Integrable maps in 4D and modified Volterra lattices

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Abstract

In recent work, we presented the construction of a family of difference equations associated with the Stieltjes continued fraction expansion of a certain function on a hyperelliptic curve of genus g. As well as proving that each such discrete system is an integrable map in the Liouville sense, we also showed it to be an algebraic completely integrable system. In the discrete setting, the latter means that the generic level set of the invariants is an affine part of an abelian variety, in this case the Jacobian of the hyperelliptic curve, and each iteration of the map corresponds to a translation by a fixed vector on the Jacobian. In addition, we demonstrated that, by combining the discrete integrable dynamics with the flow of one of the commuting Hamiltonian vector fields, these maps provide genus g algebro-geometric solutions of the infinite Volterra lattice, which justified naming them *Volterra maps*, denoted $\mathcal{V}_g$.

The original motivation behind our work was the fact that, in the particular case $g = 2$, we could recover an example of an integrable symplectic map in four dimensions found by Gubbiotti, Joshi, Tran and Viallet, who classified birational maps in 4D admitting two invariants (first integrals) with a particular degree structure, by considering recurrences of fourth order with a certain symmetry. Hence, in this particular case, the map $\mathcal{V}_2$ yields genus two solutions of the Volterra lattice. The purpose of this note is to point out how two of the other 4D integrable maps obtained in the classification of Gubbiotti et al. correspond to genus two solutions of two different forms of the modified Volterra lattice, being related via a Miura-type transformation to the $g = 2$ Volterra map $\mathcal{V}_2$.

We dedicate this work to a dear friend and colleague, Decio Levi.

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1 Introduction

This short article consists of some recollections of our colleague Decio Levi (in section 2 below), followed by a brief update on our recent results about integrable maps in four (and higher) dimensions, which provide algebro-geometric solutions of differential-difference equations of Volterra type [10]. Decio was one of the pioneers in the theory of integrability for differential-difference equations, especially in the construction of integrable lattices from Bäcklund transformations for continuous systems [13, 15], and the programme of applying the symmetry approach to the classification of such lattices, which he initiated with Yamilov [14]. Thus we like to think that Decio would have appreciated the results being presented here.

After presenting a few memories of Decio, in section 3 we begin by giving a brief overview of the 4D integrable maps which were classified by Gubbiotti et al. [7]. We then proceed to review our construction of integrable maps obtained from the Stieltjes fraction expansion of certain functions on hyperelliptic curves [10], and explain how it reproduces one of the examples from [7], denoted (P.iv), in the particular case of genus two curves. Sections 4 and 5 are devoted to the maps (P.v) and (P.vi), respectively: we show how each of these maps is related to a different form of the modified Volterra lattice, and present explicit formulae which relate their solutions to the solutions of (P.iv) via a transformation of Miura type. We end with some very short conclusions in section 6.

2 Memories of Decio Levi

Andrew Hone writes: I first met Decio in Warsaw in September 1995, when I was a PhD student participating in the 1st Non-Orthodox School on Nonlinearity and Geometry [23]. Decio was one of the lecturers, along with Orlando Ragnisco, and it was thanks to extended conversations with Orlando that I resolved to apply for postdoctoral funding to work with him when I finished my PhD. After receiving a grant from the Leverhulme Trust two years later, I finally got to be a researcher at Roma Tre, where Orlando and Decio were both professors in the Dipartimento di Fisica.

For approximately the first six months of my time in Rome, there was no available office space for postdocs, which meant that I had to share an office with Orlando. Far from being a negative aspect of my experience, this situation had many positive benefits for me, and not just scientific ones. By working in close proximity with Orlando, it meant that I was privy to the regular visits from the neighbour in the office next door, namely Decio, his long-time friend and collaborator. Apart from the pleasure of getting to know Decio, and learning many wonderful ideas about integrable systems from him, there was the fact that, by default, he would chat to Orlando in Italian, which helped me to rapidly improve my grasp of the language in those first few months. The strong bond of friendship between Orlando and Decio created a very happy atmosphere, and I have extremely fond memories of those times.

In subsequent years, I would see Decio fairly often at various international conferences, or during return visits to Rome. He had an amiable manner and a warm, cheerful smile. It was always enjoyable to talk to him, whether about technical problems, sharing family news, or just musing about life in general. Talking with Decio would leave me feeling reassured, that all was right with the world, and I liked his gentle way of concluding a long conversation with "Vabbè in somma".

It is an honour to be able to remember Decio here, both for his contributions as a scientist, and as a wonderful human being.

