	0.75
$d_\delta^5(S)$	0	0	0.0625	0.5

Table 7
Distribution $x^t(\delta)$ for $t = 1, \dots, 5$ for Example 5.3.

Worker	$x_1^t(\delta)$	$x_2^t(\delta)$	$x_3^t(\delta)$	$x_4^t(\delta)$	$x_5^t(\delta)$
1	4/3	1	2/3	1/2	1/3
2	1/3	1/4	1/6	1/8	1/12
3	1/3	1/4	1/6	1/8	1/12

of K and M nodes. So workers in set K receive more if the number of links leaving each worker ($|M|$) is greater than those of the workers in M ($|K|$) and vice versa.

Next proposition shows that $x^t(\delta)$ is stable in the sense of the core.

Proposition 5.2. *Let g be a complete bipartite network and (N, d_δ^t) the difference game in t . Then, $x^t(\delta) \in \text{Core}(N, d_\delta^t)$.*

The following example illustrates the difference games for distances $t \in \{1, 2, 3, 4, 5\}$.

Example 5.3. Consider again Example 3.3 with $K = \{1\}$, $M = \{2, 3\}$ and $\delta = \frac{1}{2}$. Table 6 shows the difference games for $t \in \{1, 2, 3, 4, 5\}$

Table 7 shows the calculation of the productivity distribution $x_i^t(\delta)$ for distances $t \in \{1, 2, 3, 4, 5\}$

We are now ready to build a productivity distribution for AN games based on the difference distribution $x^t(\delta)$. Consider g a bipartite complete network and (N, v_δ) its corresponding AN game. We define the link ratio productivity distribution (henceforth LRP distribution) as the equal distribution of the increase in productivity $\left(\frac{d_\delta^t(N)}{\delta}\right)$ with respect to the link ratio $\left(\frac{|M|}{|K|}\right)$ or $\left(\frac{|K|}{|M|}\right)$ depending of the worker considered). Formally, it is constructed by adding to 1 (the individual productivity) the sum of the difference distributions $x_i^t(\delta)$ of each distance $t \geq 1$, that is

$$\omega(\delta) := \mathbf{1}_N + \lim_{t \rightarrow \infty} \left(\sum_{u=1}^t x^u(\delta) \right).$$

Notice that, when $\delta = 0$, $d_\delta^t(S) = 0$, for all $S \subseteq N$, then $x^t(\delta) = 0_N$ and so, $\omega(\delta) = \mathbf{1}_N$.

Next proposition provides an explicit formula for LRP distribution when $\delta > 0$.

Proposition 5.4. *Let g be a complete bipartite network an (N, d_δ^t) the corresponding difference games for $t \geq 1$ and $\delta > 0$. Then, the LRP distribution $\omega(\delta)$ is given by:*

$$\omega_i(\delta) = \begin{cases} 1 + \left(\frac{|M|}{|K|} \delta + \frac{2|M|}{|N||K|} \right) \frac{|K||M|\delta}{1 - |K||M|\delta^2}, & \text{if } i \in K, \\ 1 + \left(\frac{|K|}{|M|} \delta + \frac{2|K|}{|N||M|} \right) \frac{|K||M|\delta}{1 - |K||M|\delta^2}, & \text{if } i \in M. \end{cases}$$

The following theorem shows that LRP distribution is stable in the sense of the core.

Theorem 5.5. *Let g be a complete bipartite network and (N, v_δ) the corresponding AN game. Then, $\omega(\delta) \in \text{Core}(N, v_\delta)$.*

Table 8
Shapley value vs. LRP distribution for Example 5.7.

Worker	$p^N(\delta)$	$\phi(v_\delta)$	$\omega(\delta)$
1	4	4	17/3
2	3	3	13/6
3	3	3	13/6

Table 9
AN games for Example 5.8.

S	$v_\delta(S)$	$v_{\frac{1}{2}}(S)$	$v_{\frac{1}{3}}(S)$
{i}	1	1	1
{2, 3}	2	2	2
{2, 4}			
{3, 4}			
{1, i}	$\frac{2}{1-\delta}$	4	3
{2, 3, 4}	3	3	3
{1, 2, 3}			
{1, 2, 4}	$\frac{3+4\delta}{1-2\delta^2}$	10	$\frac{39}{7}$
{1, 3, 4}			
N	$\frac{4+6\delta}{1-3\delta^2}$	28	9

To conclude this section, we present a characterization of the LRP distribution. It is based on three appealing properties for AN games. The first one, *Efficiency* means that the total benefit is divided among the workers. The second, *equality in bipartition* ensures that all workers originating the same number of links have the same productivity distribution. The last one, *link balanced productivity* property shows that the productivity of the workers in K , discounting their individual productivity, divided by the average number of links, is exactly equal to the workers in M . This guarantees an equal contribution of each link to the productivity of the network.

Formally, we consider a network g and the corresponding AN game (N, v_δ) . We define the following three properties for a single-valued solution φ on AN games (N, v_δ) :

(EF) Efficiency. $\sum_{i \in N} \varphi_i(v_\delta) = v_\delta(N)$.

(EB) Equality in bipartition. $\varphi_i(v_\delta) = \varphi_j(v_\delta)$ for all $i, j \in K$ and $\varphi_i(v_\delta) = \varphi_j(v_\delta)$ for all $i, j \in M$.

(LBP) Link balanced productivity. $\frac{1}{|M|} \sum_{i \in K} (\varphi_i(v_\delta) - 1) = \frac{1}{|K|} \sum_{j \in M} (\varphi_j(v_\delta) - 1)$.

The last theorem in this paper states that there exists a unique productivity distribution for AN games satisfying the properties EF, EB and LBP.

Theorem 5.6. *Let g be a complete bipartite network and (N, v_δ) the corresponding AN game. Then, the LRP distribution $\omega(\delta)$ is the unique productivity distribution satisfying EF, EB and LBP.*

The following examples compare the individual productivity distribution, the Shapley value and de LRP distribution.

Example 5.7. Consider again Example 3.3 with $K = \{1\}, M = \{2, 3\}$ and $\delta = \frac{1}{2}$. Table 8 compares the LRP distribution with the Shapley value and the individual productivity distribution in the grand coalition.

Example 5.8. Consider the AN game (N, v_δ) with $K = \{1\}$ and $M = \{2, 3, 4\}$, as shown in Table 9.

Table 10 compares LRP distribution with the Shapley value and the individual productivity distribution in the grand coalition for two different values of δ .

Table 10
Productivity, Shapley value and LRP distribution for Example 5.8.

Worker	$p^N\left(\frac{1}{2}\right)$	$\phi(v_{\frac{1}{2}})$	$\omega\left(\frac{1}{2}\right)$	$p^N\left(\frac{1}{3}\right)$	$\phi(v_{\frac{1}{3}})$	$\omega\left(\frac{1}{3}\right)$
1	10	9.25	19	3	3.14	4.75
2	6	6.25	3	2	1.95	1.41
3	6	6.25	3	2	1.95	1.41
4	6	6.25	3	2	1.95	1.41

Both examples show how worker 1 has higher productivity as all links emanate of him, but LRP distribution allocates a higher productivity than the Shapley value. In other words, if he leaves the network, the other workers would be disconnected. The LRP distribution compensates much more for the role of worker 1 in network connectivity.

The reader may notice that If $|K| = |M|$ by efficiency $\phi(v_\delta) = p^N(\delta) = \omega(\delta)$. If $|K| \neq |M|$, LRP distribution assigns higher productivity to those workers who have a higher number of links, recognizing their greater contribution to the interconnectedness of the network.

While the Shapley value is an effective measure for the weighted marginal productivity contribution of a node to various teams, the LRP distribution serves a distinct role in evaluating the productivity of the entire network in terms of its connections. Notably, the LRP distribution holds the advantage of being easier to calculate than the Shapley value for a complete bipartite network. To illustrate this, consider Example 5.8, where Node 1 emerges as more pivotal in the LRP due to its central role as the starting point for all network connections. In scenarios where the objective is to assess the marginal contribution of workers to different work teams, the Shapley value proves to be a valuable indicator.

