THE PARTITION ALGEBRA AND THE PLETHYSM COEFFICIENTS II: RAMIFIED PLETHYSM

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ABSTRACT. The plethysm coefficient $p(\nu, \mu, \lambda)$ is the multiplicity of the Schur function s_λ in the plethysm product $s_\nu \circ s_\mu$. In this paper we use Schur–Weyl duality between wreath products of symmetric groups and the ramified partition algebra to interpret an arbitrary plethysm coefficient as the multiplicity of an appropriate composition factor in the restriction of a module for the ramified partition algebra to the partition algebra. This result implies new stability phenomenon for plethysm coefficients when the first parts of ν, μ and λ are all large. In particular, it gives the first positive formula in the case when ν and λ are arbitrary and μ has one part. Corollaries include new explicit positive formulae and combinatorial interpretations for the plethysm coefficients $p((n-b,b),(m),(mn-r,r))$, and $p((n-b,1^b),(m),(mn-r,r))$ when m and n are large.

1. INTRODUCTION

Understanding the **plethysm coefficients** is a fundamental problem in the representation theories of symmetric and general linear groups. It was identified by R. Stanley as one of the most important open problems in algebraic combinatorics [Sta00]. Beyond pure mathematics, the plethysm coefficients arise in quantum information theory [AK08, BCI11] and are central objects in geometric complexity theory (GCT), an approach that seeks to settle the P $\neq$ NP problem. The importance of plethysm coefficients in GCT derives from their frequent appearance in formulas for multiplicities in coordinate rings: the orbit of the product of variables [Lan17, Section 9], the permanent polynomial [BLMW11, Equation (5.5.2)], the power sum polynomial and the unit tensor [FI20, Introduction]. Thus the problems of both *calculating* and *bounding* plethysm coefficients are of fundamental importance in GCT.

One of the most effective ways to study plethysm coefficients is through their stability phenomena (for example, these stabilities were used to spectacular effect in [BIP19]). Throughout this paper we set $\alpha[d] = (d - |\alpha|, \alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ and we write $p(\beta[n], \alpha[m], \kappa[mn])$ for the plethysm coefficient $\langle s_{\beta[n]} \circ s_{\alpha[m]} \, , \, s_{\kappa[mn]} \rangle$ defined using the plethysm product of Schur functions. Our original motivation was to study the *stable plethysm coefficients*, defined by

$$\lim_{m,n\to\infty} p(\beta[n], (m), \kappa[mn]).$$

(This stability is proven in [CT92, Wei90].) In the very special case when $\beta = \varnothing$, the stable coefficients are amongst the most celebrated and well-understood of all plethysm coefficients: they satisfy the stable version of Foulkes' conjecture [Man98, Theorem 4.3.1] and they were an important stepping stone in the resolution of Weintraub's conjecture [Man98, BCI11]. Moreover, there is a positive combinatorial formula for their calculation [Man98, Theorem 4.1.1] (see also [BP17, Theorem B]). Our first main result is a vast generalisation of this formula to the case when the partition β is arbitrary.

Theorem A. *Let $\beta \vdash b$ and $\kappa \vdash r$ be partitions such that $\beta[n]$ and $\kappa[mn]$ are also partitions. The plethysm coefficient $p(\beta[n], (m), \kappa[mn])$ is constant for $m \geqslant r - |\beta| + [\beta \neq \varnothing]$ and $n \geqslant r + \beta_1$*

and its value is

$$\sum_{\substack{p,\,q\,:\,p+q=r\\ \gamma=(p^{c_p},\ldots,1^{c_1})\vdash p,\,\ell(\gamma)=b\\ \beta^i\vdash c_i \text{ for } 1\leqslant i\leqslant p\\ \varepsilon\in\mathscr{P}_{>1}(q)}} c^{\beta}_{\beta^p,\ldots,\beta^1}\Big[\Big(\big(\bigotimes_{i=1}^{p}\operatorname{Inf}_{\mathfrak{S}_{c_i}}^{\mathfrak{S}_i\wr\mathfrak{S}_{c_i}}\mathbf{S}^{\beta^i}\big)\big\uparrow^{\mathfrak{S}_p}\otimes\mathbb{C}_{\mathrm{Stab}(\varepsilon)}\big\uparrow^{\mathfrak{S}_q}\Big)\Big\uparrow^{\mathfrak{S}_r}:\mathbf{S}^{\kappa}\Big]_{\mathfrak{S}_r}.$$

Unavoidably, this formula is heavy on notation:

- $[\beta\neq\varnothing]$ is an Iverson bracket, equal to 1 when $\beta\neq\varnothing$ and 0 when $\beta=\varnothing$;
- $\ell(\gamma)$ is the length of the partition γ: thus $c_p+\cdots+c_1=b$;
- $\mathscr{P}_{>1}(q)$ is the set of partitions of q having no singleton parts;
- $c^{\beta}_{\beta^p,\ldots,\beta^1}$ is a generalized Littlewood–Richardson coefficient, as defined in equation (2.3);
- $\mathbf{S}^{\kappa}$ denotes the right Specht module canonically labelled by the partition κ;
- $\mathrm{Stab}(\varepsilon)$ is the stabiliser in the symmetric group $\mathfrak{S}_q$ of a set-partition of $\{1,\ldots,q\}$ into parts of sizes $\varepsilon_1,\ldots,\varepsilon_{\ell(\varepsilon)}$, as defined in Definition 2.5.

The entire proof of Theorem A is outlined in Section 1.7 at the end of this introduction.

1.1. **Recasting plethysm in terms of diagram algebras.** We deduce Theorem A as a corollary of a number of more powerful theorems for calculating and bounding plethysm coefficients proved in this paper. The key to our approach is the see-saw pair below.

$$\begin{tikzcd}
\mathfrak{S}_{mn} \arrow[dr] & & P_r(mn) \arrow[dl] \arrow[dd, hook] \\
& (\mathbb{C}^{mn})^{\otimes r} & \\
\mathfrak{S}_m\wr\mathfrak{S}_n \arrow[uu, hook] \arrow[ur] & & R_r(m,n) \arrow[ul]
\end{tikzcd}$$

This shows the *partition algebra* $P_r(mn)$ in its well-known Schur–Weyl duality with the symmetric group $\mathfrak{S}_{mn}$: see Sections 3 and 5. Restricting to the subgroups $\mathfrak{S}_m\wr\mathfrak{S}_n\leqslant\mathfrak{S}_{mn}$, we obtain a larger algebra of invariants on the other side of Schur–Weyl duality, the *ramified partition algebras* $R_r(m,n)$: see Sections 4 and 5. By (2.7), the plethysm coefficients are the branching coefficients for the subgroups $\mathfrak{S}_m\wr\mathfrak{S}_n\leqslant\mathfrak{S}_{mn}$ and thus, by the general theory of see-saw pairs [GW98], we can recast these coefficients as branching coefficients for restriction to the subalgebra $P_r(mn)$ of the ramified partition algebra $R_r(m,n)$. In Section 6 we match up the labels of simple modules under these Schur–Weyl dualities and interpret plethysm coefficients as composition multiplicities in the partition algebra.

Theorem B. *Let α,β,κ be partitions such that $\alpha[m]$, $\beta[n]$ and $\kappa[mn]$ are partitions. Suppose that $r\geqslant|\kappa|$. The plethysm coefficient $p(\beta[n],\alpha[m],\kappa[mn])$ is equal to the composition multiplicity of the simple module $L_r(\kappa)$ for the partition algebra $P_r(mn)$ in the relevant restricted simple module for the ramified partition algebra specified on the right-hand side below:*

$$p(\beta[n],\alpha[m],\kappa[mn])=\begin{cases}\big[L_r(\varnothing^{\beta})\big\downarrow^{R_r(m,n)}_{P_r(mn)}:\,L_r(\kappa)\big]_{P_r(mn)} & \textit{if } \alpha=\varnothing, r\geqslant|\beta|,\\ \big[L_r(\alpha^{\beta[n]})\big\downarrow^{R_r(m,n)}_{P_r(mn)}:\,L_r(\kappa)\big]_{P_r(mn)} & \textit{if } \alpha\neq\varnothing, r\geqslant n|\alpha|.\end{cases}$$

A key observation in the proof of Theorem B, used at the start of each case in the proof of Theorem 6.1, is that the ramified partition algebras are quasi-hereditary and hence their simple modules $L_r(\alpha^{\beta})$, which are in general difficult to construct, arise as the simple heads of corresponding standard modules $\Delta_r(\alpha^{\beta})$, which in turn can easily be constructed using Young symmetrizers: see (3.6).

This provides a new dichotomy between the role played by the inner partition μ and the outer partition ν in a plethysm product $s_\nu \circ s_\mu$; another such dichotomy is obtained in [FI20, Theorem 3.1 versus Proposition 3.3].

1.2. **Stability and ramified branching coefficients.** The simple modules $L_r(\alpha^\beta)$ for the ramified partition algebras are quotients of the standard modules $\Delta_r(\alpha^\beta)$. We may therefore bound the plethysm coefficients appearing in Theorem B by the multiplicity of a simple module in the restriction of a standard module to the partition algebra, as in the first part of Theorem C below. In the case when $\alpha = \varnothing$ we improve on this bound. In Section 7 we prove that the standard module $\Delta_r(\varnothing^\beta)$ for the ramified partition algebra is simple whenever both $m \geqslant r - |\beta| + [\beta \neq \varnothing]$ and $n \geqslant r + \beta_1$, proving the equality in the second part of the theorem below.

Theorem C. *Let α, β, κ be partitions such that $\alpha[m], \beta[n], \kappa[mn]$ are also partitions. Suppose that $r \geqslant |\kappa|$. Then*

$$p(\beta[n], \alpha[m], \kappa[mn]) \leqslant \begin{cases} \left[\Delta_r(\varnothing^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)} \colon L_r(\kappa)\right]_{P_r(mn)} & \text{if } \alpha = \varnothing, r \geqslant |\beta|, \\ \left[\Delta_r(\alpha^{\beta[n]})\big\downarrow^{R_r(m,n)}_{P_r(mn)} \colon L_r(\kappa)\right]_{P_r(mn)} & \text{if } \alpha \neq \varnothing, r \geqslant n|\alpha|. \end{cases}$$

Moreover, if $m \geqslant r - |\beta| + [\beta \neq \varnothing]$ and $n \geqslant r + \beta_1$, then

$$p(\beta[n], (m), \kappa[mn]) = \left[\Delta_r(\varnothing^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)} \colon L_r(\kappa)\right]_{P_r(mn)}.$$

1.3. **Combinatorial formulas for ramified branching coefficients.** In light of Theorem B, we seek to calculate the ramified branching coefficients $\left[\Delta_r(\alpha^\beta)\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\right]_{P_r(mn)}$, and hence calculate and bound the plethysm coefficients. The partition algebras and ramified partition algebras arise as towers of recollement in the sense of [CMPX06]; roughly speaking this means that they arise as sequences of algebras and are equipped with idempotents which allow us to work by induction on the rank. In Proposition 8.6 and Corollary 8.7, we utilise this theory to construct quotient $P_r(mn)$-modules

$$\Delta_r(\alpha^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)} \longrightarrow \mathsf{DQ}\big(\Delta_r(\alpha^\beta)\big). \tag{1.1}$$

These quotient modules possess beautiful planar diagram bases (see Section 9.2) which are amenable to computation. By decomposing these quotient modules we are able to calculate the ramified branching coefficients explicitly, as follows.

Theorem D. *The ramified branching coefficient $[\Delta_r(\alpha^\beta)\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)]$ for $\alpha \vdash a, \beta \vdash b$ and $\kappa \vdash r$ is equal to the multiplicity of $\mathbf{S}^\kappa$ in the following $\mathbb{C}\mathfrak{S}_r$-module:*

$$\bigoplus_{\substack{p,q\,:\,p+q=r-ab \\ \gamma=(p^{c_p},\dots,1^{c_1},0^{c_0})\in\mathscr{P}_{(a^b)}(p) \\ \beta^i\vdash c_i \text{ for } 0\leqslant i\leqslant p \\ \varepsilon\in\mathscr{P}_{>1}(q)}} c^{\beta}_{\beta^p,\dots,\beta^1,\beta^0}\Big(\bigotimes_{i=0}^{p}\Big((\mathbf{S}^\alpha \otimes \mathbb{C})\big\uparrow^{\mathfrak{S}_{a+i}}_{\mathfrak{S}_a\times\mathfrak{S}_i}\big) \oslash \mathbf{S}^{\beta^i}\Big) \otimes \mathbb{C}_{\mathrm{Stab}(\varepsilon)}\Big)\Big\uparrow^{\mathfrak{S}_r}_{\mathrm{Stab}((a^b)+\gamma)\times\mathrm{Stab}(\varepsilon)}$$

where $\mathscr{P}_{>1}(q)$ is the set of partitions of q with no singleton parts, (a^b) is the empty partition if $a = 0$ and otherwise the partition with b parts all of size a, and the set $\mathscr{P}_{(a^b)}(p)$ is as defined in Definition 9.5.

By Definition 9.5, the elements $\gamma \in \mathscr{P}_{(a^b)}(p)$ are partitions having exactly b parts *including* c_0 *distinguished zero parts*. Thus $c_p + \cdots + c_1 + c_0 = b$ and the Littlewood–Richardson coefficient $c^{\beta}_{\beta^p,\dots,\beta^1,\beta^0}$ is well-defined. When $a = 0$, the partitions $\gamma \in \mathscr{P}_{(a^b)}(p)$ have exactly b non-zero parts and $c_0 = 0$, and the sum is the same as in Theorem A.

Theorem C provides a combinatorial formula for computing arbitrary ramified branching coefficients in terms of *much smaller* plethysm products and Littlewood–Richardson coefficients.

We prove Theorem D in Section 9 and then immediately deduce Theorem A as a corollary of Theorem C and Theorem D.

Using symmetric functions we prove in Section 11 that the bounds on m and n in Theorem A cannot be weakened in infinitely many cases. This is notable because, unlike the usual direction in this paper, we successfully apply symmetric functions to deduce a result about the ramified partition algebra.

1.4. **Examples and applications.** To make this paper accessible, particularly to a non-diagrammatic-algebra audience, we give plenty of examples throughout the paper. After the proof of Theorem A is complete, we begin Section 10 by giving two substantial examples showing how the decomposition of the depth quotient (see Definition 8.4) determines plethysm coefficients. We then give a more conceptual restatement of Theorem A using a functor on symmetric group modules in Proposition 10.4, before proving Proposition 10.11 which restates Theorem A and Proposition 10.4 in the language of symmetric functions. As applications we find explicitly positive formulae for the stable limits of the plethysm coefficients $p\big((n-b,b),(m),(mn-r,r)\big)$, $p\big((n-b,1^b),(m),(mn-r,r)\big)$ and $p\big((n-b,1^b),(m),(mn-r,1^r)\big)$.

1.5. **Contrasting the cases $\alpha = \varnothing$ and $\alpha \neq \varnothing$.** Since it illuminates two key points in the proof, we remark on the qualitative difference in our results on plethysm coefficients in these two cases. The reader may prefer to return to this remark after reading Section 6.

The ramified Schur functor. The outer partition in the standard module in Theorem C is β when $\alpha = \varnothing$ and $\beta[n]$ when $\alpha \neq \varnothing$. The difference arises from the behaviour of the ramified Schur functor $\mathrm{Hom}_{\mathfrak{S}_m \wr \mathfrak{S}_n}\big(-,(\mathbb{C}^{mn})^{\otimes r}\big)$ mapping left $\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n$-modules to right $R_r(m,n)$-modules. The module $\mathbf{S}_{(m)} \oslash \mathbf{S}_{\beta[n]} = \mathrm{Inf}_{\mathfrak{S}_n}^{\mathfrak{S}_m \wr \mathfrak{S}_n} \mathbf{S}_{\beta[n]}$ is acted on trivially by the base group $\mathfrak{S}_m \times \cdots \times \mathfrak{S}_m$ in the wreath product $\mathfrak{S}_m \wr \mathfrak{S}_n$ and embeds in the tensor space $(\mathbb{C}^{mn})^{\otimes r}$ whenever $r \geqslant b$; its image under the Schur functor is then the simple module $L_r(\varnothing^\beta)$ for the ramified partition algebra $R_r(m,n)$. In contrast, when $\alpha \neq \varnothing$, the base group acts non-trivially on $\mathbf{S}_\alpha \oslash \mathbf{S}_{\beta[n]}$ and it is necessary to take $r \geqslant n|\alpha|$ to get an embedding. We then obtain the simple module $L_r(\alpha^{\beta[n]})$. The distinction can be seen by comparing equations (6.1) and (6.2). This is the first and most critical point where the two cases diverge.

Standard modules versus simple modules and semisimplicity. In the case when $\alpha = \varnothing$, the standard $R_r(m,n)$-module $\Delta_r(\varnothing^\beta)$ is isomorphic to the simple module $L_r(\varnothing^\beta)$ by Theorem 7.1 for suitably large values of m and n, and so the plethysm coefficient $p(\beta[n],(m),\kappa[mn])$ is equal to the ramified branching coefficient $\big[\Delta_r(\varnothing^\beta)\big\downarrow_{P_r(mn)}^{R_r(m,n)} : L_r(\kappa)\big]_{P_r(mn)}$. Putting aside bounds for now, this leads to the formula in Theorem A for the joint limit of the plethysm coefficient when m and n independently become large. When $\alpha \neq \varnothing$, the equality of Theorem B requires $r \geqslant n|\alpha|$, but $R_r(m,n)$ is never semisimple when this condition holds. In this case we obtain only an inequality bounding $p(\beta[n],\alpha[m],\kappa[mn])$ by the ramified branching coefficient $\big[\Delta_r(\alpha^{\beta[n]})\big\downarrow_{P_r(mn)}^{R_r(m,n)} : L_r(\kappa)\big]_{P_r(mn)}$. This is the second point of divergence and explains why our results on plethysm coefficients are sharp only when $\alpha = \varnothing$.

1.6. **Analogies and motivation from Kronecker coefficients.** It is worth emphasising that all the ideas of this paper have analogues in the context of the Kronecker coefficients.

By Theorem B, the plethysm and ramified branching coefficients can be interpreted as restriction multiplicities of simple and standard modules from the ramified partition algebra to the partition algebra. By [BDO15, Equation 3.1.3] the Kronecker coefficients and stable Kronecker coefficients can be interpreted as the restriction multiplicities of simple and standard modules from the partition algebra to a certain Young subalgebra. The formula for calculating ramified branching coefficients in Theorem D of this paper has an exact analogue for the stable Kronecker coefficients, which is proven in the partition algebra context in [BDO15, Theorem 4.3].

Theorem C of this paper says that the plethysm coefficients are bounded above by their ramified branching coefficient analogues and examines when this bound is sharp. The Kronecker coefficients are bounded above by their stable analogues and analogous bounds were found by Brion in [Bri93] and reproved in the context of the partition algebra [BDO15, Corollary 3.6].

1.7. **Structure of the paper.** We intend that this paper will be found readable both by people primarily interested in plethysms of symmetric functions and by people primarily interested in diagram algebras. We therefore take some care to collect all the necessary background.

- In Section 2 we give background on symmetric functions and modules for symmetric groups and wreath products.
- In Section 3 we give a self-contained introduction to the partition algebra, showing how to use diagrams to compute its action on its standard modules in Example 3.2.
- In Section 4 we give a similar self-contained introduction to the ramified partition algebra.
- In Section 5 we finish the background material with results on Schur–Weyl duality.

The proofs of the main theorems occupy Sections 6 to 9, following the outline above: Section 6 proves Theorem B; Section 7 proves Theorem C, showing in particular that the simple module $L_r(\varnothing^\beta)$ for the ramified partition algebra is equal to the standard module $\Delta_r(\varnothing^\beta)$ provided m and n satisfy the inequalities in Theorem A; Sections 8 and 9 study the restricted module $\Delta_r(\varnothing^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)}$ and hence prove Theorem D and deduce Theorem A. Thus, when presented as a series of equations, the proof of Theorem A is

$$
\begin{aligned}
&p(\beta[n],(m),\kappa[mn]) \\
&\quad = \langle s_{\beta[n]}\circ s_\mu, s_{\kappa[mn]}\rangle && \text{(a)}\\
&\quad = \big[(\mathbf{S}_{(m)}\oslash\mathbf{S}_{\beta[n]})\big\uparrow^{\mathfrak{S}_{mn}}_{\mathfrak{S}_m\wr\mathfrak{S}_n} : \mathbf{S}_{\kappa[mn]}\big]_{\mathfrak{S}_{mn}} && \text{(b)}\\
&\quad = \big[\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\big((\mathbf{S}_{(m)}\oslash\mathbf{S}_{\beta[n]})\big\uparrow^{\mathfrak{S}_{mn}}_{\mathfrak{S}_m\wr\mathfrak{S}_n},(\mathbb{C}^{mn})^{\otimes r}\big) : \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\big(\mathbf{S}_{\kappa[mn]},(\mathbb{C}^{mn})^{\otimes r}\big)\big]_{P_r(mn)} && \text{(c)}\\
&\quad = \big[\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_m\wr\mathfrak{S}_n}\big(\mathbf{S}_{(m)}\oslash\mathbf{S}_{\beta[n]},(\mathbb{C}^{mn})^{\otimes r}\big)\big\downarrow_{\mathfrak{S}_m\wr\mathfrak{S}_n} : L_r(\kappa)\big]_{P_r(mn)} && \text{(d)}\\
&\quad = \big[L_r(\varnothing^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\big]_{P_r(mn)} && \text{(e)}\\
&\quad = \big[\Delta_r(\varnothing^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\big]_{P_r(mn)} && \text{(f)}\\
&\quad = \big[\mathsf{DQ}\big(\Delta_r(\varnothing^\beta)\big) : L_r(\kappa)\big]_{P_r(mn)} && \text{(g)}\\
&\quad = \sum c^{\beta}_{\beta^p,\ldots,\beta^1}\Big[\Big(\big(\bigotimes \mathrm{Inf}^{\mathfrak{S}_i\wr\mathfrak{S}_{c_i}}_{\mathfrak{S}_{c_i}}\mathbf{S}^{\beta^i}\big)\big\uparrow^{\mathfrak{S}_p}\otimes\mathbb{C}_{\mathrm{Stab}(\varepsilon)}\big\uparrow^{\mathfrak{S}_q}\Big)\Big\uparrow^{\mathfrak{S}_r}_{\mathfrak{S}_p\times\mathfrak{S}_q} : \mathbf{S}^\kappa\Big]_{\mathfrak{S}_r} && \text{(h)}
\end{aligned}
$$

where the outline argument for each step is as follows:

(a) Definition of plethysm coefficients;
(b) Apply (2.7) to get the equivalent restatement using left Specht modules;
(c) Apply the Schur functor for the partition algebra in Corollary 5.2 to get right modules for $P_r(mn)$; the module action is defined using the action of $P_r(mn)$ on $(\mathbb{C}^{mn})^{\otimes r}$;
(d) Apply Frobenius reciprocity on the left-hand module and use Corollary 5.2 to identify the $P_r(mn)$-module $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\big(\mathbf{S}_{\kappa[mn]},(\mathbb{C}^{mn})^{\otimes r}\big)$ with the simple module $L_r(\kappa)$;
(e) Apply the case of $\alpha=\varnothing$ of Proposition 6.1 (with the conclusion that the right-hand side equals $p(\beta[n],(m),\alpha)$ being a special case of Theorem A);
(f) Apply Theorem 7.1 using the hypotheses $m\geqslant r-|\beta|+[\beta\neq\varnothing]$ and $n\geqslant r+\beta_1$ (with the conclusion that the right-hand side equals $p(\beta[n],(m),\alpha)$ being a special case of Theorem C);
(g) Apply Corollary 8.7;

(h) Apply Theorem 9.16 decomposing the depth quotient into right Specht modules (from which we deduce Theorem D).

We end in Sections 10 and 11 with the examples and applications already outlined.

1.8. **Other diagram algebras.** In recent work Orellana, Saliola, Schilling and Zabrocki have recast the plethysm coefficients in the context of the *party algebra*, a subalgebra of the partition algebra [OSSZ]. There does not appear to be any overlap in our results, but the ideas do have a similar diagrammatic flavour.

Acknowledgements. We thank Christian Ikenmeyer, Stacey Law, Rosa Orellana, Anne Schilling, and Mike Zabrocki for interesting and informative conversations. We thank the Oberwolfach workshop 'Character Theory and Categorification' and the Institute for Mathematical Sciences workshop 'Representation Theory, Combinatorics and Geometry' held at the University of Singapore and the Heilbronn Institute for the award of a HIMR Focused Research Grant to run the workshop 'New approaches to plethysm: computation and complexity', held at the University of York in September 2023 — all three of which provided excellent environments for collaboration. The first author is grateful to financial support from EPSRC grant EP/V00090X/1. The third author is grateful to OIST (Okinawa Institute of Science and Technology) for sponsoring a visit in November 2022 during which some of this work was done.

2. Symmetric groups, wreath products and symemtric functions

2.1. **Symmetric groups.** We let $\mathfrak{S}_n$ denote the symmetric group on n letters generated by the Coxeter generators $s_i = (i, i+1)$ for $1 \leqslant i < n$. The combinatorics underlying the representation theory of symmetric groups, and also of partition algebras, is based on integer partitions. A partition of n is defined to be a sequence $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of strictly positive integers which sum to n. We write $\lambda \vdash n$ and say that the length of λ, denoted $\ell(\lambda)$, is ℓ. There is a unique partition of zero, namely the empty partition $\varnothing$. We let $\mathscr{P}(n)$ denote the set of all partitions of n and we let $\mathscr{P}_{>1}(n)$ denote the subset of those partitions with no singleton parts. (That is, all parts are strictly greater than 1.)

Given $\lambda \in \mathscr{P}(n)$, we define a tableau of shape λ to be a filling of the nodes of the Young diagram of λ with the numbers $\{1, \dots, n\}$. We define a standard tableau to be a tableau in which the entries increase along the rows read left to right and the columns read top to bottom. We let $\mathrm{Std}(\lambda)$ denote the set of all standard tableaux of shape $\lambda \in \mathscr{P}(n)$. For $\lambda \vdash n$, let t^λ denote the λ-tableau with the numbers $1, 2, \dots, n$ entered in increasing order along the rows from left to right and then from top to bottom. For example,

$$\mathsf{t}^{(3,2,1)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad .$$

We denote by $C(\mathsf{t}^\lambda)$ the subgroup of $\mathfrak{S}_n$ that preserves the set of entries in each column of t^λ and by $R(\mathsf{t}^\lambda)$ the subgroup that preserves the set of entries in each row. Recall (for example from [FH91, Chapter 4]) that the Young symmetrizer c_λ is defined by

$$c_\lambda = \Big(\sum_{\rho \in R(\mathsf{t}^\lambda)} \rho \Big) \Big(\sum_{\pi \in C(\mathsf{t}^\lambda)} \mathrm{sgn}(\pi)\pi \Big). \tag{2.1}$$

It is well-known that c_λ is a quasi-idempotent (that is c_λ^2 is equal to c_λ up to a non-zero scalar), and, for partitions of λ, ν of n, that

$$c_\lambda(\mathbb{C}\mathfrak{S}_n)c_\nu = \begin{cases} \mathbb{C}c_\lambda & \text{if } \lambda = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

The left Specht module labelled by λ is the $\mathbb{C}\mathfrak{S}_n$-module $\mathbb{C}\mathfrak{S}_n c_\lambda = \mathbf{S}_\lambda$. Later, we shall use right modules for symmetric groups to construct right modules for partition algebras. We define the right Specht $\mathbb{C}\mathfrak{S}_n$-module labelled by λ to be $\mathbf{S}^\lambda = c_\lambda^* \mathbb{C}\mathfrak{S}_n$ where

$$c_\lambda^* = \Big(\sum_{\pi \in C(\mathsf{t}^\lambda)} \operatorname{sgn}(\pi)\pi \Big) \Big(\sum_{\rho \in R(\mathsf{t}^\lambda)} \rho \Big). \tag{2.2}$$

Observe that c_λ^* is the image of c_λ under the anti-involution on $\mathbb{C}\mathfrak{S}_n$ that sends each group element to its inverse. The Specht modules are a full set of non-isomorphic irreducible $\mathbb{C}\mathfrak{S}_n$-modules.

2.2. **The Littlewood–Richardson rule.** The Littlewood–Richardson rule is a combinatorial rule for the restriction of a Specht module to a Young subgroup of the symmetric group. Through Schur–Weyl duality, the rule also computes the decomposition of a tensor product of simple representations of $\mathrm{GL}_n(\mathbb{C})$.

Theorem 2.1 (The Littlewood–Richardson Rule). *For $\lambda \vdash m+n$, $\mu \vdash m$ and $\nu \vdash n$,*

$$\mathbf{S}_\lambda \big\downarrow^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m \times \mathfrak{S}_n} \cong \bigoplus_{\mu \vdash m, \nu \vdash n} c^\lambda_{\mu,\nu} (\mathbf{S}_\mu \otimes \mathbf{S}_\nu)$$

where the $c^\lambda_{\mu,\nu}$ are the Littlewood–Richardson coefficients as defined in [JK81, Section 2.8.13].

Given a partition β and $c_1, \ldots, c_r \in \mathrm{N}_0$ such that $c_r + \cdots + c_1 = |\beta|$, we have

$$\mathbf{S}_\beta \big\downarrow_{\mathfrak{S}_{c_r} \times \cdots \times \mathfrak{S}_{c_1}} \cong \bigoplus_{\beta^i \vdash c_i} c^\beta_{\beta^r, \ldots, \beta^1} (\mathbf{S}_{\beta^r} \otimes \cdots \otimes \mathbf{S}_{\beta^1}) \tag{2.3}$$

for some coefficients $c^\beta_{\beta^r,\ldots,\beta^1} \in \mathbb{N}_0$. (As a standing convention $\beta^i \vdash c_i$ in a sum indicates that the sum is over all relevant sequences of partitions.) We call these coefficients generalized Littlewood–Richardson coefficients; they may be computed by iterative applications of Theorem 2.1. In our examples it works best to order these sequences by *decreasing* index, hence the order $\beta^r, \ldots, \beta^1$ above; since $c^\lambda_{\mu,\nu} = c^\lambda_{\nu,\mu}$, this is just a matter of notation. Later, for example in the proof of Theorem 9.16, we use the equivalent formulation of the rule for right Specht modules. The coefficients are of course the same.

2.3. **Wreath products and their modules.** Let $m, n \in \mathbb{N}$. Following [JK81, Section 4.1], we consider

$$\mathfrak{S}_m \wr \mathfrak{S}_n = \big\{ (\sigma_1, \sigma_2, \ldots, \sigma_n; \pi) \mid \sigma_i \in \mathfrak{S}_m, i = 1, \ldots, n, \pi \in \mathfrak{S}_n \big\}, \tag{2.4}$$

which we identify with a subgroup of $\mathfrak{S}_{mn}$ via the embedding

$$(\sigma_1, \sigma_2, \ldots, \sigma_n; \pi) \mapsto \begin{pmatrix} (j-1)m + i \\ (\pi(j)-1)m + \sigma_{\pi(j)}(i) \end{pmatrix}_{\substack{i=1,\ldots,m \\ j=1\ldots,n}}. \tag{2.5}$$

The representation theory of $\mathfrak{S}_m \wr \mathfrak{S}_n$ is well-developed (see [JK81, chapter 4] or [CT03]). If $\mu \vdash m$ and $\nu \vdash n$ then we can use the irreducible $\mathbb{C}\mathfrak{S}_m$-module $\mathbf{S}_\mu$ and the irreducible $\mathbb{C}\mathfrak{S}_n$-module $\mathbf{S}_\nu$ to construct an irreducible $\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n$-module

$$\mathbf{S}_\mu \oslash \mathbf{S}_\nu = (\mathbf{S}_\mu)^{\otimes n} \otimes \operatorname{Inf}^{S_m \wr S_n}_{S_n} \mathbf{S}_\nu,$$

where elements of the distinguished top group $\mathfrak{S}_n$ in the wreath product $\mathfrak{S}_m \wr \mathfrak{S}_n$ act on $(\mathbf{S}_\mu)^{\otimes n}$ by place permutation. (The symbol $\oslash$ was introduced in [CT03].) A complete set of irreducible left $\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n$-modules is obtained by inducing suitable tensor products of these modules.

2.4. **Young symmetrizers for wreath product modules.** We shall need an alternative construction of a complete set of simple $\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n$-modules using Young symmetrizers that generalises the theory for symmetric groups outlined above to wreath products of symmetric groups. Generalizing [FH91, Chapter 4] we define Young symmetrisers for the wreath product $\mathfrak{S}_m \wr \mathfrak{S}_n$. Let $L = |\mathscr{P}(m)|$ and list the L partitions of m in lexicographic order as

$$\mu^1 < \mu^2 < \ldots < \mu^L.$$

An L-partition $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(L)})$ of n is an L-tuple of partitions such that $|\nu^{(1)}| + \cdots + |\nu^{(L)}| = n$. We define $\mathscr{P}(m,n)$ to be the set of all L-partitions of n. (This is the labelling set for the simple $\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n$-modules.) We shall write elements of $\mathscr{P}(m,n)$ as

$$\boldsymbol{\mu}^{\boldsymbol{\nu}} = (\mu^1, \mu^2, \ldots, \mu^L)^{(\nu^1, \nu^2, \ldots, \nu^L)},$$

remembering that $\boldsymbol{\mu}$ is fixed and is the tuple of all partitions of m in lexicographic order. For example

$$\big((3),(2,1),(1^3)\big)^{((2),\varnothing,(3,1))} \in \mathscr{P}(3,6).$$

We shall frequently be interested in the case where there is a unique non-empty entry ν^j of $\boldsymbol{\nu}$; in this case, we shall write $(\mu^j)^{\nu^j}$ in place of $\boldsymbol{\mu}^{\boldsymbol{\nu}}$. For example, we write $(2,1)^{(3,2,1)}$ in place of $\big((3),(2,1),(1^3)\big)^{(\varnothing,(3,2,1),\varnothing)}$; this labels the simple $\mathbb{C}\mathfrak{S}_3 \wr \mathfrak{S}_6$-module $\mathbf{S}_{(2,1)} \oslash \mathbf{S}_{(3,2,1)}$.