Federico Zullo writes: The first time I met Decio was in 2003: I was a student at the Dipartimento di Fisica of Roma Tre University and needed an advisor for my last examination for my laurea triennale (bachelor's degree). I asked Orlando Ragnisco who, at that time, was very busy. He accompanied me to the office next door, where Decio was, and I asked him for a theme for my short dissertation. He very heartily introduced me to the subject of solitons, that I never heard about before, giving me books and kind advice. Later, during my laurea magistrale (master's degree), and during my PhD studies, I followed different classes taught by Decio, some with very few students. The familiar atmosphere and natural mildness of Decio's classes fostered my learning, and I'm greatly indebted to him for having taught me many topics used in

mathematical physics, like group theory, symmetries of differential equations, physics of nonlinear systems, qualitative and quantitative analysis of solutions of differential equations and others. For my own teaching, I still use some of the material that I collected from his courses.

For a period just before 2014, I was hosted by Decio in his office as a researcher. I remember the talks on disparate subjects, like religion, literature, politics, society and, obviously, our research. The talks would then continue during the lunch break, usually in Via Marconi, with Orlando and the other members of the very stimulating group of young researchers that was gathered at Roma Tre in that period, including Fabio Musso, Matteo Petrera, Christian Scimiterna, Danilo Riglioni, Riccardo Droghei, and later Giorgio Gubbiotti and Danilo Latini, all led by Decio and Orlando. I'll always keep these beautiful memories with me.

3   The map (P.iv) and the geometry of its solutions

Discrete integrable systems can be constructed by applying an appropriate discretization procedure to continuous ones, and historically this is how many examples of discrete integrability were first discovered [13, 20]. However, from both a theoretical point of view and a practical one, it is important to have a notion of integrability for discrete systems that does not require making reference to some underlying continuous system, whether this be for lattice equations [14], or for integrable maps [2, 16, 22]. While integrable maps in two and three dimensions lead to families of invariant curves (as the level sets of first integrals), the case of four dimensions can lead to new features, namely invariant tori of dimension two.

In [7], Gubbiotti et al. presented a classification of four-dimensional birational maps of recurrence type, that is

$$\varphi: \qquad (w_0, w_1, w_2, w_3) \mapsto \Big(w_1, w_2, w_3, F(w_0, w_1, w_2, w_3)\Big), \qquad (3.1)$$

for a suitable rational function F of the affine coordinates $(w_0, w_1, w_2, w_3) \in \mathbb{C}^4$, where the map φ is required to be invariant under the involution $\iota: (w_0, w_1, w_2, w_3) \mapsto (w_3, w_2, w_1, w_0)$, and to possess two independent polynomial invariants, H_1, H_2 say, with specific degree patterns $(\deg_{w_0} H_j, \deg_{w_1} H_j, \deg_{w_2} H_j, \deg_{w_3} H_j) = (1, 3, 3, 1)$ and $(2, 4, 4, 2)$ for $j = 1, 2$, respectively. The result of this classification was six maps with parameters, labelled (P.i-vi), together with six associated maps, denoted (Q.i-vi) respectively. Each of the "Q" maps arises from a corresponding "P" map, as a discrete integrating factor for linear combinations of the first integrals, so they are dual to one another in the sense of [17].

As described previously, first in [11] and then [6], the original motivation for classifying such maps was to understand autonomous versions of the fourth-order members of hierarchies of discrete Painlevé I/II equations from [5]; but, aside from the latter connection, the "P" in this nomenclature has nothing to do with the usual labelling of continuous Painlevé equations. From our point of view, the most interesting cases are the maps labelled (P.iv), (P.v) and (P.vi), since (from Table 1 in [7]) these are the only ones arising from a discrete variational principle (Lagrangian), leading to a non-degenerate Poisson bracket in four dimensions, such that the two first integrals H_1, H_2 are in involution; this means that in the real case the Liouville tori are two-dimensional. Subsequently, Gubbiotti obtained these 4D integrable maps via an alternative method, by classifying fourth-order difference equations with a discrete Lagrangian structure [8].