In a network context, readers could consider employing other off-the-shelf centrality measures to establish a ranking for different workers. It is important to note that nodes in sets K or M are indistinguishable in terms of centrality measures. Consequently, regardless of the choice of centrality measures, they would not aid in distinguishing between nodes in sets K or M . Moreover, the various allocations proposed in this work can also be viewed as centrality measures, as they are derived from distinct characteristics of nodes in each set to determine their values.

Finally, we prove that properties used in Theorem 5.6 are logically independent.

Example 5.9 (LBP Fails). Consider φ on AN game (N, v_δ) defined by $\varphi(v_\delta) := p^N(v_\delta)$ where $|K| = 1, |M| = 2$ and $\delta = \frac{1}{2}$. $\varphi(v_\delta)$ satisfies EF, EB, but not LBP since $\frac{1}{2} \sum_{i \in K} (4 - 1) = \frac{3}{2} \neq 4 = \sum_{i \in M} (3 - 1)$.

Example 5.10 (EB Fails). Consider φ on AN game (N, v_δ) given by $\varphi(v_\delta) := (0, 2, 0, 2)$ where $K = \{1, 2\}, M = \{3, 4\}$ and $\delta = 0$. $\varphi(v_\delta)$ satisfies EF, LBP but not EB.

Example 5.11 (EF Fails). Let φ on AN game (N, v_δ) defined by $\varphi(v_\delta) := p^N(v_\delta) - \mathbf{1}_N$ where $|K| = 2, |M| = 2$ and $\delta = \frac{1}{2}$. $\varphi(v_\delta)$ satisfies LBP, EB but not EF.

6. Concluding remarks

Network productivity can be considered a public good in the context of providing delivering common goods in areas such as the environment, health, and logistics. Its role in accessibility, interconnection, positive externalities, and the need for public-private collaboration supports the notion that network productivity is a crucial component for the effective provision of common goods for the benefit of society as a whole.

In this paper, we have explored both the theory of cooperative games and networks in the context of productivity measures in logistics

infrastructure, where two teams of agents/workers interact. We have focused on the structure of complete bipartite networks because they possess an interesting structural feature from the point of view of cooperative game theory, i.e., any subnetwork induced by a coalition of workers maintains the same structure and properties of the original network, allowing the results obtained to be applicable both to the whole infrastructure and to small teams of workers. From this synergy between networks and cooperative games arise finite attenuation network games (FAN games) and attenuation network games (AN games). We have shown that FAN games converge to AN games for attenuation factors below a certain threshold. Then, we have considered a coalitionally stable productivity distribution of the overall productivity of the network. In addition, we have provided an explicit formula of the Shapley value and explored an alternative productivity distribution, LRP distribution, which is easier to compute than the Shapley value, and lends itself nicely to the underlying network structure of interactions. Finally, we have characterized this distribution on the basis of three properties suitable for a realistic and functional network.

This work has implications for both academics and practitioners in this field. It is crucial to underscore that we have utilized the distinctive structure of complete bipartite networks to derive explicit formulas for both defining the games and proposing allocations. A promising avenue for future research would be to expand this investigation to more general network structures such as exploring complete multipartite networks or nested split networks. It could serve as a natural extension of our current work. Other future research could further explore the properties of AN games and their applications in various contexts. Overall, this paper contributes to the understanding of cooperative game theory and networks, and provides insights for the design and management of networks with peer effects and cooperative objectives. Finally, we propose more specific future research from the perspective of game theory, such as: (1) Analyzing the differences between the Shapley value and the individual productivity distribution in the grand coalition; (2) Extending the study to complete multipartite networks; (3) Finding alternative productivity sharing methods based on other structural features or properties of the network; (4) Analyzing other models in which the productivity of each worker depends on different types of local interactions and peer effects.

CRedit authorship contribution statement

N. Allouch: Conceptualization, Formal analysis, Investigation, Writing – original draft, Writing – review & editing. **Luis A. Guardiola:** Conceptualization, Formal analysis, Investigation, Writing – original draft, Writing – review & editing. **A. Meca:** Conceptualization, Formal analysis, Investigation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix

Proof of Proposition 3.1. Consider a (K, M, E) complete bipartite network. Take $S \subseteq N$, and the corresponding subnetwork $g(S) = (K(S), M(S), E(S))$, with matrix:

$$G(S) = \begin{pmatrix} \mathbf{0}_{k_S \times k_S} & \mathbf{1}_{k_S \times m_S} \\ \mathbf{1}_{m_S \times k_S} & \mathbf{0}_{m_S \times m_S} \end{pmatrix}_{|S| \times |S|}$$

$G^u(S)$ can be easily calculated. Indeed, if u is an even number, then $u = 2d$ with d a natural number. Then,

$$G^{2d}(S) = \begin{pmatrix} (k_S^{d-1} \cdot m_S^d)_{k_S \times k_S} & \mathbf{0}_{k_S \times m_S} \\ \mathbf{0}_{m_S \times k_S} & (k_S^d \cdot m_S^{d-1})_{m_S \times m_S} \end{pmatrix}_{|S| \times |S|}$$

If, on the other hand, it is odd $u = 2d + 1$

$$G^{2d+1}(S) = \begin{pmatrix} \mathbf{0}_{k_S \times k_S} & (k_S^d \cdot m_S^d)_{k_S \times m_S} \\ (k_S^d \cdot m_S^d)_{m_S \times k_S} & \mathbf{0}_{m_S \times m_S} \end{pmatrix}_{|S| \times |S|}$$

The expression of $m_{ij}^t(g(S), \delta) = \sum_{u=0}^t \delta^u g_{ij}^u(S)$ varies depending on which set of the bipartite graph the players are located in, we distinguish the following cases:

- If $i \in K$, then:

$$m_{ii}^t(g(S), \delta) = \begin{cases} 1 + m_S \delta^2 + k_S m_S^2 \delta^4 + \dots \\ \quad + k_S^{\frac{t}{2}-1} m_S^{\frac{t}{2}} \delta^t, & \text{if } t \text{ is even,} \\ 1 + m_S \delta^2 + k_S m_S^2 \delta^4 + \dots \\ \quad + k_S^{\frac{t-1}{2}-1} m_S^{\frac{t-1}{2}} \delta^{t-1}, & \text{if } t \text{ is odd and } t > 1, \end{cases}$$

Note that $m_{ii}^0(g(S), \delta) = m_{ii}^1(g(S), \delta) = 1$.

- If $j \in M$, then:

$$m_{jj}^t(g(S), \delta) = \begin{cases} 1 + k_S \delta^2 + k_S^2 m_S \delta^4 + \dots \\ \quad + k_S^{\frac{t}{2}} m_S^{\frac{t}{2}-1} \delta^t, & \text{if } t \text{ is even,} \\ 1 + k_S \delta^2 + k_S^2 m_S \delta^4 + \dots \\ \quad + k_S^{\frac{t-1}{2}} m_S^{\frac{t-1}{2}-1} \delta^{t-1}, & \text{if } t \text{ is odd and } t > 1, \end{cases}$$

Note that $m_{jj}^0(g(S), \delta) = m_{jj}^1(g(S), \delta) = 1$.

- If $i, j \in K$ and $i \neq j$, then:

$$m_{ij}^t(g(S), \delta) = \begin{cases} m_S \delta^2 + k_S m_S^2 \delta^4 + \dots \\ \quad + k_S^{\frac{t}{2}-1} m_S^{\frac{t}{2}} \delta^t, & \text{if } t \text{ is even,} \\ m_S \delta^2 + k_S m_S^2 \delta^4 + \dots \\ \quad + k_S^{\frac{t-1}{2}-1} m_S^{\frac{t-1}{2}} \delta^{t-1}, & \text{if } t \text{ is odd and } t > 1, \end{cases}$$

Note that $m_{ij}^0(g(S), \delta) = m_{ij}^1(g(S), \delta) = 0$.

