PROCEEDINGS A

royalsocietypublishing.org/journal/rspa

Research

Cite this article: Clarkson PA, Jordaan K, Loureiro A. 2023 Generalized higher-order Freud weights. *Proc. R. Soc. A* **479**: 20220788.
https://doi.org/10.1098/rspa.2022.0788

Received: 5 January 2023
Accepted: 21 March 2023

Subject Areas:
applied mathematics, mathematical physics

Keywords:
generalized Freud weights, semi-classical orthogonal polynomials, generalized hypergeometric functions

Author for correspondence:
Peter A. Clarkson
e-mail: P.A.Clarkson@kent.ac.uk

THE ROYAL SOCIETY PUBLISHING

Generalized higher-order Freud weights

Peter A. Clarkson[1], Kerstin Jordaan[2] and Ana Loureiro[1]

[1]School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury CT2 7FS, UK
[2]Department of Decision Sciences, University of South Africa, Pretoria 0003, South Africa

PAC, 0000-0002-8777-5284

We discuss polynomials orthogonal with respect to a semi-classical generalized higher-order Freud weight

$$\omega(x;t,\lambda) = |x|^{2\lambda+1}\exp(tx^2 - x^{2m}), \quad x\in\mathbb{R},$$

with parameters $\lambda > -1$, $t\in\mathbb{R}$ and $m = 2, 3, \ldots$. The sequence of generalized higher-order Freud weights for $m = 2, 3, \ldots$, forms a hierarchy of weights, with associated hierarchies for the first moment and the recurrence coefficient. We prove that the first moment can be written as a finite partition sum of generalized hypergeometric ${}_1F_m$ functions and show that the recurrence coefficients satisfy difference equations which are members of the first discrete Painlevé hierarchy. We analyse the asymptotic behaviour of the recurrence coefficients and the limiting distribution of the zeros as $n\to\infty$. We also investigate structure and other mixed recurrence relations satisfied by the polynomials and related properties.

1. Introduction

In this paper we consider polynomials orthogonal with respect to the *generalized higher-order Freud weight*

$$\omega(x;t,\lambda) = |x|^{2\lambda+1}\exp(tx^2 - x^{2m}), \quad x,t\in\mathbb{R},\ m = 2, 3, \ldots, \tag{1.1}$$

with $\lambda > -1$ a parameter. The main goal of this article is to bring a comprehensive self-contained analysis of these polynomials when the parameter m takes integer values higher than 1 and for any values of $\lambda > -1$ and $t\in\mathbb{R}$. The analysis for the particular cases of $m = 2, 3$ was considered in [1–6], with an emphasis on the study of the corresponding recurrence coefficients. We significantly extend existing studies on Freud type weights whilst providing a coherent and consistent approach, using techniques which are also likely to be adopted in the study of other semi-classical type weights. Throughout, we link and explain the connections to the existing theory. After giving a short mathematical background in §2, in §3, we give a closed-form expression for the moments with respect to the weight (1.1), which corresponds to a finite partition sum of generalized hypergeometric ${}_1F_m$ functions. The corresponding recurrence coefficients in the three-term recurrence relation are investigated in §4. Therein, we prove a recursive method that gives nonlinear recurrence relations satisfied by these recurrence coefficients (proposition 4.4) and give them explicitly for the cases where $m = 4, 5$, whilst recovering the already known relations for $m = 2, 3$. We prove structure relations and mixed recurrence relations satisfied by generalized higher-order Freud polynomials in §5. The asymptotic behaviour of the recurrence coefficients proved in §4 determines the limiting distribution of the zeros, and this, as well as other properties of the zeros, is investigated in §6. We conclude with the quadratic decomposition of the generalized higher-order Freud weight in §7.

2. Mathematical background

Let ν be a positive measure on the real line for which the support is not finite and all the moments

$$\mu_k = \int_{-\infty}^{\infty} x^k \,\mathrm{d}\nu(x), \quad k = 0, 1, \ldots, \tag{2.1}$$

exist. The corresponding monic orthogonal polynomial sequence $\{P_n(x)\}_{n\geq 0}$ is defined by

$$\int_{-\infty}^{\infty} P_m(x) P_n(x) \,\mathrm{d}\nu(x) = h_n \delta_{m,n}, \quad h_n > 0,$$

where $\delta_{m,n}$ denotes the Kronekar delta. A fundamental property of orthogonal polynomials is that they satisfy a three-term recurrence relation of the form

$$P_{n+1}(x) = (x - \alpha_n) P_n(x) - \beta_n P_{n-1}(x), \tag{2.2}$$

with $\beta_n > 0$ and initial values $P_{-1}(x) = 0$ and $P_0(x) = 1$. The recurrence coefficients α_n and β_n are given by the integrals

$$\alpha_n = \frac{1}{h_n} \int_{-\infty}^{\infty} x P_n^2(x) \,\mathrm{d}\nu(x) \quad \text{and} \quad \beta_n = \frac{1}{h_{n-1}} \int_{-\infty}^{\infty} x P_{n-1}(x) P_n(x) \,\mathrm{d}\nu(x).$$

Relevant for this article is the case of a measure that admits a representation via a positive weight function $\omega(x)$ on the real line as follows $\mathrm{d}\nu(x) = \omega(x)\,\mathrm{d}x$. Henceforth, we will only work with a weight function representation.

The coefficient β_n in the recurrence relation (2.2) can also be expressed in terms of the Hankel determinant

$$\Delta_n = \det\left[\mu_{j+k}\right]_{j,k=0}^{n-1} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}, \quad n \geq 1, \tag{2.3}$$

with $\Delta_0 = 1$ and $\Delta_{-1} = 0$, whose entries are given in terms of the moments (2.1) associated with the weight $\omega(x)$. Specifically

$$\beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}. \tag{2.4}$$

The monic polynomial $P_n(x)$ can be uniquely expressed as the determinant

$$P_n(x) = \frac{1}{\Delta_n}\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix},$$

and the normalization constants as

$$h_n = \frac{\Delta_{n+1}}{\Delta_n} \quad \text{and} \quad h_0 = \Delta_1 = \mu_0. \tag{2.5}$$

Also from (2.4) and (2.5), we see that the relationship between the recurrence coefficient β_n and the normalization constants h_n is given by

$$h_n = \beta_n h_{n-1}.$$

For symmetric weights, since $\omega(x) = \omega(-x)$, it follows that $\alpha_n = 0$ in (2.2). Hence, for symmetric weights, the sequence of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$, satisfy the three-term recurrence relation:

$$P_{n+1}(x) = xP_n(x) - \beta_n P_{n-1}(x). \tag{2.6}$$

The monic orthogonal polynomials $P_n(x)$ associated with symmetric weights are also symmetric, i.e. $P_n(-x) = (-1)^n P_n(x)$. This implies that each P_n contains only even or only odd powers of x, and we can write

$$P_{2n}(x) = x^{2n} + \sum_{k=1}^{n} c_{2n-2k}^{(2n)} x^{2n-2k} = x^{2n} + c_{2n-2}^{(2n)} x^{2n-2} + \cdots + c_0^{(2n)}$$

and

$$P_{2n+1}(x) = x^{2n+1} + \sum_{k=1}^{n} c_{2n-2k+1}^{(2n+1)} x^{2n-2k+1} = x^{2n+1} + c_{2n-1}^{(2n+1)} x^{2n-1} + \cdots + c_1^{(2n+1)} x.$$