Here we begin with the case of (P.iv), which is the birational map given in affine coordinates by the recurrence

$$
\begin{aligned}
&w_{n+4}w_{n+3}w_{n+2} + w_{n+2}w_{n+1}w_n + 2w_{n+2}^2(w_{n+3} + w_{n+1}) \\
&+ w_{n+2}(w_{n+3}^2 + w_{n+3}w_{n+1} + w_{n+1}^2) + w_{n+2}^3 + \nu w_{n+2}(w_{n+3} + w_{n+2} + w_{n+1}) + bw_{n+2} + a = 0.
\end{aligned}
\qquad (3.2)
$$

This map has three essential parameters a, b, ν (in the formulae from [7] we have set the parameter $d = 1$, which can be achieved by a simple rescaling), and it is of the form (3.1), with

$$F = -\frac{w_0 w_1 w_2 + w_1 w_2 w_3 + w_1^2 w_2 + w_2 w_3^2 + 2w_1 w_2^2 + 2w_2^2 w_3 + w_2^3 + \nu(w_1 w_2 + w_2 w_3 + w_2^2) + bw_2 + a}{w_2 w_3};$$

this F is the rational function of w_0, w_1, w_2, w_3 obtained by solving for w_4 in (3.2) with $n = 0$.

The first integral denoted $I_{\text{low}}^{\text{P.iv}}$ in [7] is given in affine coordinates by

$$H_1 = w_1 w_2 \Big(w_2 w_3 + w_0 w_1 - w_0 w_3 + (w_1 + w_2)^2 + \nu(w_1 + w_2) + b \Big) + a(w_1 + w_2). \tag{3.3}$$

The latter has the degree pattern $(1, 3, 3, 1)$. In particular, it is linear in w_3, which implies that, on each three-dimensional level set $H_1 = h_1 = \text{const}$, the map (3.2) reduces to a birational map in three dimensions, given by the recurrence

$$w_{n+3} w_{n+2} w_{n+1}(w_{n+2} - w_n) + w_{n+2} w_{n+1}^2 w_n + w_{n+2} w_{n+1}(w_{n+1} + w_{n+2})^2$$
$$+\nu\, w_{n+2} w_{n+1}(w_{n+1} + w_{n+2}) + b\, w_{n+2} w_{n+1} + a\, (w_{n+1} + w_{n+2}) \;\; = \;\; h_1.$$

A second independent invariant for (3.2), with degree pattern $(2, 4, 4, 2)$, is given by

$$H_2 \;\; = \;\; w_1 w_2 \begin{pmatrix} w_0^2 w_1 + w_3^2 w_2 + w_0 w_3(w_1 + w_2) + w_0(w_2^2 + 2w_1^2) + w_3(w_1^2 + 2w_2^2) \\ + 3(w_0 + w_3)w_1 w_2 + (w_1 + w_2)^3 \\ +\nu\left(w_0 w_3 + (w_0 + w_3)(w_1 + w_2) + (w_1 + w_2)^2 \right) + b\,(w_0 + w_1 + w_2 + w_3) \end{pmatrix}$$
$$+a\left(w_0 w_1 + w_3 w_2 + (w_1 + w_2)^2 \right). \tag{3.4}$$

This differs slightly from the second invariant presented in [7], which is $I_{\text{high}}^{\text{P.iv}} = H_2 - \nu H_1$.

The nondegenerate Poisson bracket between the coordinates, which was obtained in [7] by making use of a discrete Lagrangian for (3.2), is given by

$$\{\, w_n, w_{n+1} \,\} = 0, \;\; \{\, w_n, w_{n+2} \,\} = \frac{1}{w_{n+1}}, \;\; \{\, w_n, w_{n+3} \,\} = -\frac{w_n + 2w_{n+1} + 2w_{n+2} + w_{n+3} + \nu}{w_{n+1} w_{n+2}}, \tag{3.5}$$

for all n. So (3.1) is a Poisson map, in the sense that $\{\, \varphi^* G, \varphi^* H \,\} = \varphi^* \{\, G, H \,\}$ for all functions G, H on $\mathbb{C}^4$. The two independent invariants given in [7] are in involution with respect to this bracket, which is equivalent to the involutivity of functions (3.3) and (3.4), that is to say

$$\{\, H_1, H_2 \,\} = 0.$$

Hence the four-dimensional map defined by (3.2) is integrable in the Liouville sense.

Computing the Hamiltonian vector field for the first flow, generated by H_1, we find that this takes the form

$$\frac{\mathrm{d}w_n}{\mathrm{d}t} = w_n(w_{n+1} - w_{n-1}) \tag{3.6}$$

for $n = 1, 2$. However, since (3.2) is a Poisson map that commutes with this flow, it follows that the relation (3.6) extends to all $n \in \mathbb{Z}$. Thus the combined solutions of the map and the flow, which are compatible with one another, generate a sequence of functions $\big(w_n(t)\big)_{n \in \mathbb{Z}}$ satisfying (3.6), which is the Volterra lattice equation, first considered by Kac and van Moerbeke [12]. Hence, in a certain sense that can be made precise, these will turn out to be genus 2 solutions of this lattice hierarchy.