- If $i, j \in M$ and $i \neq j$, then:

$$m_{ij}^t(g(S), \delta) = \begin{cases} k_S \delta^2 + k_S^2 m_S \delta^4 + \dots \\ \quad + k_S^{\frac{t}{2}} m_S^{\frac{t}{2}-1} \delta^t, & \text{if } t \text{ is even,} \\ k_S \delta^2 + k_S^2 m_S \delta^4 + \dots \\ \quad + k_S^{\frac{t-1}{2}} m_S^{\frac{t-1}{2}-1} \delta^{t-1}, & \text{if } t \text{ is odd and } t > 1, \end{cases}$$

Note that $m_{ij}^0(g(S), \delta) = m_{ij}^1(g(S), \delta) = 0$.

- If $i \in K$ and $j \in M$ or the opposite, then:

$$m_{ij}^t(g(S), \delta)$$

$$= \begin{cases} \delta + k_S m_S \delta^3 + \dots + k_S^{\frac{t}{2}-1} m_S^{\frac{t}{2}-1} \delta^{t-1}, & \text{if } t \text{ is even,} \\ \delta + k_S m_S \delta^3 + \dots + k_S^{\frac{t-1}{2}} m_S^{\frac{t-1}{2}} \delta^t, & \text{if } t \text{ is odd and } t > 1, \end{cases}$$

Note that $m_{ij}^0(\mathbf{g}(S), \delta) = 0, m_{ij}^1(\mathbf{g}(S), \delta) = \delta$.

Therefore, to calculate the productivity of the worker have $i \in S$, we need to consider four cases:

(1) t is even and $i \in K$

$$\begin{aligned} p_i^S(\delta, t) &= 1 + k_S \cdot \left(m_S \delta^2 + k_S m_S^2 \delta^4 + \dots + k_S^{\frac{t}{2}-1} m_S^{\frac{t}{2}-1} \delta^t \right) \\ &\quad + m_S \cdot \left(\delta + k_S m_S \delta^3 + \dots + k_S^{\frac{t}{2}-1} m_S^{\frac{t}{2}-1} \delta^{t-1} \right) \\ &= 1 + \left(k_S m_S \delta^2 + k_S^2 m_S^2 \delta^4 + \dots + k_S^{\frac{t}{2}} m_S^{\frac{t}{2}} \delta^t \right) \\ &\quad + \left(m_S \delta + k_S m_S^2 \delta^3 + \dots + k_S^{\frac{t}{2}-1} m_S^{\frac{t}{2}-1} \delta^{t-1} \right) \\ &= 1 + \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u} + \sum_{u=1}^{\frac{t}{2}} k_S^{u-1} m_S^{u-1} \delta^{2u-1} \end{aligned}$$

(2) t is even and $i \in M$

$$\begin{aligned} p_i^S(\delta, t) &= 1 + m_S \cdot \left(k_S \delta^2 + k_S^2 m_S \delta^4 + \dots + k_S^{\frac{t}{2}} m_S^{\frac{t}{2}-1} \delta^t \right) \\ &\quad + k_S \cdot \left(\delta + k_S m_S \delta^3 + \dots + k_S^{\frac{t-2}{2}} m_S^{\frac{t-2}{2}} \delta^{t-1} \right) \\ &= 1 + \left(k_S m_S \delta^2 + k_S^2 m_S^2 \delta^4 + \dots + k_S^{\frac{t}{2}} m_S^{\frac{t}{2}} \delta^t \right) \\ &\quad + \left(k_S \delta + k_S^2 m_S \delta^3 + \dots + k_S^{\frac{t}{2}} m_S^{\frac{t}{2}-1} \delta^{t-1} \right) \\ &= 1 + \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u} + \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^{u-1} \delta^{2u-1} \end{aligned}$$

(3) t is odd and $i \in K$

$$\begin{aligned} p_i^S(\delta, t) &= 1 + k_S \cdot \left(m_S \delta^2 + k_S m_S^2 \delta^4 + \dots + k_S^{\frac{t-1}{2}-1} m_S^{\frac{t-1}{2}-1} \delta^{t-1} \right) \\ &\quad + m_S \cdot \left(\delta + k_S m_S \delta^3 + \dots + k_S^{\frac{t-1}{2}} m_S^{\frac{t-1}{2}} \delta^t \right) \\ &= 1 + \left(k_S m_S \delta^2 + k_S^2 m_S^2 \delta^4 + \dots + k_S^{\frac{t-1}{2}} m_S^{\frac{t-1}{2}} \delta^{t-1} \right) \\ &\quad + \left(m_S \delta + k_S m_S^2 \delta^3 + \dots + k_S^{\frac{t+1}{2}-1} m_S^{\frac{t+1}{2}-1} \delta^t \right) \\ &= 1 + \sum_{u=1}^{\frac{t-1}{2}} k_S^u m_S^u \delta^{2u} + \sum_{u=1}^{\frac{t+1}{2}} k_S^{u-1} m_S^{u-1} \delta^{2u-1} \end{aligned}$$

(4) t is odd and $i \in M$

$$\begin{aligned} p_i^S(\delta, t) &= 1 + m_S \cdot \left(k_S \delta^2 + k_S^2 m_S \delta^4 + \dots + k_S^{\frac{t-1}{2}} m_S^{\frac{t-1}{2}-1} \delta^{t-1} \right) \\ &\quad + k_S \cdot \left(\delta + k_S m_S \delta^3 + \dots + k_S^{\frac{t-1}{2}} m_S^{\frac{t-1}{2}} \delta^t \right) \\ &= 1 + \left(k_S m_S \delta^2 + k_S^2 m_S^2 \delta^4 + \dots + k_S^{\frac{t-1}{2}} m_S^{\frac{t-1}{2}} \delta^{t-1} \right) \\ &\quad + \left(k_S \delta + k_S^2 m_S \delta^3 + \dots + k_S^{\frac{t+1}{2}} m_S^{\frac{t+1}{2}-1} \delta^t \right) \\ &= 1 + \sum_{u=1}^{\frac{t-1}{2}} k_S^u m_S^u \delta^{2u} + \sum_{u=1}^{\frac{t+1}{2}} k_S^u m_S^{u-1} \delta^{2u-1} \end{aligned}$$

From the above results we can find an explicit form for the game (N, v_δ^t) . Take $S \in N$, two cases are distinguished:

- If $t = 0$. It is straightforward by definition.
- If $t > 0$ is even

$$\begin{aligned} v_\delta^t(S) &= \sum_{i \in S} p_i^S(\delta, t) = \sum_{i \in K(S)} p_i^S(\delta, t) + \sum_{i \in M(S)} p_i^S(\delta, t) \\ &= k_S \cdot \left(1 + \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u} + \sum_{u=1}^{\frac{t}{2}} k_S^{u-1} m_S^{u-1} \delta^{2u-1} \right) \\ &\quad + m_S \cdot \left(1 + \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u} + \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^{u-1} \delta^{2u-1} \right) \\ &= k_S + m_S + (k_S + m_S) \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u} \\ &\quad + \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u-1} + \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^{u-1} \delta^{2u-1} \\ &= |S| + |S| \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u} + 2 \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u-1} \\ &= |S| + (|S| \delta + 2) \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u-1} \end{aligned}$$