By substituting these expressions into the recurrence relation (2.6) and comparing the coefficients on each side, we obtain

$$\beta_{2n} = c_{2n-2}^{(2n)} - c_{2n-1}^{(2n+1)} \quad \text{and} \quad \beta_{2n+1} = -\frac{c_0^{(2n+2)}}{c_0^{(2n)}} = -\frac{P_{2n+2}(0)}{P_{2n}(0)}. \tag{2.7}$$

It follows from (2.1) that, for symmetric weights, $\mu_{2k-1} = 0, k = 1, 2, \ldots,$ and hence, it is possible to write the Hankel determinant Δ_n given by (2.3) in terms of the product of two Hankel determinants obtained by matrix manipulation, interchanging columns and rows. The product decomposition, depending on n even or odd, is given by

$$\Delta_{2n} = \mathcal{A}_n \mathcal{B}_n \quad \text{and} \quad \Delta_{2n+1} = \mathcal{A}_{n+1}\mathcal{B}_n, \tag{2.8}$$

where $\mathcal{A}_n$ and $\mathcal{B}_n$ are the Hankel determinants

$$\mathcal{A}_n = \begin{vmatrix} \mu_0 & \mu_2 & \cdots & \mu_{2n-2} \\ \mu_2 & \mu_4 & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{2n-2} & \mu_{2n} & \cdots & \mu_{4n-4} \end{vmatrix} \quad \text{and} \quad \mathcal{B}_n = \begin{vmatrix} \mu_2 & \mu_4 & \cdots & \mu_{2n} \\ \mu_4 & \mu_6 & \cdots & \mu_{2n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{2n} & \mu_{2n+2} & \cdots & \mu_{4n-2} \end{vmatrix}, \tag{2.9}$$

with $\mathcal{A}_0 = \mathcal{B}_0 = 1$. Consequently, for a symmetric weight, by substituting (2.8) into (2.4), the recurrence coefficient β_n is given by

$$\beta_{2n} = \frac{\mathcal{A}_{n+1}\mathcal{B}_{n-1}}{\mathcal{A}_n \mathcal{B}_n} \quad \text{and} \quad \beta_{2n+1} = \frac{\mathcal{A}_n \mathcal{B}_{n+1}}{\mathcal{A}_{n+1}\mathcal{B}_n}.$$

Semi-classical orthogonal polynomials are natural generalizations of classical orthogonal polynomials and were introduced by Shohat in [7]. Maroni provided a unified theory for semi-classical orthogonal polynomials (cf. [8,9]). The weights of classical orthogonal polynomials satisfy a first-order ordinary differential equation, the *Pearson equation*,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sigma(x)\omega(x)\} = \tau(x)\omega(x), \tag{2.10}$$

where $\sigma(x)$ is a monic polynomial of degree at most 2 and $\tau(x)$ is a polynomial with degree 1. For semi-classical orthogonal polynomials, the weight function $\omega(x)$ satisfies a Pearson equation (2.10) with either $\deg(\sigma(x)) > 2$ or $\deg(\tau(x)) \neq 1$ (cf. [8,10]). The generalized higher-order Freud weight given by (1.1) is a symmetric weight that satisfies the Pearson equation (2.10) with $\sigma(x) = x$ and $\tau(x) = 2(tx^2 - mx^{2m} + \lambda + 1)$ and therefore is a semi-classical weight.

3. Moments of the generalized higher-order Freud weights

The existence of the first moment $\mu_0(t;\lambda,m)$ associated with the generalized higher-order Freud weight (1.1) follows from the fact that, at ∞, the integrand behaves like $\exp(-x^2)$ and, at $x=0$, the integrand behaves like x^λ, which, for $\lambda > -1$, is integrable.

Theorem 3.1. *Let $x \in \mathbb{R}, \lambda > -1, t \in \mathbb{R}$ and $m = 2, 3, \ldots$. Then, for the generalized higher-order Freud weight (1.1), the first moment is given by*

$$\begin{aligned}
\mu_0(t;\lambda,m) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1}\exp(tx^2 - x^{2m})\,\mathrm{d}x = \int_0^{\infty} s^\lambda \exp(ts - s^m)\,\mathrm{d}s \\
&= \frac{1}{m}\sum_{k=1}^{m}\frac{t^{k-1}}{(k-1)!}\Gamma\left(\frac{\lambda+k}{m}\right) \times {}_2F_m\left(\frac{\lambda+k}{m}, 1; \frac{k}{m}, \frac{k+1}{m}, \ldots, \frac{m+k-1}{m}; \left(\frac{t}{m}\right)^m\right) \\
&= \frac{1}{m}\Gamma\left(\frac{\lambda+1}{m}\right) {}_1F_{m-1}\left(\frac{\lambda+1}{m}; \frac{1}{m}, \ldots \frac{m-1}{m}; \left(\frac{t}{m}\right)^m\right) \\
&\quad + \frac{1}{m}\sum_{k=2}^{m-1}\frac{t^{k-1}}{(k-1)!}\Gamma\left(\frac{\lambda+k}{m}\right) \\
&\quad \times {}_1F_{m-1}\left(\frac{\lambda+k}{m}; \frac{k}{m}, \frac{k+1}{m}, \ldots, \frac{m-1}{m}, \frac{m+1}{m}, \ldots, \frac{m+k-1}{m}; \left(\frac{t}{m}\right)^m\right) \\
&\quad + \frac{t^{m-1}}{m!}\Gamma\left(\frac{\lambda}{m}+1\right) {}_1F_m\left(\frac{\lambda}{m}+1; \frac{m+1}{m}, \frac{m+2}{m}, \ldots, \frac{2m-1}{m}; \left(\frac{t}{m}\right)^m\right),
\end{aligned}$$

where ${}_pF_q(a;b;z)$ is the generalized hypergeometric function (cf. [11, eq. 16.2.1].

Proof. By using the power series expansion of the exponential function, we obtain

$$\begin{aligned}
\mu_0(t;\lambda,m) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1}\exp(tx^2 - x^{2m})\,\mathrm{d}x = \int_0^{\infty} s^\lambda \exp(ts - s^m)\,\mathrm{d}s \\
&= \int_0^{\infty} s^\lambda \exp(-s^m)\sum_{n=0}^{\infty}\frac{(ts)^n}{n!}\,\mathrm{d}s \\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^{\infty} s^{n+\lambda}\exp(-s^m)\,\mathrm{d}s
\end{aligned}$$

$$= \frac{1}{m}\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^{\infty} y^{(n+\lambda-m+1)/m}\exp(-y)\,\mathrm{d}y$$

$$= \frac{1}{m}\sum_{n=0}^{\infty}\frac{t^n}{n!}\Gamma\left(\frac{\lambda+n+1}{m}\right),$$

where $\Gamma(x)$ denotes the Gamma function defined in [11, eq. 5.2.1], and the fourth equal sign follows from the Lebesgue's dominated convergence theorem. By letting $n = mk + j$ for $j = 0, 1, \ldots, m-1$, we can write

$$\mu_0(t;\lambda,m) = \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{m-1}\Gamma\left(\frac{\lambda+j+1}{m}+k\right)\frac{t^{mk+j}}{(mk+j)!}.$$