Given $\boldsymbol{\nu} \in \mathscr{P}^L(n)$ we define

$$\mathsf{t}^{\boldsymbol{\nu}} = (\mathsf{t}^{\nu_1}, \mathsf{t}^{\nu_2}, \ldots, \mathsf{t}^{\nu_L})$$

by placing the entries $\{1, \ldots, n\}$ in the tableaux t^{ν_i} in increasing order along the rows from left to right, finishing with the bottom row in each tableau, working in order of increasing i. For example

$$\mathsf{t}^{((3,2,1),(2^2,1),(3,1))} = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\ , \ \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 9 & 10 \\ \hline 11 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ , \ \begin{array}{|c|c|c|} \hline 12 & 13 & 14 \\ \hline 15 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \right).$$

We extend the notion of row and column stabilisers in the obvious fashion. For $\boldsymbol{\mu}^{\boldsymbol{\nu}} \in \mathscr{P}(m,n)$, define

$$c_{\boldsymbol{\mu}^{\boldsymbol{\nu}}} = \sum_{\substack{\pi \in C(\mathsf{t}^{\boldsymbol{\nu}}) \\ \rho \in R(\mathsf{t}^{\boldsymbol{\nu}})}} \mathrm{sgn}(\pi) \big(\underbrace{c_{\mu^1}, \ldots, c_{\mu^1}}_{|\nu^1|}, \ldots, \underbrace{c_{\mu^L}, \ldots, c_{\mu^L}}_{|\nu^L|} ;\, \rho\pi \big).$$

Here each c^i_μ should be interpreted as the appropriate sum over elements of $\mathfrak{S}_m \wr \mathfrak{S}_n$ using the notation of (2.4). In particular, if $\mu \vdash m$ and $\nu \vdash n$ then

$$c_{\mu^\nu} = \sum_{\substack{\pi \in C(\mathsf{t}^{\nu}) \\ \rho \in R(\mathsf{t}^{\nu})}} \mathrm{sgn}(\pi) \big(\underbrace{c_\mu, \ldots, c_\mu}_{n} ;\, \rho\pi \big).$$

The following proposition is well-known to experts, but we do not believe it has appeared before in print. The proof is technical but follows precisely the method of [FH91, Chapter 4].

Proposition 2.2. *For $\boldsymbol{\mu}^{\boldsymbol{\nu}}$, $\boldsymbol{\mu}^{\boldsymbol{\lambda}} \in \mathscr{P}(m,n)$,*

$$c_{\boldsymbol{\mu}^{\boldsymbol{\nu}}} (\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n) c_{\boldsymbol{\mu}^{\boldsymbol{\lambda}}} = \begin{cases} \mathbb{C} c_{\boldsymbol{\mu}^{\boldsymbol{\lambda}}} & \text{if } \boldsymbol{\mu}^{\boldsymbol{\lambda}} = \boldsymbol{\mu}^{\boldsymbol{\nu}}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $c_{\boldsymbol{\mu}^{\boldsymbol{\nu}}}$ is a quasi-idempotent and $c_{\boldsymbol{\mu}^{\boldsymbol{\nu}}} c_{\boldsymbol{\mu}^{\boldsymbol{\lambda}}} = 0$ for $\boldsymbol{\mu}^{\boldsymbol{\lambda}} \neq \boldsymbol{\mu}^{\boldsymbol{\nu}}$. The set

$$\big\{ \mathbf{S}_{\boldsymbol{\mu}^{\boldsymbol{\nu}}} = \mathbb{C}(\mathfrak{S}_m \wr \mathfrak{S}_n) c_{\boldsymbol{\mu}^{\boldsymbol{\nu}}} \mid \boldsymbol{\mu}^{\boldsymbol{\nu}} \in \mathscr{P}(m,n) \big\} \tag{2.6}$$

provides a full set of pairwise non-isomorphic left $\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n$-modules and

$$\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n c_{\boldsymbol{\mu}^{\boldsymbol{\nu}}} \cong \big[(\mathbf{S}_{\mu^1} \oslash \mathbf{S}_{\nu^1}) \otimes \cdots \otimes (\mathbf{S}_{\mu^L} \oslash \mathbf{S}_{\nu^L}) \big] \big\uparrow^{\mathfrak{S}_m \wr \mathfrak{S}_n}_{\mathfrak{S}_m \wr \mathfrak{S}_{|\boldsymbol{\nu}|}}.$$

In particular, if $\mu \vdash m$ and $\nu \vdash n$ then $(\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n)c_{\mu^\nu} \cong \mathbf{S}_\mu \oslash \mathbf{S}_\nu$. Analogous statements hold for the right modules $\mathbf{S}^{\boldsymbol{\mu}^{\boldsymbol{\nu}}} = c^*_{\boldsymbol{\mu}^\nu}\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n$. Here $c^*_{\boldsymbol{\mu}^\nu}$ is obtained from $c_{\boldsymbol{\mu}^\nu}$ by applying the anti-involution of $\mathfrak{S}_m \wr \mathfrak{S}_n$ that inverts group elements.

2.5. **Symmetric functions and plethysm.** For background on symmetric functions we refer the reader to [Sta99, Ch. 7] or [Mac95]. Here we recall that the ring Λ of symmetric functions has as an orthonormal $\mathbb{Z}$-basis the Schur functions s_λ indexed by partitions λ and is graded by degree: s_λ has degree $|\lambda|$. The plethysm product $s_\nu \circ s_\mu$ may be defined by substituting the monomials in s_μ (taken with multiplicities) for the variables in s_ν; see [Sta99, A2.6 Definition] or [Mac95, Ch. I (8.2)]. As a small example, $s_{(2)}(x_1, x_2, x_3, \ldots) = x_1^2 + x_2^2 + x_3^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots$ is the complete homogeneous symmetric function of degree 2, and so working with two variables, we have $s_{(2)}(x_1, x_2) = x_1^2 + x_2^2 + x_1x_2$ and

$$\begin{aligned}
&(s_{(2)} \circ s_{(2)})(x_1, x_2)\\
&\qquad = s_{(2)}(x_1^2, x_2^2, x_1x_2)\\
&\qquad = (x_1^2)(x_1^2) + (x_2^2)(x_2^2) + (x_1x_2)(x_1x_2) + (x_1^2)(x_2^2) + (x_1^2)(x_1x_2) + (x_2^2)(x_1x_2)\\
&\qquad = x_1^4 + x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3 + x_2^4\\
&\qquad = s_{(4)}(x_1, x_2) + s_{(2,2)}(x_1, x_2).
\end{aligned}$$

For our purposes we need two key results on the characteristic isometry from the character ring of $\mathfrak{S}_d$ to the degree d component of Λ.

Lemma 2.3. *For all partitions ν of n and μ of m we have*
(a) *the induced product $\big(\mathbf{S}_\mu \otimes \mathbf{S}_\nu\big)\big\uparrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}$ corresponds to the ordinary product $s_\mu s_\nu$.*
(b) *the $\mathfrak{S}_{mn}$-module $\big(\mathbf{S}_\mu \oslash \mathbf{S}_\nu\big)\big\uparrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}}$ corresponds to the plethysm product $s_\nu \circ s_\mu$.*

Proof. See [Sta99, Proposition 7.18.2, A2.2.3] or [Mac95, Ch. I (7.3), Appendix A (6.2)]. □

In Section 10 we also use the analogous versions of (a) and (b) for right Specht modules.

2.6. **Plethysm coefficients.** We are now ready to introduce the combinatorial objects which motivate this paper: the plethysm coefficients.

Definition 2.4. Given $\nu \vdash n$ and $\mu \vdash m$ we define the plethysm coefficient $p(\nu, \mu, \lambda)$ for $\lambda \vdash mn$ by

$$s_\nu \circ s_\mu = \sum_{\lambda \vdash mn} p(\nu, \mu, \lambda) s_\lambda.$$

Equivalently by Lemma 2.3(b), the plethysm coefficients may be defined by

$$(\mathbf{S}^\mu \oslash \mathbf{S}^\nu)\big\uparrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}} \cong \bigoplus_{\lambda \vdash mn} p(\nu, \mu, \lambda)\mathbf{S}^\lambda. \tag{2.7}$$

We invite the reader to use (2.7) to show that $p\big((2), (2), \lambda\big) \neq 0$ only in the two cases $\lambda = (4)$ or $\lambda = (2, 2)$ identified in the previous subsection by considering the 3-dimensional symmetric group module $\big(\mathbf{S}^{(2)} \oslash \mathbf{S}^{(2)}\big)\big\uparrow_{\mathfrak{S}_2 \wr \mathfrak{S}_2}^{\mathfrak{S}_4}$.

2.7. **Stabiliser subgroups and induction.** The following subgroups are used in Theorem A and Theorem D.

Definition 2.5. Given a partition ε of q we define $\mathrm{Stab}(\varepsilon)$ to be the stabiliser in $\mathfrak{S}_q$ of a set-partition of $\{1, \ldots, q\}$ into parts of sizes $\varepsilon_1, \ldots, \varepsilon_{\ell(\varepsilon)}$.

Equivalently, if ε has exactly e_j parts of size j then

$$\mathrm{Stab}(\varepsilon) = \mathfrak{S}_1 \wr \mathfrak{S}_{e_1} \times \mathfrak{S}_2 \wr \mathfrak{S}_{e_2} \cdots \times \mathfrak{S}_q \wr \mathfrak{S}_{e_q}.$$

We note that $\mathrm{Stab}(\varepsilon) \leqslant \mathfrak{S}_{e_1} \times \mathfrak{S}_{2e_2} \times \cdots \times \mathfrak{S}_{qe_q}$ and that (by transitivity of induction) the induced module $\mathbb{C}\!\uparrow_{\mathrm{Stab}(\varepsilon)}^{S_q}$ decomposes as a direct sum of Specht modules with coefficients equal to products of Littlewood–Richardson and plethysm coefficients. For example the induced module $\mathbb{C}\!\uparrow_{\mathfrak{S}_2\wr\mathfrak{S}_2}^{\mathfrak{S}_4} = \big(\mathbf{S}^{(2)} \oslash \mathbf{S}^{(2)}\big)\!\uparrow_{\mathfrak{S}_2\wr\mathfrak{S}_2}^{\mathfrak{S}_4}$ is the permutation module of $\mathfrak{S}_4$ acting on the cosets of $\mathfrak{S}_2 \wr \mathfrak{S}_2$, and so it may be written as $\mathbb{C}_{\mathrm{Stab}((2,2))}\!\uparrow_{\mathfrak{S}_2\wr\mathfrak{S}_2}^{\mathfrak{S}_4}$.

3. Partition algebras

The partition algebra was originally defined by Martin in [Mar91]. In this section we recall the definition and basic properties of this algebra, which can be found in [Mar96].

3.1. **Set-partitions.** For $r, s \in \mathbb{N}$, we consider the set $\{1, 2, \dots, r, \bar{1}, \bar{2}, \dots, \bar{s}\}$ with the total ordering

$$1 < 2 < \cdots < r < \bar{1} < \bar{2} < \cdots < \bar{s}.$$

We refer to a set-partition of $\{1, 2, \dots, r, \bar{1}, \bar{2}, \dots, \bar{s}\}$ as an (r, s)-set-partition. A subset appearing in a set-partition is called a block. For example,

$$\Lambda = \big\{\{1, 2, 4, \bar{2}, \bar{5}\}, \{3\}, \{5, 6, 7, \bar{4}, \bar{6}, \bar{7}, \bar{8}\}, \{8, \bar{3}\}, \{\bar{1}\}\big\}, \tag{3.1}$$

is an $(8, 8)$-set-partition with five blocks.

Remark 3.1. Let Λ be an (r, s)-set-partition. We order the the subsets in $\Lambda = \{\Lambda_1, \dots, \Lambda_l\}$ by *increasing minima*, so that

$$1 = \min \Lambda_1 < \min \Lambda_2 < \cdots < \min \Lambda_{l-1} < \min \Lambda_l \leqslant \bar{s}.$$

An (r, s)-set-partition, Λ, can be represented by an (r, s)-partition diagram, d_Λ, consisting of r northern and s southern vertices. We number the northern vertices from left to right by $1, 2, \dots, r$ and the southern vertices from left to right by $\bar{1}, \bar{2}, \dots, \bar{s}$ and connect two northern/southern vertices by an edge if they belong to the same block and are adjacent in the total ordering given by restriction of the above ordering to the given block; if a block contains a northern and a southern vertex, we connect the minimal such vertices. In this manner, we pick a unique representative from the equivalence class of all diagrams having the same connected components. For example, the diagram d_Λ of the $(8, 8)$-set-partition Λ in (3.1) is shown in Figure 1.

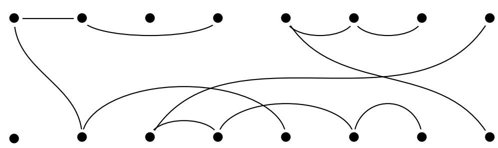


FIGURE 1. The diagram d_Λ for Λ as in (3.1).

A block of a set-partition is called a propagating block if it contains at least one northern and at least one southern vertex; a block consisting of either all southern or all northern vertices is called a non-propagating block. In (3.1), Λ has three propagating blocks and two non-propagating blocks; both non-propagating blocks are singleton blocks, consisting of a single vertex.

To each (r, s)-set-partition diagram d_Λ, we have an associated permutation π_Λ given by first deleting any node which does not belong to a propagating block, then deleting any node which is not minimal in the intersection of its block with $\{1, \dots, r\}$ or $\{\bar{1}, \dots, \bar{s}\}$, and then interpreting the diagram as a permutation. For instance, for the diagram d_Λ above, we have $\pi_\Lambda = (2, 3) \in \mathfrak{S}_3$.

We now consider a parameter $\delta \in \mathbb{C}$. We define the product $d_\Lambda d_\Gamma$ of two (r,r)-partition diagrams d_Λ and d_Γ by concatenating d_Λ above d_Γ, and identifying the southern vertices of d_Λ with the northern vertices of d_Γ. If there are t connected components consisting only of middle vertices, then the product is δ^t times the (r,r)-partition diagram equivalent to the diagram with the middle components removed.

For example, take d_Λ to be the diagram of Figure 1 and d_Γ to be the diagram of $\Gamma = \{\{1\},\{2,\bar{1},\bar{2}\},\{3,\bar{4}\},\{4,\bar{3}\},\{5,\bar{5},\bar{6}\},\{6\},\{7,8,\bar{7},\bar{8}\}\}$. Then $d_\Lambda d_\Gamma$ equals δ times the diagram of $\{\{1,2,4,\bar{1},\bar{2},\bar{5},\bar{6}\},\{3\},\{5,6,7,8,\bar{3},\bar{4},\bar{7},\bar{8}\}\}$, as shown in Figure 2.

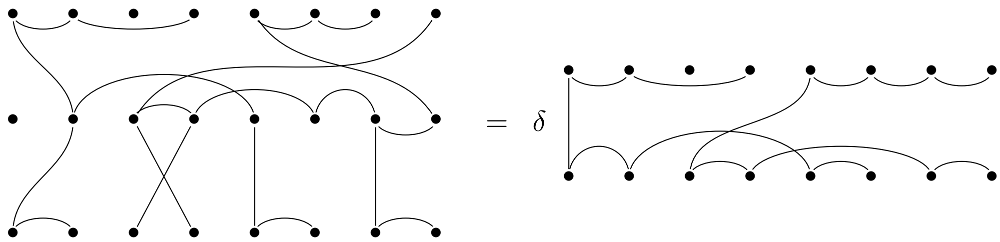


FIGURE 2. An example of a product in $P_8(\delta)$.

We let $P_r(\delta)$ denote the complex vector space with basis given by all set-partitions of $\{1,2,\dots,r,\bar{1},\bar{2},\dots,\bar{r}\}$ and with multiplication given by linearly extending the multiplication of diagrams. Then $P_r(\delta)$ is an associative $\mathbb{C}$-algebra, known as the partition algebra. The (r,r)-set-partitions with exactly r propagating blocks are just the permutations, hence $\mathbb{C}\mathfrak{S}_r$ is a subalgebra of $P_r(\delta)$.

The partition algebra is generated by the usual Coxeter generators of $\mathfrak{S}_r$ together with the two diagrams $\mathsf{p}_1 = d_\Lambda$ for $\Lambda = \{\{1\},\{\bar{1}\}\} \cup \{\{k,\bar{k}\} \mid k > 1\}$ and $\mathsf{p}_{1,2} = d_\Lambda$ for $\Lambda = \{\{1,2,\bar{1},\bar{2}\}\} \cup \{\{k,\bar{k}\} \mid 2 < k \leqslant r\}$ depicted in Figure 3. Set $\mathsf{p}_i = s_{i-1}\dots s_2 s_1 \mathsf{p}_1 s_1 s_2 \dots s_{i-1}$ for $i \geqslant 2$.

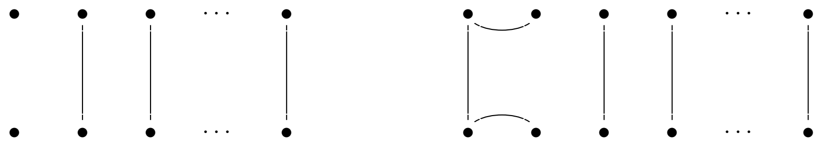


FIGURE 3. The non-Coxeter generators p_1 and $\mathsf{p}_{1,2}$ of $P_r(\delta)$

3.2. **Horizontal concatenation.** Given an (r_1,s_1)-set-partition diagram d_{Λ_1} and an (r_2,s_2)-set-partition diagram d_{Λ_2} we define their horizontal concatenation, $d_{\Lambda_1} \circledast d_{\Lambda_2}$, to be the $(r_1 + r_2, s_1 + s_2)$-set-partition obtained by placing the diagram d_{Λ_1} to the left of d_{Λ_2} and relabelling the vertices (that is, the kth northern vertex in d_{Λ_2} is relabelled by $r_1 + k$ and the kth southern vertex in d_{Λ_2} is relabelled by $s_1 + k$). This is illustrated in Figure 4.

3.3. **A filtration of the partition algebra.** Recall that a block of a set-partition is propagating if the block contains both northern and southern vertices. It is clear that multiplication in $P_r(\delta)$ cannot increase the number of propagating blocks. This leads to a filtration of the algebra $P_r(\delta)$ by the number of propagating blocks. Supposing that $\delta \neq 0$, there are idempotents $e_l = \delta^{r-l}\mathsf{p}_r\mathsf{p}_{r-1}\dots\mathsf{p}_{l+1}$ for $0 \leqslant l \leqslant r$, as depicted in Figure 5.

We have

$$\{0\} \subset P_r(\delta)e_0P_r(\delta) \subset P_r(\delta)e_1P_r(\delta) \subset \dots \subset P_r(\delta)e_{r-1}P_r(\delta) \subset P_r(\delta). \tag{3.2}$$

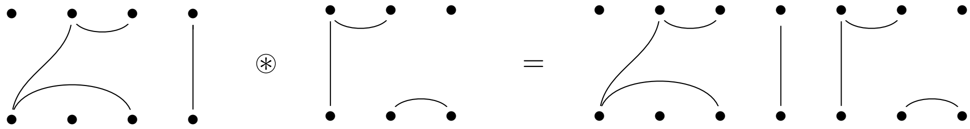


FIGURE 4. A horizontal concatenation.

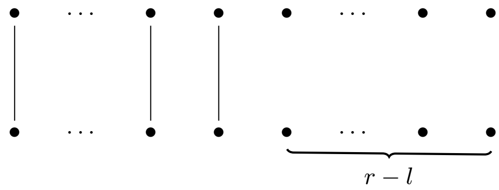


FIGURE 5. The idempotent e_l is defined to be δ^{r-l} times the diagram above having ℓ propagating blocks.

Set $J_l = P_r(\delta)e_l P_r(\delta)$ for $0 \leqslant l \leqslant r$. The ideal J_l is spanned by all (r,r)-partition-diagrams having at most l propagating blocks. It is easy to see that

$$e_{r-1}P_r(\delta)e_{r-1} \cong P_{r-1}(\delta), \tag{3.3}$$

and that this generalises to $P_l(\delta) \cong e_l P_r(\delta) e_l$ for $0 \leqslant l \leqslant r$. Moreover, since $P_r(\delta)e_{r-1}P_r(\delta)$ is the span of all (r,r)-partition diagrams with at most $r-1$ propagating blocks,

$$\frac{P_r(\delta)}{P_r(\delta)e_{r-1}P_r(\delta)} \cong \mathbb{C}\mathfrak{S}_r \tag{3.4}$$

where the left-hand side is J_r/J_{r-1}.

3.4. **Standard and simple modules for the partition algebra.** We use this filtration to construct the standard modules for the partition algebra. Since we later use the commuting left action of the symmetric group and right action of the partition algebra on tensor space (see Section 5), we require right $P_r(\delta)$-modules.

We set $V_r(k) = e_k(J_k/J_{k-1})$. Observe that $V_r(k)$ has a basis given by all (r,r)-partition diagrams with exactly k propagating blocks such that $\{j\}$ is a singleton part for all $j \geqslant k+1$. Thus the corresponding diagrams have no edges from the north vertices $k+1,\dots,r$. We identify such diagrams with the (k,r)-partition diagrams having precisely k propagating blocks. For instance, two different examples of $(3,5)$-partition diagrams with 3 propagating blocks appear as bottom halves of the concatenated diagrams in Figure 6.

Since $e_k P_r(\delta) e_k \cong P_k(\delta)$ and $P_k(\delta)$ has $\mathbb{C}\mathfrak{S}_k$ as a quotient by (3.4), $V_r(k)$ has the structure of a $(\mathbb{C}\mathfrak{S}_k, P_r(\delta))$-bimodule. We remark that $V_r(k) \otimes_{\mathfrak{S}_k} \mathbb{C}\mathfrak{S}_k \otimes_{\mathfrak{S}_k} V_r(k) \cong J_k/J_{k-1}$ where the isomorphism is defined on diagrams by $u \otimes \sigma \otimes v \mapsto u^*\sigma v$, where u^* denotes the (r,k)-partition diagram obtained from the (k,r)-partition diagram u by horizontal reflection; this makes concrete the filtration in (3.2). From (3.4), we see that any right $\mathbb{C}\mathfrak{S}_k$-module can be inflated to a $P_k(\delta)$-module. The simple right $\mathbb{C}\mathfrak{S}_k$-modules are the right Specht modules $\mathbf{S}^\kappa$ for $\kappa \vdash k$. Thus by induction using (3.3) and (3.4) we find that the simple $P_r(\delta)$-modules are indexed by $\mathscr{P}(\leqslant r) = \bigcup_{0 \leqslant i \leqslant r} \mathscr{P}(i)$. For any $\kappa \vdash k$, we define the standard (right) $P_r(\delta)$-module, $\Delta_r(\kappa)$, by

$$\Delta_r(\kappa) \cong \mathbf{S}^\kappa \otimes_{\mathfrak{S}_k} V_r(k) \tag{3.5}$$

where the action of (r,r)-diagrams $d \in P_r(\delta)$ is given as follows. Let v be a (k,r)-partition diagram in $V_r(k)$ and let $x \in \mathbf{S}^\kappa$. Concatenate v above d to get $\delta^t v'$ for some (k,r)-partition diagram v' and some non-negative integer t. If v' has fewer than k propagating blocks then we set $(x \otimes v)d = 0$. Otherwise we set $(x \otimes v)d = \delta^t x \otimes v'$.

By [Jam78, 8.4], the $\mathbb{C}\mathfrak{S}_k$-Specht module $\mathbf{S}^\kappa$ has basis $\{c^*_\kappa \sigma \mid t_\kappa \sigma \in \mathrm{Std}(\kappa)\}$, where c^*_κ is the dual Young symmetrizer defined in (2.2). By (3.5), we have $v \otimes \tau d = v\tau \otimes d$ for any $\tau \in \mathfrak{S}_k$, and so we need to multiply the basis elements of $\mathbf{S}^\kappa$ only by diagrams d such that π_d is the identity permutation. After this reduction, we obtain the basis

$$\left\{ c^*_\kappa \sigma d_\Lambda \,\middle|\, \begin{array}{l} \mathsf{t}^\kappa\sigma \in \mathrm{Std}(\kappa),\ \Lambda \text{ is a } (k,r)\text{-set-partition with } k \\ \text{propagating blocks, } \pi_\Lambda = 1 \in \mathfrak{S}_k \end{array} \right\}. \tag{3.6}$$

of $\Delta_r(\kappa)$, with the action of $P_r(\delta)$ as specified after (3.5).

In particular, taking $k = r$, we have

$$\Delta_r(\kappa) \cong \mathbf{S}^\kappa \otimes_{\mathfrak{S}_r} V_r(r) \cong \mathbf{S}^\kappa, \tag{3.7}$$

where the right-hand side is viewed as a $P_r(\delta)$-module by inflation using (3.4).

Example 3.2. Three distinct basis elements of $\Delta_5\big((2,1)\big)$ are depicted in Figure 6. The middle diagram shows $c_{(2,1)}(2,3)d_\Lambda$ where $\Lambda = \big\{\{1,\overline{1},\overline{3}\},\{2,\overline{2},\overline{5}\},\{3,\overline{4}\}\big\}$. Consider the two diagrams d_Γ and $d_{\Gamma'}$ shown in Figure 7. The product $c_{(2,1)}(2,3)d_\Lambda d_\Gamma$ is non-zero as $d_\Lambda d_\Gamma$ has 3 propagating blocks; it is computed in Figure 8 using the action described after (3.5), with a further 'untwisting' step to obtain a canonical basis element from (3.6). On the other hand since $d_\Lambda d_{\Gamma'}$ has only 2 propagating blocks, we have $c_{(2,1)}(2,3)d_\Lambda d_{\Gamma'} = 0$.

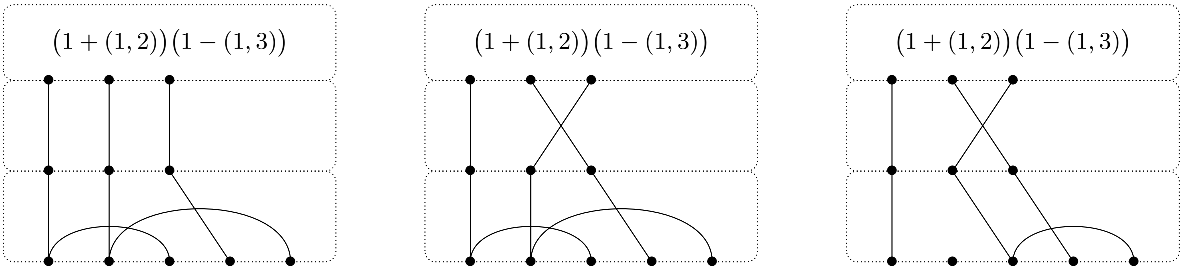


FIGURE 6. Three elements shown in the form $c_{(2,1)}\sigma d_\Lambda$ in the basis of $\Delta_5\big((2,1)\big)$ from equation (3.6). The bottom halves are $(3,5)$-diagrams lying in the basis of $V_5(3)$, regarding these halves as $(5,5)$-diagrams using the identification made at the start of Section 3.4.

FIGURE 7. The diagrams d_Γ and $d_{\Gamma'}$ of two $(5,5)$-set-partitions.

Theorem 3.3. [Mar96, Proposition 3, Proposition 9] *The partition algebra $P_r(\delta)$ is semisimple if and only if $\delta \notin \{0,1,\ldots,2r-2\}$ and, in this case, the set $\{\Delta_r(\kappa) : \kappa \in \mathscr{P}(\leqslant r)\}$ is a complete set of non-isomorphic simple right $P_r(\delta)$-modules. More generally, provided $\delta \neq 0$, the standard module $\Delta_r(\kappa)$ has a simple head, which we denote $L_r(\kappa)$, and $\{L_r(\kappa) : \kappa \in \mathscr{P}(\leqslant r)\}$ is a complete set of non-isomorphic simple right $P_r(\delta)$-modules.*

Martin showed in [Mar96] that $P_r(\delta)$ is a quasi-hereditary algebra provided $\delta \neq 0$. The partition algebra $P_r(\delta)$ is also a cellular algebra, for any value of δ, and the standard modules $\Delta_r(\kappa)$ for $\kappa \in \mathscr{P}(\leqslant r)$ are the cell modules. The following proposition tells us that certain standard modules are simple.

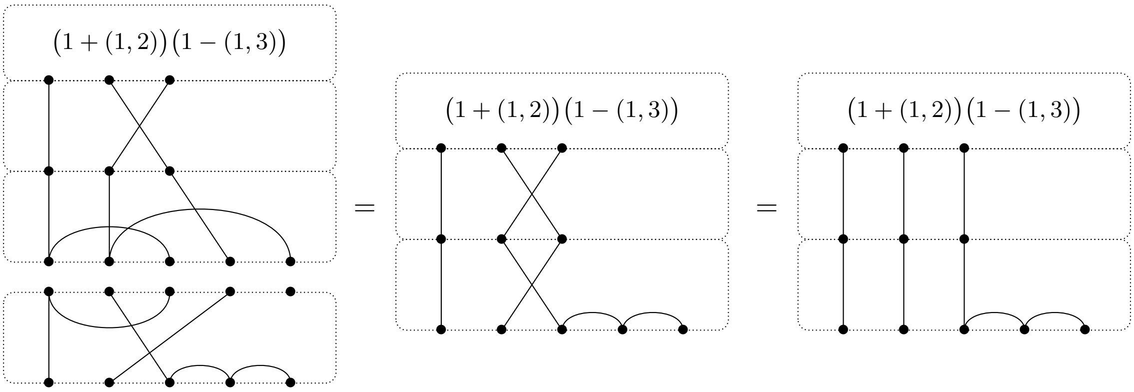


FIGURE 8. The product $c_{(2,1)}(2,3)d_\Lambda d_\Gamma$ shown first in non-canonical form as $\big(1+(1,2)\big)\big(1-(1,3)\big)(2,3)d$ where $\pi_d=(2,3)$ and then as a canonical basis element from (3.6) as $\big(1+(1,2)\big)\big(1-(1,3)\big)d'$ where $\pi_{d'}=1$.

Lemma 3.4. [Mar96, Proposition 23] *Let $\delta \neq 0$ and suppose that κ is a partition such that $\kappa[n]$ is a partition of n (i.e. $n-|\kappa| \geqslant \kappa_1$). Then the $P_r(\delta)$-standard module $\Delta_r(\kappa)=L_r(\kappa)$ if and only if $\delta \geqslant r+\kappa_1$. Moreover, in this case, the module belongs to a simple block.*

3.5. **The orbit basis of Benkart–Halverson.** The diagram basis is the most natural basis for the partition algebra. In particular, we were able to define a multiplication with respect to this basis with ease. There is another basis of the partition algebra, the orbit basis, which was studied by Benkart–Halverson in [BH19]. The advantage of this basis is that it is more intimately connected to the semisimple quotient of the partition algebra that acts *faithfully* on tensor space.

We note that the set of (r,r)-set-partitions is a lattice (a partially ordered set in which each pair of elements admits an upper and lower bound) under the partial order

$$\Lambda \leqslant \Lambda' \quad \text{if every block of } \Lambda \text{ is contained in a block of } \Lambda'. \tag{3.8}$$

If $\Lambda \leqslant \Lambda'$ we say that Λ' is coarser than Λ; equivalently Λ' is coarser than Λ if Λ' may be formed from Λ by merging some of its blocks together.

The orbit basis of the partition algebra consists of the elements x_Λ indexed by set-partitions Λ defined by the coarsening relation as follows

$$d_\Lambda = \sum_{\Lambda \leqslant \Lambda'} x_{\Lambda'}. \tag{3.9}$$

In other words, the diagram basis element d_Λ is the sum of all orbit basis elements $x_{\Lambda'}$ for which Λ' is coarser than Λ. Conversely, the elements x_Λ can be written as a sum of diagram basis elements by way of the Möbius function for the coarsening partial order; we refer to [BH19, Section 4.3] for more details. The orbit basis was so-named by Benkart and Halverson because, by [BH19, Remark 4.7], the action (see equation (5.3) below) of x_Λ on tensor space corresponds to an $\mathfrak{S}_n$-orbit on simple tensors.

In this paper, we shall only need to know that the elements x_Λ form a basis and to know how these basis elements act on tensor space (shown later in equation (5.3)). Knowing this set forms a basis of the partition algebra, and using (3.6), we deduce immediately that the standard module $\Delta_r(\kappa)$ has an orbit basis given by

$$\left\{ c_\kappa^* \sigma x_\Lambda \;\middle|\; \begin{array}{l} \mathsf{t}^\kappa\sigma \in \mathrm{Std}(\kappa),\ \Lambda \text{ is a } (k,r)\text{-set-partition with } k \\ \text{propagating blocks, } \pi_\Lambda = 1 \in \mathfrak{S}_k \end{array} \right\}. \tag{3.10}$$

4. Ramified partition algebras

The ramified partition algebra was originally defined by Martin–Elgamal in [ME04] and later rediscovered by Kennedy [Ken07], who referred to it as the class partition algebra.

4.1. **Ramified set-partitions.** We define a ramified (r, s)-set-partition to be an ordered pair (Λ, Λ') such that Λ, Λ' are set-partitions of $\{1, 2, \dots, r, \bar{1}, \bar{2}, \dots, \bar{s}\}$ and Λ' is coarser than Λ in the sense of equation (3.8). We refer to Λ as the inner set-partition and Λ' as the outer set-partition. Diagrammatically we represent these ordered pairs by drawing the partition diagram of the inner set-partition as usual and then merging parts from d_Λ to form $d_{\Lambda'}$. We continue to draw the (inner and outer) set-partitions with respect to the conventions of Remark 3.1. Examples are depicted in Figures 9 and 11.

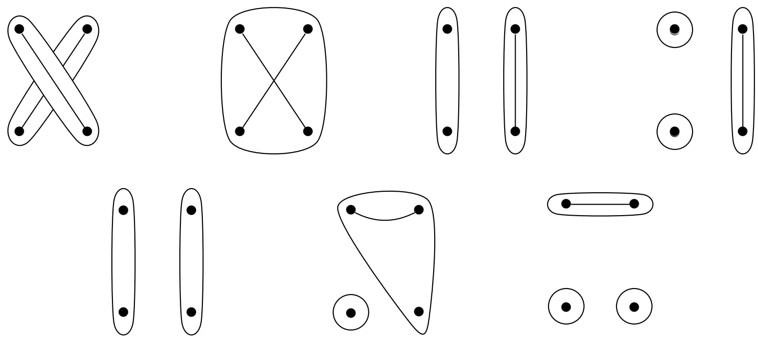


Figure 9. Some examples of ramified $(2, 2)$-set-partitions. The propagating indices (see Section 4.3) are $(1, 1), (2), (1, 0), (1), (0, 0), (0)$, and $\varnothing$ respectively.

We now consider two parameters δ_{in} and $\delta_{\text{out}} \in \mathbb{C}$. The product of ramified set-partitions is derived from the product in the two partition algebras $P_r(\delta_{\text{in}})$ and $P_r(\delta_{\text{out}})$ in a manner we shall now describe. To distinguish the products in these two algebras let us temporarily denote the product in the partition algebra $P_r(\delta_{\text{in}})$ by $\cdot_{\delta_{\text{in}}}$, and that in $P_r(\delta_{\text{out}})$ by $\cdot_{\delta_{\text{out}}}$. Then we define the product of two ramified (r, r)-set-partition diagrams as follows:

$$(d_\Lambda, d_{\Lambda'})(d_\Gamma, d_{\Gamma'}) = (\delta_{\text{in}})^t (\delta_{\text{out}})^s (d_\Delta, d_{\Delta'}), \tag{4.1}$$

where $d_\Lambda \cdot_{\delta_{\text{in}}} d_\Gamma = (\delta_{\text{in}})^s d_\Delta$ and $d_{\Lambda'} \cdot_{\delta_{\text{out}}} d_{\Gamma'} = (\delta_{\text{out}})^s d_{\Delta'}$. In other words, the multiplication of inner set-partition diagrams yields a parameter δ_{in} and the multiplication of outer set-partition diagrams a parameter δ_{out}. We let $R_r(\delta_{\text{in}}, \delta_{\text{out}})$ denote the complex vector space with basis given by all ramified set-partitions of $\{1, 2, \dots, r, \bar{1}, \bar{2}, \dots, \bar{r}\}$ and with multiplication given by linearly extending the multiplication of ramified diagrams. Then $R_r(\delta_{\text{in}}, \delta_{\text{out}})$ is an associative $\mathbb{C}$-algebra, known as the ramified partition algebra. We define an anti-involution $*$ on $R_r(\delta_{\text{in}}, \delta_{\text{out}})$ by reflecting a diagram through its horizontal axis.