- If t is odd:

$$\begin{aligned} v_\delta^t(S) &= \sum_{i \in S} p_i^S(\delta, t) = \sum_{i \in K(S)} p_i^S(\delta, t) + \sum_{i \in M(S)} p_i^S(\delta, t) \\ &= k_S \cdot \left(1 + \sum_{u=1}^{\frac{t-1}{2}} k_S^u m_S^u \delta^{2u} + \sum_{u=1}^{\frac{t+1}{2}} k_S^{u-1} m_S^{u-1} \delta^{2u-1} \right) \\ &\quad + m_S \cdot \left(1 + \sum_{u=1}^{\frac{t-1}{2}} k_S^u m_S^u \delta^{2u} + \sum_{u=1}^{\frac{t+1}{2}} k_S^u m_S^{u-1} \delta^{2u-1} \right) \\ &= k_S + m_S + (k_S + m_S) \sum_{u=1}^{\frac{t-1}{2}} k_S^u m_S^u \delta^{2u} \\ &\quad + \sum_{u=1}^{\frac{t+1}{2}} k_S^u m_S^u \delta^{2u-1} + \sum_{u=1}^{\frac{t+1}{2}} k_S^u m_S^{u-1} \delta^{2u-1} \\ &= |S| + |S| \sum_{u=1}^{\frac{t-1}{2}} k_S^u m_S^u \delta^{2u} + 2 \sum_{u=1}^{\frac{t+1}{2}} k_S^u m_S^u \delta^{2u-1} \\ &= |S| + (|S| \delta + 2) \sum_{u=1}^{\frac{t-1}{2}} k_S^u m_S^u \delta^{2u-1} + 2 k_S^{\frac{t+1}{2}} m_S^{\frac{t+1}{2}} \delta^t \quad \blacksquare \end{aligned}$$

Proof of Theorem 3.2. Consider $g = (K, M, E)$ a complete bipartite network and its corresponding FAN game (N, v_δ^t) . Take $S, T \subseteq N$ such that $S \subseteq T$ with $i \in S$, then $k_S \leq k_T$ and $m_S \leq m_T$. We have to prove that $v_\delta^t(S) - v_\delta^t(S \setminus \{i\}) \leq v_\delta^t(T) - v_\delta^t(T \setminus \{i\})$. Two cases are distinguished:

- t is even.

$$\begin{aligned} &v_\delta^t(S) - v_\delta^t(S \setminus \{i\}) \\ &= \left(|S| + (|S| \delta + 2) \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u-1} \right) \\ &\quad - \left(|S \setminus \{i\}| + (|S \setminus \{i\}| \delta + 2) \sum_{u=1}^{\frac{t}{2}} (k_S - 1)^u m_S^u \delta^{2u-1} \right) \\ &= 1 + \sum_{u=1}^{\frac{t}{2}} \left[(|S| \delta + 2) k_S^u - (|S| \delta - \delta + 2) (k_S - 1)^u \right] m_S^u \delta^{2u-1} \end{aligned}$$

$$\begin{aligned} &\leq 1 + \sum_{u=1}^{\frac{t}{2}} \left[(|\mathcal{S}| \delta + 2) k_S^u - (|\mathcal{S}| \delta - \delta + 2) (k_S - 1)^u \right] m_T^u \delta^{2u-1} \\ &\leq \sum_{u=1}^{\frac{t}{2}} \left[(|\mathcal{S}| \delta + 2) k_T^u - (|\mathcal{S}| \delta - \delta + 2) (k_T - 1)^u \right] m_T^u \delta^{2u-1} \\ &\leq \sum_{u=1}^{\frac{t}{2}} \left[(|\mathcal{T}| \delta + 2) k_T^u - (|\mathcal{T}| \delta - \delta + 2) (k_T - 1)^u \right] m_T^u \delta^{2u-1} \\ &= v'_\delta(\mathcal{T}) - v'_\delta(\mathcal{T} \setminus \{i\}) \end{aligned}$$

• t is odd. A similar argument demonstrates it. ■

Proof of Lemma 4.1. Consider $g = (K, M, E)$ a complete bipartite network and $\Lambda(g, \delta)$ the set of all possible FAN games with index $\delta \geq 0$. For each $S \subseteq N$, $g(S) = (K(S), M(S), E(S))$ is a subnetwork of g . We distinguish two cases.

If $\delta > 0$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} v'_\delta(S) &= |\mathcal{S}| + |\mathcal{S}| \sum_{u=1}^{\infty} (k_S m_S \delta^2)^u + \frac{2}{\delta} \sum_{u=1}^{\infty} (k_S m_S \delta^2)^u \\ &= |\mathcal{S}| + \left(|\mathcal{S}| + \frac{2}{\delta} \right) \sum_{u=1}^{\infty} (k_S m_S \delta^2)^u \\ &\sum_{u=1}^{\infty} (k_S m_S \delta^2)^u \text{ converges to } \frac{k_S m_S \delta^2}{1 - k_S m_S \delta^2} \text{ if and only if } k_S m_S \delta^2 < 1 \Leftrightarrow \\ \delta &< \frac{1}{\sqrt{k_S m_S}} = \frac{1}{\lambda_{\max}(S)}. \text{ Hence,} \end{aligned}$$

$$\begin{aligned} \lim_{t \rightarrow \infty} v'_\delta(S) &= |\mathcal{S}| + \left(|\mathcal{S}| + \frac{2}{\delta} \right) \frac{k_S m_S \delta^2}{1 - k_S m_S \delta^2} \\ &= k_S + m_S + \frac{(k_S + m_S) k_S m_S \delta^2}{1 - k_S m_S \delta^2} + \frac{2k_S m_S \delta}{1 - k_S m_S \delta^2} \\ &= \frac{k_S + m_S - k_S^2 m_S \delta^2 - k_S m_S \delta^2 + k_S^2 m_S \delta^2 + k_S m_S^2 \delta^2 + 2k_S m_S \delta}{1 - k_S m_S \delta^2} \\ &= \frac{k_S + m_S + 2k_S m_S \delta}{1 - k_S m_S \delta^2} =: v_\delta(S). \end{aligned}$$

If $\delta = 0$, then it is easy to check that $v'_\delta(S) = |\mathcal{S}| =: v_\delta(S)$ for all $t \in \mathbb{N}$ and for each coalition $S \subseteq N$. ■

Proof of Theorem 4.2. Consider $g = (K, M, E)$ a complete bipartite network and (N, v'_δ) his corresponding FAN game. For each $S \subseteq N$, $g(S) = (K(S), M(S), E(S))$ is a subnetwork of g . We know that $\lambda_{\max}(N) \geq \lambda_{\max}(S)$ for all $S \subseteq N$. Hence, if $\delta \in \left[0, \frac{1}{\lambda_{\max}(N)} \right]$, then $\delta \in \left[0, \frac{1}{\lambda_{\max}(S)} \right]$ for all $S \subseteq N$, and so by Lemma 4.1 we conclude that $\{v'_\delta\}_{t \in \mathbb{N}}$ converges to v_δ , defined as $v_\delta(S) = \frac{k_S + m_S + 2k_S m_S \delta}{1 - k_S m_S \delta^2}$, for any $S \subseteq N$. ■

Proof of Proposition 4.4. Take a coalition $S \subseteq N$. Then, $0 \leq \delta < \frac{1}{\lambda_{\max}(N)} \leq \frac{1}{\lambda_{\max}(S)}$. By the proof of Lemma 4.1, we know that $\sum_{u=1}^{\infty} (k_S m_S \delta^2)^u$ converges to $\frac{k_S m_S \delta^2}{1 - k_S m_S \delta^2}$. Hence, if $i \in K(S)$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} p_i^S(\delta, t) &= 1 + \sum_{u=1}^{\infty} k_S^u m_S^u \delta^{2u} + \sum_{u=1}^{\infty} k_S^{u-1} m_S^u \delta^{2u-1} \\ &= 1 + \left(k_S + \frac{1}{\delta} \right) \sum_{u=1}^{\infty} k_S^{u-1} m_S^u \delta^{2u} \\ &= 1 + \left(k_S + \frac{1}{\delta} \right) \frac{1}{k_S} \sum_{u=1}^{\infty} (k_S m_S \delta^2)^u \\ &= 1 + \left(k_S + \frac{1}{\delta} \right) \frac{1}{k_S} \frac{k_S m_S \delta^2}{1 - k_S m_S \delta^2} \\ &= 1 + \frac{k_S m_S \delta^2}{1 - k_S m_S \delta^2} + \frac{m_S \delta}{1 - k_S m_S \delta^2} \\ &= \frac{1 - k_S m_S \delta^2 + k_S m_S \delta^2 + m_S \delta}{1 - k_S m_S \delta^2} = \frac{1 + m_S \delta}{1 - k_S m_S \delta^2} =: p_i^S(\delta). \end{aligned}$$

For $j \in M(S)$, a similar argument proves that $\lim_{t \rightarrow \infty} p_j^S(\delta, t) = \frac{1 + k_S \delta}{1 - k_S m_S \delta^2} =: p_j^S(\delta)$.