Using the Gauss multiplication formula [11, eq. 5.5.6] yields

$$(mk+j)! = j!\, m^{mk}\prod_{\ell=1}^{m}\left(\frac{j+\ell}{m}\right)_k,$$

where $(a)_k$ denotes the Pochhammer symbol (cf. [11, §5.2(iii)], while it follows from [11, eq. 5.5.1] that

$$\Gamma\left(\frac{\lambda+j+1}{m}+k\right) = \left(\frac{\lambda+j+1}{m}\right)_k \Gamma\left(\frac{\lambda+j+1}{m}\right),$$

and hence, we have

$$\mu_0(t;\lambda,m) = \frac{1}{m}\sum_{k=0}^{\infty}\sum_{j=0}^{m-1}\frac{((\lambda+j+1)/m)_k\Gamma((\lambda+j+1)/m)}{m^{mk}((j+1)/m)_k((j+2)/m)_k\cdots((j+m)/m)_k}\frac{t^{mk+j}}{j!}$$

$$= \frac{1}{m}\sum_{j=0}^{m-1}\Gamma\left(\frac{\lambda+j+1}{m}\right)\frac{t^j}{j!}\sum_{k=0}^{\infty}\times\frac{((\lambda+j+1)/m)_k}{((j+1)/m)_k((j+2)/m)_k\cdots((j+m)/m)_k}\left(\frac{t}{m}\right)^{mk}$$

$$= \frac{1}{m}\sum_{j=0}^{m-1}\Gamma\left(\frac{\lambda+j+1}{m}\right)\frac{t^j}{j!}\times {}_2F_m\left(\frac{\lambda+j+1}{m},1;\frac{j+1}{m},\frac{j+2}{m},\ldots,\frac{m+j}{m};\left(\frac{t}{m}\right)^m\right),$$

as required. ■

Remark 3.2. In our earlier studies of semi-classical orthogonal polynomials, we proved special cases of theorems 3.1 and 3.3, namely, for $m = 2$ in [4] and for $m = 3, 4, 5$ in [3].

In the following theorem, we derive a differential equation satisfied by the first moment $\mu_0(t;\lambda,m)$. It is often much easier to derive properties of a function from the differential equation it satisfies rather than from an integral representation or, as in this case, a sum of generalized hypergeometric functions.

Theorem 3.3. *Let $x \in \mathbb{R}, \lambda > -1, t \in \mathbb{R}$ and $m = 2, 3, \ldots$. Then, for the generalized higher-order Freud weight (1.1), the first moment*

$$\mu_0(t;\lambda,m) = \int_{-\infty}^{\infty}|x|^{2\lambda+1}\exp(tx^2 - x^{2m})\,\mathrm{d}x = \int_0^{\infty} s^{\lambda}\exp(ts - s^m)\,\mathrm{d}s,$$

satisfies the ordinary differential equation:

$$m\frac{\mathrm{d}^m\varphi}{\mathrm{d}t^m} - t\frac{\mathrm{d}\varphi}{\mathrm{d}t} - (\lambda+1)\varphi = 0. \tag{3.1}$$

Proof. Following [3,12], we look for a solution of (3.1) in the form

$$\varphi(t) = \int_0^\infty \mathrm{e}^{st} v(s)\,\mathrm{d}s. \tag{3.2}$$

For (3.2) to satisfy (3.1), it is necessary that

$$\frac{\mathrm{d}^m\varphi}{\mathrm{d}t^m} - \frac{t}{m}\frac{\mathrm{d}\varphi}{\mathrm{d}t} - \frac{\lambda+1}{m}\varphi = \int_0^\infty \mathrm{e}^{st}\left(s^m - \frac{ts}{m} - \frac{\lambda+1}{m}\right)v(s)\,\mathrm{d}s = 0.$$

By using integration by parts, this is equivalent to

$$\int_0^\infty \mathrm{e}^{st}\left\{s^m v(s) + \frac{1}{m}v(s) + \frac{s}{m}\frac{\mathrm{d}v}{\mathrm{d}s} - \frac{\lambda+1}{m}v(s)\right\}\mathrm{d}s = 0,$$

under the assumption that $\lim_{s\to\infty} sv(s)\mathrm{e}^{st} = 0$. Hence, for $\varphi(t)$ to be a solution of (3.1), we need to choose $v(s)$ so that

$$(ms^m - \lambda)v(s) + s\frac{\mathrm{d}v}{\mathrm{d}s} = 0.$$

One solution of this equation is $v(s) = s^\lambda \exp(-s^m)$. ■

For the generalized higher-order Freud weight (1.1), the even moments can be written in terms of derivatives of the first moment, as follows:

$$\begin{aligned}\mu_{2k}(t;\lambda,m) &= \int_{-\infty}^\infty x^{2k}|x|^{2\lambda+1}\exp(tx^2 - x^{2m})\,\mathrm{d}x \\ &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\int_{-\infty}^\infty |x|^{2\lambda+1}\exp(tx^2 - x^{2m})\,\mathrm{d}x \\ &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mu_0(t;\lambda,m), \quad k = 0,1,2,\ldots,\end{aligned} \tag{3.3}$$

where the interchange of integration and differentiation is justified by Lebesgue's dominated convergence theorem. Furthermore, from the definition we have,

$$\mu_{2k+2}(t;\lambda,m) = \mu_{2k}(t;\lambda+1,m), \quad k = 0,1,2,\ldots, \tag{3.4}$$

and therefore,

$$\mu_{2k+2}(t;\lambda,m) = \mu_0(t;\lambda+k+1,m), \quad k = 0,1,2,\ldots,$$

which illustrates the importance of the first moment.

4. Recurrence coefficients for generalized higher-order Freud weights

Theorem 4.1. *For the generalized higher-order Freud weight (1.1), the recurrence coefficient β_n is given by*

$$\beta_{2n} = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\mathcal{B}_n}{\mathcal{A}_n} \quad \text{and} \quad \beta_{2n+1} = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\mathcal{A}_{n+1}}{\mathcal{B}_n}, \tag{4.1}$$

with $A_0 = B_0 = 1$ and

$$\mathcal{A}_n = \mathrm{Wr}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right) \quad \text{and} \quad \mathcal{B}_n = \mathrm{Wr}\left(\frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \frac{\mathrm{d}^2\mu_0}{\mathrm{d}t^2}, \ldots, \frac{\mathrm{d}^n\mu_0}{\mathrm{d}t^n}\right), \tag{4.2}$$

where

$$\mu_0 = \mu_0(t;\lambda,m) = \int_0^\infty x^\lambda \exp(ts - s^m)\,\mathrm{d}x,$$

and $\mathrm{Wr}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ *denotes the Wronskian given by*

$$\mathrm{Wr}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}, \quad \varphi_j^{(k)} = \frac{\mathrm{d}^k \varphi_j}{\mathrm{d}t^k}.$$

Proof. It follows from (2.9) and (3.3) that $\mathcal{A}_n$ and $\mathcal{B}_n$ can be written in terms of the Wronskians given by (4.2). Furthermore,

$$\mathcal{A}_n \frac{\mathrm{d}\mathcal{B}_n}{\mathrm{d}t} - \mathcal{B}_n \frac{\mathrm{d}\mathcal{A}_n}{\mathrm{d}t} = \mathcal{A}_{n+1}\mathcal{B}_{n-1} \quad \text{and} \quad \mathcal{B}_n \frac{\mathrm{d}\mathcal{A}_{n+1}}{\mathrm{d}t} - \mathcal{A}_{n+1} \frac{\mathrm{d}\mathcal{B}_n}{\mathrm{d}t} = \mathcal{A}_{n+1}\mathcal{B}_n \tag{4.3}$$