Example 4.1. The square of the third diagram Figure 9 is equal to δ_{in} times itself and the square of the seventh diagram in Figure 9 is equal to $\delta_{\text{in}}\delta_{\text{out}}$ times itself.

It is apparent that the ramified partition algebra $R_r(\delta_{\text{in}}, \delta_{\text{out}})$ contains a subalgebra isomorphic to the partition algebra $P_r(\delta_{\text{in}}\delta_{\text{out}})$ whose parameter is the product of the two original parameters: simply take the span of basis elements (d_Λ, d_Λ) whose inner and outer partitions are identical.

Example 4.2. The first, fourth, and seventh diagrams in Figure 9 belong to the subalgebra $P_2(\delta_{\text{in}}\delta_{\text{out}})$ of $R_2(\delta_{\text{in}}, \delta_{\text{out}})$.

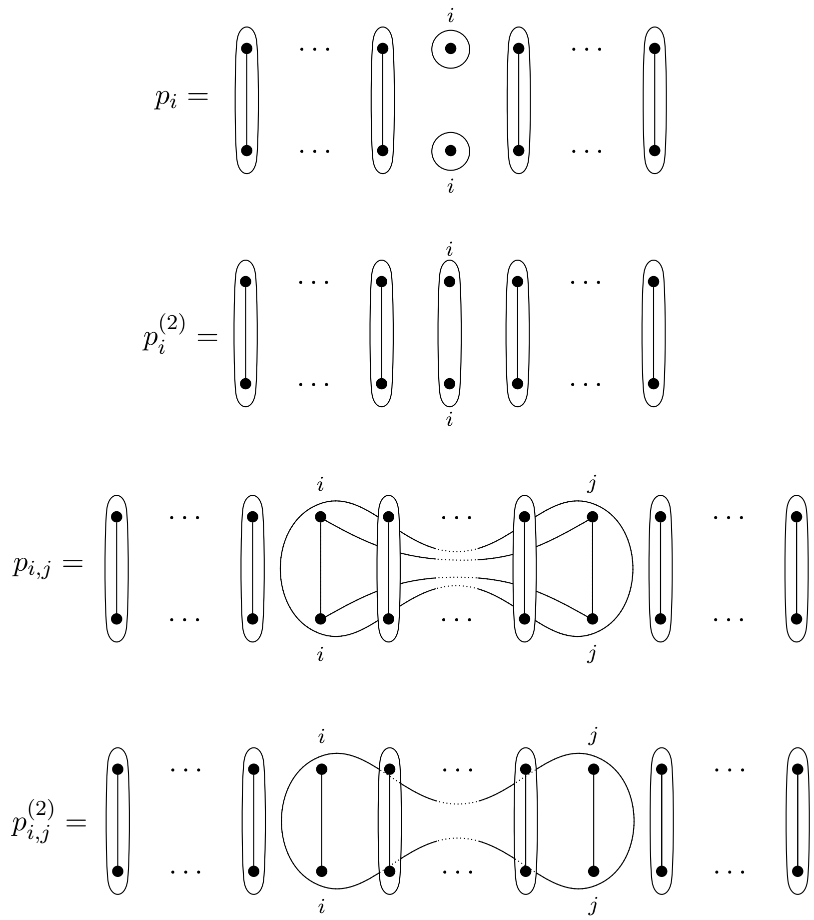


FIGURE 10. Important diagrams for $1 \leqslant i \neq j \leqslant r$. Note that the diagrams p_i and $p_{i,j}$ are simply the generators of the partition algebra embedded as a subalgebra of the ramified partition algebra.

In addition to the Coxeter generators s_i for $1 \leqslant i < r$ (embedded via the partition algebra embedding, see for example the first diagram of Figure 9), we shall also require the diagrams in Figure 10. These diagrams, together with the Coxeter generators, generate the algebra $R_r(\delta_{\text{in}}, \delta_{\text{out}})$.

We also find subalgebras of $R_r(\delta_{\text{in}}, \delta_{\text{out}})$ isomorphic to group algebras of wreath products. Suppose a, b are positive integers with $ab = r$. Then $\mathbb{C}\mathfrak{S}_a \wr \mathfrak{S}_b$ is a subalgebra of $R_{ab}(\delta_{\text{in}}, \delta_{\text{out}})$. Diagrammatically, elements of $\mathfrak{S}_a \wr \mathfrak{S}_b$ can be visualised as the ramified diagrams with b consecutive outer blocks each consisting of a inner propagating pairs. The element $(\sigma_1, \ldots, \sigma_b; \pi) \in \mathfrak{S}_a \wr \mathfrak{S}_b$ is visualised as the ramified diagram with outer blocks

$$\big\{(j-1)a+1, (j-1)a+2, \ldots, ja, \overline{(\pi(j)-1)a+1}, \overline{(\pi(j)-1)a+2} \ldots, \overline{\pi(j)a}\big\},$$

for $j = 1, \ldots b$, and inner blocks $\big\{(j-1)a+i, \overline{(\pi(j)-1)a+\sigma_j(i)}\big\}$, for $i = 1, \ldots a$. An example for $a = 2$ and $b = 3$ is depicted in Figure 11.

4.2. **Horizontal concatenation.** Given a ramified (r_1, s_1)-set-partition $(d_{\Lambda_1}, d_{\Lambda'_1})$ and a ramified (r_2, s_2)-set-partition $(d_{\Lambda_2}, d_{\Lambda'_2})$, we define their horizontal concatenation $(d_{\Lambda_1}, d_{\Lambda'_1}) \circledast (d_{\Lambda_2}, d_{\Lambda'_2})$ in the analogous fashion to the partition algebra case (see Section 3.2). Note that the resulting diagram is indeed a ramified $(r_1 + r_2, s_1 + s_2)$-set-partition.

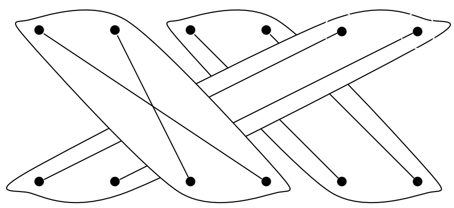


FIGURE 11. The visualisation of $\big((12), 1_{\mathfrak{S}_2}, 1_{\mathfrak{S}_2}; (1,2,3)\big) \in \mathfrak{S}_2 \wr \mathfrak{S}_3$ as a ramified $(6,6)$-set-partition.

4.3. **Propagating indices and a filtration of the ramified partition algebra.** Following [ME04], we define the propagating index of a ramified (r,s)-set-partition (Λ, Λ') to be $(a_1, a_2, \dots, a_k)$ if the outer partition Λ' has k propagating blocks and, within the i^{th} such block of Λ', the inner partition Λ has a_i propagating blocks, for $i = 1, \dots, k$. We arrange the numbers of the propagating index so that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant 0$. An example is depicted in Figure 12.

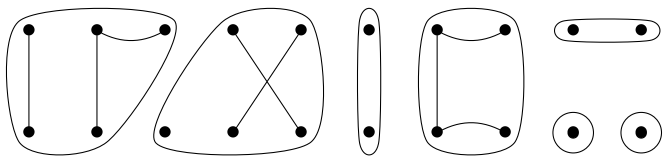


FIGURE 12. The diagram of a ramified $(10,10)$-set-partition with propagating index $(2,2,1,0)$. Note that the unique zero entry in the propagating index records that there is a unique outer propagating block containing no inner propagating blocks. No information about non-propagating outer blocks is recorded in the propagating index.

We let Θ_r denote the set of all possible propagating indices for $R_r(\delta_{\text{in}}, \delta_{\text{out}})$. For example, if $r = 2$ then $\Theta_2 = \{(1,1), (2), (1,0), (1), (0,0), (0), \varnothing\}$; an example of a ramified diagram with each propagating index is depicted in Figure 9. We write $\vartheta' < \vartheta$ if ϑ' is obtained by subtracting 1 from a single entry from ϑ, or if ϑ' is obtained from ϑ by merging two parts into a single part, or finally if $\vartheta' = \varnothing$ and $\vartheta = (0)$. Abusing notation, we let $<$ denote the transitive closure of the above relation. Then Θ_r is partially ordered by $\leqslant$. The poset Θ_3 is illustrated in the Figure 13 below; for example, $(1,0,0) < (1,1,0)$ because we have subtracted 1 from the second entry of $(1,1,0)$ and $(2,0) < (1,1,0)$ because we have merged the first two parts of $(1,1,0)$. Martin and Elgamal [ME04, Proposition 6] show that multiplication in $R_r(\delta_{\text{in}}, \delta_{\text{out}})$ preserves or decreases the propagating index under $\leqslant$.

To each element $\vartheta = (a_1, a_2, \dots a_k) \in \Theta_r$, we have a canonically associated basis element, $e_\vartheta \in R_r(\delta_{\text{in}}, \delta_{\text{out}})$ constructed as follows (an example is depicted in Figure 14). The first outer block of e_ϑ consists of the leftmost a_1 (or 1 if $a_1 = 0$) nodes from both top and bottom rows, the second outer block consists of the next a_2 nodes (or 1 if $a_2 = 0$) from both top and bottom rows, and so on until the k^{th} outer block consists of the next a_k nodes (or 1 if $a_k = 0$) from top and bottom rows, and any remaining nodes form singleton outer blocks (necessarily containing a singleton inner block). Inside the j^{th} outer propagating block, each top row node is joined to the bottom row node immediately below it, unless $a_j = 0$ when the inner blocks are singletons. The basis element e_ϑ can be scaled to an idempotent (provided $\delta_{\text{in}}, \delta_{\text{out}} \neq 0$).

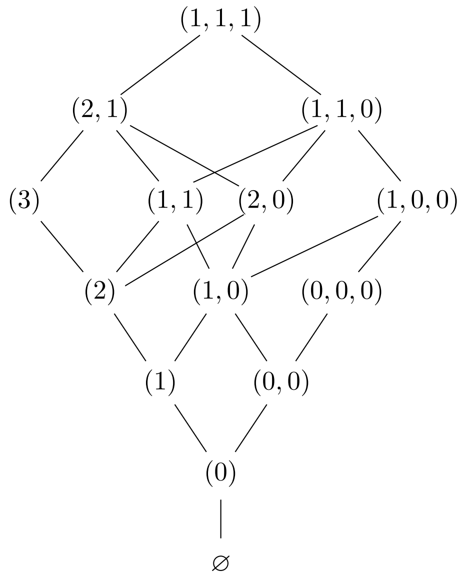


FIGURE 13. The Hasse diagram for the poset Θ_3

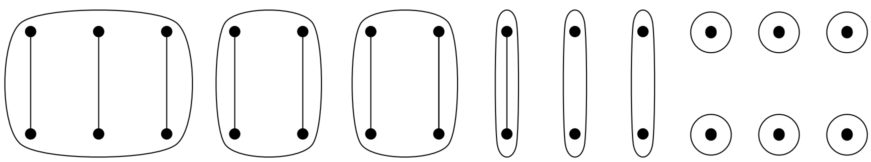


FIGURE 14. The quasi-idempotent e_ϑ for $\vartheta = (3, 2^2, 1, 0^2)$ in $R_{13}(\delta_{\text{in}}, \delta_{\text{out}})$. Note that there are two outer-propagating blocks that are not inner-propagating from the two zero entries. Since $1.3 + 2.2 + 1.1 + 2.1 = 10$, there are 3 remaining northern and southern nodes forming singleton outer blocks.

We fix $\prec$ to be any total refinement of the ordering $\leqslant$ on Θ_r. We set

$$J_{\preccurlyeq\vartheta} = \bigcup_{\vartheta' \preccurlyeq \vartheta} R_r(\delta_{\text{in}}, \delta_{\text{out}}) e_\vartheta R_r(\delta_{\text{in}}, \delta_{\text{out}})$$

and we define $J_{\prec\vartheta}$ in the obvious fashion. Similarly, we define $J_{\leqslant\vartheta}$, $J_{<\vartheta}$ in an analogous fashion in terms of the partial ordering $\leqslant$. Denote the list of all elements of Θ_r in order as

$$\varnothing = \vartheta_1 \prec \vartheta_2 \cdots \prec \vartheta_N = (1^r).$$

Then we obtain the chain of ideals

$$0 \subset J_{\preccurlyeq\vartheta_1} \subset J_{\preccurlyeq\vartheta_2} \subset \cdots \subset J_{\preccurlyeq\vartheta_N} = R_r(\delta_{\text{in}}, \delta_{\text{out}}). \tag{4.2}$$

analogous to (3.2). The quotient $J_{\preccurlyeq\vartheta_i}/J_{\preccurlyeq\vartheta_{i-1}} = J_{\preccurlyeq\vartheta_i}/J_{\prec\vartheta_i}$ has basis consisting of those ramified diagrams with propagating index ϑ_i. We observe, for $\vartheta = (a_1^{b_1}, \ldots, a_\ell^{b_\ell}) \in \Theta_r$, that

$$e_\vartheta (J_{\preccurlyeq\vartheta}/J_{\prec\vartheta}) e_\vartheta \cong \mathbb{C}\mathrm{Stab}(\vartheta) = \mathbb{C} \prod_{i=1}^{\ell} \mathfrak{S}_{a_i} \wr \mathfrak{S}_{b_i} \tag{4.3}$$

simply using the identification of group elements with ramified diagrams illustrated in Figure 11 for each $1 \leqslant i \leqslant \ell$. We set $V_r(\vartheta)$ to be the $(\mathrm{Stab}(\vartheta), R_r(\delta_{\text{in}}, \delta_{\text{out}}))$-bimodule

$$V_r(\vartheta) = e_\vartheta (J_{\preccurlyeq\vartheta}/J_{\prec\vartheta})$$

with basis given by all ramified diagrams in $e_\vartheta R_r(\delta_{\rm in},\delta_{\rm out})$ with propagating index ϑ. For $\vartheta=(a_1^{b_1},\dots,a_\ell^{b_\ell})$, we have that

$$\frac{J_{\preccurlyeq\vartheta}}{J_{\prec\vartheta}}\cong V_r(\vartheta)\otimes_{\mathbb{C}\mathrm{Stab}(\vartheta)}\mathbb{C}\mathrm{Stab}(\vartheta)\otimes_{\mathbb{C}\mathrm{Stab}(\vartheta)}V_r(\vartheta). \tag{4.4}$$

The isomorphism in equation (4.4) is the key observation needed for the following theorem.

Theorem 4.3 ([ME04, Proposition 11], [Bro23])**.** *The algebra $R_r(\delta_{\rm in},\delta_{\rm out})$ is an iterated inflation of the algebras $\mathbb{C}\mathrm{Stab}(\vartheta)$ for $\vartheta\in\Theta_r$ and thus is a cellular algebra. If $\delta_{\rm in},\delta_{\rm out}\neq 0$, then $R_r(\delta_{\rm in},\delta_{\rm out})$ is quasi-hereditary.*

Remark 4.4. The conditions for an iterated inflation in [Gre20, Theorem 1] are checked in [Bro23]. The group algebra $\mathbb{C}(\prod_{i=1}^{\ell}\mathfrak{S}_{a_i}\wr\mathfrak{S}_{b_i})$ is cellular by the work of [GG13] or [Gre20, Proposition 1, Theorem 4]. Thus $R_r(\delta_{\rm in},\delta_{\rm out})$ is an iterated inflation of cellular algebras and hence a cellular algebra. If $\delta_{\rm in},\delta_{\rm out}\neq 0$, then $J_{\preccurlyeq\vartheta}$ (for $\vartheta\in\Theta_r$) is a heredity ideal containing the quasi-idempotent e_ϑ which can be rescaled. Hence the algebra is quasi-hereditary (see also [ME04, Proposition 11]).

4.4. **Standard and simple modules for the ramified partition algebra.** By Theorem 4.3, the ramified partition algebra has distinguished cell modules. We shall assume $\delta_{\rm in},\delta_{\rm out}\neq 0$ and therefore the cell modules are standard modules. We describe these standard modules in generality here, although we require only a good understanding of the most elementary case when $\vartheta=(a^b)$. Fix $\vartheta=(a_1^{b_1},\dots,a_\ell^{b_\ell})\in\Theta_r$. The simple right modules for $\mathbb{C}\mathrm{Stab}(\vartheta)$ of equation (4.4) are the outer tensor products of the (right module analogues of) modules from equation (2.6) as follows:

$$\mathbf{S}^{\boldsymbol{\alpha}^{\boldsymbol{\beta}}}=\mathbf{S}^{\boldsymbol{\alpha}_1^{\boldsymbol{\beta}_1}}\otimes\mathbf{S}^{\boldsymbol{\alpha}_2^{\boldsymbol{\beta}_2}}\otimes\cdots\otimes\mathbf{S}^{\boldsymbol{\alpha}_\ell^{\boldsymbol{\beta}_\ell}}, \tag{4.5}$$

where $\boldsymbol{\alpha}^{\boldsymbol{\beta}}$ an ℓ-tuple of $\boldsymbol{\alpha}_i^{\boldsymbol{\beta}_i}\in\mathscr{P}(a_i,b_i)$. We define the (right) standard $R_r(\delta_{\rm in},\delta_{\rm out})$-module, $\Delta_r(\boldsymbol{\alpha}^{\boldsymbol{\beta}})$, by

$$\Delta_r(\boldsymbol{\alpha}^{\boldsymbol{\beta}})\cong\mathbf{S}^{\boldsymbol{\alpha}^{\boldsymbol{\beta}}}\otimes_{\mathrm{Stab}(\vartheta)}V_r(\vartheta) \tag{4.6}$$

where the action of $R_r(\delta_{\rm in},\delta_{\rm out})$ is given as follows. Let v be a ramified partition diagram in $V_r(\vartheta)$, $x\in S(\boldsymbol{\alpha}^{\boldsymbol{\beta}})$ and d be an (r,r)-ramified partition diagram. Concatenate v above d to get $\delta_{\rm in}^s\delta_{\rm out}^t v'$ for some ramified partition diagram v' and $s,t\in\mathbb{Z}_{\geqslant 0}$. If the propagating index of v' is not equal to ϑ (and so it has smaller index in the order $\prec$) then we set $(x\otimes v)d=0$. Otherwise we set $(x\otimes v)d=\delta_{\rm in}^s\delta_{\rm out}^t x\otimes v'$. As $R_r(\delta_{\rm in},\delta_{\rm out})$ is a cellular algebra, the simple $R_r(\delta_{\rm in},\delta_{\rm out})$-modules are quotients of the standard modules. If $\delta_{\rm in},\delta_{\rm out}\neq 0$, then each standard module $\Delta_r(\boldsymbol{\alpha}^{\boldsymbol{\beta}})$ has a simple quotient which we shall denote by $L_r(\boldsymbol{\alpha}^{\boldsymbol{\beta}})$, and these are a complete set of non-isomorphic simple modules.

We now restrict our attention to the case $\vartheta=(a^b)$ (with $ab\leqslant r$ or $b\leqslant r$ in the case $a=0$). Ramified diagrams in $V_r(a^b)=e_{(a^b)}(J_{\preccurlyeq(a^b)}/J_{\prec(a^b)})$ will be identified with (ab,r)-ramified diagrams provided $a\neq 0$ (respectively (b,r)-ramified diagrams if $a=0$) of propagating index (a^b). We simply delete the additional $r-ab$ (respectively $r-b$) northern outer singleton vertices. An example of a ramified diagram from $V_5(2^2)$ is shown in Figure 15. We let $\alpha\vdash a$ and $\beta\vdash b$ and use the right $\mathbb{C}\mathfrak{S}_a\wr\mathfrak{S}_b$-module $\mathbf{S}^\alpha\oslash\mathbf{S}^\beta$ to construct

$$\Delta_r(\alpha^\beta)=(\mathbf{S}^\alpha\oslash\mathbf{S}^\beta)\otimes_{\mathfrak{S}_a\wr\mathfrak{S}_b}V_r(a^b)=c^*_{\alpha^\beta}\mathbb{C}\mathfrak{S}_a\wr\mathfrak{S}_b\otimes_{\mathfrak{S}_a\wr\mathfrak{S}_b}V_r(a^b).$$

Let $\mathcal{S}(\alpha^\beta)$ be a set of elements of $\mathfrak{S}_m\wr\mathfrak{S}_n$ chosen so that $\{c^*_{\alpha^\beta}\sigma\mid\sigma\in\mathcal{S}(\alpha^\beta)\}$ is a basis of $c^*_{\alpha^\beta}\mathbb{C}\mathfrak{S}_m\wr\mathfrak{S}_n=\mathbf{S}^\alpha\oslash\mathbf{S}^\beta$. This set has cardinality $|\mathrm{Std}(\alpha)|^b|\mathrm{Std}(\beta)|$. We therefore obtain a basis of $\Delta_r(\alpha^\beta)$:

$$\left\{c^*_{\alpha^\beta}\sigma d_{(\Lambda,\Lambda')}\;\middle|\;\begin{array}{l}\sigma\in\mathcal{S}(\alpha^\beta)\\ (\Lambda,\Lambda')\text{ is a }(ab,r)\text{-ramified set-partition of propagating index }(a^b)\\ \pi_{\Lambda'}=1_{\mathfrak{S}_b},\pi_{\Lambda'_j\cap\Lambda}=1_{\mathfrak{S}_a},1\leqslant j\leqslant b\end{array}\right\}. \tag{4.7}$$

In the case $\alpha = \varnothing$, we need to replace (ab, r)-ramified set-partitions with (b, r)-ramified set-partitions in the above. An example is depicted in Figure 15. These standard modules $\Delta_r(\alpha^\beta)$, for $\alpha \vdash a$ and $\beta \vdash b$ (with $ab \leqslant r$ or $b \leqslant r$ in the case $a = 0$), and their simple quotients $L_r(\alpha^\beta)$, are the only ramified partition algebra modules which will be of importance to the plethysm question.

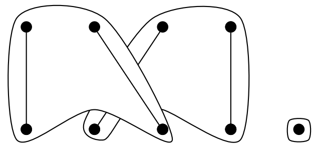


FIGURE 15. An element $d_{(\Lambda,\Lambda')}$ appearing in the basis in equation (4.7). Note that $\pi_\Lambda = (2, 3)$ but $\pi_{\Lambda'_1 \cap \Lambda} = \mathrm{id}_{\mathfrak{S}_2} = \pi_{\Lambda'_2 \cap \Lambda}$, as required.

Remark 4.5. The orbit-basis of the ramified partition algebras has not yet been studied in the literature. We do not require this basis for our purposes, but we posit that it should be worthy of study. In particular, it is natural in light of Benkart–Halverson's work [BH19] to expect that such orbit bases should exist and that they split into natural bases of the kernel and image of the ramified partition algebras acting on tensor space.

5. Partition algebras, ramified partition algebras and Schur–Weyl duality

In this section we review results from the literature regarding the Schur–Weyl dualities between group algebras of symmetric groups and partition algebras, and between group algebras of wreath products of symmetric groups and ramified partition algebras.

5.1. **Symmetric groups and partition algebras.** Let $I(d, r) = \{1, \ldots, d\}^r$ be the set of multi-indices. For a given multi-index $i = (i_1, \ldots, i_r) \in I(d, r)$, we put $e^i = e^{i_1} \otimes \cdots \otimes e^{i_r}$. Then $\{e^i \mid i \in I(d, r)\}$ is a basis of tensor space $(\mathbb{C}^d)^{\otimes r}$ over $\mathbb{C}$. We define the diagonal action $\Phi : \mathfrak{S}_d \to \mathrm{End}((\mathbb{C}^d)^{\otimes r})$ by

$$\Phi(\sigma)(e^{i_1} \otimes \cdots e^{i_r}) = e^{\sigma(i_1)} \otimes \cdots e^{\sigma(i_r)} \tag{5.1}$$

for any $\sigma \in \mathfrak{S}_d$ and $i = (i_1, \ldots, i_r) \in I(d, r)$. (The reason for using upper indices will be seen in Section 6.) Now we consider the partition algebra $P_r(d)$ with parameter d and define a right action of this algebra on tensor space. Let Λ be a (r, r)-set-partition. Following [BH19], we define $\Psi : P_r(d) \to \mathrm{End}((\mathbb{C}^d)^{\otimes r})$ on the orbit basis of equation (3.9) by first setting

$$(x_\Lambda)^{i_1,\ldots,i_r}_{i_{\overline{1}},\ldots,i_{\overline{r}}} = \begin{cases} 1 & \text{if } i_a = i_b \text{ if and only if } a \text{ and } b \text{ are in the same block of } \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

where $(i_{\overline{1}}, \ldots, i_{\overline{r}}) \in I(d, r)$ and a, b run over $\{1, \ldots, r, \overline{1}, \ldots, \overline{r}\}$. We then set

$$(e^{i_1} \otimes \cdots \otimes e^{i_r})\Psi(x_\Lambda) = \sum_{(i_{\overline{1}},\ldots,i_{\overline{r}}) \in I(d,r)} (x_\Lambda)^{i_1,\ldots,i_r}_{i_{\overline{1}},\ldots,i_{\overline{r}}} (e^{i_{\overline{1}}} \otimes \cdots \otimes e^{i_{\overline{r}}}). \tag{5.3}$$

Since the diagram basis is related to the orbit basis by the refinement relation of equation (3.8) and equation (3.9), we have as an immediate consequence, that, setting

$$(d_\Lambda)^{i_1,\ldots,i_r}_{i_{\overline{1}},\ldots,i_{\overline{r}}} = \begin{cases} 1 & \text{if } i_a = i_b \text{ when } a \text{ and } b \text{ are in the same block of } \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

the diagram basis element d_Λ acts on the right, by the rule

$$(e^{i_1} \otimes \cdots \otimes e^{i_r})\Psi(d_\Lambda) = \sum_{(i_{\overline{1}},\ldots,i_{\overline{r}}) \in I(d,r)} (d_\Lambda)^{i_1,\ldots,i_r}_{i_{\overline{1}},\ldots,i_{\overline{r}}} (e^{i_{\overline{1}}} \otimes \cdots \otimes e^{i_{\overline{r}}}). \tag{5.5}$$

We have constructed actions of the symmetric group and partition algebra on tensor space as follows:

$$\mathbb{C}\mathfrak{S}_d \xrightarrow{\Phi} \mathrm{End}_{\mathbb{C}}\big((\mathbb{C}^d)^{\otimes r}\big) \xleftarrow{\Psi} P_r(d). \tag{5.6}$$

Theorem 5.1 ([Jon94, Mar91])**.** *In the situation of* (5.6)*, the image of each representation is equal to the full centraliser algebra for the other action. That is,*

$$\Phi(\mathbb{C}\mathfrak{S}_d) = \mathrm{End}_{P_r(d)}\big((\mathbb{C}^d)^{\otimes r}\big), \qquad \Psi(P_r(d)) = \mathrm{End}_{\mathfrak{S}_d}\big((\mathbb{C}^d)^{\otimes r}\big).$$

Moreover

(i) *As a* $(\mathbb{C}\mathfrak{S}_d, P_r(d))$*-bimodule, the tensor space decomposes as*

$$(\mathbb{C}^d)^{\otimes r} \cong \bigoplus \mathbf{S}_{\kappa[d]} \otimes L_r(\kappa)$$

where the sum is over all partitions $\kappa \in \mathscr{P}(\leqslant r)$ *with* $\kappa_1 \leqslant d - |\kappa|$*.*

(ii) *For* $d \geqslant 2r$*, the partition algebra* $P_r(d)$ *is isomorphic to* $\mathrm{End}_{\mathbb{C}\mathfrak{S}_d}\big((\mathbb{C}^d)^{\otimes r}\big)$ *and acts faithfully on tensor space. Thus* $(\mathbb{C}^d)^{\otimes r}$ *is a semisimple* $\mathbb{C}$*-algebra and the modules* $\{\Delta_r(\kappa) \mid \kappa \in \mathscr{P}(\leqslant r)\}$ *provide a complete set of non-isomorphic simple* $P_r(d)$*-modules.*

As an immediate corollary of the decomposition of tensor space we obtain a Schur functor from left symmetric group modules to right modules for the partition algebra. This step is very familiar to experts but we give full details as we need an analogous argument as part of the proof of Proposition 6.1 on the ramified partition algebra.

Corollary 5.2. *Let* κ *be a partition of* r *such that* $\kappa[d]$ *is a partition. The functor*

$$\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_d}(-, (\mathbb{C}^d)^{\otimes r}) : \mathbb{C}(\mathfrak{S}_d)\text{−mod} \to \text{mod−}P_r(d)$$

sends the left Specht module $\mathbf{S}_{\kappa[d]}$ *to the simple module* $L_r(\kappa)$ *for the partition algebra* $P_r(d)$*.*

Proof. Let $\rho \in \mathscr{P}(\leqslant r)$ be such that $\rho[d]$ is a partition. Considered as a left $\mathbb{C}\mathfrak{S}_d$-module, $\mathbf{S}_{\rho[d]} \otimes L_r(\rho)$ is a direct sum of $\dim L_r(\rho)$ copies of $\mathbf{S}_{\rho[d]}$. Therefore $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_d}\big(\mathbf{S}_{\kappa[d]}, \mathbf{S}_{\rho[d]} \otimes L_r(\rho)\big) = 0$ unless $\rho = \kappa$; in the remaining case then, considering the right $P_r(d)$-action on the bimodule, we have $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_d}\big(\mathbf{S}_{\kappa[d]}, \mathbf{S}_{\kappa[d]} \otimes L_r(\kappa)\big) \cong L_r(\kappa)$ as $P_r(d)$-modules. The corollary now follows from the decomposition of tensor space in Theorem 5.1(i). □

For subsequent work with tensor space it will be convenient to associate a set-partition of $\{1, 2, \dots, r\}$ to each pure tensor. Given a set partition P, we write $i \sim_P i'$ if i and i' are in the same part of P.

Definition 5.3. We say that the basis vector

$$e = e^{j_1} \otimes e^{j_2} \otimes \cdots \otimes e^{j_r} \in (\mathbb{C}^d)^{\otimes r}$$

has value-type S if $k \sim_S l$ if and only if $j_k = j_l$. We write $\mathrm{val}(e) = S$.

For example, $e = e^1 \otimes e^1 \otimes e^1 \otimes e^2 \otimes e^3 \otimes e^2 \otimes e^3$ has $\mathrm{val}(v) = \{\{1, 2, 3\}, \{4, 6\}, \{5, 7\}\}$.

5.2. **Wreath product groups and ramified partition algebras.** Recall from Section 2.3 that, following [JK81, Section 4.1], we have defined

$$\mathfrak{S}_m \wr \mathfrak{S}_n = \big\{(\sigma_1, \sigma_2, \dots, \sigma_n; \pi) \mid \sigma_i \in \mathfrak{S}_m, i = 1, \dots, n, \pi \in \mathfrak{S}_n\big\},$$

which we identify with a subgroup of $\mathfrak{S}_{mn}$ via the embedding equation (2.5). The diagonal action (5.1) of $\mathfrak{S}_{mn}$ on tensor space $(\mathbb{C}^{mn})^{\otimes r}$ restricts to an action $\Phi : \mathfrak{S}_m \wr \mathfrak{S}_n \to \mathrm{End}\big((\mathbb{C}^{mn})^{\otimes r}\big)$ of $\mathfrak{S}_m \wr \mathfrak{S}_n$. Having chosen our wreath product subgroup in the fashion above, we let this guide our choice of a new labelling set for the basis of tensor space as follows. For $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, we set

$$v_i^j = e^{(j-1)m+i},$$

and we note that

$$(\sigma_1,\sigma_2,\dots,\sigma_n;\pi)(v_{i_1}^{j_1}\otimes v_{i_2}^{j_2}\otimes\cdots\otimes v_{i_r}^{j_r}) = v^{\pi(j_1)}_{\sigma_{\pi(j_1)}(i_1)}\otimes v^{\pi(j_2)}_{\sigma_{\pi(j_2)}(i_2)}\otimes\cdots\otimes v^{\pi(j_r)}_{\sigma_{\pi(j_r)}(i_r)}. \tag{5.7}$$

On the other hand, there is an action $\Psi R_r(m,n)\to \mathrm{End}\big((\mathbb{C}^{mn})^{\otimes r}\big)$ of the ramified partition algebra $R_r(m,n)$ with parameters $\delta_{\mathrm{in}}=m$ and $\delta_{\mathrm{out}}=n$ on the right on tensor space. This is given by identifying a ramified diagram $d_{(\Lambda,\Lambda')}\in P_r(m,n)$ with the pair of elements $d_\Lambda\in P_r(m)$ and $d_{\Lambda'}\in P_r(n)$ and then acting by these elements as in equation (5.4) on the subscripts and superscripts, respectively:

$$(v_{i_1}^{j_1}\otimes\cdots\otimes v_{i_r}^{j_r})\Psi(d_{(\Lambda,\Lambda')}) = \sum_{\substack{(i_{\overline{1}},\dots,i_{\overline{r}})\in I(m,r)\\ (j_{\overline{1}},\dots,j_{\overline{r}})\in I(n,r)}} (d_\Lambda)^{i_1,\dots,i_r}_{i_{\overline{1}},\dots,i_{\overline{r}}}(d_\Lambda)^{j_1,\dots,j_r}_{j_{\overline{1}},\dots,j_{\overline{r}}}\,(v^{j_{\overline{1}}}_{i_{\overline{1}}}\otimes\cdots\otimes v^{j_{\overline{r}}}_{j_{\overline{1}}}). \tag{5.8}$$

The left action Φ of $\mathfrak{S}_m\wr\mathfrak{S}_n$ and the right action Ψ of $R_r(m,m)$ on $(\mathbb{C}^{mn})^{\otimes r}$ commute and we have the following analogue of Theorem 5.1.

Theorem 5.4 ([Ken07, Corollary 3.3.3])**.** *In the situation outlined above, the image of each representation is equal to the full centraliser algebra for the other action. That is,*

$$\Phi\big(\mathbb{C}\mathfrak{S}_m\wr\mathfrak{S}_n\big)=\mathrm{End}_{R_r(m,n)}\big((\mathbb{C}^{mn})^{\otimes r}\big),\qquad \Psi\big(R_r(m,n)\big)=\mathrm{End}_{\mathbb{C}(\mathfrak{S}_m\wr\mathfrak{S}_n)}\big((\mathbb{C}^{mn})^{\otimes r}\big).$$

For parameters $m\geqslant 2r$ and $n\geqslant 2r$, the ramified partition algebra $R_r(m,n)$ is isomorphic to $\mathrm{End}_{\mathbb{C}(\mathfrak{S}_m\wr\mathfrak{S}_n)}\big((\mathbb{C}^{mn})^{\otimes r}\big)$ and acts faithfully on tensor space. Therefore $R_r(m,n)$ is a semisimple $\mathbb{C}$-algebra and the modules

$$\big\{\Delta_r(\boldsymbol{\alpha}^{\boldsymbol{\beta}})\mid \vartheta=(a_1^{b_1},\dots,a_\ell^{b_\ell})\in\Theta_r \textit{ and } \boldsymbol{\alpha}^{\boldsymbol{\beta}} \textit{ an } \ell\textit{-tuple of } \boldsymbol{\alpha}_i^{\boldsymbol{\beta}_i}\in\mathscr{P}(a_i,b_i) \textit{ for } 1\leqslant i\leqslant \ell\big\}$$

provide a complete set of non-isomorphic simple $R_r(m,n)$-modules.