It is easy to prove that $p_i^N(\delta) \geq p_i^S(\delta)$ for all $S \subseteq N$. Indeed, if $i \notin S$ then $p_i^S(\delta) = 0$, and the inequality holds. If $i \in S$, the inequality is satisfied because $k_N \geq k_S$ and $m_N \geq m_S$. Therefore, if we take a coalition $S \subseteq N$, it satisfies that:

$$\begin{aligned} \sum_{i \in S} p_i^N(\delta) &\geq \sum_{i \in S} p_i^S(\delta) = \sum_{i \in K(S)} p_i^S(\delta) + \sum_{i \in M(S)} p_i^S(\delta) \\ &= k_S \cdot \frac{1 + m_S \delta}{1 - k_S m_S \delta^2} + m_S \cdot \frac{1 + k_S \delta}{1 - k_S m_S \delta^2} = v_\delta(S). \end{aligned}$$

It is straightforward to prove that $\sum_{i \in N} p_i^N(\delta) = v_\delta(N)$. Hence, $p^N(\delta) \in \text{Core}(N, v_\delta)$.

If $\delta = 0$ then, $v_\delta(S) = |\mathcal{S}|$ for all team $S \subseteq N$ and $p_i^N(0) = 1$ for each worker $i \in N$. Therefore, $p^N(0) \in \text{Core}(N, v_\delta)$. We then conclude that $p^N(\delta) \in \text{Core}(N, v_\delta)$, for any $\delta \geq 0$. ■

Proof of Lemma 4.5. If $i \in K(S)$, then $v_\delta(S) - v_\delta(S \setminus \{i\})$ is equal to:

$$\begin{aligned} &\frac{k_S + m_S + 2k_S m_S \delta}{1 - k_S m_S \delta^2} - \frac{(k_S - 1) + m_S + 2(k_S - 1) m_S \delta}{1 - (k_S - 1) m_S \delta^2} \\ &= \frac{k_S + m_S + 2k_S m_S \delta}{1 - k_S m_S \delta^2} - \frac{k_S + m_S + 2k_S m_S \delta - 2m_S \delta - 1}{1 - k_S m_S \delta^2 + m_S \delta^2} \\ &= \frac{P}{Q} - \frac{P}{Q + m_S \delta^2} + \frac{2m_S \delta + 1}{Q + m_S \delta^2} \\ &= \frac{P(Q + m_S \delta^2) - PQ}{Q(Q + m_S \delta^2)} + \frac{2m_S \delta + 1}{Q + m_S \delta^2} = \frac{P \cdot m_S \delta^2}{Q(Q + m_S \delta^2)} + \frac{2m_S \delta + 1}{Q + m_S \delta^2} \\ &= \frac{k_S m_S \delta^2 + m_S^2 \delta^2 + 2k_S m_S^2 \delta^3 + 2m_S \delta + 1 - 2k_S m_S^2 \delta^3 - k_S m_S \delta^2}{(1 - k_S m_S \delta^2)(1 - k_S m_S \delta^2 + m_S \delta^2)} \\ &= \frac{1 + 2m_S \delta + m_S^2 \delta^2}{(1 - k_S m_S \delta^2)(1 - k_S m_S \delta^2 + m_S \delta^2)} \\ &= \frac{(1 + m_S \delta)^2}{(1 - k_S m_S \delta^2)(1 - k_S m_S \delta^2 + m_S \delta^2)} \end{aligned}$$

where $P := k_S + m_S + 2k_S m_S \delta$ and $Q := 1 - k_S m_S \delta^2$.

If $i \in M(S)$, then $v_\delta(S) - v_\delta(S \setminus \{i\})$ is equal to:

$$\begin{aligned} &\frac{k_S + m_S + 2k_S m_S \delta}{1 - k_S m_S \delta^2} - \frac{k_S + m_S - 1 + 2k_S (m_S - 1) \delta}{1 - k_S (m_S - 1) \delta^2} \\ &= \frac{k_S + m_S + 2k_S m_S \delta}{1 - k_S m_S \delta^2} - \frac{k_S + m_S + 2k_S m_S \delta - 2k_S \delta - 1}{1 - k_S m_S \delta^2 + k_S \delta^2} \\ &= \frac{P}{Q} - \frac{P}{Q + k_S \delta^2} + \frac{2k_S \delta + 1}{Q + k_S \delta^2} \\ &= \frac{P(Q + k_S \delta^2) - PQ}{Q(Q + k_S \delta^2)} + \frac{2k_S \delta + 1}{Q + k_S \delta^2} = \frac{P \cdot k_S \delta^2}{Q(Q + k_S \delta^2)} + \frac{2k_S \delta + 1}{Q + k_S \delta^2} \\ &= \frac{k_S^2 \delta^2 + k_S m_S \delta^2 + 2k_S^2 m_S \delta^3 + 2k_S \delta + 1 - 2k_S^2 m_S \delta^3 - k_S m_S \delta^2}{(1 - k_S m_S \delta^2)(1 - k_S m_S \delta^2 + k_S \delta^2)} \\ &= \frac{k_S^2 \delta^2 + 2k_S \delta + 1}{(1 - k_S m_S \delta^2)(1 - k_S m_S \delta^2 + k_S \delta^2)} \\ &= \frac{(1 + k_S \delta)^2}{(1 - k_S m_S \delta^2)(1 - k_S m_S \delta^2 + k_S \delta^2)} \quad \blacksquare \end{aligned}$$

Proof of Theorem 4.6. Consider the AN game (N, v_δ) . Let us demonstrate that for all $i \in S \subseteq T \subseteq N$, $v_\delta(T) - v_\delta(T \setminus \{i\}) \geq v_\delta(S) - v_\delta(S \setminus \{i\})$.

Indeed, take $i \in S \subseteq T \subseteq N$. If $i \in K(S)$

$$\begin{aligned} v_\delta(T) - v_\delta(T \setminus \{i\}) &= \frac{(1 + m_T \delta)^2}{(1 - k_T m_T \delta^2)(1 - k_T m_T \delta^2 + m_T \delta^2)} \\ &= \frac{(1 + m_T \delta)^2}{(1 - k_T m_T \delta^2)(1 - (k_T - 1) m_T \delta^2)} \end{aligned}$$

$$\begin{aligned} &\geq \frac{(1 + m_S \delta)^2}{(1 - k_S m_S \delta^2)(1 - (k_S - 1) m_S \delta^2)} \\ &= \frac{(1 + m_S \delta)^2}{(1 - k_S m_S \delta^2)(1 - k_S m_S \delta^2 + m_S \delta^2)} \\ &= v_\delta(S) - v_\delta(S \setminus \{i\}) \end{aligned}$$

since $k_T \geq k_S$ and $m_T \geq m_S$. For $i \in M(S)$ the proof is similar. ■