(cf. [13, §6.5.1]) and (4.3), together with (2) yields (4.1). ∎

Theorem 4.2. *Let* $\omega_0(x)$ *be a symmetric positive weight on the real line for which all the moments exist, and let* $\omega(x;t) = \exp(tx^2)\omega_0(x)$*, with* $t \in \mathbb{R}$*, is a weight such that all the moments of exist. Then the recurrence coefficient* $\beta_n(t)$ *satisfies the Volterra, or the Langmuir lattice, equation:*

$$\frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\beta_{n+1} - \beta_{n-1}).$$

Proof. See, e.g. Van Assche [14, Theorem 2.4]. ∎

Theorem 4.3. *For the generalized higher-order Freud weight* (1.1)*, the associated monic polynomials* $P_n(x)$ *satisfy the recurrence relation:*

$$P_{n+1}(x) = xP_n(x) - \beta_n(t;\lambda)P_{n-1}(x), \quad n = 0, 1, 2, \ldots, \tag{4.4}$$

with $P_{-1}(x) = 0$ *and* $P_0(x) = 1$*, where*

$$\beta_{2n}(t;\lambda) = \frac{\mathcal{A}_{n+1}(t;\lambda)\mathcal{A}_{n-1}(t;\lambda+1)}{\mathcal{A}_n(t;\lambda)\mathcal{A}_n(t;\lambda+1)} = \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\mathcal{A}_n(t;\lambda+1)}{\mathcal{A}_n(t;\lambda)}$$

and

$$\beta_{2n+1}(t;\lambda) = \frac{\mathcal{A}_n(t;\lambda)\mathcal{A}_{n+1}(t;\lambda+1)}{\mathcal{A}_{n+1}(t;\lambda)\mathcal{A}_n(t;\lambda+1)} = \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\mathcal{A}_{n+1}(t;\lambda)}{\mathcal{A}_n(t;\lambda+1)},$$

where $\mathcal{A}_n(t;\lambda)$ *is the Wronskian given by* (4.2) *with*

$$\begin{aligned} \mu_0(t;\lambda,m) = \frac{1}{m} \sum_{k=1}^{m} \frac{t^{k-1}}{(k-1)!} \Gamma\left(\frac{\lambda+k}{m}\right) \\ \times {}_2F_m\left(\frac{\lambda+k}{m}, 1; \frac{k}{m}, \frac{k+1}{m}, \ldots, \frac{m+k-1}{m}; \left(\frac{t}{m}\right)^m\right). \end{aligned}$$

Proof. It follows from substituting (3.4) into the expression for $\mathcal{B}_n(t;\lambda)$ given in (4.2) that $\mathcal{B}_n = \mathcal{A}_n(t;\lambda+1)$ and then the result immediately follows from (2) and (4.1). ∎

(a) Nonlinear recursive relations

We follow the approach found in [9, §7] whose key results are summarized in [15, Proposition 3.1].

Note that for a given $m \geq 1$, we can write

$$x^{2m}P_n(x) = \sum_{\ell=-m}^{m} C^{(2m)}_{n,n+2\ell} P_{n+2\ell}, \tag{4.5}$$

where

$$C^{(2m)}_{n,n+2\ell} = \frac{1}{h_{n+2\ell}} \int_{-\infty}^{\infty} x^{2m} P_{n+2\ell}(x) P_n(x) \omega(x)\,\mathrm{d}x \quad \text{for } \ell = -m, \ldots, m.$$

Observe that $C^{(2m)}_{n,n+k} = C^{(2m)}_{n+k,n} = 0$ for $|k| \geq 2m+1$ and

$$C^{(2m)}_{n,n+2\ell} = \frac{h_n}{h_{n+2\ell}} C^{(2m)}_{n+2\ell,n} = \frac{1}{\beta_{n+1} \cdots \beta_{n+2\ell}} C^{(2m)}_{n+2\ell,n} \quad \text{for } \ell = 1, \ldots, m.$$

From the recurrence relation (2.6), it follows

$$x^2 P_n(x) = P_{n+2} + (\beta_n + \beta_{n+1}) P_n + \beta_{n-1}\beta_n P_{n-2}, \quad n \geq 0. \tag{4.6}$$

In particular, one has $C^{(2)}_{n,n} = \beta_n + \beta_{n+1}$, $C^{(2)}_{n,n-2} = \beta_{n-1}\beta_n$ and $C^{(2)}_{n,n+2} = 1$. The computation of the coefficients $\{C^{(2m+2)}_{n-2\ell,n}\}_{\ell=0}^{m+1}$ can be derived from the coefficients $\{C^{(2m)}_{n-2\ell,n}\}_{\ell=0}^{m}$ as follows:

$$C^{(2m+2)}_{n-2\ell,n} = \beta_{n+2}\beta_{n+1} C^{(2m)}_{n-2\ell,n+2} + (\beta_n + \beta_{n+1}) C^{(2m)}_{n-2\ell,n} + C^{(2m)}_{n-2\ell,n-2}, \quad \ell = 0, \ldots, m, \tag{4.7}$$

which is a direct consequence of (4.5) multiplied by x^2 and (4.6).

Proposition 4.4. *The recurrence coefficient β_n for the generalized higher-order Freud weight* (1.1) *satisfies the discrete equation:*

$$2m V^{(2m)}_n - 2t\beta_n = n + \left(\lambda + \frac{1}{2}\right)[1 - (-1)^n], \tag{4.8}$$

where

$$V^{(2m)}_n = C^{(2m-2)}_{n,n-2} + \beta_n C^{(2m-2)}_{n,n}. \tag{4.9}$$

Alternatively, (4.9) can be written as follows:

$$V^{(2m)}_n = \frac{1}{h_{n-2}} \int_{-\infty}^{\infty} x^{2m-2} P_{n-2}(x) P_n(x) \omega(x)\,\mathrm{d}x + \frac{\beta_n}{h_n} \int_{-\infty}^{\infty} x^{2m-2} P_n^2(x) \omega(x)\,\mathrm{d}x.$$

Proof. For any monic polynomial sequence $\{P_n(x)\}_{n\geq 0}$, one can always write

$$x \frac{\mathrm{d}P_n}{\mathrm{d}x}(x) = \sum_{j=0}^{n} \rho_{n,j} P_{n-j}(x), \quad \text{for } n \geq 1,$$

with $\rho_{n,0} = n$. The assumption that $\{P_n(x)\}_{n\geq 0}$ is orthogonal with respect to the semi-classical weight $\omega(x)$ satisfying the differential equation (2.10) with $\sigma(x) = x$ and $\tau(x) = 2(tx^2 - mx^{2m} + \lambda + 1)$ gives, using integration by parts,

$$\begin{aligned}\rho_{n,j} h_{n-j} &= \int_{-\infty}^{\infty} x \frac{\mathrm{d}P_n}{\mathrm{d}x}(x) P_{n-j}(x) \omega(x)\,\mathrm{d}x \\ &= -\int_{-\infty}^{\infty} \{\tau(x) P_{n-j}(x) + x \frac{\mathrm{d}P_{n-j}}{\mathrm{d}x}(x)\} P_n(x) \omega(x)\,\mathrm{d}x,\end{aligned}$$

where $h_k = \int_{-\infty}^{\infty} P_k^2(x)\omega(x)\,\mathrm{d}x > 0$. Therefore, $\rho_{n,j} = 0$ for any $j \geq 2m+1$ and the symmetry of the weight implies $\rho_{n,j} = 0$ for any j odd. Therefore, we have

$$x \frac{\mathrm{d}P_n}{\mathrm{d}x}(x) = \sum_{\ell=0}^{m} \rho_{n,2\ell}\, P_{n-2\ell}(x), \quad \text{for } n \geq 0. \tag{4.10}$$