We now delve a little deeper into the combinatorics of tensor space. In the following definition we associate a ramified set-partition of $\{1,2,\dots,r\}$ to each pure tensor.

Definition 5.5. We say that the pure tensor

$$v=v_{i_1}^{j_1}\otimes v_{i_2}^{j_2}\otimes\cdots\otimes v_{i_r}^{j_r}\in(\mathbb{C}^{mn})^{\otimes r}$$

has ramified value-type (R,S) if $k\sim_S l$ if and only if $j_k=j_l$ and $k\sim_R l$ if and only if $j_k=j_l$ and $i_k=i_l$. We write $\mathrm{ramval}(v)=(R,S)$. Note that $R\leqslant S$.

Example 5.6. For example, the pure tensor $v=v_2^1\otimes v_1^1\otimes v_1^1\otimes v_3^2\otimes v_2^3\otimes v_3^2\otimes v_3^3$ has

$$\mathrm{ramval}(v)=(R,S)=\big(\{\{1\},\{2,3\},\{4,6\},\{5\},\{7\}\},\{\{1,2,3\},\{4,6\},\{5,7\}\}\big).$$

To obtain $S=\{\{1,2,3\},\{4,6\},\{5,7\}\}$ note that the superscripts match in positions 1,2,3 and they match in positions 4 and 6 and they also match in positions 5 and 7. Although the subscripts match in positions 1 and 5, the superscripts do not match and so $1\nsim_R 5$.

Definition 5.7. Fix a ramified value-type (R,S). We define the associated minimal R value-type tuple $(i_1^*,\dots,i_r^*)$ by specifying the numbers from left to right so that each is the minimal possible value such that

$$v_{i_1^*}^{j_1}\otimes v_{i_2^*}^{j_2}\otimes\dots v_{i_r^*}^{j_r}$$

has value-type (R,S) for any $j_1,\dots,j_r$ such that $j_k=j_l$ if and only if $k\sim_S l$.

This definition is best understood via an example.

Example 5.8. Fix $(R,S)=\big(\{\{1\},\{2,3\},\{4,6\},\{5\},\{7\}\},\big\{\{1,2,3\},\{4,6\},\{5,7\}\big\}\big)$. The minimal R value-type tuple is given by $i^*=(1,2,2,1,1,1,2)$. In particular, note that

$$v_1^{j_1}\otimes v_2^{j_1}\otimes v_2^{j_1}\otimes v_1^{j_2}\otimes v_1^{j_3}\otimes v_1^{j_2}\otimes v_2^{j_3}$$

has value-type (R,S), for any distinct values j_1,j_2,j_3.

6. The ramified Schur functor

To prove Theorem A, we shall use the Schur–Weyl duality seen in Theorem 5.1 between $\mathbb{C}\mathfrak{S}_d$ and the partition algebra $P_r(d)$, together with the Schur–Weyl duality seen in Theorem 5.4 between $\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n$ and the ramified partition algebra $R_r(m,n)$. The former is well understood: as seen in the proof of Corollary 5.2, the bimodule decomposition of tensor space in Theorem 5.1 immediately provides the correspondence between simple left modules for $\mathbb{C}S_d$ and simple right modules for $P_r(d)$. In the latter case, however, we need to establish the correspondence between simple modules.

We shall need this correspondence only for simple $\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n$-modules of the special form $\mathbf{S}_\mu \oslash \mathbf{S}_\nu$, with μ a partition of m and ν a partition of n. We remark that the case where $\mu = (m)$ and $\nu = (n)$ was studied in [BP17, Theorem 6.1] entirely in the language of classical partition algebras: $\Delta_r(\varnothing^\varnothing)\downarrow^{R_r(m,n)}_{P_r(mn)}$ is the stable Foulkes module, denoted $\mathbb{F}^r(m,n)$ in [BP17]. Our proof includes this in its first case.

Proposition 6.1. *The ramified Schur functor*

$$\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_m\wr\mathfrak{S}_n}\big(-, (\mathbb{C}^{mn})^{\otimes r}\big) : \mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n\text{–mod} \to \text{mod–}R_r(m,n)$$

satisfies

$$\mathbf{S}_{\alpha[m]} \oslash \mathbf{S}_{\beta[n]} \mapsto \begin{cases} L_r(\varnothing^\beta) & \text{if } \alpha = \varnothing, r \geqslant |\beta| \\ L_r(\alpha^{\beta[n]}) & \text{if } \alpha \neq \varnothing, r \geqslant n|\alpha| \\ 0 & \text{if either } \alpha = \varnothing, \beta \neq \varnothing, r < |\beta| \text{ or } \alpha \neq \varnothing, r < n|\alpha|. \end{cases}$$

Proof. The proof splits into two parts considering the top two cases separately. Each part ends by showing that the image is zero when the condition for the third case holds.

The case $\alpha = \varnothing$. Suppose that $\beta \vdash b$. Assume first that $r \geqslant b$. We must show that

$$\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_m\wr\mathfrak{S}_n}\big(\mathbf{S}_{(m)} \oslash \mathbf{S}_{\beta[n]}, (\mathbb{C}^{mn})^{\otimes r}\big) \cong L_r(\varnothing^\beta).$$

The simple module on the left-hand side is

$$\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_m\wr\mathfrak{S}_n}\big(\mathbb{C}(\mathfrak{S}_m \wr \mathfrak{S}_n)c_{(m)^{\beta[n]}}, (\mathbb{C}^{mn})^{\otimes r}\big) \cong c_{(m)^{\beta[n]}}(\mathbb{C}^{mn})^{\otimes r}.$$

Therefore, since the standard module has a simple head, it suffices to construct a non-zero homomorphism of right $R_r(m,n)$-modules from the standard module

$$\Delta_r(\varnothing^\beta) \to c_{(m)^{\beta[n]}}(\mathbb{C}^{mn})^{\otimes r}.$$

As $R_r(m,n)$-modules, we have that

$$\Delta_r(\varnothing^\beta) \cong c^*_{\varnothing^\beta} e_{(0^b)}(J_{\preccurlyeq(0^b)}/J_{\prec(0^b)}).$$

where $e_{(0^b)} \in R_r(m,n)$ is the quasi-idempotent constructed in Section 4.3 and exemplified in Figure 14. Observe here that $e_{(0^b)}$ and $c^*_{\varnothing^\beta}$ commute, and that $e_{(0^b)}(J_{\preccurlyeq(0^b)}/J_{\prec(0^b)})$ is spanned by all ramified diagrams whose propagating index in the sense of Section 4.3 is (0^b). Thus the north vertices $b+1,\ldots,r$ are outer singletons; if $b = 0$ then $e_{(0^0)} = e_\varnothing$ is the diagram in which all r northern and southern vertices are outer singletons. Our aim is to define a non-zero homomorphism of right $R_r(m,n)$-modules,

$$c^*_{\varnothing^\beta} e_{(0^b)}(J_{\preccurlyeq(0^b)}/J_{\prec(0^b)}) \to c_{(m)^{\beta[n]}}(\mathbb{C}^{mn})^{\otimes r}.$$

Firstly, we define

$$\chi : e_{(0^b)}(J_{\preccurlyeq(0^b)}/J_{\prec(0^b)}) \to c_{(m)^{\beta[n]}}(\mathbb{C}^{mn})^{\otimes r},$$

by setting $\chi(e_{(0^b)}) = c_{(m)^{\beta[n]}} z$, where

$$z = \sum_{\substack{1 \leqslant i_1,\ldots,i_r \leqslant m \\ 1 \leqslant j_{b+1},\ldots,j_r \leqslant n}} (v_{i_1}^{n-b+1} \otimes v_{i_2}^{n-b+2} \otimes \cdots \otimes v_{i_{b-1}}^{n-1} \otimes v_{i_b}^{n}) \otimes (v_{i_{b+1}}^{j_{b+1}} \otimes \cdots \otimes v_{i_r}^{j_r}). \tag{6.1}$$

Note that we have assumed that $r \geqslant b$. In the first b places, the superscripts are distinct and equal the b entries lying outside the first row of the tableau $\mathsf{t}^{\beta[n]}$.

We must show that χ is well-defined. In general, given an idempotent e in an algebra A, there is a well-defined homomorphism $eA \to U$, to a right A-module U, with $e \mapsto u$ provided $ue = u$. If I is an ideal then we obtain a well-defined map from the quotient $eA/I \to U$ provided, in addition, $ueI = 0$, or equivalently, $uI = 0$.

Using the diagrammatic right action of $R_r(m,n)$, we have $ze_{(0^b)} = m^b \times (m^{r-b}n^{r-b})z = m^r n^{r-b} z$, so the first condition is certainly satisfied. We now check the second. Thus we verify that $c_{(m)^{\beta[n]}} z J_{\prec(0^b)} = c_{(m)^{\beta[n]}} z e_{(0^b)} J_{\prec(0^b)} = 0$. Now, $e_{(0^b)} J_{\prec(0^b)}$ is generated by two types of ramified diagrams: (a) those obtained by merging two outer propagating blocks of $e_{(0^b)}$, and (b) those obtained by replacing an outer propagating block of $e_{(0^b)}$ with two non-propagating outer parts, i.e., those which factor through $e_{(0^{b-1})}$. (Thus if $b = 0$ there are no outer propagating blocks and there is nothing to prove; correspondingly $(0^0) = \varnothing$ appears at the bottom of the Hasse diagram in Figure 13.)

(a) This case is easy: if $d_{(\Lambda_1,\Lambda_1')}$ is obtained by merging the i^{th} and j^{th} outer propagating blocks of $e_{(0^b)}$ (with $1 \leqslant i \neq j \leqslant b$), then $c_{(m)^{\beta[n]}} z d_{(\Lambda_1,\Lambda_1')} = 0$ because the vectors in the i^{th} and j^{th} tensor places differ in their superscripts and therefore are killed by a ramified partition with i and j in the same outer block.

(b) This case requires a little more work. We shall show that $c_{(m)^{\beta[n]}} z e_{(0^{b-1})} = 0$. Let k be the length of the final row of β. Then entries k and n appear in the same column of $\mathsf{t}^{\beta[n]}$ and the transposition (k,n) lies in $C(\mathsf{t}^{\beta[n]})$. We may take cosets and write

$$\begin{aligned} c_{(m)^{\beta[n]}} z = \tilde{c} \sum \big(&(v_{i_1}^{n-b+1} \otimes \cdots \otimes v_{i_{b-1}}^{n-1} \otimes v_{i_b}^{n}) \otimes (v_{i_{b+1}}^{j_{b+1}} \otimes \cdots \otimes v_{i_r}^{j_r}) \\ &- (v_{i_1}^{n-b+1} \otimes \cdots \otimes v_{n-1}^{n-1} \otimes v_{i_b}^{k}) \otimes (v_{i_{b+1}}^{j_{b+1}} \otimes \cdots \otimes v_{i_r}^{j_r}) \big) \end{aligned}$$

for some $\tilde{c}$. The vectors appearing in all tensor positions except the b^{th} are equal, and, when we act from the right by the ramified diagram $e_{(0^{b-1})}$, all terms cancel.

Having shown that the map χ is well-defined, we may now define the homomorphism of right $R_r(m,n)$-modules that we require by restriction: we define

$$\widetilde{\chi} : c^*_{\varnothing\beta} e_{(0^b)} (J_{\preccurlyeq(0^b)}/J_{\prec(0^b)}) \to c_{(m)^{\beta[n]}} (\mathbb{C}^{mn})^{\otimes r},$$

by setting

$$\widetilde{\chi}(c^*_{\varnothing\beta} e_{(0^b)}) = c_{(m)^{\beta[n]}} z c^*_{\varnothing\beta}.$$

Here the left action on z is that of the wreath product and the right action is the diagrammatic action on tensor space.

Finally, we must show that $\widetilde{\chi}$ is non-zero. To do this we show there is a strictly positive coefficient of the basis vector

$$\mathbf{v}_0 = (v_1^{n-b+1} \otimes v_1^{n-b+2} \otimes \cdots \otimes v_1^{n}) \otimes (v_1^1 \otimes \cdots \otimes v_1^1)$$

in $\widetilde{\chi}(c^*_{\varnothing\beta} e_{(0^b)}) = c_{(m)^{\beta[n]}} z c^*_{\varnothing\beta}$. To see this, first note that, expressed as a sum of diagrams,

$$c^*_{\varnothing\beta} e_{(0^b)} = \sum_{\substack{\sigma \in C(\mathsf{t}^\beta) \\ \tau \in R(\mathsf{t}^\beta)}} \operatorname{sgn}(\sigma)(e_{(0^b)} \sigma\tau).$$

Here $e_{(0^b)}\sigma\tau$ is the ramified diagram with $1,\dots,b$ outer propagating and those outer propagating blocks permuted according to the permutation $\sigma\tau$. Hence, $c_{(m)^{\beta[n]}} z c^*_{\varnothing^\beta}$ equals

$$\sum \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)(c_{(m)},\dots,c_{(m)};\rho\pi)(v_{i_1}^{n-b+1}\otimes v_{i_2}^{n-b+2}\otimes\cdots\otimes v_{i_b}^{n}\otimes v_{i_{b+1}}^{j_{b+1}}\otimes\cdots\otimes v_{i_r}^{j_r})(e_{(0^b)}\sigma\tau)$$

where the sum is over all $\pi\in C(\mathsf{t}^{\beta[n]})$, $\rho\in R(\mathsf{t}^{\beta[n]})$, $\sigma\in C(\mathsf{t}^{\beta})$, $\tau\in R(\mathsf{t}^{\beta})$ and indices $1\leqslant i_1,\dots,i_r\leqslant m$ and $1\leqslant j_{b+1},\dots,j_r\leqslant n$.

Taking π,ρ,σ,τ all to be identity permutations and all $i_k=j_k=1$, we have a contribution of $+1$ towards the coefficient of $\mathbf{v}_0$. Now suppose π,ρ,σ,τ contribute to the coefficient of $\mathbf{v}_0$. Then let us first see that π must preserve the set $\{n-b+1,\dots,n\}$. If not then some $k\in\{n-b+1,\dots,n\}$ has $\pi(k)\leqslant n-b$ and (as $\rho\in R(\mathsf{t}^{\beta[n]})\leqslant\mathfrak{S}_{n-b}\times\mathfrak{S}_b$) there is a vector with superscript at most $n-b$ among the first b tensors. But the diagrammatic action of $e_{(0^b)}\sigma\tau$ then only changes its position, not its value, and therefore there is no contribution to the coefficient of $\mathbf{v}_0$.

Assume now that $\pi\in C(\mathsf{t}^{\beta[n]})$ preserves the set $\{n-b+1,\dots,n\}$; i.e. π fixes the first row of $\mathsf{t}^{\beta[n]}$. By assumption, the left action by π gives a pure tensor of vectors having superscripts $\{n-b+1,\dots,n\}$ in the first b places, in some order, and to obtain $\mathbf{v}_0$ the right action must permute them into increasing order. To do this we require $\sigma(k)=\pi(n-b+k)$ for all $k=1\dots,b$. Therefore $\operatorname{sgn}(\sigma)=\operatorname{sgn}(\pi)$ and the contribution is strictly positive.

To complete the part of the proof where $\alpha=\varnothing$, we must show that if $r<b$ then

$$\operatorname{Hom}_{\mathbb{C}\mathfrak{S}_m\wr\mathfrak{S}_n}\big(\mathbf{S}_{(m)}\oslash\mathbf{S}_{\beta[n]},(\mathbb{C}^{mn})^{\otimes r}\big)\cong c_{(m)^{\beta[n]}}(\mathbb{C}^{mn})^{\otimes r}=0.$$

Taking any pure tensor $v_{i_1}^{j_1}\otimes\cdots v_{i_r}^{j_r}$, the condition on r ensures that there exist $x\neq y\in\{1,\dots,n\}$ such that the entries x,y lie in the same column of the $\beta[n]$-tableau $\mathsf{t}^{\beta[n]}$ but neither v_i^x nor v_i^y appear in the pure tensor for any $i\in\{1,\dots,m\}$. Taking cosets of the subgroup generated by the transposition (x,y) in $C(\mathsf{t}^{\beta[n]})$ as a subgroup of the top group of $\mathfrak{S}_m\wr\mathfrak{S}_n$, we may factorise $c_{(m)^{\beta[n]}}$ to see that $c_{(m)^{\beta[n]}}(v_{i_1}^{j_1}\otimes\cdots v_{i_r}^{j_r})=0$.

The case $\alpha\neq\varnothing$. Suppose that $r\geqslant|\alpha|n$. We set $a=|\alpha|$. We follow the same method as above and construct a non-zero homomorphism of right $R_r(m,n)$-modules

$$\Delta(\alpha^{\beta[n]})\to c_{\alpha[m]^{\beta[n]}}(\mathbb{C}^{mn})^{\otimes r}.$$

As $R_r(m,n)$-modules, we have that

$$\Delta(\alpha^{\beta[n]})\cong c^*_{\alpha^{\beta[n]}}e_{(a^n)}(J_{\preccurlyeq(a^n)}/J_{\prec(a^n)}),$$

so we must define a non-zero homomorphism of right $R_r(m,n)$-modules

$$c^*_{\alpha^{\beta[n]}}e_{(a^n)}(J_{\preccurlyeq(a^n)}/J_{\prec(a^n)})\to c_{\alpha[m]^{\beta[n]}}(\mathbb{C}^{mn})^{\otimes r}.$$

Firstly, define

$$\chi: e_{(a^n)}(J_{\preccurlyeq(a^n)}/J_{\prec(a^n)})\to c_{\alpha[m]^{\beta[n]}}(\mathbb{C}^{mn})^{\otimes r},$$

by setting $\chi(e_{(a^n)})=c_{\alpha[m]^{\beta[n]}}z$, where

$$z=\sum_{\substack{1\leqslant i_k\leqslant m\\1\leqslant j_k\leqslant n}}(v^1_{m-a+1}\otimes v^1_{m-a+2}\otimes\cdots\otimes v^1_m)\otimes\cdots\otimes(v^n_{m-a+1}\otimes v^n_{m-a+2}\otimes\cdots\otimes v^n_m)\\ \otimes v^{j_{an+1}}_{i_{an+1}}\otimes\cdots\otimes v^{j_r}_{i_r}. \tag{6.2}$$

(In the first a tensors the superscripts are are 1 and the subscripts are distinct, and in the next a tensors the superscripts are 2 and the subscripts are distinct, and the pattern continues until the subscript is n; this is possible as $r\geqslant an$.)

To check that χ is well-defined, again we verify routinely that multiplication by the quasi-idempotent $e_{(a^n)}$ scales z and then we show that acting on $c_{\alpha[m]^{\beta[n]}}z$ by ramified diagrams of the following three types all give zero: (a) ramified diagrams $d_{(\Lambda,\Lambda')}$ obtained by merging two outer propagating blocks of $e_{(a^n)}$; (b) ramified diagrams $d_{(\Lambda,\Lambda')}$ obtained by merging two inner propagating blocks of $e_{(a^n)}$; (c) ramified diagrams $d_{(\Lambda,\Lambda')}$ which replace an inner propagating block of $e_{(a^n)}$ with two inner singleton parts.

(a) We see that $c_{\alpha[m]^{\beta[n]}} z d_{(\Lambda,\Lambda')} = 0$ because the vectors in the corresponding tensor positions differ in their superscripts.

(b) Similarly, $c_{\alpha[m]^{\beta[n]}} z d_{(\Lambda,\Lambda')} = 0$ because the vectors in the corresponding tensor positions differ in their subscripts.

(c) This case again requires more work. The ramified diagram factors via $e_{(a^n)} p_1^{(2)}$ so it suffices to consider $d_{(\Lambda,\Lambda')} = e_{(a^n)} p_1^{(2)}$. As the transposition $(1, m-a+1)$ lies in $C(\mathsf{t}^{\alpha[m]})$, we may take cosets and write

$$c_{\alpha[m]^{\beta[n]}} = \tilde{c}\left(1_{\mathfrak{S}_m \wr \mathfrak{S}_n} - ((1, m-a+1), 1_{\mathfrak{S}_m}, \ldots, 1_{\mathfrak{S}_m}; 1_{\mathfrak{S}_n})\right)$$

so that $c_{\alpha[m]^{\beta[n]}} z$ equals

$$\begin{aligned}\tilde{c}\big((&v^1_{m-a+1} \otimes v^1_{m-a+2} \otimes \cdots \otimes v^1_m) \otimes \cdots \otimes (v^n_{m-a+1} \otimes \cdots \otimes v^n_m) \otimes v^{j_{an+1}}_{i_{an+1}} \otimes \cdots \otimes v^{j_r}_{i_r} \\ &- (v^1_1 \otimes v^1_{m-a+2} \otimes \cdots \otimes v^1_m) \otimes \cdots \otimes (v^n_{m-a+1} \otimes \cdots \otimes v^n_m) \otimes v^{j_{an+1}}_{i_{an+1}} \otimes \cdots \otimes v^{j_r}_{i_r}.)\end{aligned}$$

The vectors appearing in all tensor positions except the first are equal, and, when we act from the right by the diagram $d_{(\Lambda,\Lambda')} = e_{(a^n)} p_1^{(2)}$, all terms cancel.

Having shown that the map χ is well-defined; restriction provides a homomorphism of right $R_r(m,n)$-modules:

$$\widetilde{\chi} : c^*_{\alpha^{\beta[n]}} e_{(a^n)} J_{\preccurlyeq (a^n)}/J_{\prec(a^n)} \to c_{\alpha[m]^{\beta[n]}} (\mathbb{C}^{mn})^{\otimes r}, \widetilde{\chi}(c^*_{\alpha^{\beta[n]}} e_{(a^n)}) = c_{\alpha[m]^{\beta[n]}} z c^*_{\alpha^{\beta[n]}}.$$

It remains to show that $\widetilde{\chi}$ is non-zero, which we do by considering the coefficient of

$$\mathbf{v}_0 = (v^1_{m-a+1} \otimes \cdots \otimes v^1_m) \otimes (v^2_{m-a+1} \otimes \cdots \otimes v^2_m) \otimes \cdots \otimes (v^n_{m-a+1} \otimes \cdots \otimes v^n_m) \otimes v^1_1 \otimes \cdots \otimes v^1_1$$

in $\widetilde{\chi}(c^*_{\alpha^{\beta[n]} e_{(a^n)}}) = c_{\alpha[m]^{\beta[n]}} z c^*_{\alpha^{\beta[n]}}$. Acting on the right of z is $c^*_{\alpha^{\beta[n]}}$, a sum of ramified diagrams which will permute the places of the first an tensors. On the left of z is

$$c_{\alpha[m]^{\beta[n]}} = \sum \operatorname{sgn}(\pi)(c_{\alpha[m]}, \ldots, c_{\alpha[m]}\,;\, \rho\pi)$$

where the sum is over all $\pi \in C(\mathsf{t}^{\beta[n]})$ and $\rho \in R(\mathsf{t}^{\beta[n]})$. (We emphasise that this is a linear combination of permutation matrices acting diagonally on tensor space.) Each $c_{\alpha[m]}$ is itself a Young symmetrizer in $\mathfrak{S}_m$, but we observe that, since the right action only permutes the tensor places, the only contributions to the coefficient of $\mathbf{v}_0$ come from those terms in $\mathfrak{S}_{\{m-a+1,\ldots,m\}}$ and (up to translation by $m-a$) we look only at $c_{\alpha^{\beta[n]}}$. For every permutation summand in $c_{\alpha[m]^{\beta[n]}}$ that contributes to the coefficient of $\mathbf{v}_0$, the inverse permutation appears in $c^*_{\alpha^{\beta[n]}}$ and undoes its effect. The signs of those permutations on the left and right agree and the coefficient of $\mathbf{v}_0$ is strictly positive.

To complete this part of the proof where $\alpha \neq \varnothing$, we now suppose that $r < an$. In this case we claim that

$$\operatorname{Hom}_{\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n}\left(\mathbf{S}_{\alpha[m]} \oslash \mathbf{S}_{\beta[n]}, (\mathbb{C}^{mn})^{\otimes r}\right) \cong c_{\alpha[m]^{\beta[n]}} (\mathbb{C}^{mn})^{\otimes r} = 0.$$

Take any pure tensor $v^{j_1}_{i_1} \otimes \cdots v^{j_r}_{i_r}$. The condition on r ensures that for some $j \in \{1, \ldots, n\}$ there exist entries $x \neq y \in \{1, \ldots, m\}$ lying in the same column of the standard $\alpha[m]$-tableau $\mathsf{t}^{\alpha[m]}$ but such that neither of v^j_x nor v^j_y appear in the pure tensor. Then taking cosets of the subgroup generated by $(x, y) \in C(\mathsf{t}^{\alpha[m]})$ in the j^{th} copy of $\mathfrak{S}_m$ inside $\mathfrak{S}_m \wr \mathfrak{S}_n$ we may factorise $c_{\alpha[m]^{\beta[n]}}$ to see that $c_{\alpha[m]^{\beta[n]}}(v^{j_1}_{i_1} \otimes \cdots v^{j_r}_{i_r}) = 0$. $\square$

Theorem B now follows easily by adapting steps (a) to (d) in the outline proof of Theorem A in Section 1.7 as follows. For $\kappa \in \mathscr{P}(\leqslant r)$, we choose r sufficiently large (see the final equality) so that

$$\begin{aligned}
p\big(\beta[n], \alpha[m], \kappa[mn]\big) &= \langle s_{\beta[n]} \circ s_{\alpha[m]}, s_{\kappa[mn]} \rangle \\
&= \Big[(\mathbf{S}_{\alpha[m]} \oslash \mathbf{S}_{\beta[n]})\big\uparrow^{\mathfrak{S}_{mn}}_{\mathfrak{S}_m \wr \mathfrak{S}_n} : \mathbf{S}_{\kappa[mn]}\Big]_{\mathbb{C}\mathfrak{S}_{mn}} \\
&= \Big[\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\Big((\mathbf{S}_{\alpha[m]} \oslash \mathbf{S}_{\beta[n]})\big\uparrow^{\mathfrak{S}_{mn}}_{\mathfrak{S}_m \wr \mathfrak{S}_n}, (\mathbb{C}^{mn})^{\otimes r}\Big) \,:\, L_r(\kappa)\Big]_{P_r(mn)} \\
&= \Big[\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n}\Big(\mathbf{S}_{\alpha[m]} \oslash \mathbf{S}_{\beta[n]}, (\mathbb{C}^{mn})^{\otimes r}\big\downarrow^{\mathfrak{S}_{mn}}_{\mathfrak{S}_m \wr \mathfrak{S}_n}\Big) \,:\, L_r(\kappa)\Big]_{P_r(mn)} \\
&= \Big[\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n}\big(\mathbf{S}_{\alpha[m]} \oslash \mathbf{S}_{\beta[n]}, (\mathbb{C}^{mn})^{\otimes r}\big)\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\Big]_{P_r(mn)} \\
&= \begin{cases} \Big[L_r(\varnothing^{\beta})\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\Big]_{P_r(mn)} & \text{if } \alpha = \varnothing, r \geqslant |\beta| \\ \Big[L_r(\alpha^{\beta[n]})\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\Big]_{P_r(mn)} & \text{if } \alpha \neq \varnothing, r \geqslant n|\alpha|. \end{cases}
\end{aligned}$$

Here, the third equality is obtained by the Schur–Weyl duality between the group algebra of the symmetric group and the partition algebra using Corollary 5.2, and the final step comes from the Schur–Weyl duality between the group algebra of the wreath product of symmetric groups and the ramified partition algebra using Theorem 5.4 and the corollary analogous to Corollary 5.2.

This proves Theorem B. and we also obtain an upper bound on plethysm coefficients. By choosing r sufficiently large (i.e., if $\alpha = \varnothing$ then choose $r \geqslant |\beta|$, or if $\alpha \neq \varnothing$ then choose $r \geqslant n|\alpha|$), we have

$$p\big(\beta[n], \alpha[m], \kappa[mn]\big) = \begin{cases} \Big[L_r(\varnothing^{\beta})\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\Big] \leqslant \Big[\Delta_r(\varnothing^{\beta})\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\Big] & \text{if } \alpha = \varnothing \\ \Big[L_r(\alpha^{\beta[n]})\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\Big] \leqslant \Big[\Delta_r(\alpha^{\beta[n]})\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\Big] & \text{if } \alpha \neq \varnothing. \end{cases}$$

In the next section we shall show that this is an equality when $\alpha = \varnothing$.

7. Stability phenomena when the inner partition is trivial

The $\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n$-modules of the form $\mathbf{S}_{(m)} \oslash \mathbf{S}_{\nu}$ are obtained by inflation from the $\mathbb{C}\mathfrak{S}_n$-modules $\mathbf{S}_{\nu}$. Thus certain questions about these modules can be simplified to questions for only the 'outer' symmetric group structure, for example $\dim(\mathbf{S}_{(m)} \oslash \mathbf{S}_{\nu}) = |\mathrm{Std}(\nu)|$. In this section, we start by restricting our focus to the relationship between this 'outer' symmetric group and the 'outer' partition algebra, via Schur–Weyl duality. Our aim is to prove the following theorem.

Theorem 7.1. *Let $\beta \vdash b \leqslant r$ and suppose that $n \geqslant r + \beta_1$ and $m \geqslant r - b + [b \neq 0]$. Then, as a $(\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n, R_r(m,n))$-bimodule, the tensor space $(\mathbb{C}^{mn})^{\otimes r}$ has a direct summand isomorphic to*

$$(\mathbf{S}_{(m)} \oslash \mathbf{S}_{\beta[n]}) \otimes \Delta_r(\varnothing^{\beta}).$$

In particular, under these assumptions, $\Delta_r(\varnothing^{\beta}) = L_r(\varnothing^{\beta})$.

We reiterate that the analogous question for the trivial module $\mathbf{S}_{(m)} \oslash \mathbf{S}_{(n)}$ was already considered in [BP17] and so Theorem 7.1 completes our understanding for the case where the inner partition is trivial.

Before proving the theorem, we need to understand of a basis of the $P_r(n)$-module $\Delta_r(\beta)$ and a basis of the $R_r(m,n)$-module $\Delta_r(\varnothing^{\beta})$.

We begin with the partition algebra, by delving into the orbit basis of the $P_r(n)$-module $\Delta_r(\beta)$ from equation (3.10). Let $P \cup Q$ be a set-partition of $\{\overline{1}, \overline{2}, \ldots, \overline{r}\}$ with $|P| = b$. In other words, we pick a set of b distinguished blocks of the set-partition $P \cup Q$ of $\{\overline{1}, \overline{2}, \ldots, \overline{r}\}$. We now define a (b,r)-set-partition $\Lambda(P,Q)$ where the b distinguished blocks from P provide the propagating blocks. We set $\Lambda(P,Q)$ to be the (b,r)-set-partition

$$\Lambda(P,Q)=\big\{\{i\}\cup P_i \mid 1\leqslant i\leqslant b\big\}\cup Q$$

where P is written according to the conventions of Remark 3.1. It follows that there are no crossings between the b propagating strands of $d_{\Lambda(P,Q)}$, that is $\pi_P=1_{\mathfrak{S}_b}$. We set

$$\mathcal{B}_b=\big\{(P,Q)\mid (\bigcup P)\cap(\bigcup Q)=\varnothing, (\bigcup P)\cup(\bigcup Q)=\{\overline{1},\overline{2},\dots,\overline{r}\}, |P|=b\big\}.$$

The orbit basis of the $P_r(n)$-module $\Delta_r(\beta)$ can now be rewritten as follows

$$\{c^*_\beta\sigma x_{\Lambda(P,Q)}\mid \mathsf{t}^\beta\sigma\in \mathrm{Std}(\beta), (P,Q)\in\mathcal{B}_b\}. \tag{7.1}$$

We now turn our attention to modules of the ramified partition algebra with the aim of describing a diagram basis of the $R_r(m,n)$-module $\Delta_r(\varnothing^\beta)$. Suppose R is a set-partition of $\{\overline{1},\overline{2},\dots,\overline{r}\}$ that is a refinement of $P\cup Q$, i.e., $R\leqslant P\cup Q$. We let

$$\Lambda(R)=\big\{\{i\}\mid 1\leqslant i\leqslant b\big\}\cup R,$$

so $\Lambda(R)$ has no propagating blocks. Then $\Lambda(R)\leqslant\Lambda(P,Q)$ and the ramified diagram $d_{(\Lambda(R),\Lambda(P,Q))}$ has its inner set-partition specified by R, with no inner propagating blocks, and its outer set-partition $\Lambda(P,Q)$ which has b propagating blocks specified by P.

Now we can write down a diagram basis of the $R_r(m,n)$-module $\Delta_r(\varnothing^\beta)=c^*_{\varnothing^\beta}\mathbb{C}\mathfrak{S}_b\otimes_{\mathfrak{S}_b}V_r(0^b)$ as follows:

$$\big\{c^*_{\varnothing^\beta}\sigma d_{(\Lambda(R),\Lambda(P,Q))}\mid \mathsf{t}^\beta\sigma\in\mathrm{Std}(\beta),(P,Q)\in\mathcal{B}_b, R\leqslant P\cup Q\big\}. \tag{7.2}$$

Proof of Theorem 7.1. We suppose that $\beta_{[n]}$ is a partition of n such that $n\geqslant r+\beta_1$. Then, by Lemma 3.4, the $P_r(n)$-module $L_r(\beta)=\Delta_r(\beta)$ is alone in its block. Thus by Theorem 5.1 we have that

$$\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}\big(\mathbf{S}_{\beta_{[n]}},(\mathbb{C}^n)^{\otimes r}\big)\cong c_{\beta_{[n]}}(\mathbb{C}^n)^{\otimes r}\cong L_r(\beta)=\Delta_r(\beta) \tag{7.3}$$

as right $P_r(n)$-modules.

The remainder of this proof will consist of three steps. Firstly, we write down a basis of $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(\mathbf{S}_{\beta_{[n]}},(\mathbb{C}^n)^{\otimes r})$. Secondly, we use the previously identified basis to construct $\dim(\Delta_r(\varnothing^\beta))$ elements of $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_m\wr\mathfrak{S}_n}(\mathbf{S}_{(m)}\oslash\mathbf{S}_{\beta_{[n]}},(\mathbb{C}^n)^{\otimes r})\cong L_r(\varnothing^\beta)$ (where the isomorphism is provided by Proposition 6.1). Thirdly, we prove that these $\dim(\Delta_r(\varnothing^\beta))$ distinct $\mathbb{C}\mathfrak{S}_m\wr\mathfrak{S}_n$-homomorphisms are linearly independent. As $L_r(\varnothing^\beta)$ is a quotient of $\Delta_r(\varnothing^\beta)$, this will show that $L_r(\varnothing^\beta)=\Delta_r(\varnothing^\beta)$ as required.