Proof of Theorem 4.7. We have that for all coalitions $S, R \subseteq N$ such that $k_S = k_R$ and $m_S = m_R$ then $v_\delta(S) = v_\delta(R)$. Moreover we can consider $|S| = m_S + k_S$, therefore $\gamma(S) = \frac{(s-1)!(n-s)!}{n!} = \frac{(k_S+m_S-1)!(n-k_S-m_S)!}{n!} = \gamma(k_S, m_S)$. If $i \in K$:

$$\begin{aligned} \phi_i(v_\delta) &= \sum_{i \in S \subseteq N} \gamma(S) \cdot (v_\delta(S) - v_\delta(S \setminus \{i\})) \\ &= \sum_{m=0}^{|M|} \sum_{\substack{i \in S \subseteq N: \\ k_S = 1 + m_S = m}} \gamma(S) \cdot (v_\delta(S) - v_\delta(S \setminus \{i\})) + \dots \\ &\quad + \sum_{m=0}^{|M|} \sum_{\substack{i \in S \subseteq N: \\ k_S = |K| + m_S = m}} \gamma(S) \cdot (v_\delta(S) - v_\delta(S \setminus \{i\})) \\ &= \sum_{k=1}^{|K|} \sum_{m=0}^{|M|} \sum_{\substack{i \in S \subseteq N: \\ k_S = k + m_S = m}} \gamma(S) \cdot \frac{(1 + m\delta)^2}{(1 - km\delta^2)(1 - km\delta^2 + m\delta^2)} \\ &= \sum_{k=1}^{|K|} \sum_{m=0}^{|M|} \frac{(1 + m\delta)^2}{(1 - km\delta^2)(1 - km\delta^2 + m\delta^2)} \sum_{\substack{i \in S \subseteq N: \\ k_S = k + m_S = m}} \gamma(k_S, m_S) \\ &= \sum_{k=1}^{|K|} \sum_{m=0}^{|M|} \frac{(1 + m\delta)^2}{(1 - km\delta^2)(1 - km\delta^2 + m\delta^2)} \cdot \binom{|M|}{m} \\ &\quad \cdot \binom{|K| - 1}{k - 1} \cdot \gamma(k, m) \\ &= \sum_{k=1}^{|K|} \sum_{m=0}^{|M|} \Pi_M^K(k, m) \cdot \frac{(1 + m\delta)^2}{(1 - km\delta^2)(1 - km\delta^2 + m\delta^2)} \end{aligned}$$

If $i \in M$:

$$\begin{aligned} \phi_i(v_\delta) &= \sum_{i \in S \subseteq N} \gamma(S) \cdot (v_\delta(S) - v_\delta(S \setminus \{i\})) \\ &= \sum_{k=0}^{|K|} \sum_{\substack{i \in S \subseteq N: \\ k_S = k + m_S = 1}} \gamma(S) \cdot (v_\delta(S) - v_\delta(S \setminus \{i\})) + \dots \\ &\quad + \sum_{k=0}^{|K|} \sum_{\substack{i \in S \subseteq N: \\ k_S = k + m_S = |M|}} \gamma(S) \cdot (v_\delta(S) - v_\delta(S \setminus \{i\})) \\ &= \sum_{k=0}^{|K|} \sum_{m=1}^{|M|} \sum_{\substack{i \in S \subseteq N: \\ k_S = k + m_S = m}} \gamma(S) \cdot \frac{(1 + k\delta)^2}{(1 - km\delta^2)(1 - km\delta^2 + k\delta^2)} \\ &= \sum_{k=0}^{|K|} \sum_{m=1}^{|M|} \frac{(1 + k\delta)^2}{(1 - km\delta^2)(1 - km\delta^2 + k\delta^2)} \sum_{\substack{i \in S \subseteq N: \\ k_S = k + m_S = m}} \gamma(k_S, m_S) \\ &= \sum_{k=0}^{|K|} \sum_{m=1}^{|M|} \frac{(1 + k\delta)^2}{(1 - km\delta^2)(1 - km\delta^2 + k\delta^2)} \cdot \binom{|M| - 1}{m - 1} \\ &\quad \cdot \binom{|K|}{k} \cdot \gamma(k, m) \\ &= \sum_{k=0}^{|K|} \sum_{m=1}^{|M|} \Pi_K^M(m, k) \cdot \frac{(1 + k\delta)^2}{(1 - km\delta^2)(1 - km\delta^2 + k\delta^2)} \quad \blacksquare \end{aligned}$$

Proof of Proposition 5.1. Consider $g = (K, M, E)$ a complete bipartite network and (N, v_δ^i) , (N, v_δ^{i-1}) its corresponding FAN games. We distinguish two cases.

Case 1: t is even, then

$$\begin{aligned} d_\delta^t(S) &= v_\delta^t(S) - v_\delta^{t-1}(S) = \left(|S| + (|S| \delta + 2) \sum_{u=1}^{\frac{t}{2}} k_S^u m_S^u \delta^{2u-1} \right) \\ &\quad - \left(|S| + (|S| \delta + 2) \sum_{u=1}^{\frac{t}{2}-1} (k_S^u m_S^u \delta^{2u-1}) + 2k_S^{\frac{t}{2}} m_S^{\frac{t}{2}} \delta^{t-1} \right) \\ &= (|S| \delta + 2) k_S^{\frac{t}{2}} m_S^{\frac{t}{2}} \delta^{t-1} - 2k_S^{\frac{t}{2}} m_S^{\frac{t}{2}} \delta^{t-1} = |S| k_S^{\frac{t}{2}} m_S^{\frac{t}{2}} \delta^t \end{aligned}$$

Case 2: t is odd, then

$$\begin{aligned} d_\delta^t(S) &= v_\delta^t(S) - v_\delta^{t-1}(S) \\ &= \left(|S| + (|S| \delta + 2) \sum_{u=1}^{\frac{t-1}{2}} (k_S^u m_S^u \delta^{2u-1}) + 2k_S^{\frac{t+1}{2}} m_S^{\frac{t+1}{2}} \delta^t \right) \\ &\quad - \left(|S| + (|S| \delta + 2) \sum_{u=1}^{\frac{t-1}{2}} k_S^u m_S^u \delta^{2u-1} \right) = 2k_S^{\frac{t+1}{2}} m_S^{\frac{t+1}{2}} \delta^t \end{aligned}$$

this can be rewritten as

$$d_\delta^t(S) = \begin{cases} \frac{k_S + m_S}{2} \cdot 2 (\lambda_{\max}(S) \delta)^t, & \text{if } t \text{ is even,} \\ \sqrt{k_S m_S} \cdot 2 (\lambda_{\max}(S) \delta)^t, & \text{if } t \text{ is odd,} \end{cases}$$

We wonder if we can express both expressions for even and odd t in a single algebraic expression that depends on the eigenvalues. If this were possible, we should be able to write both expressions in the form:

$$\begin{aligned} \frac{k_S + m_S}{2} \cdot 2 (\lambda_{\max}(S) \delta)^t &= [A (\lambda_{\max}(S))^t + B (-\lambda_{\max}(S))^t] \delta^t; \\ A + B &= k_S + m_S \end{aligned} \tag{1}$$

and

$$\begin{aligned} \sqrt{k_S m_S} \cdot 2 (\lambda_{\max}(S) \delta)^t &= [A (\lambda_{\max}(S))^t + B (-\lambda_{\max}(S))^t] \delta^t \\ A - B &= 2\sqrt{k_S m_S} \end{aligned} \tag{2}$$

Solving the system (1)–(2), we obtain:

$$\begin{aligned} d_\delta^t(S) &= \left[\left(\frac{k_S + m_S}{2} + \sqrt{k_S m_S} \right) (\lambda_{\max}(S))^t + \left(\frac{k_S + m_S}{2} - \sqrt{k_S m_S} \right) \right. \\ &\quad \left. \times (-\lambda_{\max}(S))^t \right] \delta^t \\ &= \frac{1}{2} \left[(k_S + m_S + 2\sqrt{k_S m_S}) (\lambda_{\max}(S))^t + (k_S + m_S - 2\sqrt{k_S m_S}) \right. \\ &\quad \left. \times (-\lambda_{\max}(S))^t \right] \delta^t \\ &= \frac{1}{2} \left[(\sqrt{k_S} + \sqrt{m_S})^2 (\lambda_{\max}(S))^t + (\sqrt{k_S} - \sqrt{m_S})^2 (-\lambda_{\max}(S))^t \right] \\ &\quad \times \delta^t \quad \blacksquare \end{aligned}$$