The complex geometry of the solutions of the map defined by (3.2) is related to a family of hyperelliptic curves of genus 2, given by the Weierstrass quintic

$$\Gamma: \quad y^2 = (1 + \nu x + bx^2)^2 + 4a(1 + \nu x + bx^2)x^3 + 4h_1 x^4 + 4(h_2 + \nu h_1)x^5. \tag{3.7}$$

On any genus 2 curve Γ of the above form, we take the meromorphic function F given by

$$F = \frac{y + \mathcal{P}(x)}{\mathcal{Q}(x)} = \frac{\mathcal{R}(x)}{y - \mathcal{P}(x)}, \tag{3.8}$$

where $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are polynomials in the spectral parameter x, given by

$$\mathcal{P}(x) = 1 + p_1 x + p_2 x^2, \quad \mathcal{Q}(x) = 2 + q_1 x + q_2 x^2, \quad \mathcal{R}(x) = r_1 x + r_2 x^2 + r_3 x^3, \tag{3.9}$$

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which are required to satisfy

$$\mathcal{P}(x)^2 + \mathcal{Q}(x)\mathcal{R}(x) = f(x), \tag{3.10}$$

with $f(x) = (1 + \nu x + bx^2)^2 + 4a(1 + \nu x + bx^2)x^3 + 4h_1x^4 + 4(h_2 + \nu h_1)x^5$ being the quintic on the right-hand side of (3.7). Then the key to the construction in [10] is to expand the function F as a continued fraction of Stieltjes type (S-fraction), that is

$$F = 1 - \cfrac{w_1 x}{1 - \cfrac{w_2 x}{1 - \cfrac{w_3 x}{1 - \cdots}}}, \tag{3.11}$$

and by iterating from one line of the fraction to the next we find that we obtain a recurrence for the coefficients w_j. More precisely, the non-trivial coefficients of the polynomials (3.9) are given in terms of w_j and the parameters by

$$p_1 = 2w_0 + \nu, \; p_2 = 2w_0(w_1 + w_0 + w_{-1}) + b, \; \tfrac{1}{2}q_1 = w_0 + w_1 + \nu, \; \tfrac{1}{2}q_2 = w_0w_{-1} + w_1w_2 + (w_1 + w_0)^2 + \nu(w_0 + w_1) + b,$$

and $r_1 = -2w_0$; there are similar (but slightly more unwieldy) expressions for r_2 and r_3, which are omitted here, but are easily obtained from the relation (3.10). With these identifications, the iteration of the S-fraction (3.11) for F becomes precisely the map (P.iv) in terms of the affine coordinates w_j, as given by (3.2).

In [10] it was also shown that each iteration of the continued fraction is equivalent to the discrete Lax equation

$$\mathbf{L}(x)\mathbf{M}(x) = \mathbf{M}(x)\widetilde{\mathbf{L}}(x) \,,$$

where

$$\mathbf{L}(x) := \begin{pmatrix} \mathcal{P}(x) & \mathcal{R}(x) \\ \mathcal{Q}(x) & -\mathcal{P}(x) \end{pmatrix}, \qquad \mathbf{M}(x) := \begin{pmatrix} 1 & -w_1 x \\ 1 & 0 \end{pmatrix}.$$

Furthermore, we found that each generic common level set of the two invariants H_1, H_2 is isomorphic to an affine part of the Jacobian of the associated spectral curve Γ (or rather, of its completion), and each iteration of the map corresponds to a translation on the Jacobian by the divisor class $[(0, -1) - \infty]$. Thus, in addition to being integrable in the Liouville sense, the map (3.2) is an algebraic completely integrable system, being a discrete analogue of an a.c.i. system (see [1, 21]).

The map (3.2) can also be rewritten in terms of tau functions τ_n, related to w_n via

$$w_n = \frac{\tau_n \tau_{n+3}}{\tau_{n+1}\tau_{n+2}}.$$

These tau functions satisfy a Somos-9 recurrence, that is

$$\alpha_1 \, \tau_{n+9}\tau_n + \alpha_2 \, \tau_{n+8}\tau_{n+1} + \alpha_3 \, \tau_{n+7}\tau_{n+2} + \alpha_4 \, \tau_{n+6}\tau_{n+3} + \alpha_5 \, \tau_{n+5}\tau_{n+4} = 0, \tag{3.12}$$

with coefficients α_j that depend on a, b, ν and the values of H_1, H_2 along each orbit of (3.2); for details see Proposition 2.1 in [10]. Using the S-fraction (3.11), we were also able to write explicit Hankel determinant formulae for these tau functions τ_n, analogous to results for Somos sequences in genus 1 [3], and other Hankel determinant formulae for solutions of the Volterra lattice [4]. Furthermore, we found a Miura map relating the solutions of (P.iv) to one of the maps derived from J-fractions in [9], using the classical method of contraction of continued fractions due to Stieltjes [19] (see also [18]), which in this case turned out to provide solutions of the infinite Toda lattice.