We start by considering $\Delta_r(\beta)$. A generator of the standard module is $c^*_\beta e_b$, and we let $Z\in c_{\beta_{[n]}}(\mathbb{C}^n)^{\otimes r}$ be the image of this generator under the isomorphism of equation (7.3). Now we have a basis of $\Delta_r(\beta)$ from equation (7.1), and the image of this orbit basis under the first isomorphism of (7.3) provides us with a basis of $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}\big(\mathbf{S}_{\beta_{[n]}},(\mathbb{C}^n)^{\otimes r}\big)$:

$$\{\vartheta_{\sigma,P,Q}\mid \mathsf{t}^\beta\sigma\in\mathrm{Std}(\beta),(P,Q)\in\mathcal{B}_b\},$$

where the $\mathbb{C}\mathfrak{S}_n$-homomorphism $\vartheta_{\sigma,P,Q}$ is defined on the generator $c_{\beta_{[n]}}\in\mathbf{S}_{\beta_{[n]}}$ by

$$\vartheta_{\sigma,P,Q}(c_{\beta_{[n]}})=Z\Psi(\sigma x_{\Lambda(P\cup Q)})=\sum\alpha^{j_1,\dots,j_r}_{\sigma,P,Q}(v^{j_1}\otimes\cdots\otimes v^{j_r}) \tag{7.4}$$

for some coefficients $\alpha^{j_1,\dots,j_r}_{\sigma,P,Q}\in\mathbb{C}$. Observe that, by the action of the orbit basis on tensor space specified in (5.3) the inequality $\alpha^{j_1,\dots,j_r}_{\sigma,P,Q}\neq 0$ implies that $(j_1,\dots,j_r)$ has value-type $P\cup Q$ (as in Definition 5.5).

The second step is to consider inflations of these homomorphisms to construct elements of $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_n}(\mathbf{S}_{\beta_{[n]}}, (\mathbb{C}^n)^{\otimes r})$. We now assume that $m \geqslant r - b + [b \geqslant 1]$. Given R any refinement of the set-partition $P \cup Q$, we define a map $\vartheta_{\sigma,P,Q,R} : \mathbf{S}_{(m)} \oslash \mathbf{S}_{\beta_{[n]}} \to (\mathbb{C}^{mn})^{\otimes r}$ given by

$$\vartheta_{\sigma,P,Q,R}\big(c_{{(m)}^{\beta_{[n]}}}\big) = \sum_{\substack{j_1,\dots,j_r \in \{1,\dots,n\} \\ i_1,\dots i_r \in \{1,\dots,m\}}} \alpha^{j_1,\dots,j_r}_{\sigma,P,Q}(v^{j_1}_{i_1} \otimes \cdots \otimes v^{j_r}_{i_r}) \tag{7.5}$$

where the coefficients $\alpha^{j_1,\dots,j_r}$ are those equation (7.4) and the sum is over *all indices* $i_1, \dots i_r \in \{1, \dots, m\}$, $j_1, \dots, j_r \in \{1, \dots, n\}$ satisfying: if $j_x = j_y$ then $i_x = i_y$ if and only if x, y belong to the same part of R. Observe that if the tensor $v^{j_1}_{i_1} \otimes v^{j_2}_{i_2} \otimes \cdots \otimes v^{j_r}_{i_r}$ appears with non-zero coefficient in equation (7.5), then its ramified value-type is $(R, P \cup Q)$.

We claim that $\vartheta_{\sigma,R,P,Q}$ is an $\mathfrak{S}_m \wr \mathfrak{S}_n$-homomorphism. It is clear from equation (7.4) $\vartheta_{\sigma,R,P,Q}$ is a homomorphism for modules of the distinguished top group $\mathfrak{S}_n$ in $\mathfrak{S}_m \wr \mathfrak{S}_n$. Therefore, we need only check the action of the base group fixes the right-hand side of equation (7.5). The action of the base group is as follows:

$$(\sigma_1, \dots, \sigma_n; 1_{\mathfrak{S}_n})(v^{j_1}_{i_1} \otimes \cdots \otimes v^{j_r}_{i_r}) = (v^{j_1}_{\sigma_{j_1}(i_1)} \otimes \cdots \otimes v^{j_r}_{\sigma_{j_r}(i_r)}).$$

Only the subscripts have changed and this action preserves ramified value-type as we are applying the same permutation to the subscripts where the superscripts agree. The base group acts trivially and $\vartheta_{\sigma,R,P,Q}$ is an $\mathfrak{S}_m \wr \mathfrak{S}_n$-homomorphism.

We have constructed $\dim \Delta_r(\varnothing^\beta)$ homomorphisms and the third step is to prove that

$$\{\vartheta_{\sigma,P,Q,R} \mid \mathsf{t}^\beta\sigma \in \mathrm{Std}(\beta), (P,Q) \in B, R \leqslant P \cup Q\} \tag{7.6}$$

is a linearly independent set. Assume, for a contradiction, that the set of equation (7.6) is linearly dependent and, in particular, that

$$\sum_{\sigma,P,Q,R} (\beta_{\sigma,P,Q,R})(\vartheta_{\sigma,P,Q,R}) = 0, \tag{7.7}$$

for some coefficients $\beta_{\sigma,P,Q,R} \in \mathbb{C}$ which are not all zero. This implies that

$$\sum_{\sigma,P,Q,R} \beta_{\sigma,P,Q,R} \sum \alpha^{j_1,\dots,j_r}_{\sigma,P,Q}(v^{j_1}_{i_1} \otimes \cdots \otimes v^{j_r}_{i_r}) = 0 \tag{7.8}$$

where the second summation is over the same indexing set as that of equation (7.5).

We now fix a ramified value-type (R, S) and restrict our attention to the tensor summand of equation (7.8) whose vectors have this fixed ramified value-type. We consider those P, Q with $P \cup Q = S$. Because the summation is over *all indices* of the form in equation (7.5), we are able to restrict further and only consider the tensor summand (for a given ramified value-type (R, S)) in the image of the projection onto the minimal R value-type tuple i^* (as defined in Definition 5.7). This is possible due to our assumption that $m \geqslant r - b + [b \geqslant 1]$: either $b = 0$ and we have $m \geqslant r$, so there are r distinct subscripts available, or $b \geqslant 1$ and there are b parts in the set-partition P, so the maximal number of parts of R in any part of $P \cup Q$ is $r - (b - 1)$ (obtained when $Q = \varnothing$, P has $b - 1$ singleton parts and one part of $r - (b - 1)$ vertices, and R is all singletons). It then follows from equation (7.8) that the following holds for some coefficients $\beta_{\sigma,P,Q,R}$ that are not all zero:

$$\sum_{\substack{\sigma,P,Q,R \\ P\cup Q = S,}} \beta_{\sigma,P,Q,R} \sum_{j_1,\dots,j_r \in \{1,\dots,n\}} \alpha^{j_1,\dots,j_r}_{\sigma,P,Q}(v^{j_1}_{i^*_1} \otimes \cdots \otimes v^{j_r}_{i^*_r}) = 0. \tag{7.9}$$

This implies that

$$\sum_{\substack{\sigma,P,Q \\ P\cup Q = S}} \beta_{\sigma,P,Q,R} \sum_{j_1,\dots,j_r \in \{1,\dots,n\}} \alpha^{j_1,\dots,j_r}_{\sigma,P,Q}(v^{j_1} \otimes \cdots \otimes v^{j_r}) = 0 \tag{7.10}$$

for the same coefficients $\beta_{\sigma,P,Q,R}$. Since the $\mathbb{C}\mathfrak{S}_n$-homomorphisms in equation (7.4) are linearly independent, this is a contradiction. Thus the set of $\dim \Delta_r(\varnothing^\beta)$ homomorphisms in equation (7.6) is linearly independent, and

$$\dim \operatorname{Hom}_{\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n}\big(\mathbf{S}_{(m)} \oslash \mathbf{S}_{\beta_{[n]}}, (\mathbb{C}^{mn})^{\otimes r}\big) \geqslant \dim \Delta_r(\varnothing^\beta).$$

Since

$$\operatorname{Hom}_{\mathbb{C}\mathfrak{S}_m \wr \mathfrak{S}_n}\big(\mathbf{S}_{(m)} \oslash \mathbf{S}_{\beta_{[n]}}, (\mathbb{C}^{mn})^{\otimes r}\big) \cong L_r(\varnothing^\beta),$$

and $L_r(\varnothing^\beta)$ is a quotient of $\Delta_r(\varnothing^\beta)$, we conclude that equation (7.6) specifies a basis of $\Delta_r(\varnothing^\beta)$ and $\Delta_r(\varnothing^\beta) \cong L_r(\varnothing^\beta)$. □

We show later in Corollary 11.5 that the bounds on m and n are tight. We now complete the proof of Theorem B.

Corollary 7.2. *Provided $n \geqslant r + \beta_1$ and $m \geqslant r - |\beta| + [\beta \neq \varnothing]$ the plethysm coefficient $p(\beta[n], (m), \kappa[mn])$ satisfies*

$$p(\beta[n], (m), \kappa[mn]) = \big[\Delta_r(\varnothing^\beta){\downarrow}^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\big]_{P_r(mn)}.$$

Proof. By Theorem 7.1 we have $\Delta_r(\varnothing^\beta) = L_r(\varnothing^\beta)$. The result now follows from the final displayed equation at the end of Section 6, which states in the case $\alpha = \varnothing$ that $p(\beta[n], \varnothing, \kappa[mn]) = \big[L_r(\varnothing^\beta){\downarrow}^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\big]_{P_r(mn)}$. □

In §9 we derive a combinatorial formula for the calculation of the right-hand side which shows that it is independent of m and n for sufficiently large r, and therefore Corollary 7.2 provides the new stability of plethysm coefficients stated in Theorem A.

8. Restricting our attention to a layer of fixed depth

In this section, we consider the restriction of the ramified partition algebra modules $\Delta_r(\alpha^\beta)$ to the partition algebra. We show the restriction to the partition algebra has a standard module filtration with well-defined filtration multiplicities. We prove that these multiplicities provide upper bounds for plethysm coefficients. In the case that the inner partition is trivial, we obtain new closed formulas for plethysm coefficients by way of Theorem 7.1.

Definition 8.1. Assume that $\delta_{\text{in}}, \delta_{\text{out}} \neq 0$. Given partitions α, β and κ, we define the ramified branching coefficient of the simple module $L_r(\kappa)$ for $P_r(\delta_{\text{in}}\delta_{\text{out}})$ in the standard module $\Delta_r(\alpha^\beta)$ for $R_r(\delta_{\text{in}}, \delta_{\text{out}})$ to be the filtration multiplicity

$$\big[\Delta_r(\alpha^\beta){\downarrow}^{R_r(\delta_{\text{in}},\delta_{\text{out}})}_{P_r(\delta_{\text{in}}\delta_{\text{out}})} : \Delta_r(\lambda)\big]_{P_r(\delta_{\text{in}}\delta_{\text{out}})}.$$

We shall see that the filtration multiplicities are well-defined in Corollary 8.8 regardless of the non-zero values of $\delta_{\text{in}}, \delta_{\text{out}}$. We have already seen that $R_r(\delta_{\text{in}}, \delta_{\text{out}})$ is semisimple for sufficiently large parameters in Theorem 5.4, and so the reader may prefer to consider only the semisimple case where these filtrations are direct sums.

8.1. **The action of the partition algebra by restriction.** The right module $\mathbf{S}^{\alpha^\beta}$ for the wreath product $\mathfrak{S}_a \wr \mathfrak{S}_b$ was defined in (4.5) to be $\mathbf{S}^\alpha \oslash \mathbf{S}^\beta$. We shall describe the action of the generators of the partition algebra $\mathrm{P}_r(\delta_{\text{in}}\delta_{\text{out}})$ on a basis of $\Delta_r(\alpha^\beta) = \mathbf{S}^{\alpha^\beta} \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} V_r(a^b)$ for $\alpha \vdash a$ and $\beta \vdash b$, with $ab \leqslant r$ or, in the case $a = 0$, with $b \leqslant r$. (If $b = 0$ then $(0^0) = \varnothing$ and $V_r(\varnothing)$ is generated by the idempotent $e_\varnothing$; recall that this is the diagram in which all r northern and southern vertices are outer singletons.)

Consider first the right $\mathrm{P}_r(\delta_{\text{in}}\delta_{\text{out}})$ action on the usual diagram basis of $V_r(a^b)$. Let $(\Lambda, \Lambda') \in V_r(a^b)$ be a ramified diagram with inner blocks $\Lambda = \{S_1, S_2, \dots, S_p\}$ and outer blocks $\Lambda' =$

$\{\Sigma_1, \Sigma_2, \ldots, \Sigma_q\}$ written using the convention of Remark 3.1 so that the blocks are ordered by increasing minima. We set

$$d_{(\Lambda,\Lambda')}\mathsf{p}_{1,2} = \begin{cases} d_{(\{S_1,S_2,S_3,\ldots S_p\},\{\Sigma_1,\Sigma_2,\ldots,\Sigma_q\})} & \text{if } 1,2 \in S_1 \subseteq \Sigma_1 \\ d_{(\{S_1\cup S_2,S_3,\ldots S_p\},\{\Sigma_1\cup\Sigma_2,\ldots,\Sigma_q\})} & \text{if } 1 \in S_1 \subseteq \Sigma_1, 2 \in S_2 \subseteq \Sigma_2 \\ d_{(\{S_1\cup S_2,S_3,\ldots S_p\},\{\Sigma_1,\Sigma_2,\ldots,\Sigma_q\})} & \text{if } 1 \in S_1 \subseteq \Sigma_1, 2 \in S_2 \subseteq \Sigma_1, \end{cases} \tag{8.1}$$

providing the resulting diagram belongs to $V_r(a^b)$ and we leave $d_{(\Lambda,\Lambda')}\mathsf{p}_{1,2}$ undefined otherwise. (The diagram does not belong to $V_r(a^b)$ if taking the product with $\mathsf{p}_{1,2}$ decreases the number of propagating outer-blocks or the number of propagating inner-blocks.) We also set

$$d_{(\Lambda,\Lambda')}\mathsf{p}_{1} = \begin{cases} \delta_{\text{in}}\delta_{\text{out}}d_{(\Lambda,\Lambda')} & \text{if } \{1\} = S_1 = \Sigma_1 \\ d_{(\{\{1\},S_1-\{1\},S_2,S_3,\ldots S_p\},\{\{1\},\Sigma_1-\{1\},\Sigma_2,\ldots,\Sigma_q\})} & \text{if } \{1\} \subset S_1 \subseteq \Sigma_1 \\ \delta_{\text{in}}d_{(\{\{1\},S_2,S_3,\ldots S_p\},\{\{1\},\Sigma_1-\{1\},\Sigma_2,\ldots,\Sigma_q\})} & \text{if } \{1\} = S_1 \subset \Sigma_1, \end{cases} \tag{8.2}$$

providing the resulting diagram belongs to $V_r(a^b)$ and we leave $d_{(\Lambda,\Lambda')}\mathsf{p}_1$ undefined otherwise. It is worth noting that the elements on the right of equation (8.2) are not necessarily written in the form specified by Remark 3.1. For $\mathsf{p} = \mathsf{p}_1$ or $\mathsf{p} = \mathsf{p}_{1,2}$ and $x \in \mathbf{S}^{\alpha^\beta}$, we observe that

$$(x \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} d_{(\Lambda,\Lambda')})\mathsf{p} = \begin{cases} x \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} (d_{(\Lambda,\Lambda')}\mathsf{p}) & \text{if } d_{(\Lambda,\Lambda')}\mathsf{p} \in V_r(a^b) \text{ is defined} \\ 0 & \text{otherwise.} \end{cases}$$

The generators $\mathsf{s}_{i,i+1}$ for $1 \leqslant i < r$ act in the usual fashion by permuting $\{1, 2, \ldots, r\}$. For ease of notation, we do not write these actions out explicitly.

8.2. **The depth quotient.** In this section we identify a quotient of the standard module $\Delta_r(\alpha^\beta)$ that contains all simple modules $L_r(\kappa)$ in which the partition κ has the maximum possible size r.

Definition 8.2. Let $r \in \mathbb{N}$ and $(a^b) \in \Theta_r$. We define the depth-radical of $V_r(a^b)$ to be the subspace $W_r(a^b) \subseteq V_r(a^b)$ spanned by the ramified diagrams $d_{(\Lambda,\Lambda')}$ satisfying either of the following two conditions:

(i) the set-partition Λ contains two southern vertices in the same block;
(ii) the set-partition Λ' contains a singleton southern block.

We define the depth-radical of $\Delta_r(\alpha^\beta)$ to be the subspace

$$\mathsf{DR}\big(\Delta_r(\alpha^\beta)\big) = \mathrm{S}(\alpha^\beta) \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} W_r(a^b) \subseteq \mathrm{S}(\alpha^\beta) \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} V_r(a^b) = \Delta_r(\alpha^\beta).$$

This construction will allow us to study the smallest possible modules in which we can see the ramified branching coefficients.

Proposition 8.3. *Given $r \in \mathbb{N}$, the depth radical $\mathsf{DR}\big(\Delta_r(\alpha^\beta)\big)$ is a $P_r(\delta_{\text{in}}\delta_{\text{out}})$-submodule of $\Delta_r(\alpha^\beta)\downarrow^{R_r(\delta_{\text{in}}}_{P_r(\delta_{\text{in}}\delta_{\text{out}})}$.*

Proof. It is clear that the generators $s_{i,i+1}$ for $1 \leqslant i < r$ preserve the space $\mathsf{DR}\big(\Delta_r(\alpha^\beta)\big)$ as both conditions of Definition 8.2 are invariant under the permutation action. By equation (8.2), the generator p_1 acts on a given diagram $d_{(\Lambda,\Lambda')}$ either by scalar multiplication, or by removing an edge from Λ *at the expense* of introducing a singleton into Λ'. Therefore the generator p_1 preserves $\mathsf{DR}\big(\Delta_r(\alpha^\beta)\big)$ by Definition 8.2(ii). By equation (8.1) the generator $\mathsf{p}_{1,2}$ acts on a given diagram $d_{(\Lambda,\Lambda')}$ either trivially or by introducing an edge in Λ. Therefore the generator $\mathsf{p}_{1,2}$ preserves $\mathsf{DR}\big(\Delta_r(\alpha^\beta)\big)$ by Definition 8.2(i). $\square$

Definition 8.4. Define the depth quotient $\mathsf{DQ}(\Delta_r(\alpha^\beta))$ to be the quotient $P_r(\delta_{\rm in}\delta_{\rm out})$-module

$$\Delta_r(\alpha^\beta) = \mathbf{S}^{\alpha^\beta} \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} V_r(a^b) \twoheadrightarrow \mathbf{S}^{\alpha^\beta} \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} V_r^0(a^b) = \mathsf{DQ}\big(\Delta_r(\alpha^\beta)\big),$$

where $V_r^0(a^b)$ is the quotient vector space $V_r(a^b)/W_r(a^b)$, that is the span of those ramified diagrams in $V_r^0(a^b)$ that do not lie in $W_r(a^b)$.

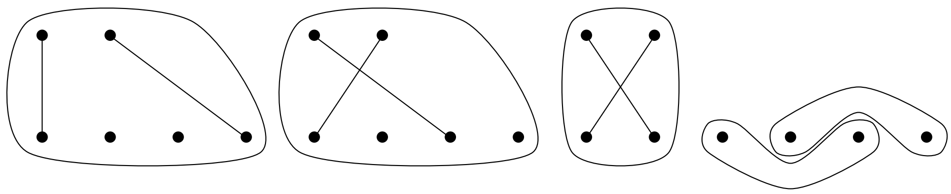


FIGURE 16. An element of an element of $V_{14}^0(2^3)$. There are no inner southern arcs and there are no outer southern singletons.

Example 8.5. An example element of $V_{14}^0(2^3)$ is depicted in Figure 16.

The next proposition is an elementary application of idempotent truncation (see for example [Gre07, Section 6.2]) which will allow us to decompose the restriction of $\Delta_r(\alpha^\beta)$ to the partition algebra. Recall that for $\delta_{\rm in}\delta_{\rm out} \neq 0$ we have defined the idempotent $e_{r-1} = \frac{1}{\delta_{\rm in}\delta_{\rm out}}\mathsf{p}_r \in P_r(\delta_{\rm in}\delta_{\rm out}) \subseteq R_r(\delta_{\rm in}, \delta_{\rm out})$, and, that in (3.3) and (3.4), we saw that $e_{r-1}P_r(\delta_{\rm in}\delta_{\rm out})e_{r-1} \cong P_{r-1}(\delta_{\rm in}\delta_{\rm out})$ and $P_r(\delta_{\rm in}\delta_{\rm out})/P_r(\delta_{\rm in}\delta_{\rm out})e_{r-1}P_r(\delta_{\rm in}\delta_{\rm out}) \cong \mathbb{C}\mathfrak{S}_r$.

Proposition 8.6. *For $r \geqslant 2$,*

$$\mathsf{DR}\big(\Delta_r(\alpha^\beta)\big)e_{r-1}P_r(\delta_{\rm in}\delta_{\rm out}) = \mathsf{DR}\big(\Delta_r(\alpha^\beta)\big), \qquad \mathsf{DQ}\big(\Delta_r(\alpha^\beta)\big)e_{r-1} = 0,$$

and moreover

$$\mathsf{DR}\big(\Delta_r(\alpha^\beta)\big)e_{r-1} \cong \Delta_{r-1}(\alpha^\beta)$$

as an $e_{r-1}P_r(\delta_{\rm in}\delta_{\rm out})e_{r-1} \cong P_{r-1}(\delta_{\rm in}\delta_{\rm out})$-module if the right-hand side is defined, and otherwise $\mathsf{DR}(\Delta_r(\alpha^\beta))e_{r-1} = 0$. When $r = 1$,

$$\mathsf{DR}\big(\Delta_1((1)^{(1)})\big) = 0 \qquad \mathsf{DR}\big(\Delta_1(\varnothing^{(1)})\big) = 0 \qquad \mathsf{DR}\big(\Delta_1(\varnothing^{\varnothing})\big) = \Delta_1(\varnothing^{\varnothing})$$

Proof. We consider the first statement. We let $d_{(\Lambda,\Lambda')}$ be a ramified diagram basis element of $W_r(a^b)$ and $x \in \mathrm{S}(\alpha^\beta)$. We shall write $x \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} d_{(\Lambda,\Lambda')}$ in the form

$$x \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} d_{(\Lambda,\Lambda')} = x \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} d_{(\bar\Lambda,\bar\Lambda')}e_{r-1}d$$

for some ramified diagram $d_{(\bar\Lambda,\bar\Lambda')} \in W_r(a^b)$ and some partition diagram $d \in P_r(\delta_{\rm in}\delta_{\rm out})$ and hence deduce the result. First, suppose that Λ' contains a singleton block $\{i\}$ for $1 \leqslant i \leqslant r$. In this case we set

$$d_{(\bar\Lambda,\bar\Lambda')} = d_{(\Lambda,\Lambda')}s_{i,r},$$

where $s_{i,r} = s_{i,i+1}\cdots s_{r-1,r-2}s_{r-1,r}s_{r-1,r-2}\cdots s_{i,i+1}$. We set

$$d_{(\Lambda,\Lambda')} = d_{(\bar\Lambda,\bar\Lambda')}e_{r-1}s_{i,r}$$

as required. Now suppose that Λ' contains a block J with two southern vertices $j_1 < j_2$. We suppose that j_2 is maximal with respect to this property. In this case we set

$$d_{(\bar\Lambda,\bar\Lambda')} = d_{(\Lambda,\Lambda')}s_{j_2,r}s_{j_1,r-1}.$$

We easily observe that

$$d_{(\Lambda,\Lambda')} = d_{(\bar\Lambda,\bar\Lambda')}e_{r-1,r}(\mathsf{p}_{r-1,r}s_{j_1,r-1}s_{j_2,r}).$$

We further observe that

$$x \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} d_{(\Lambda,\Lambda')} = (x \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} d_{(\bar{\Lambda},\bar{\Lambda'})})e_{r-1}d$$

because of our maximality assumption on j_2; as there is no inner block to the right of J containing a pair of southern vertices, thus $s_{j_1,r-1}s_{j_2,r}$ does not swap the order of blocks which can be permuted by the left action of $\mathfrak{S}_a \wr \mathfrak{S}_b$. The first statement follows.

We now consider the second and third statements. Again, consider $x \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} d_{(\Lambda,\Lambda')}$ a basis element of $\Delta_r(\alpha^\beta)$ and consider $d_{(\Lambda,\Lambda')}e_{r-1}$ using equation (8.2) and conjugation. In all three cases the resulting outer partition contains a singleton block and therefore $d_{(\Lambda,\Lambda')}e_{r-1} \in \mathsf{DR}(\Delta_r(\alpha^\beta))$. Therefore the second statement holds. Finally, we see that all possible $(\Pi, \Pi') \in V_r(a^b)$ with a singleton part $\{r\}$ in both Π and Π' can occur as $d_{(\Lambda,\Lambda')}e_{r-1}$, thus the third statement holds. For $r = 1$ modules are all 1-dimensional and the statement is easily verified. $\square$

The calculations of Proposition 8.6, together with equation (3.3) and equation (3.4) provide the following corollary.

Corollary 8.7. *There is a short exact sequence of $P_r(\delta_{\text{in}}\delta_{\text{out}})$-modules*

$$0 \to \mathsf{DR}\big(\Delta_r(\alpha^\beta)\big) \to \Delta_r(\alpha^\beta)\big\downarrow_{P_r(\delta_{\text{in}}\delta_{\text{out}})} \to \mathsf{DQ}\big(\Delta_r(\alpha^\beta)\big) \to 0,$$

where

$$\mathsf{DR}\big(\Delta_r(\alpha^\beta)\big) \cong \Delta_{r-1}(\alpha^\beta) \otimes e_{r-1}P_r(\delta_{\text{in}}\delta_{\text{out}}) \tag{8.3}$$

and $\mathsf{DQ}\big(\Delta_r(\alpha^\beta)\big)$ decomposes as a direct sum of inflated simple $\mathbb{C}\mathfrak{S}_r$-modules.

Corollary 8.8. *For arbitrary parameters $\delta_{\text{in}}, \delta_{\text{out}} \neq 0$, the restriction $\Delta_r(\alpha^\beta)\downarrow^{R_r(\delta_{\text{in}},\delta_{\text{out}})}_{P_r(\delta_{\text{in}}\delta_{\text{out}})}$ has a standard filtration with the following equality of filtration multiplicities:*

$$\begin{aligned}&\left[\Delta_r(\alpha^\beta)\downarrow^{R_r(\delta_{\text{in}},\delta_{\text{out}})}_{P_r(\delta_{\text{in}}\delta_{\text{out}})} : \Delta_r(\lambda)\right]_{P_r(\delta_{\text{in}}\delta_{\text{out}})} \\ &\qquad = \begin{cases} \left[\mathsf{DQ}(\Delta_r(\alpha^\beta)) : \Delta_r(\lambda)\right]_{P_r(\delta_{\text{in}}\delta_{\text{out}})} & \text{if } |\lambda| = r, \\ \left[\Delta_{r-1}(\alpha^\beta)\downarrow^{R_r(\delta_{\text{in}}\delta_{\text{out}})}_{P_r(\delta_{\text{in}}\delta_{\text{out}})} : \Delta_{r-1}(\lambda)\right]_{P_{r-1}(\delta_{\text{in}}\delta_{\text{out}})} & \text{if } |\lambda| < r. \end{cases}\end{aligned}$$

We remark that, since $P_r(\delta_{\text{in}}\delta_{\text{out}})$ is quasi-hereditary [Mar96] when $\delta_{\text{in}}, \delta_{\text{out}} \neq 0$, standard filtration multiplicities are well-defined (see, for example [Don98, Appendix, A1(8)]). The corollary describes these well-defined multiplicities.

Proof. The existence of a standard module filtration is proved by induction. The base case $r = 1$ is clear from Proposition 8.6: the three (1-dimensional) standard modules for the ramified partition algebra restrict to standard modules for the partition algebra. For $r > 1$, we use the short exact sequence above. The quotient is a direct sum of simple $\mathbb{C}\mathfrak{S}_r$-modules, which are $P_r(\delta_{\text{in}}\delta_{\text{out}})$-standard modules by inflation. If the submodule $\mathsf{DR}(\Delta_r(\alpha^\beta))$ is non-zero then a standard filtration of $\mathsf{DR}(\Delta_r(\alpha^\beta)) \cong \Delta_{r-1}(\alpha^\beta) \otimes e_{r-1}P_r(\delta_{\text{in}}\delta_{\text{out}})$ is obtained from the $P_{r-1}(\delta_{\text{in}}\delta_{\text{out}})$-standard filtration of $\Delta_{r-1}(\alpha^\beta)\downarrow^{R_{r-1}(\delta_{\text{in}},\delta_{\text{out}})}_{P_{r-1}(\delta_{\text{in}}\delta_{\text{out}})}$ by globalisation using the isomorphism $\Delta_{r-1}(\kappa) \otimes e_{r-1}P_r(\delta_{\text{in}}\delta_{\text{out}}) \cong \Delta_r(\kappa)$. $\square$

9. General formula for ramified branching coefficients

We now consider the decomposition of the module

$$\mathsf{DQ}\big(\Delta_r(\alpha^\beta)\big) = \mathbf{S}^{\alpha^\beta} \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} V_r^0(a^b).$$

Using the canonical quotient map $P_r(mn) \to \mathbb{C}\mathfrak{S}_r$, we regard $V_r^0(a^b)$ as a $(\mathbb{C}\mathfrak{S}_a \wr \mathfrak{S}_b, \mathbb{C}\mathfrak{S}_r)$-bimodule. Thus, for the remainder of this paper, we need not consider the partition algebra structure, we can simply discuss bimodules for wreath products of symmetric groups.

9.1. **Types of diagrams.** Fix a, b, r with $ab \leqslant r$ or $b \leqslant r$ if $a = 0$. We wish to understand the left action of $\mathfrak{S}_a \wr \mathfrak{S}_b$ and the right action $\mathfrak{S}_r$ on $V_r^0(a^b)$. Recall that $V_r^0(a^b) \subset V_r(a^b)$ has a basis given by the ramified diagrams $d_{(\Lambda,\Lambda')}$ of propagating index (a^b), as defined in Section 4.3, such that

- the outer set-partition Λ has no singleton blocks;
- the inner set-partition Λ' consists of propagating pairs (that is, pairs $\{i, \overline{j}\}$ for i a northern vertex and $\overline{j}$ a southern vertex) and southern singletons.

Examples are depicted in Figures 16 to 22. The purpose of this section is to define the propagating type and the non-propagating type of such a ramified diagram $d_{(\Lambda,\Lambda')} \in V_r^0(a^b)$ in such a way as to decompose this module and to determine a direct sum decomposition of $\mathsf{DQ}(\Delta_r(\alpha^\beta))$ as a $\mathbb{C}\mathfrak{S}_r$-module.

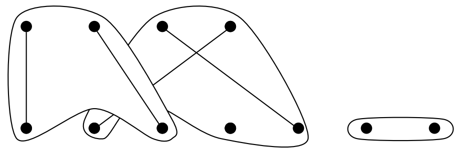


FIGURE 17. A ramified diagram $d_{(\Lambda,\Lambda')} \in V_7^0(2^2)$. Here the outer set-partition is $\Lambda = \{\{1, 2, \overline{1}, \overline{3}\}, \{3, 4, \overline{2}, \overline{4}, \overline{5}\}, \{\overline{6}, \overline{7}\}\}$ and the inner set-partition is $\Lambda' = \{\{1, \overline{1}\}, \{2, \overline{3}\}, \{3, \overline{5}\}, \{4, \overline{2}\}, \{\overline{4}\}, \{\overline{6}\}, \{\overline{7}\}\}$. Compare with Figure 18.

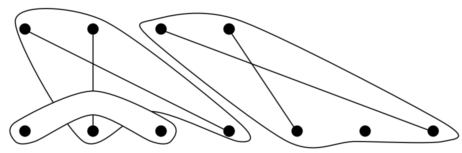


FIGURE 18. Another ramified diagram in $V_7^0(2^2)$. This diagram can be obtained from that of Figure 17 by acting on the right by the permutation $(1, 4, 6)(2, 5, 7, 3) \in \mathfrak{S}_7$.

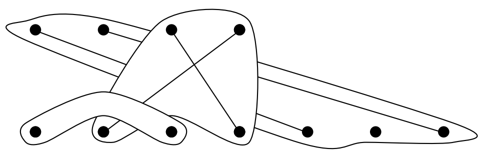


FIGURE 19. Another diagram in $V_7^0(2^2)$. This ramified diagram can be obtained from that of Figure 18 by acting on the left by $((1, 2), 1_{\mathfrak{S}_2}; (1, 2)) \in \mathfrak{S}_2 \wr \mathfrak{S}_2$. Observe that it cannot be obtained from the ramified diagram of Figure 18 via a right action.

Our aim is to decompose $V_r^0(a^b)$ and so we shall first focus on the properties of the ramified diagram basis elements $d_{(\Lambda,\Lambda')}$ which are preserved by the left and right actions. Roughly speaking, this means we shall need to describe the number of singletons and their positioning within the blocks of (Λ, Λ'). For example, the diagrams in Figures 17 to 19 are all obtained from one another by left action by $\mathfrak{S}_a \wr \mathfrak{S}_b$ and/or right action by $\mathfrak{S}_r$. On the other hand, those in Figures 17, 20 and 21 *cannot* be obtained from one another by these actions. In order to discuss this in more detail, we first note that within the ramified diagram basis element $d_{(\Lambda,\Lambda')} \in V_r^0(a^b)$ we have that:

- there are precisely b propagating outer-blocks $P_1, \dots, P_b$. The refinement of each block, $\Lambda \cap P_i$ for $1 \leqslant i \leqslant b$, consists of precisely a propagating pairs and some number, γ_i say, of southern singletons;
- there are some non-propagating southern outer-blocks, say $Q_1, \dots, Q_l$ for some $l \geqslant 0$. The refinement of each block, $\Lambda \cap Q_i$ for $1 \leqslant i \leqslant \ell$, consists of some number, $\varepsilon_i \geqslant 2$ say, of singleton southern blocks.

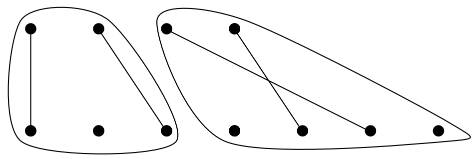


FIGURE 20. A diagram in $V_7^0(2^2)$. This diagram cannot be obtained from any of Figures 17 to 19 or Figure 21 via the $\mathfrak{S}_2 \wr \mathfrak{S}_2$-left-action or $\mathfrak{S}_7$-right-action (although it can be obtained from any of these diagrams by right multiplication by a ramified partition diagram).

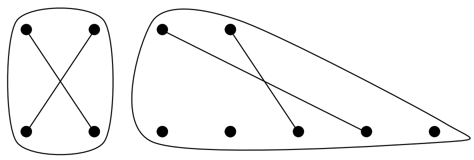


FIGURE 21. A diagram in $V_7^0(2^2)$. This diagram cannot be obtained from any of Figures 17 to 20 via the $\mathfrak{S}_2 \wr \mathfrak{S}_2$-left-action or $\mathfrak{S}_7$-right-action.