Proof of Proposition 5.2. Consider $g = (K, M, E)$ a complete bipartite network and (N, d_δ^i) its corresponding difference game. Let us prove that $\sum_{i \in N} x_i^t(\delta) = d_\delta^t(N)$ and $\sum_{i \in S} x_i^t(\delta) \geq d_\delta^t(S)$. It is easy to check that $x^t(\delta)$ satisfy efficiency:

$$\begin{aligned} \sum_{i \in N} x_i^t(\delta) &= \sum_{i \in K} \left(\frac{d_\delta^t(N)}{|N|} \cdot \frac{|M|}{|K|} \right) + \sum_{i \in M} \left(\frac{d_\delta^t(N)}{|N|} \cdot \frac{|K|}{|M|} \right) \\ &= d_\delta^t(N) \cdot \left(\frac{|M|}{|N|} + \frac{|K|}{|N|} \right) = d_\delta^t(N) \end{aligned}$$

It is straightforward to check that productivity distribution $x^t(\delta)$ has the following explicit formula:

If t is even, then

$$x_i^t(\delta) = \begin{cases} |K|^{\frac{t}{2}-1} |M|^{\frac{t}{2}+1} \delta^t, & \text{if } i \in K, \\ |K|^{\frac{t}{2}+1} |M|^{\frac{t}{2}-1} \delta^t, & \text{if } i \in M. \end{cases}$$

if t is odd, then

$$x_i^t(\delta) = \begin{cases} \frac{2}{|N|} |K|^{\frac{t-1}{2}} |M|^{\frac{t+3}{2}} \delta^t, & \text{if } i \in K, \\ \frac{2}{|N|} |K|^{\frac{t+3}{2}} |M|^{\frac{t-1}{2}} \delta^t, & \text{if } i \in M. \end{cases}$$

In order to demonstrate coalitional stability for a coalition $S \subset N$, we distinguish two cases.

Case 1: t is even, then

$$\begin{aligned} \sum_{i \in S} x_i^t(\delta) &= k_S \cdot |K|^{\frac{t}{2}-1} |M|^{\frac{t}{2}+1} \delta^t + m_S \cdot |K|^{\frac{t}{2}+1} |M|^{\frac{t}{2}-1} \delta^t \\ &\geq k_S \cdot k_S^{\frac{t}{2}-1} m_S^{\frac{t}{2}+1} \delta^t + m_S \cdot k_S^{\frac{t}{2}+1} m_S^{\frac{t}{2}-1} \delta^t \\ &= k_S^{\frac{t}{2}} m_S^{\frac{t}{2}+1} \delta^t + k_S^{\frac{t}{2}+1} m_S^{\frac{t}{2}} \delta^t = |S| k_S^{\frac{t}{2}} m_S^{\frac{t}{2}} \delta^t = d_\delta^t(S). \end{aligned}$$

Case 2: t is odd, then

$$\begin{aligned} \sum_{i \in S} x_i^t(\delta) &= k_S \cdot \frac{2}{|N|} |K|^{\frac{t-1}{2}} |M|^{\frac{t+3}{2}} \delta^t + m_S \cdot \frac{2}{|N|} |K|^{\frac{t+3}{2}} |M|^{\frac{t-1}{2}} \delta^t \\ &= k_S \cdot \frac{2|M|}{|N|} |K|^{\frac{t-1}{2}} |M|^{\frac{t+1}{2}} \delta^t + m_S \cdot \frac{2|K|}{|N|} |K|^{\frac{t+1}{2}} |M|^{\frac{t-1}{2}} \delta^t \\ &\geq k_S \cdot \frac{2|M|}{|N|} k_S^{\frac{t-1}{2}} m_S^{\frac{t+1}{2}} \delta^t + m_S \cdot \frac{2|K|}{|N|} k_S^{\frac{t+1}{2}} m_S^{\frac{t-1}{2}} \delta^t \\ &= \frac{2|M|}{|N|} k_S^{\frac{t+1}{2}} m_S^{\frac{t+1}{2}} \delta^t + \frac{2|K|}{|N|} k_S^{\frac{t+1}{2}} m_S^{\frac{t+1}{2}} \delta^t = 2k_S^{\frac{t+1}{2}} m_S^{\frac{t+1}{2}} \delta^t = d_\delta^t(S). \quad \blacksquare \end{aligned}$$

Proof of Proposition 5.4. Consider $g = (K, M, E)$ a complete bipartite network and $\Lambda(\delta)$ the set of all possible FAN games with index $\delta > 0$. We distinguish two cases.

Case 1: $i \in K$, then

$$\begin{aligned} \omega_i(\delta) &= 1 + \lim_{t \rightarrow \infty} \left(\sum_{u=1}^{\frac{t}{2}} (|K|^{u-1} |M|^{u+1} \delta^{2u}) \right. \\ &\quad \left. + \sum_{u=0}^{\frac{t-1}{2}} \left(\frac{2}{|N|} |K|^u |M|^{u+2} \delta^{2u+1} \right) \right) \\ &= 1 + \frac{|M|}{|K|} \lim_{t \rightarrow \infty} \left(\sum_{u=1}^{\frac{t}{2}} (|K| |M| \delta^2)^u \right) \\ &\quad + \frac{2|M|^2 \delta}{|N|} \lim_{t \rightarrow \infty} \left(\sum_{u=0}^{\frac{t-1}{2}} (|K| |M| \delta^2)^u \right) \\ &= 1 + \frac{|M|}{|K|} \sum_{u=1}^{\infty} (|K| |M| \delta^2)^u + \frac{2|M|^2 \delta}{|N|} \sum_{u=0}^{\infty} (|K| |M| \delta^2)^u \\ &= 1 + \frac{|M|}{|K|} \frac{|K| |M| \delta^2}{1 - |K| |M| \delta^2} + \frac{2|M|^2 \delta}{|N|} \frac{1}{1 - |K| |M| \delta^2} \\ &= 1 + \left(\frac{|M|}{|K|} \delta + \frac{2|M|}{|N| |K|} \right) \frac{|K| |M| \delta}{1 - |K| |M| \delta^2} \end{aligned}$$

Case 2: $i \in M$, then

$$\begin{aligned} \omega_i(\delta) &= 1 + \lim_{t \rightarrow \infty} \left(\sum_{u=1}^{\frac{t}{2}} (|K|^{u+1} |M|^{u-1} \delta^{2u}) \right. \\ &\quad \left. + \sum_{u=0}^{\frac{t-1}{2}} \left(\frac{2}{|N|} |K|^{u+2} |M|^u \delta^{2u+1} \right) \right) \\ &= 1 + \frac{|K|}{|M|} \lim_{t \rightarrow \infty} \left(\sum_{u=1}^{\frac{t}{2}} (|K| |M| \delta^2)^u \right) \end{aligned}$$

$$\begin{aligned} &+ \frac{2|K|^2 \delta}{|N|} \lim_{t \rightarrow \infty} \left(\sum_{u=0}^{\frac{t-1}{2}} (|K| |M| \delta^2)^u \right) \\ &= 1 + \frac{|K|}{|M|} \sum_{u=1}^{\infty} (|K| |M| \delta^2)^u + \frac{2|K|^2 \delta}{|N|} \sum_{u=0}^{\infty} (|K| |M| \delta^2)^u \\ &= 1 + \frac{|K|}{|M|} \frac{|K| |M| \delta^2}{1 - |K| |M| \delta^2} + \frac{2|K|^2 \delta}{|N|} \frac{1}{1 - |K| |M| \delta^2} \\ &= 1 + \left(\frac{|K|}{|M|} \delta + \frac{2|K|}{|N| |M|} \right) \frac{|K| |M| \delta}{1 - |K| |M| \delta^2} \quad \blacksquare \end{aligned}$$