royalsocietypublishing.org/journal/rspa Proc. R. Soc. A 479: 20220788

Recall (4.6) to write

$$\frac{1}{h_n}\int_{-\infty}^{\infty} x^2 P_n^2(x)\omega(x)\,\mathrm{d}x = (\beta_n + \beta_{n+1})$$

and
$$\frac{1}{h_{n-2}}\int_{-\infty}^{\infty} x^2 P_{n-2}(x)P_n(x)\omega(x)\,\mathrm{d}x = \beta_n\beta_{n-1},$$

and hence,

$$\rho_{n,2\ell} = \begin{cases} \dfrac{2m}{h_n}\displaystyle\int_{-\infty}^{\infty} x^{2m} P_n^2(x)\omega(x)\,\mathrm{d}x & \\ \quad -2t(\beta_n+\beta_{n+1}) - (2\lambda+2+n), & \text{if } \ell = 0, \\ \dfrac{2m}{h_{n-2}}\displaystyle\int_{-\infty}^{\infty} x^{2m} P_{n-2}(x)P_n(x)\omega(x)\,\mathrm{d}x - 2t\,\beta_n\beta_{n-1}, & \text{if } \ell = 1, \\ \dfrac{2m}{h_{n-2\ell}}\displaystyle\int_{-\infty}^{\infty} x^{2m} P_{n-2\ell}(x)P_n(x)\omega(x)\,\mathrm{d}x, & \text{if } 2 \le \ell \le m-1, \\ 2m\,\beta_n\cdots\beta_{n-2m+1}, & \text{if } \ell = m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.11}$$

Take $\ell = 0$ in (4.7) and note that $C_{n,n-2}^{(2m-2)} = C_{n-2,n}^{(2m-2)}\beta_n\beta_{n-1}$ to obtain

$$C_{n,n}^{(2m)} = \beta_{n+1}(C_{n,n+2}^{(2m-2)}\beta_{n+2} + C_{n,n}^{(2m-2)}) + \beta_n(C_{n-2,n}^{(2m-2)}\beta_{n-1} + C_{n,n}^{(2m-2)}).$$

The symmetric orthogonality recurrence relation (2.6) implies that

$$P_{n+2}(x)P_n(x) = P_{n+1}^2(x) + \beta_n P_{n-1}(x)P_{n+1}(x) - \beta_{n+1}P_n^2(x),$$

which gives the relation

$$C_{n,n+2}^{(2m-2)}\beta_{n+2} + C_{n,n}^{(2m-2)} = C_{n-1,n+1}^{(2m-2)}\beta_n + C_{n+1,n+1}^{(2m-2)}, \tag{4.12}$$

and consequently, we have

$$C_{n,n}^{(2m)} = V_{n+1}^{(2m)} + V_n^{(2m)} \quad \text{where } V_n^{(2m)} = \beta_n(\beta_{n-1}C_{n-2,n}^{(2m-2)} + C_{n,n}^{(2m-2)}). \tag{4.13}$$

On the other hand, expressions for the coefficients $\rho_{n,2j}$ can be obtained through a purely algebraic way and therefore expressed recursively. For that, we differentiate with respect to x the recurrence relation (2.6) and use the structure relation (4.10) to obtain

$$P_n(x) + \sum_{\ell=0}^{m}\rho_{n,2\ell}P_{n-2\ell}(x) = \frac{\mathrm{d}P_{n+1}}{\mathrm{d}x}(x) + \beta_n\frac{\mathrm{d}P_{n-1}}{\mathrm{d}x}(x).$$

We multiply the latter by x and use again (4.10) and then (2.6) to obtain a linear combination of terms of $\{P_n(x)\}_{n\ge 0}$, and this gives

$$P_{n+1}(x) + \beta_n P_{n-1}(x) = \sum_{\ell=0}^{m+1}(\rho_{n+1,2\ell} - \rho_{n,2\ell} + \beta_n\rho_{n-1,2\ell-2} - \beta_{n-2\ell+2}\,\rho_{n,2\ell-2})P_{n-2\ell+1}(x).$$

Since the terms are linearly independent, we equate the coefficients of $P_{n+1}, P_n, \ldots, P_{n-2m-1}$ to obtain

$$\begin{cases} \rho_{n,0} = n, & \\ \rho_{n+1,2} - \rho_{n,2} = 2\beta_n, & \\ \rho_{n+1,2\ell} - \rho_{n,2\ell} = \beta_{n-2\ell+2}\,\rho_{n,2\ell-2} - \beta_n\,\rho_{n-1,2\ell-2}, & \text{for } \ell = 2,\ldots,m-1, \\ \beta_{n-2m}\,\rho_{n,2m} = \beta_n\,\rho_{n-1,2m}, & \text{for } j = m-1. \end{cases} \tag{4.14}$$

We combine (4.11) with (4.14) to conclude that the first equation (when $\ell = 0$) gives

$$mV_{n+1}^{(2m)} + mV_n^{(2m)} - t(\beta_n + \beta_{n+1}) = n + (\lambda + 1),$$

which implies (4.8). ■

The expressions for $V_n^{(2m)}$ can then be obtained recursively using (4.9), (4.7) and (4.13) to write

$$\begin{aligned} V_n^{(2m)} = \beta_n \left(V_{n+1}^{(2m-2)} + V_n^{(2m-2)}\right) + (\beta_n + \beta_{n+1}) V_n^{(2m-2)} \\ - \beta_n(\beta_n + \beta_{n+1}) \left(V_n^{(2m-4)} + V_{n+1}^{(2m-4)}\right) \\ + \beta_n \beta_{n-1} \left(V_{n-2}^{(2m-4)} + V_{n-1}^{(2m-4)}\right) \\ + \beta_n \beta_{n-1} \beta_{n+1} \beta_{n+2} C_{n-2,n+2}^{(2m-4)}. \end{aligned} \tag{4.15}$$

Combining (4.12) with (4.13) gives

$$V_n^{(2m)} - V_{n-1}^{(2m)} = \beta_n \left(V_{n+1}^{(2m-2)} + V_n^{(2m-2)}\right) - \beta_{n-1} \left(V_{n-1}^{(2m-2)} + V_{n-2}^{(2m-2)}\right).$$

By using the latter relation, we replace the term $(\beta_n + \beta_{n+1})V_n^{(2m-2)}$ in (4.15) to obtain

$$\begin{aligned} V_n^{(2m)} = \beta_n \left(V_{n+1}^{(2m-2)} + V_n^{(2m-2)} + V_{n-1}^{(2m-2)}\right) + \beta_{n+1} V_{n-1}^{(2m-2)} \\ - \beta_{n+1}\beta_{n-1} \left(V_{n-1}^{(2m-4)} + V_{n-2}^{(2m-4)}\right) + \beta_{n+2}\beta_{n+1}\beta_n\beta_{n-1} C_{n-2,n+2}^{(2m-4)}. \end{aligned} \tag{4.16}$$

Consider $n \to n-1$ and $m \to m-1$ in the latter expression, and replace it in (4.16) and this yields

$$\begin{aligned} V_n^{(2m)} = \beta_n \left(V_{n+1}^{(2m-2)} + V_n^{(2m-2)} + V_{n-1}^{(2m-2)}\right) + \beta_{n-1}\beta_{n+1} V_n^{(2m-4)} + \beta_n\beta_{n+1} V_{n-2}^{(2m-4)} \\ - \beta_{n+1}\beta_n\beta_{n-2} \left(V_{n-2}^{(2m-6)} + V_{n-3}^{(2m-6)}\right) \\ + \beta_{n+1}\beta_n\beta_{n-1} \left(\beta_{n+1}\beta_{n-2} C_{n-3,n+1}^{(2m-6)} + \beta_{n+2} C_{n-2,n+2}^{(2m-4)}\right). \end{aligned}$$