Example 9.1. The diagrams in Figures 17 to 19 each have $a = b = 2$ and $r = 7$. Each has two outer propagating blocks, P_1 and P_2, one of which contains a singleton and the other contains no singletons. Each has a unique non-propagating outer block, Q_1, containing 2 singletons.

Example 9.2. Consider the ramified diagram in Figure 16. Here $a = 2$, $b = 3$ and $r = 14$. We call the propagating outer-blocks P_1, P_2, P_3 reading from left to right, and observe that P_1 and P_2 each contain 2 inner singleton southern blocks and P_3 contains no inner singleton southern blocks (in addition to the $a = 2$ inner propagating pairs). There are also two non-propagating southern outer-blocks, Q_1, Q_2, each of which contains precisely two singleton southern blocks.

Before continuing further, we now introduce some notation for recording the inner-singleton blocks. We let p denote the total number of southern singletons belonging to an outer *propagating* block and q denote the total number of southern singletons belonging to an outer *non-propagating* block. We note that every southern vertex belongs to either a propagating pair inner-block (of which there are ab in total) or a singleton inner-block and therefore $p+q = r - ab$.

Example 9.3. The ramified diagrams in Figures 17 to 19 each have $p = 1$ and $q = 2$.

We next consider how these singleton vertices are partitioned into propagating and non-propagating blocks. We remind the reader that the non-propagating outer-blocks each contain *at least 2* inner-singleton vertices. Recall that $\mathscr{P}_{>1}(q)$ denotes the set of all integer partitions of q whose parts are all strictly greater than 1.

Definition 9.4. We suppose that $d_{(\Lambda,\Lambda')}$ has non-propagating outer blocks $Q_1,\dots,Q_\ell$ such that, for $1 \leqslant i \leqslant \ell$, $\Lambda \cap Q_i$ consists of some number $\varepsilon_i \geqslant 2$ of singleton southern blocks. We define the non-propagating type of $d_{(\Lambda,\Lambda')}$ to be the partition $\varepsilon = (\varepsilon_1,\varepsilon_2,\varepsilon_3,\dots,\varepsilon_\ell) \in \mathscr{P}_{>1}(q)$ if $\ell \neq 0$ and to be $\varnothing$ if $\ell = 0$.

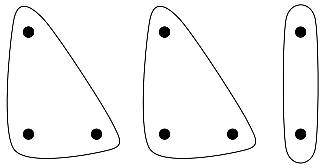


FIGURE 22. A diagram in $V_5^0(0^3)$ with propagating type $(2,2,1)$ and non-propagating type $\varnothing$.

We remind the reader that the propagating outer-blocks must each contain at least one southern vertex. When $a = 0$ this implies that every propagating outer-block must contain at least one southern inner-singleton (see for example, Figure 22). When $a > 0$ a propagating outer-block is allowed to contain zero southern inner-singletons (as already seen in Figures 17 to 19). In what follows, we use γ to denote the numbers of southern inner-singletons in the propagating outer-blocks, ordering these numbers so that they are weakly decreasing. Thus $\gamma \in \mathscr{P}_{(a^b)}(p)$ where $\mathscr{P}_{(a^b)}$ is specified as follows.

Definition 9.5. Let $b \in \mathbb{N}_0$.

(i) We define $\mathscr{P}_{(0^b)}(p)$ to be the set of all weakly decreasing sequences of positive integers γ of length b which sum to p.
(ii) For $a \in \mathbb{N}$, we define $\mathscr{P}_{(a^b)}(p)$ to be the set of all weakly decreasing sequences of non-negative integers γ of length b which sum to $p - ab$.

Definition 9.6. We suppose that $d_{(\Lambda,\Lambda')}$ has propagating outer-blocks $P_1,\dots,P_b$ such that, for $1 \leqslant i \leqslant b$, $\Lambda \cap P_i$ consists of precisely a propagating pairs and some number, γ_i say, of southern singletons. We define the propagating type of $d_{(\Lambda,\Lambda')}$ to be the sequence $\gamma = (p^{c_p},\dots,1^{c_1},0^{c_0}) \in \mathscr{P}_{(a^b)}(p)$.

Note that by the remarks before Definition 9.5, $\gamma \in \mathscr{P}_{(a^b)}(p)$. In particular, when $a = 0$, since each propagating outer-block must contain at least one southern inner-singleton, the sequence γ has no zero terms, and the zero multiplicity c_0 is 0. In case (ii) when $a > 0$, elements of $\mathscr{P}_{(a^b)}(\gamma)$ may have zero parts: these are counted by c_0 in the sums in Theorem D.

Definition 9.7. We say that a ramified diagram $d_{(\Lambda,\Lambda')}$ as in Definitions 9.4 and 9.6 has type (γ,ε).

We shall see that the $(\mathbb{C}\mathfrak{S}_a \wr \mathfrak{S}_b, \mathbb{C}\mathfrak{S}_r)$-bimodule $V_r^0(a^b)$ decomposes as a direct sum in which each summand is spanned by the diagrams of a fixed type.

Example 9.8. The diagrams in Figures 17 to 19 all have propagating-type $(1,0)$ and non-propagating-type (2).

Example 9.9. The diagrams in Figures 20 and 21 have propagating-type $(2,1)$ and $(3,0)$ respectively. Both diagrams in Figures 20 and 21 have non-propagating-type $\varnothing$.

9.2. **Elementary diagrams.** Let $x, y \in \mathbb{Z}_{\geqslant 0}$. For each pair $(0,0) \neq (x,y) \in \mathbb{Z}^2_{\geqslant 0}$ we define a ramified $(\max\{1,x\}, x+y)$-set-partition diagram $v_{x,y}$ by setting the first x inner blocks to be of the form $\{k, \overline{k}\}$ for $1 \leqslant k \leqslant x$ and the remaining inner blocks to be singletons; there is a single outer block that is the union of all $\max\{1,x\}+x+y$ vertices. Examples are depicted in Figure 23. We set $(\varnothing, y)$ to be $(0,y)$-set-partition consisting of precisely one outer block, and whose inner blocks are all singletons (see Figure 24 for an example).

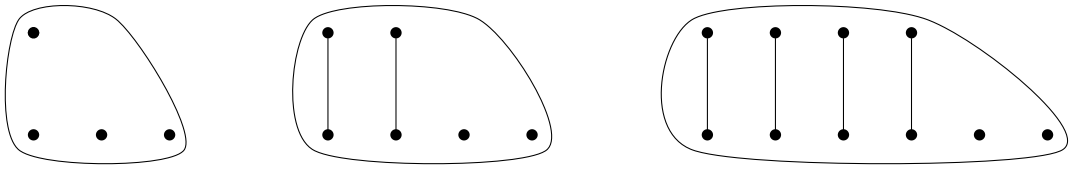


FIGURE 23. Diagrams $v_{x,y} = v_{0,3}$, $v_{2,2}$, and $v_{4,2}$ respectively. Note that x is the number of propagating strands and y is the number of southern inner-singletons.

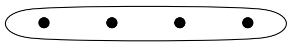


FIGURE 24. The diagram $v_{(\varnothing,4)}$. Here $\varnothing$ records that this a diagram consisting purely of southern vertices.

Definition 9.10. For $\gamma = (p^{c_p}, \dots, 1^{c_1}, 0^{c_0}) \in \mathscr{P}_{(a^b)}(p)$ and $\varepsilon \in \mathscr{P}_{>1}(q)$, the diagram $v_{\gamma,\varepsilon}$, is defined by horizontal concatenation as follows:

$$v_{\gamma,\varepsilon} = \big((v_{a,p})^{\circledast c_p} \circledast \cdots \circledast (v_{a,1})^{\circledast c_1} \circledast (v_{a,0})^{\circledast c_0}\big) \circledast (v_{\varnothing,\varepsilon_1} \circledast v_{\varnothing,\varepsilon_2} \circledast \cdots \circledast v_{\varnothing,\varepsilon_\ell}).$$

Let $V^0_r(a^b : \gamma, \varepsilon) \subseteq V^0_r(a^b)$ denote the $(\mathbb{C}\mathfrak{S}_a \wr \mathfrak{S}_b, \mathbb{C}\mathfrak{S}_r)$-bimodule generated by $v_{\gamma,\varepsilon} \in V^0_r(a^b)$.

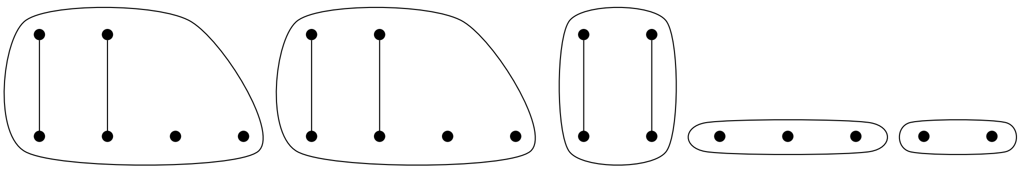


FIGURE 25. The diagram $v_{\gamma,\varepsilon}$ for $\gamma = (2^2, 0)$ and $\varepsilon = (3,2)$ with $a = 2$ and $b = 3$, obtained via horizontal concatenation as $(v_{(2,2)})^{\circledast 2} \circledast v_{(2,0)} \circledast v_{(\varnothing,3)} \circledast v_{(\varnothing,2)}$. This diagram is discussed in Example 9.12.

Since the propagating type and non-propagating type of a ramified diagram are invariant under both the left-$\mathfrak{S}_a \wr \mathfrak{S}_b$ and the right-$\mathfrak{S}_r$ actions, the following proposition follows.

Proposition 9.11. *There is a direct sum decomposition of the $(\mathfrak{S}_a \wr \mathfrak{S}_b, \mathfrak{S}_r)$-module $V^0_r(a^b)$ as follows:*

$$V^0_r(a^b) = \bigoplus_{\substack{p+q=r-ab \\ \gamma \in \mathscr{P}_{(a^b)}(p) \\ \varepsilon \in \mathscr{P}_{>1}(q)}} V^0_r\big((a^b) : \gamma, \varepsilon\big).$$

Example 9.12. Suppose that $r = 15$, $a = 2$, $b = 3$ and $r - ab = 9$. We let $p = 4$ and $q = 5$. There are 4 possible choices of $\gamma \in \mathscr{P}_{(2^3)}(4)$, namely $(4, 0^2)$, $(3,1,0)$, $(2^2, 0)$ and $(2, 1^2)$. There are two choices of $\varepsilon \in \mathscr{P}_{>1}(5)$, namely (5) and $(3,2)$. The diagram $v_{\gamma,\varepsilon}$ for $\gamma = (2^2, 0)$ and $\varepsilon = (3,2)$ is depicted in Figure 25.

9.3. **The direct sum decomposition of the depth quotient.** We are now ready to provide the complete decomposition of $\mathsf{DQ}(\Delta_r(\alpha^\beta)) = \mathbf{S}^{\alpha^\beta} \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} V_r^0(a^b)$ into irreducible summands. Examples 10.1 and 10.2 and the discussion before Definition 10.3 illustrate the key ideas in the proof. In light of Proposition 9.11, we focus our attention on a fixed summand $V_r^0(a^b : \gamma, \varepsilon) \subseteq V_r^0(a^b)$. We shall consider two extremal cases first.

Lemma 9.13. *For any $\varepsilon \in \mathscr{P}_{>1}(q)$, there is an isomorphism of right $\mathfrak{S}_q$-modules*

$$V_q^0\big(\varnothing : \varnothing, \varepsilon\big) \cong \mathbb{C}\big\uparrow_{\mathrm{Stab}(\varepsilon)}^{\mathfrak{S}_q}.$$

Proof. The diagrammatic module is a transitive permutation $\mathfrak{S}_q$-module with the required stabiliser, and so the result follows. □

We remark that this lemma is also proved in [BP17, Theorem 7.11].

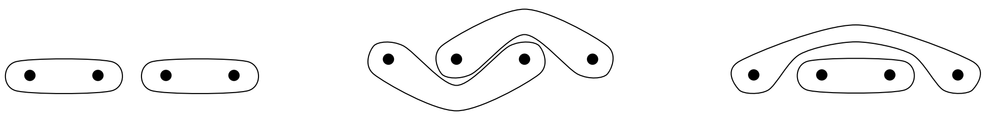


FIGURE 26. The diagrammatic basis of $V_4^0(\varnothing, (2^2))$. This right $\mathfrak{S}_4$-module is the transitive permutation module isomorphic to $\mathbb{C}\!\uparrow_{\mathfrak{S}_2 \wr \mathfrak{S}_2}^{\mathfrak{S}_4}$. Observe that the stabiliser of the first diagram shown is the usual copy of $\mathfrak{S}_2 \wr \mathfrak{S}_2 \leqslant \mathfrak{S}_4$.

Example 9.14. Let $q = 4$ and $\varepsilon = (2^2)$. The module $V_4^0(\varnothing, \varepsilon)$ is 3-dimensional with basis as depicted in Figure 26. This module is isomorphic to $\mathbb{C}\!\uparrow_{\mathfrak{S}_2 \wr \mathfrak{S}_2}^{\mathfrak{S}_4}$.

We next consider another extreme case, where $q = 0$ and $\gamma = (s^b)$ so $r = (a+s)b$.

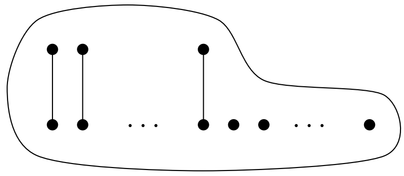
Lemma 9.15. *Let $r = (a+s)b$. There is an isomorphism of right $\mathfrak{S}_r$-modules*

$$\big(\mathbf{S}^\alpha \oslash \mathbf{S}^\beta\big) \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} V_r^0\big((a^b) : (s^b), \varnothing\big) \cong \big(\mathbf{S}^\alpha \otimes \mathbb{C}_{\mathfrak{S}_s}\big\uparrow_{\mathfrak{S}_a \times \mathfrak{S}_s}^{\mathfrak{S}_{a+s}} \oslash \mathbf{S}^\beta\big)\big\uparrow_{\mathfrak{S}_{a+s} \wr \mathfrak{S}_b}^{\mathfrak{S}_{(a+s)b}}$$

Proof. By [Alp86, page 56, Corollary 3], it suffices to find a right $\mathbb{C}\mathfrak{S}_{a+s} \wr \mathfrak{S}_b$-submodule X of $\big(\mathbf{S}^\alpha \oslash \mathbf{S}^\beta\big) \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} V_r^0\big((a^b) : (s^b), \varnothing\big)$ that is isomorphic to $\big(\mathbf{S}^\alpha \otimes \mathbb{C}_{\mathfrak{S}_s}\big)\big\uparrow_{\mathfrak{S}_a \times \mathfrak{S}_s}^{\mathfrak{S}_{a+s}} \oslash \mathbf{S}^\beta$ and such that

$$\dim\,(\mathbf{S}^\alpha \oslash \mathbf{S}^\beta) \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_b} V_r^0\big((a^b) : (s^b), \varnothing\big) = |\mathfrak{S}_{(a+s)b} : \mathfrak{S}_{a+s} \wr \mathfrak{S}_b| \dim X.$$

Let $k = [\mathfrak{S}_{a+s} : \mathfrak{S}_a \times \mathfrak{S}_s]$ and let $\vartheta_1, \ldots, \vartheta_k$ be right-coset representatives for $\mathfrak{S}_a \times \mathfrak{S}_s$ in $\mathfrak{S}_{a+s}$; thus $\mathfrak{S}_{a+s} = \bigsqcup_{j=1}^k (\mathfrak{S}_a \times \mathfrak{S}_s)\vartheta_j$. Recall that $v_{(a,s)}$ denotes the diagram



where there are a northern and $a+s$ southern vertices. As a vector space, we define X by

$$X = \big\langle (x \otimes y) \otimes v_{a,s}\vartheta_{i_1} \circledast \cdots \circledast v_{a,s}\vartheta_{i_b} \mid i_1, \ldots, i_b \in \{1, \ldots, k\}, x \in (\mathbf{S}^\alpha)^{\otimes b}, y \in \mathbf{S}^\beta \big\rangle$$

Note that the ramified $\big(ab, (a+s)b\big)$ diagrams appearing in the final tensor factor in each chosen basis element of X have outer blocks

$$\big\{1, \ldots, a, \overline{1}, \ldots, \overline{a+s}\big\}, \big\{a+1, \ldots, 2a, \overline{(a+s)+1}, \ldots, \overline{2(a+s)}\big\}, \ldots$$

Thus there are no outer crossings. Observe that any $\pi \in \mathfrak{S}_a \leqslant \mathfrak{S}_{a+s}$ satisfies $v_{(a,s)}\pi = \pi v_{(a,s)}$; this is the $(a, a+s)$-diagram with the permutation π on the first a strings, followed by s isolated souther vertices.

We now give X the structure of a right $\mathbb{C}\mathfrak{S}_{a+s} \wr \mathfrak{S}_b$-module. For the action of the base group we must first understand how $\mathfrak{S}_{a+s}$ acts on each $v_{(a,s)}\vartheta_{i_c}$. Let $\sigma \in \mathfrak{S}_{a+s}$ and suppose that $\vartheta_{i_c}\sigma \in (\mathfrak{S}_a \times \mathfrak{S}_s)\vartheta_j$, so that $\vartheta_{i_c}\sigma = \pi_1\pi_2\vartheta_j$ where $\pi_1 \in \mathfrak{S}_a$ and $\pi_2 \in \mathfrak{S}_s$. We then have

$$v_{(a,s)}\vartheta_{i_c} \cdot \sigma = v_{(a,s)} \cdot \pi_1\pi_2\vartheta_j = \pi_1 v_{(a,s)}\vartheta_j \tag{9.1}$$

since π_2 acts trivially on the isolated southern dots. We now define the action of an arbitrary element $(\sigma_1, \ldots, \sigma_b)$ of the base group $\mathfrak{S}_{a+s} \times \cdots \times \mathfrak{S}_{a+s}$ of $\mathfrak{S}_{a+s} \wr \mathfrak{S}_b$ in the obvious way by

$$(x \otimes y) \otimes v_{(a,s)}\vartheta_{i_1} \circledast \cdots \circledast v_{(a,s)}\vartheta_{i_s} \cdot (\sigma_1, \ldots, \sigma_b) = (x \otimes y) \otimes (v_{(a,s)}\vartheta_{i_1} \cdot \sigma_1) \circledast \cdots \circledast (v_{(a,s)}\vartheta_{i_s} \cdot \sigma_s)$$

where each factor in the concatenation of the right-hand side is defined by (9.1). Now let $\tau \in \mathfrak{S}_b$. We define

$$v_{(a,s)}\vartheta_{i_1} \circledast \cdots \circledast v_{(a,s)}\vartheta_{i_b}(1_{\mathfrak{S}_a}, \ldots, 1_{\mathfrak{S}_a}; \tau) = (1_{\mathfrak{S}_a}, \ldots, 1_{\mathfrak{S}_a}; \tau)(v_{(a,s)}\vartheta_{i_{\tau(1)}} \circledast \cdots \circledast v_{(a,s)}\vartheta_{i_{\tau(b)}}).$$

Let w denote a basis vector spanning the trivial $\mathfrak{S}_{\mathfrak{S}_s}$-module. We now claim that as right-$\mathbb{C}(\mathfrak{S}_{a+s} \times \mathfrak{S}_b)$-modules there is an isomorphism

$$X \cong \big(\mathbf{S}^\alpha \otimes \mathbb{C}_{\mathfrak{S}_s}\big)\big\uparrow_{\mathfrak{S}_a \times \mathfrak{S}_s}^{\mathfrak{S}_{a+s}} \oslash \mathbf{S}^\beta$$

defined by

$$(x_1 \otimes \cdots \otimes x_b \otimes y) \otimes (v_{(a,s)}\vartheta_{i_1} \circledast \cdots \circledast v_{(a,s)}\vartheta_{i_b}) \mapsto \big((x_1 \otimes w) \otimes \vartheta_{i_1}\big) \otimes \cdots \otimes \big((x_b \otimes w) \otimes \vartheta_{i_b}\big) \otimes y. \tag{9.2}$$

To see that this commutes with the action of the base group it suffices to check this for

$$(\sigma, 1_{\mathfrak{S}_{a+s}}, \ldots, 1_{\mathfrak{S}_{a+s}})$$

where $\sigma \in \mathfrak{S}_{a+s}$. Suppose that $\vartheta_{i_1} \cdot \sigma = \pi_1\pi_2\vartheta_j$ where, as before $\pi_1 \in \mathfrak{S}_a$, $\pi_2 \in \mathfrak{S}_s$. Acting on the left-hand side of (9.2) we obtain

$$(x_1\pi_1 \otimes x_2 \otimes \cdots \otimes x_b \otimes y) \otimes (v_{(a,s)}\vartheta_{i_1} \circledast \cdots \circledast v_{(a,s)}\vartheta_{i_b}).$$

Acting on the right-hand side of (9.2) we obtain

$$\begin{aligned}(x_1 \otimes w) &\otimes \vartheta_1\sigma \otimes (x_2 \otimes w) \otimes \vartheta_{i_2} \cdots \otimes (x_b \otimes w) \otimes \vartheta_{i_b} \otimes y \\ &= (x_1 \otimes w) \otimes \pi_1\pi_2\vartheta_j \otimes (x_2 \otimes w) \otimes \vartheta_{i_2} \cdots \otimes (x_b \otimes w) \otimes \vartheta_{i_b} \otimes y \\ &= (x_1\pi_1 \otimes w\pi_2) \otimes \vartheta_j \otimes (x_2 \otimes w) \otimes \vartheta_{i_2} \cdots \otimes (x_b \otimes w) \otimes \vartheta_{i_b} \otimes y\end{aligned}$$

and we see that the actions are compatible with (9.2). For the top group, again let $\tau \in \mathfrak{S}_b$. Acting on the left-hand side of (9.2) we obtain

$$(x_{\tau(1)} \otimes \cdots \otimes x_{\tau(b)} \otimes \tau y) \otimes (v_{(a,s)}\vartheta_{i_{\tau(1)}} \circledast \cdots \circledast v_{(a,s)}\vartheta_{i_{\tau(b)}})$$

and on the right-hand side

$$(x_{\tau(1)} \otimes w \otimes \vartheta_{i_{\tau(1)}}) \otimes \cdots \otimes (x_{\tau(b)} \otimes w \otimes \vartheta_{i_{\tau(b)}}) \otimes \tau y$$

and again the actions agree.

Finally we check the dimensions. We have

$$\begin{aligned}\dim\big((\mathbf{S}^\alpha &\otimes \mathbb{C}_{\mathfrak{S}_a})\big\uparrow_{\mathfrak{S}_a \times \mathfrak{S}_s}^{\mathfrak{S}_{a+s}} \oslash \mathbf{S}^\beta\big) \times |\mathfrak{S}_{(a+s)b} : \mathfrak{S}_{a+s} \wr \mathfrak{S}_b| \\ &= \big(\dim \mathbf{S}^\alpha \times |\mathfrak{S}_{a+s} : \mathfrak{S}_a \times \mathfrak{S}_s|\big)^b \dim \mathbf{S}^\beta \times |\mathfrak{S}_{(a+s)b} : \mathfrak{S}_{a+s} \wr \mathfrak{S}_b| \\ &= \big(\dim \mathbf{S}^\alpha\big)^b \frac{(a+s)!^b}{a!^b s!^b} \dim \mathbf{S}^\beta \frac{\big((a+s)b\big)!}{(a+s)!^b b!} \\ &= \big(\dim \mathbf{S}^\alpha\big)^b \dim \mathbf{S}^\beta \frac{\big(((a+s)b\big)!}{a!^b s!^b b!}.\end{aligned}$$

The number of $\mathfrak{S}_a \times \mathfrak{S}_b$-coset representatives required for the diagrams in $V_r^0((a^b):(s^b),\varnothing)$ is

$$\frac{((a+s)b)!}{(a+s)!^b b!}\frac{(a+s)!^b}{a!^b s!^b} = \frac{(a+s)b)!}{a!^b s!^b b!}.$$

To see this, note that we can choose the b blocks of $a+s$ bottom row dots that will be the blocks in

$$\binom{(a+s)b}{a+s,\dots,a+s}\frac{1}{b!}$$

ways and we then choose the s dots within each block to be singletons in $\binom{a+s}{s}^b$ ways. Therefore

$$\dim\big(\mathbf{S}^\alpha \oslash \mathbf{S}^\beta \otimes V_r^0(s^b,\varnothing)\big) = (\dim \mathbf{S}^\alpha)^b \dim \mathbf{S}^\beta \times \frac{\big((a+s)b\big)!}{a!^b s!^b b!}$$

and the dimensions agree. □

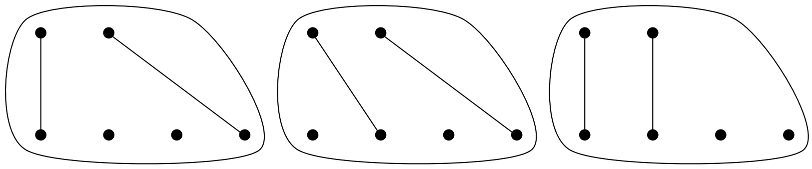


FIGURE 27. A prototypical diagram appearing in an element of X. Here $a=2$ and $b=3$ and $s=2$. The cosets (of minimal length) are $\vartheta_1=(2,4)$, $\vartheta_2=(1,2,4)$ and $\vartheta_3=\mathrm{id}_{\mathfrak{S}_4}$.

We may now use the previous two lemmas to describe the decomposition in the general case. We remind the reader that the set $\mathscr{P}_{(a^b)}(p)$ was defined in Definition 9.5; splitting into the two cases $a>0$, in which $c_0>0$ is permitted, and $a=0$ in which case, as we observed after Definition 9.6, $c_0=0$. We remind the reader of our standing convention that $\beta^i \vdash c_i$ in a sum indicates that the sum is over all relevant sequences of partitions indexed by i.

Theorem 9.16. *Suppose that $r-ab=p+q$, $\gamma=(p^{c_p},\dots,1^{c_1},0^{c_0})\in\mathscr{P}_{(a^b)}(p)$ and $\varepsilon\in\mathscr{P}_{>1}(q)$. Then the $P_r(\delta_{\mathrm{in}}\delta_{\mathrm{out}})$-submodule*

$$\mathbf{S}^{\alpha^\beta} \otimes_{\mathfrak{S}_a\wr\mathfrak{S}_b} V_r^0\big((a^b):\gamma,\varepsilon\big) \subseteq \mathsf{DQ}\big(\Delta_r(\alpha^\beta)\big)$$

decomposes as follows

$$\bigoplus_{\beta^i\vdash c_i} c^{\beta}_{\beta^p,\dots,\beta^1,\beta^0}\Big(\bigotimes_{i=0}^{p}\Big((\mathbf{S}^\alpha\otimes\mathbb{C})\big\uparrow^{\mathfrak{S}_{a+i}}_{\mathfrak{S}_a\times\mathfrak{S}_i}\big)\oslash\mathbf{S}^{\beta^i}\Big)\otimes\mathbb{C}_{\mathrm{Stab}(\varepsilon)}\Big)\Big\uparrow^{\mathfrak{S}_r}_{\mathrm{Stab}((a^b)+\gamma)\times\mathrm{Stab}(\varepsilon)}$$

via the canonical map $P_r(\delta_{\mathrm{in}}\delta_{\mathrm{out}})\to\mathbb{C}\mathfrak{S}_r$.

Proof. We first note that as an $\mathbb{C}\mathfrak{S}_r$-module,

$$V_r^0\big((a^b):\gamma,\varepsilon\big) \cong \big(V_{p+ab}^0\big((a^b):\gamma,\varnothing\big)\otimes V_q^0\big(\varnothing:\varnothing,\varepsilon\big)\big)\big\uparrow^{\mathfrak{S}_r}_{\mathfrak{S}_{p+ab}\times\mathfrak{S}_q},$$

simply because the action of $\mathfrak{S}_r$ permutes propagating outer blocks amongst themselves and permutes non-propagating outer blocks amongst themselves. Thus, by Lemma 9.13 and the transitivity of induction, it suffices to show that $\mathbf{S}^{\alpha^\beta}\otimes V_{ab+p}^0\big((a^b):\gamma,\varnothing\big)$ decomposes as follows:

$$\bigoplus_{\beta^i\vdash c_i} c^{\beta}_{\beta^p,\dots,\beta^1,\beta^0}\Big(\bigotimes_{i=0}^{p}(\mathbf{S}^\alpha\otimes\mathbb{C})\big\uparrow^{\mathfrak{S}_{a+i}}_{\mathfrak{S}_a\times\mathfrak{S}_i}\oslash\mathbf{S}^{\beta^i}\Big)\Big\uparrow^{\mathfrak{S}_{ab+p}}_{\mathrm{Stab}((a^b)+\gamma)}. \tag{9.3}$$

We prove this statement. Firstly, for any right $\mathbb{C}\mathfrak{S}_a\wr\mathfrak{S}_b$-module X,

$$X\otimes_{\mathfrak{S}_a\wr\mathfrak{S}_b} V_{ab+p}^0\big((a^b):\gamma,\varnothing\big)\cong X\big\downarrow_{\mathfrak{S}_a\wr\mathfrak{S}_{\mathbf{c}}}\otimes_{\mathfrak{S}_a\wr\mathfrak{S}_{\mathbf{c}}}\Big(\bigotimes_{i=0}^{p} V^0_{(a+i)c_i}\big((a^{c_i}):(i^{c_i}),\varnothing\big)\Big)\Big\uparrow^{\mathfrak{S}_{ab+p}}_{\prod_i\mathfrak{S}_{(a+i)c_i}},$$

where $\mathbf{c} = (c_p, \ldots, c_1, c_0)$. To see this, observe that a ramified diagram $v \in V^0_{ab+p}((a^b) : \gamma, \varnothing)$ may be written as

$$v = \vartheta\big((v_{a,p})^{\circledast c_p} \circledast \cdots \circledast (v_{a,1})^{\circledast c_1} \circledast (v_{a,0})^{\circledast c_0}\big)\sigma,$$

for $\vartheta \in \mathfrak{S}_a \wr \mathfrak{S}_b$ and $\sigma \in \mathfrak{S}_{ab+p}$. (Recall that c_i is the multiplicity of i as a part of γ.) Then, for $x \in X$, the isomorphism sends

$$x \otimes v \mapsto x\vartheta \otimes \big((v_{a,p})^{\circledast c_p} \circledast \cdots \circledast (v_{a,1})^{\circledast c_1} \circledast (v_{a,0})^{\circledast c_0}\big)\sigma.$$

Therefore $\mathbf{S}^{\alpha^\beta} \otimes V^0_{ab+p}\big((a^b) : \gamma, \varnothing)$ is isomorphic to

$$(\mathbf{S}^\alpha \oslash \mathbf{S}^\beta)\big\downarrow_{\mathfrak{S}_a \wr \mathfrak{S}_\mathbf{c}} \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_\mathbf{c}} \Big(\bigotimes_{i=0}^{p} V^0_{(a+i)c_i}\big((a^{c_i}) : (i^{c_i}), \varnothing\big)\Big)\big\uparrow^{\mathfrak{S}_{ab+p}}_{\prod_{i=0}^p \mathfrak{S}_{(a+i)c_i}}.$$

Using [CT03, Lemma 3.3(2)], this is isomorphic to

$$\bigoplus_{\beta^i \vdash c_i} c^\beta_{\beta^p,\ldots,\beta^1,\beta^0} \Big(\bigotimes_{i=0}^{p} \mathbf{S}^\alpha \oslash \mathbf{S}^{\beta^i}\Big) \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_\mathbf{c}} \Big(\bigotimes_{i=0}^{p} V^0_{(a+s)c_i}\big((a^{c_i}) : (i^{c_i}), \varnothing\big)\Big)\big\uparrow^{\mathfrak{S}_{ab+p}}_{\prod_{i=0}^p \mathfrak{S}_{(a+i)c_i}}.$$

where $c^\beta_{\beta^p,\ldots,\beta^1,\beta^0}$ is the generalized Littlewood–Richardson coefficient defined in (2.3). Regrouping terms, this becomes

$$\bigoplus_{\beta^i \vdash c_i} c^\beta_{\beta^p,\ldots,\beta^1,\beta^0} \Big(\bigotimes_{i=0}^{p} \Big((\mathbf{S}^\alpha \oslash \mathbf{S}^{\beta^i}) \otimes_{\mathfrak{S}_a \wr \mathfrak{S}_{c_i}} V^0_{(a+i)c_i}\big((a^{c_i}) : (i^{c_i}), \varnothing\big)\Big)\Big)\big\uparrow^{\mathfrak{S}_{ab+p}}_{\prod_{i=0}^p \mathfrak{S}_{(a+i)c_i}}.$$

Lemma 9.15 provides the isomorphism to

$$\bigoplus_{\beta^i \vdash c_i} c^\beta_{\beta^p,\ldots,\beta^1,\beta^0} \Big(\bigotimes_{i=0}^{p} \big((\mathbf{S}^\alpha \otimes \mathbb{C}_{\mathfrak{S}_i})\big\uparrow^{\mathfrak{S}_{a+i}}_{\mathfrak{S}_a \times \mathfrak{S}_i} \oslash \mathbf{S}^{\beta^i}\big)\Big)\big\uparrow^{\mathfrak{S}_{ab+p}}_{\prod_{i=0}^p \mathfrak{S}_{(a+i)c_i}},$$

and then transitivity of induction yields the desired statement. □

9.4. **Proofs of Theorems A and D.** We are now ready to prove Theorem D and as a corollary, Theorem A.

Proof of Theorem D. The full decomposition of the depth quotient is obtained from Proposition 9.11 and Theorem 9.16 by summing over all partitions $\gamma \in \mathscr{P}_{(a^b)}(p)$ and $\varepsilon \vdash q$ such that $p + q = r - ab$, using Corollary 7.2 and the first case of Corollary 8.8, applied with $\lambda = \kappa$. Theorem D then follows from Corollary 8.8. □

Proof of Theorem A. This follows immediately from the case $\alpha = \varnothing$ of Theorem C and Theorem D. □

10. Examples and applications

We shall write $\overline{rc}$ for the ramified branching coefficients formally defined in Definition 8.1 and determined in Theorem D. Thus

$$\overline{rc}(\alpha^\beta, \kappa) = \big[\Delta_r(\alpha^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\big]_{P_r(mn)}. \tag{10.1}$$

The values of m, n and r will be clear from context. By the final part of Theorem C, when $\alpha = \varnothing$ we have $p(\beta[n], (m), \kappa[mn]) = \overline{rc}(\varnothing^\beta, \kappa)$ provided $m \geqslant r - |\beta| + [\beta \neq \varnothing]$ and $n \geqslant r + \beta_1$.

10.1. **Examples of Theorems A, C and D.** We consider

$$\Delta_5(\varnothing^{(2,1)})\big\downarrow^{R_5(m,n)}_{P_5(mn)}, \quad \Delta_5\big((1)^{(2,1)}\big)\big\downarrow^{R_5(m,n)}_{P_5(mn)}$$

and find all the composition factors $L_5(\kappa)$ for $\kappa \vdash 5$ of these modules by decomposing the depth quotient. We have chosen to change only the partition α (from $\varnothing$ to (1)) as this minor change results in big changes in the ramified branching coefficient $[\Delta_r(\alpha^\beta)\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)]_{P_5(mn)}$ and even bigger changes in the stable plethysm coefficients. Indeed, as we discussed in Section 1.5, in the first case we obtain the stable values of $p\big((n-3,2,1),(m),\kappa[mn]\big)$ for $\kappa \vdash 5$ and arbitrary m *and* n, whereas in the second case we obtain the stable values of $p\big((2,1),(m-1,1),\kappa[mn]\big)$ for $\kappa \vdash 5$ and arbitrary m; the outer partition $(2,1)$ is now fixed.