Proof of Theorem 5.5. Consider $g = (K, M, E)$ a complete bipartite network and (N, v_δ) its corresponding AN game. We prove first that $\omega(\delta)$ satisfies efficiency. Indeed,

$$\begin{aligned} &\sum_{i \in N} \omega_i(\delta) \\ &= \sum_{i \in K} \omega_i(\delta) + \sum_{i \in M} \omega_i(\delta) \\ &= |K| \left[1 + \left(\frac{|M|}{|K|} \delta + \frac{2|M|}{|N| |K|} \right) \frac{|K| |M| \delta}{1 - |K| |M| \delta^2} \right] \\ &\quad + |M| \left[1 + \left(\frac{|K|}{|M|} \delta + \frac{2|K|}{|N| |M|} \right) \frac{|K| |M| \delta}{1 - |K| |M| \delta^2} \right] \\ &= |K| + |M| + \left(|K| \delta + |M| \delta + \frac{2|M|}{|N|} + \frac{2|K|}{|N|} \right) \frac{|K| |M| \delta}{1 - |K| |M| \delta^2} \\ &= |K| + |M| + (|K| \delta + |M| \delta + 2) \frac{|K| |M| \delta}{1 - |K| |M| \delta^2} \\ &= \frac{|K| - |K|^2 |M| \delta^2 + |M| - |K| |M|^2 \delta^2 + |K|^2 |M| \delta^2 + |K| |M|^2 \delta^2 + 2|K| |M| \delta}{1 - |K| |M| \delta^2} \\ &= \frac{|K| + |M| + 2|K| |M| \delta}{1 - |K| |M| \delta^2} = v_\delta(N) \end{aligned}$$

Consider now the set of all possible FAN games with index δ . We know that $x^t(\delta)$ is a core allocation of the game (N, d_δ^t) for all $t \geq 1$. Moreover, $\sum_{u=1}^t d_\delta^u = v_\delta^t - v_\delta^0$ and $\sum_{u=1}^t \text{Core}(N, d_\delta^u) \not\subseteq \text{Core}(N, v_\delta^t - v_\delta^0)$. Hence,

$$\sum_{i \in S} \sum_{u=1}^t x_i^u(\delta) \geq v_\delta^t(S) - v_\delta^0(S).$$

Then we take as t tends to infinity,

$$\begin{aligned} \sum_{i \in S} (\omega_i(\delta) - 1) &\geq v_\delta(S) - v_\delta^0(S); \\ \left(\sum_{i \in S} \omega_i(\delta) \right) - |S| &\geq v_\delta(S) - |S|; \\ \sum_{i \in S} \omega_i(\delta) &\geq v_\delta(S) \end{aligned}$$

Hence, we conclude that $\omega(\delta)$ is a core allocation of (N, v_δ) . \blacksquare

Proof of Theorem 5.6. It is clear that the LRP distribution $\omega(\delta)$ satisfies EF, EB and LBP.

To show the converse, take a productivity distribution φ on the class of AN games, that satisfies EF, EB and LBP.

By EF and EB we have that $|K| \varphi_i(v_\delta) + |M| \varphi_j(v_\delta) = v_\delta(N)$ for any $i \in K$ and $j \in M$.

Moreover, by EB and LBP: $\frac{|K|}{|M|} (|K| \varphi_i(v_\delta) - |K|) = |M| \varphi_j(v_\delta) - |M|$ for any $i \in K$ and $j \in M$. Substituting the second equation into the first equation, we obtain that:

$$\begin{aligned} |K| \varphi_i(v_\delta) &= v_\delta(N) - |M| \varphi_j(v_\delta) \\ |K| \varphi_i(v_\delta) &= v_\delta(N) - \frac{|K|}{|M|} (|K| \varphi_i(v_\delta) - |K|) - |M|; \\ \varphi_i(v_\delta) &= \frac{1}{|K|} v_\delta(N) - \frac{1}{|M|} (|K| \varphi_i(v_\delta) - |K|) - \frac{|M|}{|K|}; \\ \varphi_i(v_\delta) + \frac{|K|}{|M|} \varphi_i(v_\delta) &= \frac{1}{|K|} v_\delta(N) + \frac{|K|}{|M|} - \frac{|M|}{|K|}; \\ \varphi_i(v_\delta) &= \left(\frac{1}{|K|} v_\delta(N) + \frac{|K|}{|M|} - \frac{|M|}{|K|} \right) : \left(1 + \frac{|K|}{|M|} \right). \end{aligned}$$

Developing the last expression, we obtain:

$$\begin{aligned} & \varphi_i(v_\delta) \\ &= \left(\frac{1}{|K|} v_\delta(N) + \frac{|K|}{|M|} - \frac{|M|}{|K|} \right) : \left(1 + \frac{|K|}{|M|} \right) \\ &= \left(\frac{|K| + |M| + 2|K||M|\delta}{(1 - |K||M|\delta^2)|K|} + \frac{|K|^2 - |M|^2}{|M||K|} \right) : \frac{|N|}{|M|} \\ &= \left(\frac{|N| + 2|K||M|\delta}{(1 - |K||M|\delta^2)|K|} + \frac{|N|(|K| - |M|)}{|M||K|} \right) \cdot \frac{|M|}{|N|} \\ &= \frac{\frac{|M||N| + 2|K||M|^2\delta}{(1 - |K||M|\delta^2)|K|} + \frac{|K| - |M|}{|K|}}{(1 - |K||M|\delta^2)|K||N|} \\ &= \frac{|M||N| + 2|K||M|^2\delta + |K||N| - |K|^2|M|\delta^2|N| - |M||N| + |K||M|^2\delta^2|N|}{(1 - |K||M|\delta^2)|K||N|} \\ &= \frac{2|K||M|^2\delta + |K||N| - |K|^2|M|\delta^2|N| + |K||M|^2\delta^2|N|}{(1 - |K||M|\delta^2)|K||N|} \\ &= \frac{|K||N| - |K|^2|M|\delta^2|N|}{(1 - |K||M|\delta^2)|K||N|} + \frac{|K||M|^2\delta^2|N|}{(1 - |K||M|\delta^2)|K||N|} \\ &\quad + \frac{2|K||M|^2\delta}{(1 - |K||M|\delta^2)|K||N|} \\ &= \frac{(1 - |K||M|\delta^2)|N||K|}{(1 - |K||M|\delta^2)|N||K|} + \frac{|K||M|^2\delta^2}{(1 - |K||M|\delta^2)|K|} + \frac{2|K||M|^2\delta}{(1 - |K||M|\delta^2)|N||K|} \\ &= 1 + \left(\frac{|M|}{|K|} \delta + \frac{2|M|}{|N||K|} \right) \frac{|K||M|\delta}{1 - |K||M|\delta^2} = \omega_i(\delta) \end{aligned}$$

Finally,

$$\begin{aligned} \frac{|K|}{|M|} (|K| \varphi_i(v_\delta) - |K|) &= |M| \varphi_j(v_\delta) - |M|; \\ \varphi_j(v_\delta) &= 1 + \frac{|K|^2}{|M|^2} (\varphi_i(v_\delta) - 1); \end{aligned}$$

Developing the expression, we obtain:

$$\begin{aligned} \varphi_j(v_\delta) &= 1 + \frac{|K|^2}{|M|^2} \left[\left(\frac{|M|}{|K|} \delta + \frac{2|M|}{|N||K|} \right) \frac{|K||M|\delta}{1 - |K||M|\delta^2} \right] \\ &= 1 + \left(\frac{|K|}{|M|} \delta + \frac{2|K|}{|N||M|} \right) \frac{|K||M|\delta}{1 - |K||M|\delta^2} = \omega_j(\delta) \quad \blacksquare \end{aligned}$$

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