If we replace the term $V_{n-2}^{(2m-4)}$ by the corresponding expression given by the latter relation and successively continuing the process, then one can deduce the following expressions for $V_n^{(2m)}$ as follows:

$$\begin{aligned} V_n^{(2)} &= \beta_n, \\ V_n^{(4)} &= V_n^{(2)} \left(V_{n+1}^{(2)} + V_n^{(2)} + V_{n-1}^{(2)}\right), \\ V_n^{(6)} &= V_n^{(2)} \left(V_{n+1}^{(4)} + V_n^{(4)} + V_{n-1}^{(4)} + V_{n+1}^{(2)} V_{n-1}^{(2)}\right). \end{aligned}$$

For higher orders, we compute the coefficients $V_n^{(2m)}$ recursively as stated below. We opted for not giving the expressions in terms of β_n since those are rather long. For $m = 4, 5$, we have

$$\begin{aligned} V_n^{(8)} = V_n^{(2)} \left(V_{n+1}^{(6)} + V_n^{(6)} + V_{n-1}^{(6)}\right) + V_n^{(4)} V_{n+1}^{(2)} V_{n-1}^{(2)} \\ + V_{n+1}^{(2)} V_n^{(2)} V_{n-1}^{(2)} \left(V_{n+2}^{(2)} + V_{n-2}^{(2)}\right) \end{aligned}$$

and

$$\begin{aligned} V_n^{(10)} = V_n^{(2)} \left(V_{n+1}^{(8)} + V_n^{(8)} + V_{n-1}^{(8)}\right) + V_n^{(6)} V_{n+1}^{(2)} V_{n-1}^{(2)} \\ + V_{n+1}^{(2)} V_n^{(2)} V_{n-1}^{(2)} \left(V_{n+2}^{(4)} + V_{n-2}^{(4)}\right) + V_{n+1}^{(2)} V_n^{(2)} V_{n-1}^{(2)} \\ \times \left\{\left(V_n^{(2)} + V_{n-1}^{(2)}\right) V_{n+2}^{(2)} + \left(V_{n+1}^{(2)} + V_n^{(2)}\right) V_{n-2}^{(2)} + V_{n+2}^{(2)} V_{n-2}^{(2)}\right\}. \end{aligned}$$

Remark 4.5.

(i) For the case when $\lambda = -\frac{1}{2}$, proposition 4.4 was proved by Benassi & Moro [16], using a result in [17]. Although it is straightforward to modify the proof presented therein for the case when $\lambda \neq -\frac{1}{2}$, we hereby present an alternative approach purely depending on the structure relation of the semi-classical polynomials.

(ii) Equations such as (4.8) for recurrence relation coefficients are sometimes known as *Laguerre-Freud equations* [18,19]; see also [6,20–23].

(iii) When $m = 2$, the discrete equation is

$$4\beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) - 2t\beta_n = n + \left(\lambda + \frac{1}{2}\right)[1 - (-1)^n)], \tag{4.17}$$

which is $\mathrm{dP_I}$, and when $m = 3$, the discrete equation is

$$\begin{aligned} 6\beta_n(\beta_{n-2}\beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1}\beta_n + \beta_{n-1}\beta_{n+1} + \beta_n^2 + 2\beta_n\beta_{n+1} \\ + \beta_{n+1}^2 + \beta_{n+1}\beta_{n+2}) - 2t\beta_n = n + (\lambda + \tfrac{1}{2})[1 - (-1)^n)], \end{aligned} \tag{4.18}$$

which is a special case of $\mathrm{dP_I^{(2)}}$, the second member of the discrete Painlevé I hierarchy. For further information about the discrete Painlevé I hierarchy, see [24,25]. Equations (4.17) and (4.18) with $t = 0$ were derived by Freud [18]; see also [5,14]. Further, equations (4.17) and (4.18) with $\lambda = -\frac{1}{2}$ are also known as 'string equations' and arise in important physical applications such as two-dimensional quantum gravity, cf. [26–31].

(b) Asymptotics for the recurrence coefficients as $n \to \infty$

In 1976, Freud [18] conjectured that the asymptotic behaviour of recurrence coefficients β_n in the recurrence relation (2.6) satisfied by monic polynomials $\{P_n(x)\}_{n\geq 0}$ orthogonal with respect to the weight

$$\omega(x) = |x|^\rho \exp(-|x|^m),$$

with $x \in \mathbb{R}$, $\rho > -1$, $m > 0$ could be described by

$$\lim_{n\to\infty} \frac{\beta_n}{n^{2/m}} = \left[\frac{\Gamma((1/2)m)\Gamma(1 + (1/2)m)}{\Gamma(m+1)}\right]^{2/m}. \tag{4.19}$$

Freud stated the conjecture for orthonormal polynomials, proved it for $m = 2, 4, 6$ and also showed that (4.19) is valid whenever the limit on the left-hand side exists. Magnus [5] proved Freud's conjecture for the case when m is an even positive integer and also for weights

$$w(x) = \exp\{-Q(x)\},$$

where $Q(x)$ is an even degree polynomial with positive leading coefficient. We refer the readers to [32, §4.18] for a detailed history of solutions to Freud's conjecture up to that point. The conjecture was settled by Lubinsky *et al.* in [33] as a special case of a more general result for recursion coefficients of exponential weights, see also [34]. In [35], Lubinsky & Saff introduced the class of *very smooth Freud weights* of order α with conditions on Q that are satisfied when Q is of the form x^α, $\alpha > 0$. Associated with each weight in this class, one can define a_n as the unique, positive root of the equation (cf. [33, p. 67] and the references therein)

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n s Q'(a_n s)}{\sqrt{1-s^2}}\,\mathrm{d}s. \tag{4.20}$$

Theorem 4.6. *Consider the generalized higher-order Freud weight (1.1). Then the recurrence coefficients β_n associated with this weight satisfy*

$$\lim_{n\to\infty} \frac{\beta_n(t;\lambda)}{n^{1/m}} = \frac{1}{4}\left(\frac{(m-1)!}{(1/2)_m}\right)^{1/m}.$$

Proof. Let $Q(x) = \frac{1}{2}x^{2m}$, then evaluating (4.20) yields $n = a_n^{2m}(\frac{1}{2})_m/(m-1)!$ and the result is a straightforward consequence of the more general result in [33, Theorem 2.3] taking $W(x) = \exp\{-Q(x)\}$, $w = |x|^{\lambda+1/2}$, $P(x) = \frac{1}{2}tx^2$ and $\Psi(x) = 1$. ∎

Remark 4.7. Taking $m=2$ in theorem 4.6, we recover [1, Corollary 4.2 (ii)] for the recurrence coefficients associated with the generalized quartic Freud weight $|x|^{2\lambda+1}\exp(tx^2-x^4)$, which satisfy

$$\lim_{n\to\infty}\frac{\beta_n(t;\lambda)}{\sqrt{n}}=\frac{1}{\sqrt{12}},$$

while, for $m=3$, the recurrence coefficients associated with the generalized sextic Freud weight $|x|^{2\lambda+1}\exp(tx^2-x^6)$ satisfy

$$\lim_{n\to\infty}\frac{\beta_n(t;\lambda)}{\sqrt[3]{n}}=\frac{1}{\sqrt[3]{60}},$$

as shown in [3, Corollary 4.8].