In what follows, we will present analogous properties for the maps (P.v) and (P.vi), and point out how they are closely connected to (P.iv).

4 The map (P.v)

The map (P.v) is given by the recurrence

$$w_{n+4}w_{n+3}^2w_{n+2}^2 + w_{n+2}^2w_{n+1}^2w_n + w_{n+2}^3(w_{n+1} + w_{n+3})^2 + \tilde{\nu}w_{n+2}^2(w_{n+1} + w_{n+3}) + \tilde{c}w_{n+2} + \tilde{a} = 0, \quad (4.1)$$

with three essential parameters $\tilde{a}, \tilde{c}, \tilde{\nu}$ (compared with [7] we have put tildes here to distinguish them from the parameters in (3.2), and rescaled so that the parameter $d \to 1$).

The lowest degree first integral of the map defined by (4.1), with degree pattern $(1, 3, 3, 1)$, is given by

$$H_1 = w_3w_2^3w_1^2 + w_2^2w_1^3w_0 - w_3w_2^2w_1^2w_0 + w_2^3w_1^3 + \tilde{\nu}w_2^2w_1^2 + \tilde{c}w_2w_1 + \tilde{a}(w_2 + w_1), \quad (4.2)$$

and this is the same as $I_{\text{low}}^{\text{P.v}}$ in [7]. Another first integral, with degree pattern $(2, 4, 4, 2)$, is

$$
\begin{aligned}
H_2 &= w_2^2w_1^2\Big((w_3w_2 + w_1w_0 + w_2w_1)^2 + \tilde{\nu}(w_3 + w_1)(w_2 + w_0)\Big) \\
&\quad + \tilde{c}w_2w_1(w_3w_2 + w_1w_0 + w_2w_1) + \tilde{a}(w_3w_2^2 + w_1^2w_0 + w_2^2w_1 + w_2w_1^2).
\end{aligned}
\quad (4.3)
$$

The second invariant presented in [7] is $I_{\text{high}}^{\text{P.v}} = H_2 - \tilde{\nu}H_1$.

The nondegenerate Poisson bracket between the coordinates is given by

$$\{\, w_n, w_{n+1} \,\} = 0, \ \{\, w_n, w_{n+2} \,\} = \frac{1}{w_{n+1}^2}, \ \{\, w_n, w_{n+3} \,\} = -\frac{2(w_nw_{n+1} + w_{n+1}w_{n+2} + w_{n+2}w_{n+3}) + \tilde{\nu}}{w_{n+1}^2 w_{n+2}^2}.$$

The independent first integrals (4.2) and (4.3) are in involution with respect to this bracket, which shows that the map (4.1) is Liouville integrable.

Computing the Hamiltonian vector field for the first flow, generated by H_1, we find that this takes the form

$$\frac{\mathrm{d}w_n}{\mathrm{d}t} = w_n^2(w_{n+1} - w_{n-1}) \quad (4.4)$$

for $n = 1, 2$. However, since the map (4.1) is Poisson and commutes with the flow $\{\cdot, H_1\}$, the equation (4.4) holds for all $n \in \mathbb{Z}$. Thus the compatible solutions of the map and the flow together provide a sequence of functions $\big(w_n(t)\big)_{n \in \mathbb{Z}}$ which satisfy (4.4), which is a degenerate case of the modified Volterra lattice equation [25].