Example 10.1. We take $\alpha = \varnothing$ and $\beta = (2,1)$ and $\kappa \vdash 5$. By Theorem C, provided $n \geqslant 7$ and $m \geqslant 3$ we have $p\big((n-3,2,1),(m),\kappa[mn]\big) = \overline{rc}\big(\varnothing^{(2,1)},\kappa\big)$. We shall derive below the stable plethysm and ramified branching coefficients

$$\begin{aligned}
p\big((n-3,2,1),(m),(mn-5,5)\big) &= 2 = \overline{rc}\big(\varnothing^{(2,1)},(5)\big),\\
p\big((n-3,2,1),(m),(mn-5,4,1)\big) &= 5 = \overline{rc}\big(\varnothing^{(2,1)},(4,1)\big),\\
p\big((n-3,2,1),(m),(mn-5,3,2)\big) &= 4 = \overline{rc}\big(\varnothing^{(2,1)},(3,2)\big),\\
p\big((n-3,2,1),(m),(mn-5,3,1^2)\big) &= 3 = \overline{rc}\big(\varnothing^{(2,1)},(3,1^2)\big),\\
p\big((n-3,2,1),(m),(mn-5,2^2,1)\big) &= 2 = \overline{rc}\big(\varnothing^{(2,1)},(2^2,1)\big),\\
p\big((n-3,2,1),(m),(mn-5,2,1^3)\big) &= 0 = \overline{rc}\big(\varnothing^{(2,1)},(2,1^3)\big),\\
p\big((n-3,2,1),(m),(mn-5,1^5)\big) &= 0 = \overline{rc}\big(\varnothing^{(2,1)},(1^5)\big)
\end{aligned}$$

for m and n satisfying these bounds. We decompose the depth quotient $\mathsf{DQ}\big(\Delta_5(\varnothing^{(2,1)})\big)$ as in Theorem 9.16 thereby computing the coefficients above by the formula in Theorems A and D. There are three summands of $\mathsf{DQ}\big(\Delta_5(\varnothing^{(2,1)})\big)$ which are of interest. These are generated by the diagrams $v_{(\gamma,\varepsilon)}$ depicted in Figure 28. To see this from the formulae, note that $|\gamma|+|\varepsilon| = 5-0\times 3 = 5$ and since $\alpha=\varnothing$ and $|\beta|=3$, the partition γ has three non-zero parts. As always, ε has no parts of size 1 since non-propagating blocks may not be singletons (such basis elements lie in the depth radical). Each outer block has at least one southern dot, so we have two further dots to place. Our options are as follows:

- place both extra dots in the *same* propagating block (set $\gamma=(3,1^2)$) leaving no extra dots to place in a non-propagating block (that is, $\varepsilon=\varnothing$).
- place each extra dot in a *separate* propagating block (set $\gamma=(2^2,1)$) leaving no extra dots to place in a non-propagating block (that is, $\varepsilon=\varnothing$).
- place both extra dots in the *same* non-propagating block (set $\varepsilon=(2)$ and $\gamma=(1^3)$)

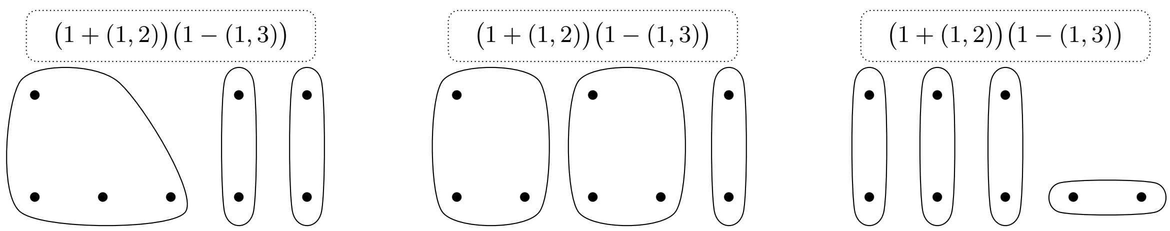


FIGURE 28. The generators $c^*_{(2,1)} \otimes v_{(\gamma,\varepsilon)}$ for (γ,ε) equal to $\big((3,1^2),\varnothing\big)$ and $\big((2^2,1),\varnothing\big)$ and $\big((1^3),(2)\big)$ respectively. These generate the direct summands, which we denote by M_1, M_2, and M_3 of $\mathsf{DQ}\big(\Delta_5(\varnothing^{(2,1)})\big)$.

For M_1 we have $\gamma = (3,1^2)$ so the multiplicities of the parts (read as throughout in decreasing order) are 1 and 2 and we restrict the Specht module $\mathbf{S}^{(2,1)}$ to $\mathfrak{S}_1 \times \mathfrak{S}_2$, obtaining

$$\mathbf{S}^{(2,1)}\big\downarrow^{\mathfrak{S}_3}_{\mathfrak{S}_1\times\mathfrak{S}_2} = \mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)} \oplus \mathbf{S}^{(1)} \otimes \mathbf{S}^{(1^2)}. \tag{10.2}$$

Since $\varepsilon = \varnothing$ we have

$$M_1 \cong \big((\mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)} \oplus \mathbf{S}^{(1)} \otimes \mathbf{S}^{(1^2)}\big)\big\uparrow^{\mathfrak{S}_3\times\mathfrak{S}_2}_{\mathfrak{S}_1\times\mathfrak{S}_2}\big\uparrow^{\mathfrak{S}_5}_{\mathfrak{S}_3\times\mathfrak{S}_2} \cong \mathbf{S}^{(5)} \oplus 2\mathbf{S}^{(4,1)} \oplus \mathbf{S}^{(3,2)} \oplus \mathbf{S}^{(3,1^2)}.$$

For M_2 we have $\gamma = (2^2,1)$ so the multiplicities of the parts are now 2 and 1 and we take a similar restriction

$$\mathbf{S}^{(2,1)}\big\downarrow^{\mathfrak{S}_3}_{\mathfrak{S}_2\times\mathfrak{S}_1} = \mathbf{S}^{(2)} \otimes \mathbf{S}^{(1)} \oplus \mathbf{S}^{(1^2)} \otimes \mathbf{S}^{(1)}. \tag{10.3}$$

Since $\varepsilon = \varnothing$ we have

$$\begin{aligned} M_2 &\cong \big((\mathbf{S}^{(2)} \oslash \mathbf{S}^{(1^2)}) \otimes \mathbf{S}^{(1)}\big)\big\uparrow^{\mathfrak{S}_5}_{\mathfrak{S}_2\wr\mathfrak{S}_2\times\mathfrak{S}_1} \oplus \big((\mathbf{S}^{(2)} \oslash \mathbf{S}^{(2)}) \otimes \mathbf{S}^{(1)}\big)\big\uparrow^{\mathfrak{S}_5}_{\mathfrak{S}_2\wr\mathfrak{S}_2\times\mathfrak{S}_1} \\ &\cong \mathbf{S}^{(5)} \oplus 2\mathbf{S}^{(4,1)} \oplus 2\mathbf{S}^{(3,2)} \oplus \mathbf{S}^{(3,1^2)} \oplus \mathbf{S}^{(2^2,1)} \end{aligned}$$

For M_3 we have $\gamma = (1^3)$ and so we do not restrict the Specht module $\mathbf{S}^{(2,1)}$. Since $\varepsilon = (2)$ we have

$$M_3 \cong (\mathbf{S}^{(2,1)} \otimes \mathbb{C}_{\mathfrak{S}_2})\big\uparrow^{\mathfrak{S}_5}_{\mathfrak{S}_3\times\mathfrak{S}_2} \cong \mathbf{S}^{(4,1)} \oplus \mathbf{S}^{(3,2)} \oplus \mathbf{S}^{(3,1^2)} \oplus \mathbf{S}^{(2^2,1)}.$$

Summing over the coefficients appearing in the decompositions of M_1, M_2 and M_3 we obtain the ramified branching coefficients as stated at the beginning of this example.

Example 10.2. We now take $\alpha = (1)$ and keep $\beta = (2,1)$ and $\kappa \vdash 5$. Now $n = |\beta| = 3$ is fixed. By Theorem C we have

$$p\big((2,1),(m-1,1),\kappa[3m]\big) \leqslant \overline{rc}\big((1)^{(2,1)},\kappa\big).$$

(Note that the hypothesis $r \geqslant n|\alpha|$ holds because $5 \geqslant 3 \times 1$.) By Theorem 1.2 in [dBPW21] or the theorem proved in §2.1 of [Bri93], given any partition $\kappa \vdash 5$ the plethysm coefficient $p\big((2,1),(m-1,1),\kappa[12] + (3m-12)\big)$ is constant for all $m \geqslant 4$.

We shall derive the following stable plethysm and ramified branching coefficients:

$$\begin{aligned} p\big((2,1),(m-1,1),(3m-5,5)\big) &= 1 \leqslant 2 = \overline{rc}\big((1)^{(2,1)},(5)\big), \\ p\big((2,1),(m-1,1),(3m-5,4,1)\big) &= 3 \leqslant 6 = \overline{rc}\big((1)^{(2,1)},(4,1)\big), \\ p\big((2,1),(m-1,1),(3m-5,3,2)\big) &= 4 \leqslant 7 = \overline{rc}\big((1)^{(2,1)},(3,2)\big), \\ p\big((2,1),(m-1,1),(3m-5,3,1^2)\big) &= 4 \leqslant 6 = \overline{rc}\big((1)^{(2,1)},(3,1^2)\big), \\ p\big((2,1),(m-1,1),(3m-5,2^2,1)\big) &= 3 \leqslant 6 = \overline{rc}\big((1)^{(2,1)},(2^2,1)\big), \\ p\big((2,1),(m-1,1),(3m-5,2,1^3)\big) &= 2 \leqslant 3 = \overline{rc}\big((1)^{(2,1)},(2,1^3)\big), \\ p\big((2,1),(m-1,1),(3m-5,1^5)\big) &= 0 \leqslant 1 = \overline{rc}\big((1)^{(2,1)},(1^5)\big) \end{aligned}$$

for $m \geqslant 4$. Note that in contrast to the previous case, the outer partition in the plethysm is fixed as $(2,1)$ and only the inner partition $(m-1,1)$ varies. We notice that none of the bounds are sharp in this case. The stable values of the plethysm coefficients can easily be calculated using computer algebra. We now calculate the ramified branching coefficients. Again using Theorem 9.16, there are three summands of $\mathsf{DQ}\big(\Delta_5((1)^{(2,1)})\big)$ which are of interest. These are generated by the diagrams $v_{(\gamma,\varepsilon)}$ depicted in Figure 29.

Arguing as in the previous example, we have that

$$N_3 \cong (\mathbf{S}^{(2,1)} \otimes \mathbb{C}_{\mathfrak{S}_2})\big\uparrow^{\mathfrak{S}_5}_{\mathfrak{S}_3\times\mathfrak{S}_2} \cong \mathbf{S}^{(4,1)} \oplus \mathbf{S}^{(3,2)} \oplus \mathbf{S}^{(3,1^2)} \oplus \mathbf{S}^{(2^2,1)}.$$

However, the other two direct summands behave very differently.

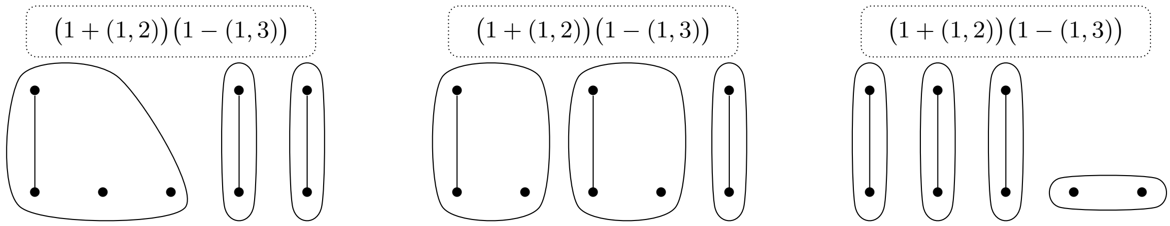


FIGURE 29. The generators $c^*_{(2,1)} \otimes v_{(\gamma,\varepsilon)}$ for (γ,ε) equalling $((1^2,0),\varnothing)$ and $((2,0^2),\varnothing)$ and $(\varnothing,(2))$ respectively. The distinguished zero parts for each $\gamma \in \mathscr{P}_{(1^3)}$ are indicated. These diagrams generate the direct summands, which we denote by N_1, N_2, and N_3 of $\mathsf{DQ}(\Delta_5((1)^{(2,1)}))$.

For N_1 we have $\gamma = (2,0^2)$ and so the multiplicities of the parts are again 1 and 2 and the restriction of $\mathbf{S}^{(2,1)}$ is given by (10.2). Following Theorem 9.16, from the first summand $\mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)}$ we obtain $\mathbf{S}^{(1)} \otimes \mathbb{C}{\uparrow}^{\mathfrak{S}_3}_{\mathfrak{S}_1\times\mathfrak{S}_2} \oslash \mathbf{S}^{(2)}$ which we may write as $M^{(2,1)} \oslash \mathbf{S}^{(2)}$, where the first tensor factor is the Young permutation module $M^{(2,1)}$ for $\mathbb{C}\mathfrak{S}_3$; similarly from the second summand $\mathbf{S}^{(1)} \otimes \mathbf{S}^{(1^2)}$ we obtain $M^{(2,1)} \oslash \mathbf{S}^{(1^2)}$. Thus

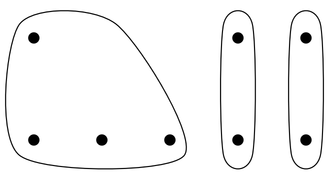
$$\begin{aligned} N_1 &\cong (M^{(2,1)} \otimes \mathbf{S}^{(2)}){\big\uparrow}^{\mathfrak{S}_5}_{\mathfrak{S}_3\times\mathfrak{S}_2} \oplus (M^{(2,1)} \otimes \mathbf{S}^{(1^2)}){\big\uparrow}^{\mathfrak{S}_5}_{\mathfrak{S}_3\times\mathfrak{S}_2} \\ &\cong \big(\mathbf{S}^{(5)} \oplus 2\mathbf{S}^{(4,1)} \oplus 2\mathbf{S}^{(3,2)} \oplus \mathbf{S}^{(3,1^2)} \oplus \mathbf{S}^{(2^2,1)}\big) \\ &\qquad\qquad \oplus \big(\mathbf{S}^{(4,1)} \oplus \mathbf{S}^{(3,2)} \oplus 2\mathbf{S}^{(3,1^2)} \oplus \mathbf{S}^{(2^2,1)}) \oplus \mathbf{S}^{(2,1^3)}\big) \\ &\cong \mathbf{S}^{(5)} \oplus 3\mathbf{S}^{(4,1)} \oplus 3\mathbf{S}^{(3,2)} \oplus 3\mathbf{S}^{(3,1^2)} \oplus 2\mathbf{S}^{(2^2,1)} \oplus \mathbf{S}^{(2,1^3)}. \end{aligned}$$

For N_2 we have $\gamma = (1^2,0)$ and so the restriction is as in equation (10.3). Again the induction function differs, and we have

$$\begin{aligned} N_2 &\cong \big((\mathbf{S}^{(1^2)} \oslash \mathbf{S}^{(2)} \oplus \mathbf{S}^{(2)} \oslash \mathbf{S}^{(2)} \oplus \mathbf{S}^{(1^2)} \oslash \mathbf{S}^{(1^2)} \oplus \mathbf{S}^{(2)} \oslash \mathbf{S}^{(1^2)}) \otimes \mathbf{S}^{(1)}\big){\big\uparrow}^{\mathfrak{S}_5}_{\mathfrak{S}_2\wr\mathfrak{S}_2\times\mathfrak{S}_1} \\ &= \big((\mathbf{S}^{(3,1)} + \mathbf{S}^{(2^2)} + \mathbf{S}^{(4)} \oplus \mathbf{S}^{(2,1^2)} + \mathbf{S}^{(2^2)} + \mathbf{S}^{(1^4)}) \otimes \mathbf{S}^{(1)}\big){\big\uparrow}^{\mathfrak{S}_5}_{\mathfrak{S}_4\times\mathfrak{S}_1} \\ &= \mathbf{S}^{(5)} \oplus 2\mathbf{S}^{(4,1)} \oplus 3\mathbf{S}^{(3,2)} \oplus 2\mathbf{S}^{(3,1^2)} \oplus 3\mathbf{S}^{(2^2,1)} \oplus 2\mathbf{S}^{(2,1^3)} \oplus \mathbf{S}^{(1^5)}. \end{aligned}$$

Summing over the coefficients appearing in the decompositions of N_1, N_2 and N_3 we obtain the ramified branching coefficients as stated at the beginning of this example.

10.2. **IndInfRes.** As motivation for the following definition, we return to Example 10.1. In this example we computed the summand of $\mathsf{DQ}\big(\Delta_r(\varnothing^\beta){\downarrow}^{R_r(m,n)}_{P_r(mn)}\big)$ corresponding to a diagram $v_{(\gamma,\varepsilon)}$ in four steps. In the first step we restricted $\mathbf{S}^\beta$ to $S_\mathbf{c}$ where $\mathbf{c}$ is the composition recording the number of propagating outer parts in the diagram $v_{(\gamma,\varepsilon)}$ (see Definition 9.10) having each possible number of southern dots between 1 and r. (Thus c_i is the number of parts of γ equal to i and since there are $|\beta| = b$ outer propagating parts, $\mathbf{c}$ is a composition of b.) This gives us a sum of tensor products $\bigotimes_{i=1}^r \mathbf{S}^{\beta^i}$ where each β^i is a partition of c_i. For instance, for the first diagram in the example, reproduced below,



we have $c = (c_5, c_4, c_3, c_2, c_1) = (0,0,1,0,2)$ recording the multiplicities of the parts in $\gamma = (3,1,1)$. We saw in (10.2) that the restricted module satisfies $\mathbf{S}^{(2,1)}{\downarrow}^{\mathfrak{S}_3}_{\mathfrak{S}_1\times\mathfrak{S}_2} = \mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)} \oplus \mathbf{S}^{(1)} \otimes \mathbf{S}^{(1^2)}$. As seen in Lemma 9.15, the outer propagating parts having i southern dots are permuted amongst themselves by the wreath product $\mathfrak{S}_i \wr \mathfrak{S}_{c_i}$. Since $\alpha = \varnothing$ and so there are no inner propagating parts, the base group acts trivially. In the second step we obtain the action of the base group by inflating each tensor factor $\mathbf{S}^{\beta^i}$ from $\mathfrak{S}_{c_i}$ to $\mathfrak{S}_i \wr \mathfrak{S}_{c_i}$, obtaining

$$\mathrm{Inf}_{\mathfrak{S}_1}^{\mathfrak{S}_3\wr\mathfrak{S}_1} \mathbf{S}^{(1)} \otimes \mathrm{Inf}_{\mathfrak{S}_2}^{\mathfrak{S}_1\wr\mathfrak{S}_2} \mathbf{S}^{(2)} \oplus \mathrm{Inf}_{\mathfrak{S}_1}^{\mathfrak{S}_3\wr\mathfrak{S}_1} \mathbf{S}^{(1)} \otimes \mathrm{Inf}_{\mathfrak{S}_2}^{\mathfrak{S}_1\wr\mathfrak{S}_2} \mathbf{S}^{(1,1)}.$$

Setting $p = \sum_i ic_i$, the full action of the symmetric group $\mathfrak{S}_p$ is then given, as seen in the proof of Theorem 9.16 by a third step in which we induce from $\prod_i \mathfrak{S}_i \wr \mathfrak{S}_{c_i}$ to $\mathfrak{S}_p$. (Note that here $\alpha = \varnothing$ so $a = 0$ and the module $(\mathbf{S}^\alpha \otimes \mathbb{C})\uparrow_{\mathfrak{S}_a\times\mathfrak{S}_i}^{\mathfrak{S}_{a+i}}$ is simply the trivial $\mathfrak{S}_i$-module.) Using transitivity of induction we then finish by tensoring with the trivial module $\mathbb{C}_{\mathrm{Stab}(\varepsilon)}$ and inducing from $\mathfrak{S}_p \times \mathrm{Stab}(\varepsilon)$ to $\mathfrak{S}_r$. The following functor performs the first three steps.

Definition 10.3. Let $\gamma = (p^{c_p}, \dots, 1^{c_1})$ be a partition of p having exactly b parts. We define $\mathrm{IndInfRes}_\gamma : \text{mod}-\mathbb{C}\mathfrak{S}_b \to \text{mod}-\mathbb{C}\mathfrak{S}_p$ on each right $\mathbb{C}\mathfrak{S}_b$-module W by

$$\mathrm{IndInfRes}_\gamma W = \Big(\prod_{i=1}^{p} \mathrm{Inf}_{\mathfrak{S}_{c_i}}^{\mathfrak{S}_i\wr\mathfrak{S}_{c_i}} \big(W\big\downarrow_{\mathfrak{S}_{c_p}\times\cdots\times\mathfrak{S}_{c_1}}\big)\Big)\Big\uparrow_G^{\mathfrak{S}_p}$$

where the subgroup G in the induction functor is $\mathfrak{S}_p \wr \mathfrak{S}_{c_p} \times \cdots \times \mathfrak{S}_1 \wr \mathfrak{S}_{c_1}$. Given $b,\, p \in \mathbb{N}$ we define $\mathrm{IndInfRes}_{\mathfrak{S}_b}^{\mathfrak{S}_p} : \text{mod}-\mathbb{C}\mathfrak{S}_b \to \text{mod}-\mathbb{C}\mathfrak{S}_p$ by $\mathrm{IndInfRes}_{\mathfrak{S}_b}^{\mathfrak{S}_p} = \sum_\gamma \mathrm{IndInfRes}_\gamma$, where the sum is over all $\gamma \vdash p$ having exactly b parts.

Proposition 10.4. *Let $\kappa \vdash r$. For any partition β of b, the ramified branching coefficient $\overline{rc}(\varnothing^\beta, \kappa)$ satisfies*

$$\overline{rc}(\varnothing^\beta,\kappa) = \sum_{\substack{p,q:p+q=r\\ \varepsilon\in\mathscr{P}_{>1}(q)}} \big[\big(\mathrm{IndInfRes}_{\mathfrak{S}_b}^{\mathfrak{S}_p} \mathbf{S}^\beta \otimes \mathbb{C}_{\mathrm{Stab}(\varepsilon)}\big\uparrow^{\mathfrak{S}_q}\big)\big\uparrow_{\mathfrak{S}_p\times\mathfrak{S}_q}^{\mathfrak{S}_r} : \mathbf{S}^\kappa\big]_{\mathfrak{S}_r}.$$

Moreover if $m \geqslant r - |\beta| + [\beta \neq \varnothing]$ and $n \geqslant r + \beta_1$ then either side is equal to the plethysm coefficient $p(\beta[n], (m), \kappa[mn])$.

Proof. Fix a partition $\gamma = (p^{c_p}, \dots, 1^{c_1}) \vdash p$ having exactly b parts. We have

$$\mathbf{S}^\beta\big\downarrow_{\mathfrak{S}_{c_p}\times\cdots\times\mathfrak{S}_{c_1}} = \bigoplus_{\beta^i\vdash c_i} c^\beta_{\beta^p,\dots,\beta^1} \mathbf{S}^{\beta^p} \otimes \cdots \otimes \mathbf{S}^{\beta^1}$$

where $c^\beta_{\beta^p,\dots,\beta^1}$ is a generalized Littlewood–Richardson coefficient as defined in (2.3). By transitivity of induction it follows that for each $\varepsilon \in \mathscr{P}_{>1}(q)$, the composition multiplicity

$$\big[\big(\mathrm{IndInfRes}_\gamma \mathbf{S}_\beta \otimes \mathbb{C}_{\mathrm{Stab}(\varepsilon)}\big\uparrow^{\mathfrak{S}_q}\big)\big\uparrow_{\mathfrak{S}_p\times\mathfrak{S}_q}^{\mathfrak{S}_r} : \mathbf{S}^\kappa\big]_{\mathfrak{S}_r}$$

is precisely the contribution to the sum in Theorem D coming from the partitions γ and ε. The result now follows from the definition of $\mathrm{IndInfRes}_\gamma$ and $\mathrm{IndInfRes}_{\mathfrak{S}_b}^{\mathfrak{S}_p}$ by summing over all partitions γ and ε. The result on $p(\beta[n], (m), \kappa[mn])$ follows similarly from Theorem A. □

10.3. **Marked partitions and plethysm coefficients when β has one row and $\alpha = \varnothing$.** In this subsection we apply Proposition 10.4 to give an elegant *and clearly positive* formula for the ramified branching coefficients when β has a single part. We require the following definition.

Definition 10.5. Let $b \in \mathbb{N}$. A b-marked partition of $r \in \mathbb{N}$ is a pair of partitions (γ, ε) such that $\ell(\gamma) = b$, $\varepsilon \in \mathscr{P}_{>1}(|\varepsilon|)$ and $|\gamma| + |\varepsilon| = r$. Let $\mathscr{MP}_b(r)$ denote the set of b-marked partitions of r.

Thus a b-marked partition of r may be regarded as an ordinary partition of r having b distinguished parts, such that only the distinguished parts may have size 1. Marked partitions (γ, ε) are the types, in the sense of Definition 9.7, of ramified diagrams when $a = 0$.

Proposition 10.6. *Let κ be a partition of r and let $b \in \mathbb{N}_0$. Then*
$$\overline{rc}(\varnothing^{(b)}, \kappa) = \sum_{(\gamma,\varepsilon)\in \mathscr{MP}_b(r)} \big[\mathbb{C}_{\mathrm{Stab}(\gamma)\times\mathrm{Stab}(\varepsilon)}\big\uparrow^{S_r} : \mathbf{S}^\kappa\big]_{\mathfrak{S}_r}.$$
Moreover if $m \geqslant r - b + [b \neq 0]$ and $n \geqslant r + b$ then either side is the plethysm coefficient $p\big((n-b,b),(m),\kappa[mn]\big)$.

Proof. It is easy to see that $\mathrm{IndInfRes}^{\mathfrak{S}_b}_{\mathfrak{S}_p} \mathbf{S}^{(b)} = \sum_\gamma \mathbb{C}_{\mathrm{Stab}(\gamma)}\big\uparrow^{\mathfrak{S}_p}$ where the sum is over all $\gamma \vdash p$ such that $\ell(\gamma) = b$. The result now follows from Proposition 10.4 using transitivity of induction. □

The special case $\kappa = (r)$ is worth noting. For any partitions γ and ε, it follows from Frobenius reciprocity that $\big[\mathbb{C}_{\mathrm{Stab}(\gamma)\times\mathrm{Stab}(\varepsilon)}\big\uparrow^{S_r} : \mathbf{S}^{(r)}\big]_{\mathfrak{S}_r} = 1$ and so
$$\overline{rc}\big(\varnothing^{(b)}, (r)\big) = \big|\mathscr{MP}_b(r)\big|. \tag{10.4}$$
This leads to a simple closed form for the generating function of the stable limit of the corresponding plethysm coefficients. Let $P(z) = \prod_{i=1}^\infty (1-z^i)^{-1}$ be the generating function for the sequence of partition numbers.

Proposition 10.7. *Let $b \in \mathbb{N}_0$. We have*
$$\sum_{r=0}^\infty \lim_{m,n\to\infty} p\big((n-b,b),(m),(mn-r,r)\big)z^r = \frac{z^b}{(1-z^2)\dots(1-z^b)}P(z).$$

Proof. By (10.4) it is equivalent to show that the generating function for b-marked partitions is the right-hand side in the proposition. In turn this follows because partitions with exactly b parts are enumerated by $z^b/(1-z)\dots(1-z^b)$ and partitions with no singleton parts are enumerated by $(1-z)P(z)$. □

One reason for the interest in Proposition 10.7 is that, via Euler's Pentagonal Number Theorem (see for instance [And98, Corollary 1.7]), it gives an efficient recurrence relation for the stable limits of the plethysm coefficients $p\big((n-b,b),(m),(mn-r,r)\big)$. When $b = 0$ the generating function in the proposition enumerates partitions of r into non-singleton parts; this is OEIS [OEI23] sequence A002865. When $b = 1$ the generating function is $zP(z)$ enumerating partitions, with a shift by 1. This is OEIS sequence A000041. When $b = 2$ the generating function is
$$\frac{z^2P(z)}{1-z^2} = \frac{z^2}{(1-z^2)^2}\frac{1}{(1-z)(1-z^3)\dots}.$$
Since $z^2/(1-z^2)^2 = \sum_{k=1}^\infty kz^{2k}$ and the remaining part of the right-hand side enumerates partitions into parts not of size 2, the coefficient of z^r in the right-hand side is the total number of parts of size 2 in all partitions of r. The coefficients of $z^2P(z)/(1-z^2)$ form sequence A024786 in OEIS. The sequences for greater b do not, at the time of writing, appear in OEIS.

10.4. **Symmetric functions.** We finish this section by restating Theorems A and D in the language of symmetric functions and using this restatement to prove three new stability results. We remind the reader of our standing convention that $\beta^i \vdash i$ in a sum indicates that the sum is over all relevant sequences of partitions.

Definition 10.8. Let $\beta \vdash b$ and let $\gamma = (p^{c_p}, \dots, 1^{c_1}) \vdash p$ be a partition such that $\ell(\gamma) \leqslant b$. Given a non-empty partition α we set $c_0 = |\beta| - \ell(\gamma)$ and define
$$G^\alpha_{\beta,\gamma} = \sum_{\beta^i \vdash c_i} c^\beta_{\beta^p,\dots,\beta^1,\beta^0} \prod_{i=0}^p s_{\beta^i} \circ (s_\alpha s_{(i)}).$$

If $\ell(\gamma) = b$ we define

$$G^{\varnothing}_{\beta,\gamma} = \sum_{\beta^i \vdash c_i} c^{\beta}_{\beta^p,\ldots,\beta^1} \prod_{i=1}^{p} s_{\beta^i} \circ s_{(i)}.$$

We set $G^{\alpha}_{\beta,\gamma} = 0$ in all other cases.

Note that in the first product $s_{(0)}$ should be interpreted as $s_{\varnothing} = 1$. Thus whenever $G^{\alpha}_{\beta,\gamma}$ is non-zero its degree is $|\gamma| + |\alpha||\beta|$. For example $G^{\alpha}_{\beta,\varnothing} = s_\beta \circ s_\alpha$ for any partitions α and β and $G^{\varnothing}_{\beta,(1^b)} = s_\beta \circ s_{(1)} = s_\beta$ for any partition β.

Definition 10.9. Given a partition $\varepsilon = (q^{e_q}, \ldots, 2^{e_2}, 1^{e_1})$, we define $H_\varepsilon = \prod_{j=1}^{q} s_{(e_j)} \circ s_{(j)}$.

In our application we have $\varepsilon \in \mathscr{P}_{>1}(q)$ for some q and so $e_1 = 0$. It is worth noting that if ε has at most one part of any given size then H_ε is the complete homogeneous symmetric function denoted h_ε in the standard notation. By the following lemma, H_ε corresponds to the permutation module of $\mathfrak{S}_q$ acting on the set-partitions into parts of size specified by ε.

Lemma 10.10. *For each partition $\varepsilon \vdash q$, the symmetric function corresponding under the characteristic isometry to the module $\mathbb{C}_{\mathrm{Stab}(\varepsilon)}\uparrow^{\mathfrak{S}_q}$ is H_ε.*

Proof. By Lemma 2.3(b) the plethysm $s_{(e_j)} \circ s_{(j)}$ corresponds under the characteristic isometry to the induced module $\mathbb{C}\uparrow^{\mathfrak{S}_{je_j}}_{\mathfrak{S}_j \wr \mathfrak{S}_{e_j}}$. Using that $\mathrm{Stab}(\varepsilon) = \mathfrak{S}_q \wr \mathfrak{S}_{e_q} \times \cdots \times \mathfrak{S}_1 \wr \mathfrak{S}_{e_1}$ where e_j is the multiplicity of j as a part of ε, the lemma now follows from Lemma 2.3(a), that the induced product of modules corresponds to the ordinary product of symmetric functions. □

Proposition 10.11. *Let $\alpha \vdash a$, $\beta \vdash b$ and $\kappa \vdash r$ be partitions. The ramified branching coefficient $\overline{rc}(\alpha^\beta, \kappa)$ satisfies*

$$\overline{rc}(\alpha^\beta, \kappa) = \sum_{\substack{p,\,q\,:\,p+q\,=\,r-ab \\ \gamma \vdash p,\, \varepsilon \in \mathscr{P}_{>1}(q)}} \langle G^{\alpha}_{\beta,\gamma} H_\varepsilon, s_\kappa \rangle.$$

Moreover if $\alpha = \varnothing$, $m \geqslant r - b + [b \neq 0]$ and $n \geqslant r + b$ then either side is the plethysm coefficient $p(\beta[n], (m), \kappa[mn])$.

Proof. By Lemma 10.10, H_ε is the symmetric function corresponding to $\mathbb{C}_{\mathrm{Stab}(\varepsilon)}\uparrow^{\mathfrak{S}_q}$. Therefore, by Lemma 2.3(b) and Theorem D, to prove the claim on the ramified branching coefficient it suffices to show that $G^{\alpha}_{\beta,\gamma}$ is the symmetric function corresponding under the characteristic isometry to

$$\bigoplus_{\substack{\gamma = (p^{c_p},\ldots,1^{c_1},0^{c_0}) \in \mathscr{P}_{(a^b)}(p) \\ \beta^i \vdash c_i \text{ for } 1 \leqslant i \leqslant p}} c^{\beta}_{\beta^p,\ldots,\beta^1,\beta^0} \Big(\bigotimes_{i=0}^{p} \Big((\mathbf{S}^\alpha \otimes \mathbb{C}) \big\uparrow^{\mathfrak{S}_{a+i}}_{\mathfrak{S}_a \times \mathfrak{S}_i} \big) \oslash \mathbf{S}^{\beta^i} \Big) \big\uparrow^{\mathfrak{S}_p}_{\mathrm{Stab}((a^b)+\gamma)} \Big)$$

(The set $\mathscr{P}_{(a^b)}(p)$ was defined in Definition 9.5.) By Lemma 2.3(b), $\mathbf{S}^\alpha \otimes \mathbb{C}\uparrow^{\mathfrak{S}_{a+i}}_{\mathfrak{S}_a \times \mathfrak{S}_i}$ corresponds to $s_\alpha s_{(i)}$ and hence, using both parts of this lemma, the tensor product corresponds to the symmetric function $\prod_{i=1}^{p} s_{\beta^i} \circ \big(s_\alpha s_{(i)}\big)$. The proposition now follows from the definition of $G^{\alpha}_{\beta,\gamma}$, noting that if $\alpha \neq \varnothing$ then $\gamma \in \mathscr{P}_{(a^b)}(p)$ and γ has $c_0 = b - \ell(\gamma)$ distinguished zero parts, while if $\alpha = \varnothing$ then $\gamma \in \mathscr{P}_{(0^b)}(p)$ and so $\ell(\gamma) = b$. The result on $p(\beta[n], (m), \kappa[mn])$ follows as in the proof of Proposition 10.4. □

Note that each G_γ and H_ε can be expressed as a linear combination of Schur functions using the Littlewood–Richardson rule and plethysm coefficients $p(\beta^k, (k), \lambda)$ for varying partitions λ. A further application of the Littlewood–Richardson rule then expresses each $G_\gamma H_\varepsilon$ as a linear combination of Schur functions. This makes precise the claim in the introduction that Corollary D allows stable plethysm coefficients to be computed using *much smaller* Littlewood–Richardson and plethysm coefficients.