5. Generalized higher-order Freud polynomials

(a) Differential equations

The second-order differential equations satisfied by generalized higher-order Freud polynomials can be obtained by using ladder operators as was done for the special cases $m=2$ and $m=3$ in [4, Theorem 6] and [2, Theorem 4.3], respectively. An alternative approach is given by Maroni in [8,9].

Proposition 5.1. *The polynomial sequence $\{P_n(x)\}_{n\geq 0}$ orthogonal with respect to the generalized higher-order Freud weight (1.1) is a solution to the differential equation:*

$$J(x;n)\frac{\mathrm{d}^2P_{n+1}}{\mathrm{d}x^2}(x)+K(x;n)\frac{\mathrm{d}P_{n+1}}{\mathrm{d}x}(x)+L(x;n)P_{n+1}(x)=0,$$

where

$$\begin{aligned}J(x;n)&=xD_{n+1}(x),\\ K(x;n)&=C_0(x)D_{n+1}(x)-x\frac{\mathrm{d}D_{n+1}}{\mathrm{d}x}(x)+D_{n+1}(x),\\ L(x;n)&=\mathcal{W}(\tfrac{1}{2}(C_{n+1}(x)-C_0(x)),D_{n+1}(x))-D_{n+1}(x)\sum_{j=0}^{n}\frac{1}{\beta_j}D_j(x),\end{aligned}$$

with

$$C_{n+1}(x)=-C_n(x)+\frac{2x}{\beta_n}D_n(x)$$

and

$$D_{n+1}(x)=-x+\frac{\beta_n}{\beta_{n-1}}D_{n-1}(x)+\frac{x^2}{\beta_n}D_n(x)-xC_n(x),$$

subject to the initial conditions $C_0(x)=-1+2(tx^2-mx^{2m}+\lambda+1)$, $D_{-1}(x)=0$ and

$$D_0(x)=2x\left\{m\sum_{j=1}^{m}\mu_{2j-2}(t,\lambda)x^{2m-2j}-t\mu_0(t,\lambda)\right\}.$$

(b) Mixed recurrence relations

We first consider the connection formula between the corresponding sequences of generalized higher-order Freud orthogonal polynomials in the framework of Christoffel transformations when the measure is modified by multiplying with a polynomial. In our case, the measure is modified by a quadratic factor.

Theorem 5.2. *Let $\{P_n(x;\lambda)\}_{n\geq 0}$ be the sequence of monic generalized higher-order Freud polynomials orthogonal with respect to the weight* (1.1)*, then, for m,n fixed,*

$$xP_{2n}(x;\lambda+1) = P_{2n+1}(x;\lambda) \tag{5.1a}$$

and

$$x^2P_{2n-1}(x;\lambda+1) = xP_{2n}(x;\lambda) - \left\{\beta_{2n}(\lambda) + \frac{P'_{2n+1}(0;\lambda)}{P'_{2n-1}(0;\lambda)}\right\} P_{2n-1}(x;\lambda). \tag{5.1b}$$

Proof. Let $P_n(x;\lambda+1)$ be the polynomials associated with the even weight function:

$$\omega(x;\lambda+1) = |x|^{2\lambda+3}\exp\left(tx^2 - x^{2m}\right) = x^2\omega(x;\lambda), \quad m=2,3,\ldots.$$

The factor x^2 by which the weight $\omega(x;\lambda)$ is modified has a double zero at the origin, and therefore, Christoffel's formula (cf. [36, Theorem 2.5], [37, Theorem 2.7.1]), applied to the monic polynomials $P_n(x;\lambda+1)$, is

$$x^2P_n(x;\lambda+1) = \frac{1}{P_n(0;\lambda)P'_{n+1}(0;\lambda) - P'_n(0;\lambda)P_{n+1}(0;\lambda)}\begin{vmatrix} P_n(x;\lambda) & P_{n+1}(x;\lambda) & P_{n+2}(x;\lambda) \\ P_n(0;\lambda) & P_{n+1}(0;\lambda) & P_{n+2}(0;\lambda) \\ P'_n(0;\lambda) & P'_{n+1}(0;\lambda) & P'_{n+2}(0;\lambda)\end{vmatrix}.$$

Since the weight $\omega(x;\lambda)$ is even, we have that $P_{2n+1}(0;\lambda) = P'_{2n}(0;\lambda)$, while $P_{2n}(0;\lambda)\neq 0$ and $P'_{2n+1}(0;\lambda)\neq 0$, and hence,

$$x^2P_n(x;\lambda+1) = \frac{-1}{P'_n(0;\lambda)P_{n+1}(0;\lambda)}\begin{vmatrix} P_n(x;\lambda) & P_{n+1}(x;\lambda) & P_{n+2}(x;\lambda) \\ 0 & P_{n+1}(0;\lambda) & 0 \\ P'_n(0;\lambda) & 0 & P'_{n+2}(0;\lambda)\end{vmatrix},$$

for n odd, while, for n even,

$$x^2P_n(x;\lambda+1) = \frac{1}{P_n(0;\lambda)P'_{n+1}(0;\lambda)}\begin{vmatrix} P_n(x;\lambda) & P_{n+1}(x;\lambda) & P_{n+2}(x;\lambda) \\ P_n(0;\lambda) & 0 & P_{n+2}(0;\lambda) \\ 0 & P'_{n+1}(0;\lambda) & 0\end{vmatrix}.$$

This yields

$$x^2P_n(x;\lambda+1) = P_{n+2}(x;\lambda) - a_nP_n(x;\lambda), \tag{5.2}$$

where

$$a_n = \begin{cases} \dfrac{P_{n+2}(0;\lambda)}{P_n(0;\lambda)}, & \text{for } n \text{ even,} \\[2ex] \dfrac{P'_{n+2}(0;\lambda)}{P'_n(0;\lambda)}, & \text{for } n \text{ odd.}\end{cases}$$

By using the three-term recurrence relation (2.6) to eliminate $P_{n+2}(x;\lambda)$ in (5.2), we obtain

$$x^2P_n(x;\lambda+1) = xP_{n+1}(x;\lambda) - (\beta_{n+1}(\lambda) + a_n)P_n(x;\lambda).$$

It follows from (2.7) that, for n even, $\beta_{n+1}(\lambda) + a_n = 0$, and the result follows. ∎

Theorem 5.3. *For a fixed $m = 2,3,\ldots$, let $\{P_n(x;\lambda)\}_{n\geq 0}$ be the sequence of monic generalized higher-order Freud polynomials orthogonal with respect to the weight* (1.1)*. Then, for n fixed,*

$$P_{2n+1}(x;\lambda) = P_{2n+1}(x;\lambda+1) + \beta_{2n}(\lambda+1)P_{2n-1}(x;\lambda+1) \tag{5.3a}$$

and

$$P_{2n}(x;\lambda) = P_{2n}(x;\lambda+1) - \frac{\beta_{2n}(\lambda)\beta_{2n-1}(\lambda+1)P'_{2n-1}(0;\lambda)}{P'_{2n+1}(0;\lambda)}P_{2n-2}(x;\lambda+1). \tag{5.3b}$$