If we make the tau function substitution

$$w_n = \frac{\tau_n \tau_{n+2}}{\tau_{n+1}^2} \quad (4.5)$$

for (P.v), then we find that the sequence (τ_n) satisfies a Somos-8 relation. More precisely, by direct computer algebra calculations we can show the following:

Proposition 4.1. *Whenever w_n is a solution of (4.1), the sequence (τ_n) satisfies the following Somos-8 recurrence, with coefficients that are functions of the Hamiltonians H_1, H_2 as in (4.2) and (4.3) above (constant along each orbit):*

$$\alpha_1\,\tau_{n+8}\tau_n + \alpha_2\,\tau_{n+7}\tau_{n+1} + \alpha_3\,\tau_{n+6}\tau_{n+2} + \alpha_4\,\tau_{n+5}\tau_3 + \alpha_5\,\tau_{n+4}^2 = 0, \quad (4.6)$$

where the coefficients are given by

$$\alpha_1 = H_1, \qquad \alpha_2 = \tilde{a}H_2, \qquad \alpha_3 = \tilde{a}^2H_2 - H_1^3,$$
$$\alpha_4 = \tilde{a}\Big(H_2^2 + \tilde{\nu}H_1H_2 + \tilde{c}H_1^2 + \tilde{a}^2H_1\Big), \qquad \alpha_5 = -H_1\Big(H_2^2 + \tilde{\nu}H_1H_2 + \tilde{c}H_1^2 + \tilde{a}^2H_1\Big).$$

Let us denote a solution of the Volterra lattice (3.6) by $\hat{w}_n$. Then the Miura map from the modified Volterra lattice (4.4) takes the form

$$\hat{w}_n = w_{n+1}w_n. \quad (4.7)$$

This Miura map remains valid at the level of the maps (3.2) and (4.1), in the following sense.

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Theorem 4.2. *Let w_n be a solution of (4.1) with parameters $\tilde{a}, \tilde{c}, \tilde{\nu}$, lying on the level set $H_1 = \tilde{h}_1$, $H_2 = \tilde{h}_2$, of the first integrals (4.2) and (4.3). Then $\hat{w}_n$ given by the Miura map (4.7) is a solution of (3.2) with parameters*

$$\nu = \tilde{\nu}, \qquad b = \tilde{c}, \qquad a = \tilde{h}_1.$$

Furthermore, on this solution $\hat{w}_n$, the values h_1, h_2 of the first integrals (3.3) and (3.4) for the map (3.2) are given by

$$h_1 = \tilde{h}_2, \qquad h_2 = -\tilde{a}^2 - \tilde{\nu}\tilde{h}_2 - \tilde{c}\tilde{h}_1.$$

Proof: The first part of this result is verified by substituting the Miura formula (4.7) directly into (3.2), using (4.1) to eliminate w_{n+5} followed by w_{n+4}, and then using the formula for H_1 in (4.2) to eliminate w_{n+3} on the level set $H_1 = \tilde{h}_1$. Analogous calculations, rewriting (3.3) and (3.4) in terms of w_n satisfying (4.1) and comparing with $\tilde{h}_2$, the value of the first integral (4.3) for the latter map, yield the above expressions for h_1, h_2. $\qquad\square$

It is worth commenting on the meaning of the Miura formula (4.7), restricted to this finite-dimensional setting. Given initial data w_0, w_1, w_2, w_3 for the map (4.1), we can fix a level set $H_1 = \tilde{h}_1$ to write

$$\hat{w}_0 = w_0 w_1, \ \hat{w}_1 = w_1 w_2, \ \hat{w}_2 = w_2 w_3, \ \hat{w}_3 = w_3 \, G(w_1, w_2, w_3, \tilde{h}_1),$$

for some rational function G, obtained by using the formula (4.2) for H_1 to eliminate w_4. Similarly, we can use H_1 to eliminate w_0 above in terms of w_1, w_2, w_3 and $\tilde{h}_1$, and after taking resultants we can do further elimination to solve for each of w_0, w_1, w_2, w_3 as algebraic functions of $\hat{w}_0, \hat{w}_1, \hat{w}_2, \hat{w}_3$ and $\tilde{h}_1$. So this leads to an explicit inverse of (4.7), at least in the form of an algebraic correspondence.

5 The map (P.vi)

The map (P.vi) is given by

$$
\begin{aligned}
&w_{n+4}(w_{n+3}^2 - \delta^2)(w_{n+2}^2 - \delta^2) + w_n(w_{n+1}^2 - \delta^2)(w_{n+2}^2 - \delta^2) \\
&+ w_{n+2}\Big((w_{n+2}^2 - \delta^2)(w_{n+3} + w_{n+1})^2 + \bar{c} - \delta^4\Big) + \bar{\nu}(w_{n+2}^2 - \delta^2)(w_{n+3} + w_{n+1}) + \bar{a} = 0.
\end{aligned}
\tag{5.1}
$$

This depends on only three essential parameters $\bar{a}, \bar{c}, \bar{\nu}$; compared with [7] we have replaced $a \to \bar{a}$, $c \to \bar{c}$, $d \to -\bar{\nu}$ and $\delta \to \delta^2$. Note the map P(v) in the previous section arises from P(vi) in the limit $\delta \to 0$, while for $\delta \neq 0$ the map can always be rescaled so that $\delta \to 1$, but it will be convenient to retain this parameter which has the same weight as w_n in (5.1).