10.5. **Applications of Proposition 10.11.** As a warm up we give the symmetric functions proof of Proposition 10.6. It is well known that the Littlewood–Richardson coefficient $c^{\lambda}_{\mu\nu}$ is non-zero only if $\ell(\lambda) \geqslant \ell(\mu)$. It follows that the generalized Littlewood–Richardson coefficient $c^{(b)}_{\beta^p,\ldots,\beta^1}$ in the sum defining $G^{\varnothing}_{(b),\gamma}$ is non-zero if and only if $\beta^i = (c_i)$ for each i, and in this case its value is 1. Therefore

$$G^{\varnothing}_{(b),(p^{c_p},\ldots,1^{c_1})} = \prod_{i=1}^{p} s_{(c_i)} \circ s_{(i)}.$$

Observe that this is the symmetric function H_γ in Definition 10.9. Therefore, by Proposition 10.11, provided $m \geqslant r - b + [b \neq 0]$ and $n \geqslant r + b$, we have

$$p\big((n-b,b),(m),\kappa\big) = \sum_{\substack{p,q:p+q=r\\ \gamma\vdash p,\ell(\gamma)=b\\ \varepsilon\in\mathscr{P}_{>1}(q)}} \langle H_\gamma H_\varepsilon, s_\kappa\rangle$$

in which we may simplify the condition defining the summation to $(\gamma,\varepsilon) \in \mathscr{M}\mathscr{P}_b(r)$. This expresses the plethysm coefficient $p\big((n-b,b),(m),\kappa[mn]\big)$ for $m \geqslant r$ and $n \geqslant r$ as a *clearly positive* sum of generalized plethysm coefficients of smaller degrees.

Remark 10.12. Taking the special case $b = 0$ of this result and substituting the definition of H_ε, we obtain

$$p\big((n),(m),\kappa[mn]\big) = \sum_{\varepsilon\in\mathscr{P}_{>1}(r)} \Big\langle \prod_{j\geqslant 2} s_{(e_j)} \circ s_{(j)}, s_\kappa \Big\rangle$$

where e_j is the multiplicity of j as a part of the partition ε. This recovers the main result, Theorem B, of [BP17], originally proved as the main theorem in [Man98].

We now give three further applications of Proposition 10.11 that prove new results in the case when $\alpha = \varnothing$.

The case $\beta = (1^b)$. We require the following basic results on symmetric functions.

Lemma 10.13. *Let* $p \in \mathbb{N}$.

(a) *The plethysm* $s_\pi \circ s_{(i)}$ *of degree* p *has* $s_{(p)}$ *as a constituent if and only if* π *has exactly one part; in this case the multiplicity is* 1.

(b) *The plethysm* $s_\pi \circ s_{(i)}$ *has* $s_{(1^p)}$ *as a constituent if and only if* $i = 1$ *and* $\pi = (1^p)$.

Proof. Part (a) follows from Corollary 9.1 of [PW19], which implies as a special case that the lexicographically greatest constituent of $s_\pi \circ s_{(i)}$ is $(|\pi|(i-1)+\pi_1,\pi_2,\ldots,\pi_{\ell(\pi)})$. For (b), we use that every constituent of $s_\pi \circ s_{(i)}$ appears in $s_{(i)} \times \cdots \times s_{(i)}$ where there are $|\pi|$ factors in the product. By the Littlewood–Richardson rule (or its simpler special case, Young's rule), if s_ρ appears in this product then $\ell(\rho) \leqslant |\pi|$. Therefore $s_\pi \circ s_{(i)}$ has $s_{(1^p)}$ as a constituent only if $i = 1$, and then since $s_\pi \circ s_{(1)} = s_\pi$, we have $\pi = (1^p)$. □

Let $\mathscr{M}\mathscr{P}^\star_b(r)$ be the set of b-marked partitions (γ,ε) of r such that the parts of γ are distinct.

Proposition 10.14. *Let* $m, n \in \mathbb{N}$ *and let* $b < n$. *Suppose that* $m \geqslant r - b + 1$ *and* $n \geqslant r + 1$.

(i) *We have* $p\big((n-b,1^b),(m),(mn-r,r)\big) = |\mathscr{M}\mathscr{P}^\star_b(r)|$.

(ii) *We have* $p\big((n-b,1^b),(m),(mn-r,1^r)\big) = [r = b]$.

Proof. Observe that, dually to the case $\beta = (b)$, the generalized Littlewood–Richardson coefficient $c^{(1^b)}_{\beta^p,\ldots,\beta^1}$ in the sum defining $G^{\varnothing}_{(1^b),\gamma}$ where $\gamma = (p^{c_p},\ldots,1^{c_1})$ is non-zero if and only if $\beta^i = (1^{c_i})$ for each i, and in this case its value is 1. Therefore

$$G^{\varnothing}_{(1^b),\gamma} = \prod_{i=1}^{p} s_{(1^{c_i})} \circ s_{(i)}.$$

For (i), it follows easily from Lemma 10.13(a) that $G^{\varnothing}_{(1^b),\gamma} H_\varepsilon$ has $s_{(r)}$ as a constituent if and only if γ has distinct parts, and in this case the multiplicity is 1. Therefore by Proposition 10.11 we have $p\big((n-b,1^b),(m),(r)\big) = |\mathscr{MP}^\star_b(r)|$, as required. Similarly for (ii), it follows easily from Lemma 10.13(b) that $G^{\varnothing}_{(1^b),\gamma} H_\varepsilon$ has $s_{(1^r)}$ as a constituent only if $\gamma = (1^p)$ and then, since $\varepsilon \in \mathscr{P}(>1)$, $\varepsilon = \varnothing$. Since we require $\ell(\gamma) = |(1^b)|$ and $|\gamma| + |\varepsilon| = r$, it then follows that $r = b$. Hence by Proposition 10.11, $p\big((n-b,1^b),(m),(mn-r,1^r)\big) \neq 0$ if and only if $r = b$, and in this case, the coefficient is 1. $\square$

We remark that (ii) is a special case of Theorem 3.1(2) in [LR04]; the short proof given here and the explicit positive formula in (i) are both new.

The case $\kappa = (b)$. We now generalize the argument for Proposition 10.14(i). Given a partition β, let $\mathcal{S}_\beta(p)$ be the set of semistandard β-tableaux having entries from $\mathbb{N}$ whose sum of entries is p. For example,

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \in \mathcal{S}_{(4,2,1)}(15)$$

Proposition 10.15. *Let $\beta \vdash b$ be a partition. Then*

$$\overline{rc}\big(\varnothing^\beta,(r)\big) = \sum_{p=b}^{r} \big|\mathcal{S}_\beta(p)\big|\,\big|\mathscr{P}_{>1}(r-p)\big|$$

Moreover if $m \geqslant r - b + [b \neq 0]$ and $n \geqslant r + b$ then either side is the plethysm coefficient $p(\beta[n],(m),\kappa[mn])$.

Proof. Let $\gamma = (p^{c_p},\ldots,1^{c_1})$ be a partition with $\ell(\gamma) = b$. By Lemma 10.13(a), the contribution to $G^{\varnothing}_{\beta,\gamma}$ from partitions $\beta^p,\ldots,\beta^1$ in the sum in Definition 10.8 is non-zero if and only if each β^i has at most one part, and in this case the contribution is 1. Since $c^{\beta}_{(c_1),(c_2),\ldots,(c_p)}$ is the number of semistandard tableaux of shape β having exactly c_i entries equal to i for each $1 \leqslant i \leqslant p$, and by the hypothesis $|\gamma| = p$ we have $\sum_{i=1}^{p} i c_i = p$, it follows that

$$\sum_{\substack{\gamma \vdash p \\ \ell(\gamma)=b}} \langle G^{\varnothing}_{\beta,\gamma}, s_{(p)}\rangle = |\mathcal{S}_\beta(p)|.$$

Note that this set is empty unless $p \geqslant |\beta| = b$. Again by Lemma 10.13(a), $\langle H_\varepsilon, s_{(q)}\rangle = 1$ for each partition ε. Now by Proposition 10.11 we have

$$\begin{aligned}
\overline{rc}\big(\varnothing^\beta,(r)\big) &= \sum_{\substack{p,\,q\,:\,p+q=r \\ \gamma\vdash p,\,\varepsilon\in\mathscr{P}_{>1}(q)}} \langle G^{\varnothing}_{\beta,\gamma} H_\varepsilon, s_{(r)}\rangle \\
&= \sum_{p,,q\,:\,p+q=r} \ \sum_{\gamma\vdash p:\ell(\gamma)=b} \langle G^{\varnothing}_{\beta,\gamma}, s_{(p)}\rangle \sum_{\varepsilon\in\mathscr{P}_{>1}(q)} \langle H_\varepsilon, s_{(q)}\rangle \\
&= \sum_{\substack{p,\,q\,:\,p+q=r \\ p\geqslant b}} \big|\mathcal{S}_\beta(p)\big|\big|\mathscr{P}_{>1}(q)\big|
\end{aligned}$$

where the second line is a final application of Lemma 10.13(a), and that $G^{\varnothing}_{\beta,\gamma} = 0$ unless $\ell(\gamma) = b$, the third line substitutes the results on $G^{\varnothing}_{\beta,\gamma}$ and H_ε just obtained. $\square$

For instance, to deduce Proposition 10.14(i) from this proposition, observe that each $t \in \mathcal{S}_{(1^b)}(p)$ has b distinct entries summing to p, and so there is an obvious bijection between the set $\mathcal{S}_{(1^b)}(p) \times \mathscr{P}_{>1}(q)$ and the subset of $\mathscr{MP}_b(r)$ of those marked partitions (γ,ε) such that $|\gamma| = p$, $|\varepsilon| = q$ and γ has distinct parts. Again it is a notable feature of Proposition 10.15 that the formula is explicit and clearly positive.

Cases where $|\kappa| \leqslant |\beta| + 2$. We end by showing how Theorem A determines the plethysm coefficients $p(\beta[n], (m), \kappa[mn])$ when $|\kappa| \leqslant |\beta| + 1$ and giving an illustrative example of how it can be used to compute this plethysm coefficient when $|\kappa| = |\beta| + 2$. We require the following lemma; in (iii), $\beta - \square$ denotes a partition obtained from β by removing a single removable box from its Young diagram.

Lemma 10.16. *Let* $\beta \vdash b$ *and let* $\kappa \vdash p$.

(i) *Let* $p < b$. *Then* $G^{\varnothing}_{\beta,\gamma} = 0$.

(ii) *Let* $p = b$. *Then* $G^{\varnothing}_{\beta,\gamma} \neq 0$ *if and only if* $\gamma = (1^b)$ *and* $G^{\varnothing}_{\beta,(1^b)} = s_\beta$.

(iii) *Let* $p = b+1$. *Then* $G^{\varnothing}_{\beta,\gamma} \neq 0$ *if and only if* $\gamma = (2, 1^{b-1})$ *and* $G^{\varnothing}_{\beta,(2,1^{b-1})} = \sum_{\pi=\beta-\square} s_\pi s_{(2)}$.

Proof. That $G^{\varnothing}_{\beta,\gamma} = 0$ in each case follows easily from the remark after Definition 10.8 that $G^{\varnothing}_{\beta,\gamma}$ has degree p whenever it is non-zero. For (ii) we use that a generalized Littlewood–Richardson coefficient with a single bottom factor c^{β}_{π} is non-zero if and only if $\pi = \beta$ and for (iii) that if $\pi \vdash b-1$ then $c^{\beta}_{\pi,(1)}$ is non-zero if and only if π is obtained from β by removing a single box; the product in $G^{\varnothing}_{\beta,\gamma}$ is then $(s_\pi \circ s_{(1)})(s_{(1)} \circ s_{(2)}) = s_\pi s_{(2)}$, as required. $\square$

Proposition 10.17. *Let* $\beta \vdash b$ *and let* $\kappa \vdash r$. *Suppose that* $m \geqslant r - |\beta| + [\beta \neq \varnothing]$ *and* $n \geqslant r + \beta_1$.

(i) *If* $r < b$ *then* $p(\beta[n], (m), \kappa[mn]) = 0$.

(ii) *If* $r = b$ *then* $p(\beta[n], (m), \kappa[mn]) \neq 0$ *if and only if* $\kappa = \beta$ *and then the coefficient is* 1.

(iii) *If* $r = b + 1$ *then* $p(\beta[n], (m), \kappa[mn]) \neq 0$ *if and only if* κ *is obtained from* β *by first removing a box and then adding two boxes, not both in the same column.*

Proof. Each part follows easily from the corresponding part in Lemma 10.16, using Proposition 10.11, noting that in each case that since $\varepsilon \in \mathscr{P}_{>1}$ and $|\varepsilon| \leqslant 1$, we have $\varepsilon = \varnothing$ in the sum in this proposition; for (iii) we use Young's rule to multiply s_π by $s_{(2)}$. $\square$

We remark that the multiplicity in case (iii) can be arbitrarily high in the case when κ is obtained from β by adding a single box: for example if β is the staircase partition $(\ell, \ell-1, \ldots, 1)$ and $\kappa = (\ell+1, \ell-1, \ldots, 1)$, then κ can be obtained by removing *any* of the b removable boxes from β, and then adding two boxes, not both in the same column. Therefore $p\big((\ell, \ell-1, \ldots, 1)[n], (m), (\ell+1, \ell-1, \ldots, 1)[mn]\big) = \ell$ whenever m and n satisfy $m \geqslant 2$ and $n \geqslant \binom{\ell+1}{2} + \ell + 1 = \binom{\ell+2}{2}$.

We conclude with an example illustrative of the case when $r = b + 2$.

Example 10.18. We take $\beta = (3, 3, 3)$ and $r = 11$. There are three non-zero products $G^{\varnothing}_{\beta,\gamma} H_\varepsilon$ in the right-hand side of Proposition 10.11.

- $\gamma = (3, 1^8)$ and $\varepsilon = \varnothing$: the multiplicities of the parts of γ are $c_3 = 1$, $c_2 = 0$ and $c_1 = 8$, and since $(3, 3, 3)$ has a unique removable box we must then take $\beta^3 = (1)$ and $\beta^1 = (3, 3, 2)$ to obtain a non-zero Littlewood–Richardson coefficient $c^{(3,3,3)}_{\beta^3,\varnothing,\beta^1}$. Thus in this case the product is $(s^{(1)} \circ s^{(3)})(s_{(3,3,2)} \circ s_{(1)}) = s_{(3)} s_{(3,3,2)}$.
- $\gamma = (2, 2, 1^7)$ and $\varepsilon = \varnothing$: the multiplicities of the parts of γ are $c_2 = 2$ and $c_1 = 7$, and we may take either $\beta^2 = (2)$ and $\beta^1 = (3, 3, 1)$ or $\beta^2 = (1, 1)$ and $\beta^1 = (3, 2, 2)$ to obtain a non-zero Littlewood–Richardson coefficient. The product is

$$(s^{(2)} \circ s_{(2)})(s_{(3,3,1)} \circ s_{(1)}) + (s_{(1,1)} \circ s_{(2)})(s_{(3,2,2)} \circ s_{(1)}) \\ = s_{(4)} s_{(3,3,1)} + s_{(2,2)} s_{(3,3,1)} + s_{(3,1)} s_{(3,2,2)}.$$

- $\gamma = (1^9)$ and $\varepsilon = (2)$: the product is $s_{(3,3,3)} s_{(2)}$.

Therefore

$$p\big((n-9, 3, 3, 3), (m), \kappa[mn]\big) \\ = \langle s_{(3)} s_{(3,3,2)} + s_{(4)} s_{(3,3,1)} + s_{(2,2)} s_{(3,3,1)} + s_{(3,1)} s_{(3,2,2)} + s_{(3,3,3)} s_{(2)}, s_\kappa \rangle.$$

It is now routine to use the Littlewood–Richardson rule to calculate the plethysm coefficients $p\big((n-9,3,3,3),(m),\kappa[mn]\big)$ for $n \geqslant 14$ and $m \geqslant 3$ for each $\kappa \vdash 11$. For instance when $\kappa = (3,3,3,2)$, the plethysm coefficient is 4, with contributions of 1, 2 and 1 from the three products. In fact it suffices to take $n = 13$ and $m = 3$ to get the stable value.

11. The bounds in Theorems A and D when $\alpha = \varnothing$ cannot be weakened

In this section we show that when $\beta = (b)$ the bound on n in Theorems A and C cannot be weakened for infinitely many κ, and when $\beta = \varnothing$, similarly the bound on m cannot be weakened when $\kappa = (r)$.

We require the following proposition and lemma, which generalize the Cayley–Sylvester formula (see [Cay54, §20] or, for an elegant modern proof using the symmetric group, [Gia13, Corollary 2.12]) that the plethysm coefficient $p\big((n),(m),(mn-r,r)\big)$ is the difference between the number of partitions of r and $r-1$ contained in the $n \times m$ box. To prove Proposition 11.1, we shall use the combinatorial model for a general plethysm $s_\nu \circ s_\mu$ from [dBPW21]; we give enough details to make this section self-contained provided the reader takes one basic lemma from [dBPW21] on trust.

Proposition 11.1. *For k, $r \in \mathbb{N}$ satisfying $n-k \geqslant k$ and $mn-r \geqslant r$, we define $T\big((n-k,k),m\big)_r$ to be the set of semistandard Young tableaux of shape $(n-k,k)$ with entries from $\{0,1,\ldots,m\}$ whose sum of all n entries is r. Then*

$$p\big((n-b,b),(m),(mn-r,r)\big) = \big|T\big((n-b,b),m\big)_r\big| - \big|T\big((n-b,b),m\big)_{r-1}\big|$$

Proof. The set of (m)-tableaux with entries from $\mathbb{N}_0$ is totally ordered by comparing their entries, read left-to-right, by the lexicographic order. Let $<$ denote this total order. For example when $m = 3$ we have $\begin{array}{|c|c|c|}\hline 0&1&2\\\hline\end{array} < \begin{array}{|c|c|c|}\hline 0&1&3\\\hline\end{array} < \begin{array}{|c|c|c|}\hline 0&3&3\\\hline\end{array}$. Define a plethystic semistandard tableaux of shape $\big((n-b,b),(m)\big)$ to be an $(n-b,b)$-tableau T whose entries are (m)-tableaux, arranged in T so that they are weakly increasing under $<$ read along each row, and strictly increasing under $<$ read down each column. If for each $i \in \mathbb{N}$, the plethystic semistandard tableau T has exactly ω_i entries of i in its (m)-tableau entries, then we say T has weight ω and write $x^T = x_1^{\omega_1}x_2^{\omega_2}\ldots$. (Note that, by this definition, zero entries are not considered when computing the weight.) Let $\mathrm{PSSYT}\big((n-b,b),(m)\big)_\omega$ denote the set of plethystic semistandard tableau of shape $\big((n-b,b),(m)\big)$ and weight ω. For example

$$\begin{array}{|c|c|}\hline \begin{array}{|c|c|c|}\hline 0&0&0\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 0&0&1\\\hline\end{array} \\\hline \begin{array}{|c|c|c|}\hline 0&1&1\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 0&1&1\\\hline\end{array} \\\hline\end{array}\,, \quad \begin{array}{|c|c|}\hline \begin{array}{|c|c|c|}\hline 0&0&0\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 0&0&1\\\hline\end{array} \\\hline \begin{array}{|c|c|c|}\hline 0&1&1\\\hline\end{array} & \begin{array}{|c|c|c|}\hline 0&1&2\\\hline\end{array} \\\hline\end{array}$$

are in $\mathrm{PSSYT}\big((2,2),(3)\big)_{(5)}$ and $\mathrm{PSSYT}\big((2,2),(3)\big)_{(4,1)}$, respectively. By [dBPW21, Lemma 3.1], we have

$$s_{(n-b,b)} \circ s_{(m)} = \sum_T x^T$$

where the sum is over all plethystic semistandard tableaux T of shape $\big((n-b,b),(m)\big)$. Hence the coefficient of the monomial symmetric function labelled by the partition $(mn-r,r)$ in $s_{(n-b,b)} \circ s_{(m)}$ is $\big|\mathrm{PSSYT}\big((n-b,b),(m)\big)_{(r)}\big|$. By the duality between the complete homogeneous and monomial symmetric functions (see [Sta99, (7.30)]), this coefficient is $\langle s_{(n-b,b)} \circ s_{(m)}, h_{(mn-r,r)}\rangle$. Hence, by the relation $s_{(mn-r,r)} = h_{(mn-r,r)} - h_{(mn-r+1,r-1)}$, we get

$$\langle s_{(n-b,b)} \circ s_{(m)}, s_{(mn-r,r)}\rangle = \big|\mathrm{PSSYT}\big((n-b,b),(m)\big)_{(r)}\big| - \big|\mathrm{PSSYT}\big((n-b,b),(m)\big)_{(r-1)}\big|.$$

Finally observe that each (m)-tableau entry in a plethystic semistandard tableaux of weight (r) has entries from 0 and 1, with exactly r entries of 1, and so the set $\mathrm{PSSYT}\big((n-b,b),(m)\big)_{(r)}$ and the set $T\big((n-b,b),m\big)_r$ are in bijection by the map that replaces each (m)-tableau entry in a plethystic semistandard tableau by its number of 1s. Note that the map preserves the semistandard condition thanks to our choice of the order $<$ on (m)-tableaux. For example, the image of the plethystic semistandard tableaux shown left above is $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & 2 \\ \hline \end{array}$. $\square$

Let $\mathscr{PP}^m_b(r)$ denote the set of pairs of partitions (γ, π) such that $\gamma_1, \pi_1 \leqslant m$, $\ell(\gamma) = b$ and $|\gamma| + |\pi| = r$. Note there is no restriction on $\ell(\pi)$.

Lemma 11.2.
(i) *If* $n \geqslant r + b$ *and* $n \geqslant 2b$ *then* $\big|T\big((n-b,b),m\big)_r\big| = \big|\mathscr{PP}^m_b(r)\big|$.
(ii) *We have* $\big|T\big((r-1,b),m\big)_r\big| = \big|\mathscr{PP}^m_b(r)\big| - 1$.

Proof. For (i) we suppose that $n \geqslant r + b$. Let $t_{(i,j)}$ be the entry in box (i,j) of the tableau $t \in T\big((n-b,b),m\big)_r$. Suppose, for a contradiction, that $t_{(1,c)} \geqslant 1$ for some $c \leqslant b$; that is, there is an entry of 1 above a box in the second row of t. Since $t_{(2,j)} > t_{(1,j)}$ for each j, the sum of the entries of t, namely r, is at least

$$(n-b-c+1) + (c-1) + 2(b-c+1) = n + b - 2(c-1)$$

where $n-b-c+1$ counts entries in the boxes $(1,c), \ldots, (1,n-b)$, $c-1$ counts entries in the boxes $(2,1), \ldots, (2,c-1)$ and $2(b-c+1)$ counts entries in the boxes $(2,c), \ldots, (2,b)$. Therefore $r \geqslant (n+b) - 2(c-1)$, which given the hypothesis $n \geqslant r+b$, implies that $c > b$, a contradiction. Therefore $t_{(1,1)} = \ldots = t_{(1,b)} = 0$ and t is determined by the pair of partitions $\gamma = (t_{(2,b)}, \ldots t_{(2,1)})$ and $\pi = (t_{(1,n-b)}, \ldots, t_{(1,b+1)})$. Note that since $t_{(2,1)} \geqslant 1$, we have $\ell(\gamma) = b$. Therefore $(\gamma, \pi) \in \mathscr{PP}^m_b(r)$. For example, taking $n = 8$, $b = 3$, $r = 5$ and any $m \geqslant 2$,

$$\begin{array}{|c|c|c|c|c|}\hline 0&0&0&1&1\\ \hline 1&1&1 \\ \cline{1-3}\end{array}\,,\quad \begin{array}{|c|c|c|c|c|}\hline 0&0&0&0&2\\ \hline 1&1&1 \\ \cline{1-3}\end{array}\,,\quad \begin{array}{|c|c|c|c|c|}\hline 0&0&0&0&1\\ \hline 1&1&2 \\ \cline{1-3}\end{array}$$

are in bijection with $\big((1,1,1),(1,1)\big), \big((1,1,1),(2)\big), \big((2,1,1),(1)\big) \in \mathscr{PP}^m_3(5)$, respectively. Conversely, given $(\gamma,\pi) \in \mathscr{PP}^m_b(r)$, since $\ell(\pi) \leqslant |\pi| = r - |\gamma| \leqslant r - \ell(\gamma) = r - b \leqslant (n-b) - b = n - 2b$, we may reverse the process just to described to define a tableau $t \in T\big((n-b,b),m\big)_r$ whose image is (λ, π). This gives a bijection proving (i).

If instead $n = r + b - 1$ then the inverse map fails to be well-defined when $\ell(\pi) = r - b$ and $\ell(\gamma) = b$ and $\pi_1 \leqslant \gamma_{\ell(\gamma)}$. Thus the partition pair $(\gamma, \pi) = ((1^b),(1^{r-b})) \in \mathscr{PP}^m_b(r)$ is not the image of a tableau $t \in T\big((r+b-1-b,b),m\big)_r$. Since this is the only case where $\pi_1 \leqslant \gamma_{\ell(\gamma)}$, (ii) now follows from (i). $\square$

The subset of $\mathscr{PP}^m_b(r)$ of those partition pairs (γ,π) in which π has a singleton part is in bijection with $\mathscr{PP}^m_b(r-1)$ by removing the final part of π. Recall from Definition 10.5 that a b-marked partition of r is a pair (γ, ε) such that $|\gamma| + |\varepsilon| = r$, $\ell(\gamma) = b$ and ε has no singleton parts. Let $\mathscr{MP}^m_b(r)$ be the subset of $\mathscr{MP}_b(r)$ consisting of those marked partitions (γ, ε) for which $\gamma_1 \leqslant m$ and $\varepsilon_1 \leqslant m$. By this bijection we have

$$\big|\mathscr{PP}^m_b(r)\big| - \big|\mathscr{PP}^m_b(r-1)\big| = \big|\mathscr{MP}^m_b(r)\big|. \tag{11.1}$$

Recall that, by Theorem C, the ramified branching coefficient $\overline{rc}\big(\varnothing^{(b)},(r)\big)$ is the stable limit of the plethysm coefficients $p\big((n-b,b),(m),(mn-r,r)\big)$ for large m and n.

Corollary 11.3. *Let* m, $n \in \mathbb{N}$. *Let* $b \in \mathbb{N}_0$ *and let* $r \in \mathbb{N}$ *with* $r > b$.
(i) *If* $n \geqslant r + b$ *then* $p\big((n-b,b),(m),(mn-r,r)\big) = \big|\mathscr{MP}^m_b(r)\big|$ *and if* $m \geqslant r - b + [b \neq 0]$ *then each side is* $\overline{rc}\big(\varnothing^{(b)},(r)\big)$.
(ii) *If* $b > 0$ *then* $p\big((r,b),(r-b),((r-b)(r+b)-r,r)\big) = \overline{rc}\big(\varnothing^{(b)},(r)\big) - 1$.
(iii) *We have* $p\big((r),(r-1),(r^2-r,r)\big) = \overline{rc}\big(\varnothing^{\varnothing},(r)\big) - 1$.
(iv) *We have* $p\big((r-1,b),(m),m(r+b-1)-r,r\big) = \big|\mathscr{MP}^m_b(r)\big| - 1$ *and if* $m \geqslant r - b + [b \neq 0]$, *then each side is* $\overline{rc}\big(\varnothing^{(b)},(r)\big) - 1$.

Proof.

(i) By Proposition 11.1 and Lemma 11.2(i), when $n \geqslant r + b$ we have

$$\begin{aligned} p\big((n-b,b),(m),(mn-b,b)\big) &= \big|T\big((n-b,b),m\big)_r\big| - \big|T\big((n-b,b),m\big)_{r-1}\big| \\ &= \big|\mathscr{P}\mathscr{P}^m_b(r)\big| - \big|\mathscr{P}\mathscr{P}^m_b(r-1)\big|. \end{aligned}$$

The first claim now follows using (11.1). Suppose first of all that $b = 0$. If $(\gamma,\pi) \in \mathscr{M}\mathscr{P}^m_b(r)$ then $\gamma = \varnothing$ and the largest possible part in π is r. Since by hypothesis, $m \geqslant r$, this restriction has no force, and so $\mathscr{M}\mathscr{P}^m_0(r) = \mathscr{M}\mathscr{P}_0(r)$. Thus the second claim follows from $\overline{rc}\big(\varnothing^{\varnothing},(r)\big) = \big|\mathscr{M}\mathscr{P}_0(r)\big|$. Now suppose $b > 0$. If $(\gamma,\pi) \in \mathscr{M}\mathscr{P}^m_b(r)$ then since $\ell(\gamma) = b$, the largest possible part in γ is $r-(b-1)$ and the largest possible part in π is $r-b$. Since $m \geqslant r-b+[b\neq 0] = r-b+1$ by hypothesis, again the restriction that $\gamma_1 \leqslant m$ and $\pi_1 \leqslant m$ has no force. Arguing in the same way as the case $b = 0$ we now have $\mathscr{M}\mathscr{P}^m_b(r) = \mathscr{M}\mathscr{P}_b(r)$ and $\overline{rc}\big(\varnothing^{(b)},(r)\big) = \big|\mathscr{M}\mathscr{P}_b(r)\big|$ and the second claim follows in the same way.

(ii) We have $n = r+b$. The unique pair in $\mathscr{M}\mathscr{P}^{r-b+1}_b(r)$ not present in $\mathscr{M}\mathscr{P}^{r-b}_b(r) = \mathscr{M}\mathscr{P}_b(r)$ is $\big((r-b+1,1^{b-1}),\varnothing\big)$, therefore (ii) follows from both parts of (i).

(iii) Again we have $n = r+b$, since now $b = 0$. The unique pair in $\mathscr{M}\mathscr{P}^{r-1}_0(r)$ not present in $\mathscr{M}\mathscr{P}^r_0(r) = \mathscr{M}\mathscr{P}_0(r)$ is $\big((r),\varnothing\big)$, therefore (iii) follows similarly from both parts of (i).

(iv) We have $n = r+b-1$. Arguing similarly to (i) using Proposition 11.1 and then Lemma 11.2(ii) for the first summand and Lemma 11.2(i) for the second, we have

$$\begin{aligned} p\big((r-1,b),(m),(m(r+b-1)-r,r)\big) &= \big|T\big((r-1,b),m\big)_r\big| - 1 - \big|T\big((r-1,b),m\big)_{r-1}\big| \\ &= \big|\mathscr{P}\mathscr{P}^m_b(r)\big| - \big|\mathscr{P}\mathscr{P}^m_b(r-1)\big| - 1. \end{aligned}$$

The first claim now follows as in (i) using (11.1). As seen in (i), when $m \geqslant r-b+[b\neq 0]$ we have $\mathscr{M}\mathscr{P}^m_b(r) = \mathscr{M}\mathscr{P}_b(r)$ and so the second claim follows from $\overline{rc}\big(\varnothing^{(b)},(r)\big) = \big|\mathscr{M}\mathscr{P}_b(r)\big|$. □

We summarise the results of this section in the following corollary.

Corollary 11.4. *Let $b \in \mathbb{N}_0$. Let $r, b \in \mathbb{N}$ with $r > b$. When $\beta = (b)$ and $\kappa = (r)$ the bounds $m \geqslant r-b+[b\geqslant 1]$ and $n \geqslant r+\beta_1$ in Theorems A and C cannot be weakened.*

Proof. By Corollary 11.3(ii), if $b > 0$, then $p\big((r,b),(r-b),(m(r+b)-r,r)\big)$ is one less than the stable value, which is attained when m increases from $r-b$ to $r-b+[b\neq 0] = r-b+1$. By Corollary 11.3(iii) $p\big((r),(r-1),(r^2-r,r)\big)$ is again one less than the stable value, which is attained when m increases from $r-1$ to $r-b+[b\neq 0] = r$. (Note that here $b = 0$.) By Corollary 11.3(iv), $p\big((r-1,b),(m),m(r+b-1)-r,r\big)$ is one less than the stable value, attained when n increases from $r+b-1$ to $r+b$. □

We finish with a corollary for modules for the ramified partition algebra, showing that the bounds $m \geqslant r-|\beta|+[\beta\neq 0]$ and $n \geqslant r+\beta_1$ in Theorem C cannot be weakened in infinitely many cases. We remark that an alternative proof is given by reading the outline in Section 1.7, noting that the only use of these bounds is in step (f): thus if we assume, for a contradiction, that $L_r(\varnothing^\beta) = \Delta_r(\varnothing^\beta)$ then we obtain that the plethysm coefficient $p\big(\beta[n],(m),\kappa[mn]\big)$ is equal to its stable value; this contradicts Corollary 11.3 for appropriately chosen m, n and partitions β, κ.

Corollary 11.5. *There exist infinitely many partitions β such that if $n < r+\beta_1$ or $m < r-|\beta|+[\beta\neq\varnothing]$ then $L_r(\varnothing^\beta)$ is a proper quotient of $\Delta_r(\varnothing^\beta)$ and the inequality in Theorem C is strict.*

Proof. By Theorem B we have $p(\beta[n], \varnothing, \kappa[mn]) = \left[L_r(\varnothing^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\right]_{P_r(mn)}$ for each $\kappa \vdash r$. Hence by Theorem C,

$$p(\beta[n], \varnothing, \kappa[mn]) \leqslant \left[\Delta_r(\varnothing^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\right]_{P_r(mn)}$$

with equality if every composition factor $L_r(\kappa)$ of $\Delta_r(\varnothing^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)}$ appears in $L_r(\varnothing^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)}$, with equal multiplicity. Therefore, taking the contrapositive of the case for equality, if

$$p(\beta[n], \varnothing, \kappa[mn]) < \left[\Delta_r(\varnothing^\beta)\big\downarrow^{R_r(m,n)}_{P_r(mn)} : L_r(\kappa)\right]_{P_r(mn)}$$

then $\Delta_r(\varnothing^\beta)\downarrow^{R_r(m,n)}_{P_r(mn)}$ is not simple. The right-hand side in the two displayed equation above is the ramified branching coefficient $\overline{rc}(\varnothing^\beta, \kappa)$ found in Theorem C, or its equivalent restatement, Theorem 10.11. Since by Theorem C the ramified branching coefficient is the stable limit of the plethysm coefficients $p(\beta[n], (m), \kappa[mn])$ for large m and n, the result now follows from Corollary 11.4. □

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