The lowest degree first integral of the map defined by (5.1), with degree pattern $(1, 3, 3, 1)$, is given by

$$
\begin{aligned}
H_1 &= \big(w_1^2 w_2^2 - \delta^2(w_1^2 + w_2^2)\big)\Big(w_3 w_2 + w_0 w_1 + w_1 w_2 - w_3 w_0 + \bar{\nu}\Big) \\
&\quad \delta^4(w_3 w_2 + w_0 w_1 - w_0 w_3) + \bar{c} w_2 w_1 + \bar{a}(w_2 + w_1).
\end{aligned}
\tag{5.2}
$$

A nondegenerate Poisson bracket between the coordinates is given by

$$
\{w_n, w_{n+1}\} = 0, \ \{w_n, w_{n+2}\} = \frac{1}{w_{n+1}^2 - \delta^2}, \ \{w_n, w_{n+3}\} = -\frac{2(w_n w_{n+1} + w_{n+1} w_{n+2} + w_{n+2} w_{n+3}) + \bar{\nu}}{(w_{n+1}^2 - \delta^2)(w_{n+2}^2 - \delta^2)},
$$

and was derived in [7] using a discrete Lagrangian structure for (5.1). A second independent first integral H_2 was given in [7], which is in involution with H_1 with respect to this bracket. Here we take the second independent quantity as

$$
\begin{aligned}
H_2 &= (w_1^2 - \delta^2)(w_2^2 - \delta^2)^2\, w_3^2 + (w_1^2 - \delta^2)^2(w_2^2 - \delta^2)\, w_0^2 + (2w_1 w_2 + \bar{\nu})(w_1^2 - \delta^2)(w_2^2 - \delta^2)\, w_3 w_0 \\
&+ \big(2w_1^3 w_2^2 + \bar{\nu} w_1^2 w_2 + \bar{c} w_1 + \bar{a} - (2w_1 w_2^2 + \bar{\nu} w_2)\delta^2 - w_1 \delta^4\big)(w_2^2 - \delta^2)\, w_3 \\
&+ \big(2w_1^2 w_2^3 + \bar{\nu} w_1 w_2^2 + \bar{c} w_2 + \bar{a} - (2w_1^2 w_2 + \bar{\nu} w_1)\delta^2 - w_2 \delta^4\big)(w_1^2 - \delta^2)\, w_0 \\
&+ w_1^4 w_2^4 + \bar{\nu} w_1^3 w_2^3 + \bar{c} w_1^2 w_2^2 + \bar{a} w_1 w_2(w_1 + w_2) \\
&- \Big((w_1^2 w_2^2 + \bar{\nu} w_1 w_2)(w_1^2 + w_2^2) + \bar{a}(w_1 + w_2)\Big)\delta^2 + (w_1^2 w_2^2 + \bar{\nu} w_1 w_2 - \bar{c})\delta^4 - (w_1^2 + w_2^2)\delta^6;
\end{aligned}
\tag{5.3}
$$

so the map (5.1) is Liouville integrable.

The Hamiltonian vector field for the first flow, generated by H_1, takes the form

$$\frac{\mathrm{d}w_n}{\mathrm{d}t} = (w_n^2 - \delta^2)(w_{n+1} - w_{n-1}) \tag{5.4}$$

for $n = 1, 2$, and once again, since the Poisson map (5.1) is compatible with the flow $\{\cdot, H_1\}$, the equation (5.4) holds for all $n \in \mathbb{Z}$, and thus the map and the flow together produce a sequence of functions $(w_n(t))_{n \in \mathbb{Z}}$ satisfying (5.4), which (up to rescaling) is the general form of the modified Volterra lattice equation. If we set $\delta \to 0$ in (5.4), then the equation (4.4) is recovered, corresponding to the same limit that reproduces (4.1) as a degenerate case of (5.1). However, the behaviour of the degenerate map (4.1) is sufficiently different compared with (5.1) e.g. with respect to singularity structure, that it is worth giving it a separate analysis as we have done here.

Let us denote a solution of the Volterra lattice (3.6) by $\hat{w}_n$. Then the Miura map from the modified Volterra lattice (5.4) takes the form

$$\hat{w}_n = (w_{n+1} \mp \delta)(w_n \pm \delta), \tag{5.5}$$

(so there are effectively two maps, with an opposite choice of sign in each factor on the right-hand side above). Moreover, this persists at the level of the maps (3.2) and (5.1), in the following sense.

Theorem 5.1. *Let w_n be a solution of (5.1) with parameters $\bar{a}, \bar{c}, \bar{\nu}$, lying on the level set $H_1 = \bar{h}_1$, $H_2 = \bar{h}_2$ of the first integrals (5.2) and (5.3). Then for either choice of signs, $\hat{w}_n$ given by the Miura map (5.5) is a solution of (3.2) with parameters*

$$\nu = \bar{\nu} + 6\delta^2, \qquad b = \bar{c} + 4\bar{\nu}\delta^2 + 7\delta^4, \qquad a = \bar{h}_1 + \bar{c}\delta^2 + \bar{\nu}\delta^4 - \delta^6.$$

Moreover, on either solution $\hat{w}_n$, the values h_1, h_2 of the first integrals (3.3) and (3.4) for the map (3.2) are given by

$$h_1 = \bar{h}_2 + 2\delta^8, \qquad h_2 = -\bar{a}^2 - \bar{\nu}\bar{h}_2 - \bar{c}\bar{h}_1 - 2\bar{h}_2\delta^2 + (\bar{h}_1 - \bar{\nu}\bar{c})\delta^4 - \bar{\nu}\delta^8 - 4\delta^{10}.$$

Proof: The first part of this result is verified by substituting the Miura formula (5.5) directly into (3.2), using (5.1) to eliminate w_{n+5} followed by w_{n+4}, and then using (5.2) to eliminate w_{n+3} on the level set $H_1 = \bar{h}_1$. After the initial substitution of the Miura map and eliminating, all the final results are quadratic in δ, so do not depend on the choice of sign in (5.5). Similar calculations using the same substitutions in the formulae (3.3) and (3.4), together with the expression (5.3) on the level set $H_2 = \bar{h}_2$, produce the expressions for h_1, h_2, which are the corresponding values of the first integrals for (3.2). $\qquad \square$

We can also malke use of a tau function substitution for (P.vi), which has the more complicated structure

$$w_n + \delta = \rho_n \frac{\sigma_{n+2}\tau_n}{\sigma_{n+1}\tau_{n+1}}, \tag{5.6}$$

$$w_n - \delta = \frac{1}{\rho_{n+1}} \frac{\sigma_n \tau_{n+2}}{\sigma_{n+1}\tau_{n+1}}, \tag{5.7}$$

with

$$\rho_{n+2} = \rho_n.$$

This implies that

$$\hat{w}_n^{(+)} = (w_n - \delta)(w_{n+1} + \delta) = \frac{\sigma_n \sigma_{n+3}}{\sigma_{n+1}\sigma_{n+2}}, \tag{5.8}$$

$$\hat{w}_n^{(-)} = (w_n + \delta)(w_{n+1} - \delta) = \frac{\tau_n \tau_{n+3}}{\tau_{n+1}\tau_{n+2}} \tag{5.9}$$

are both solutions of (3.2), and both sequences (σ_n) and (τ_n) satisfy the same Somos-9 relation. Thus the two different formulae for the Miura map in (5.5) can be regarded as defining a Bäcklund transformation for the discrete equation (3.2) with parameter δ, since given $\hat{w}_n^{(-)}$ and a solution w_n of (5.1), a new solution $\hat{w}_n^{(+)}$ of (3.2) is generated by taking

$$\hat{w}_n^{(+)} = \hat{w}_n^{(-)} + 2\delta(w_{n+1} - w_n).$$

6 Conclusion

We have shown that the integrable maps (P.iv), (P.v) and (P.vi) from [7] are closely related to one another, via Miura-type transformations, and they provide genus two solutions of Volterra and modified Volterra lattices, respectively. So far we do not have a complete understanding of what the relations between these maps mean geometrically, particularly from the Poisson and algebro-geometric points of view. However, since the construction of the integrable maps $\mathcal{V}_g$ presented in [10] is valid for any $g \geq 1$, this strongly suggests that (P.v) and (P.vi) should each be the $g = 2$ members of a family of maps defined for any g. In the elliptic case ($g = 1$) we have constructed elliptic solutions of the modified Volterra and Volterra lattices, and showed how they are linked by the Miura transformation, essentially recovering the solutions found in [24], which can be interpreted in terms of integrable maps in the plane (QRT type). The complete description of these results, together with the proposed extension to families of maps for all $g \geq 1$, is planned for future work.

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