Proof. Substitute (5.1*a*) into the three-term recurrence relation

$$P_{2n+1}(x;\lambda) = xP_{2n}(x;\lambda) - \beta_{2n}(\lambda)P_{2n-1}(x;\lambda), \tag{5.4}$$

to eliminate $P_{2n+1}(x;\lambda)$ and obtain

$$xP_{2n}(x;\lambda+1) = xP_{2n}(x;\lambda) - \beta_{2n}(\lambda)P_{2n-1}(x;\lambda).$$

Let $a_{2n} = P'_{2n+1}(0;\lambda)/P'_{2n-1}(0;\lambda)$. Substitute (5.1*b*) into (5.4) to eliminate $P_{2n-1}(x;\lambda)$ and obtain

$$xP_{2n}(x;\lambda+1) = xP_{2n}(x;\lambda) - \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda)+a_{2n}}(xP_{2n}(x;\lambda) - x^2P_{2n-1}(x;\lambda+1)). \tag{5.5}$$

Simplification and rearrangement of terms in (5.5) yields

$$\left(1 - \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda)+a_{2n}}\right)P_{2n}(x;\lambda) = P_{2n}(x;\lambda+1) - \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda)+a_{2n}}xP_{2n-1}(x;\lambda+1),$$

then, by using the three-term recurrence relation to eliminate $xP_{2n-1}(x;\lambda+1)$, we obtain

$$\begin{aligned}\left(1 - \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda)+a_{2n}}\right)P_{2n}(x;\lambda) &= \left(1 - \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda)+a_{2n}}\right)P_{2n}(x;\lambda+1)\\ &\quad + \frac{\beta_{2n}(\lambda)}{\beta_{2n}(\lambda)+a_{2n}}\beta_{2n-1}(\lambda+1)P_{2n-2}(x;\lambda+1),\end{aligned}$$

which simplifies to (5.3*b*). Substituting (5.1*a*) into the three-term recurrence relation

$$P_{2n+1}(x;\lambda+1) = xP_{2n}(x;\lambda+1) - \beta_{2n}(\lambda+1)P_{2n-1}(x;\lambda+1),$$

yields (5.3*a*). ■

Theorem 5.3 gives the connection formula between the corresponding sequences of generalized higher-order Freud polynomials in the framework of Geronimus transformations, the inverse of a Christoffel transformation. For more on quadratic Geronimus transformations of a weight $\omega(x)$, where $(x^2 - c)v(x) = \omega(x)$, see [38]. The generalized Christoffel formula, where the weight is modified by a rational function, often referred to as an Uvarov transformation, can also be considered as the Darboux transformation of an integrable system (cf. [37,39]) and is considered in the framework of Gaussian quadrature rules in [40,41].

(c) Quasi-orthogonality for $\lambda \in (-2, -1)$

Theorem 5.3 yields the quasi-orthogonality of generalized higher-order Freud polynomials for $-2 < \lambda < -1$.

Theorem 5.4. *Suppose $-2 < \lambda < -1$. For each fixed $m = 2, 3, \ldots$, the generalized higher-order Freud polynomial $P_n(x;\lambda)$ is quasi-orthogonal of order 2 on $\mathbb{R}$ with respect to the weight*

$$|x|^{2\lambda+3}\exp\left(tx^2 - x^{2m}\right), \quad t \in \mathbb{R}.$$

Proof. Suppose $-2 < \lambda < -1$, then $\lambda + 1 > -1$. When n is even, we have from (5.3*b*) that

$$\begin{aligned}&\int_{-\infty}^{\infty} x^k P_n(x;\lambda)|x|^{2\lambda+3}\exp\left(tx^2 - x^{2m}\right)\mathrm{d}x\\ &\quad= \int_{-\infty}^{\infty} x^k P_n(x;\lambda+1)|x|^{2\lambda+3}\exp\left(tx^2 - x^{2m}\right)\mathrm{d}x\\ &\qquad - \frac{\beta_n(\lambda)\beta_{n-1}(\lambda+1)P'_{n-1}(0;\lambda)}{P'_{n+1}(0;\lambda)}\\ &\qquad \times \int_{-\infty}^{\infty} x^k P_{n-2}(x;\lambda+1)|x|^{2\lambda+3}\exp\left(tx^2 - x^{2m}\right)\mathrm{d}x,\end{aligned} \tag{5.6}$$

while, for n is odd, it follows from (5.3*a*) that

$$\begin{aligned}
&\int_{-\infty}^{\infty} x^k P_n(x;\lambda)|x|^{2\lambda+3}\exp\left(tx^2 - x^{2m}\right)\mathrm{d}x \\
&\quad = \int_{-\infty}^{\infty} x^k P_n(x;\lambda+1)|x|^{2\lambda+3}\exp\left(tx^2 - x^{2m}\right)\mathrm{d}x \\
&\qquad + \beta_n(\lambda+1)\int_{-\infty}^{\infty} x^k P_{n-2}(x;\lambda+1)|x|^{2\lambda+3}\exp\left(tx^2 - x^{2m}\right)\mathrm{d}x.
\end{aligned} \tag{5.7}$$

Since $\lambda+1>-1$, it follows from the orthogonality of the generalized higher-order Freud polynomials that

$$\int_{-\infty}^{\infty} x^k P_n(x;\lambda+1)|x|^{2\lambda+3}\exp\left(tx^2 - x^{2m}\right) = 0 \quad \text{for } k=0,\dots,n-1,$$

and we see that all the integrals on the right-hand side of (5.6) and (5.7) are equal to zero for $k=0,\dots,n-3$. ■

6. Zeros of generalized higher-order Freud polynomials

(a) Asymptotic zero distribution

The asymptotic behaviour of the recurrence coefficients of generalized higher-order Freud polynomials orthogonal with respect to (1.1), satisfying Freud's conjecture, given by (4.19), is independent of the values of t and λ. The asymptotic behaviour implies that the recurrence coefficients are regularly varying, irrespective of t and λ. To consider the asymptotic distribution of the zeros of generalized higher-order Freud polynomials orthogonal with respect to the weight (1.1) as $n\to\infty$, we use an appropriate scaling and apply the property of regular variation as detailed in [42].

Theorem 6.1. *Let $\phi(n)=n^{1/(2m)}$ and assume that n,N tend to infinity in such a way that the ratio $n/N\to\ell$. Then, for the sequence of scaled monic polynomials $P_{n,N}(x)=(\phi(N))^{-n}P_n(\phi(N)x)$ associated with the generalized higher-order Freud weight (1.1), the asymptotic zero distribution, as $n\to\infty$, has density*

$$a_m(\ell) = \frac{2m}{c\pi(2m-1)}\left(1-\frac{x^2}{c^2}\right)^{1/2} {}_2F_1\left(1, 1-m; \frac{3}{2}-m; \frac{x^2}{c^2}\right), \tag{6.1}$$

where

$$c = 2a\ell^{1/(2m)} \quad \textit{with } a = \frac{1}{2}\left(\frac{(m-1)!}{(1/2)_m}\right)^{1/(2m)},$$

defined on the interval $(-2a\ell^{1/(2m)}, 2a\ell^{1/(2m)})$.

Proof. The scaled monic polynomials $P_{n,N}(x)=(\phi(N))^{-n}P_n(\phi(N)x)$ associated with the generalized higher-order Freud weight (1.1) have recurrence coefficient $\beta_{n,N}(t;\lambda)=\beta_n(t;\lambda)/(\phi(N))^2$. Since $\phi:\mathbb{R}^+\to\mathbb{R}^+$ and, for every $\ell>0$, we have

$$\lim_{x\to\infty}\frac{\phi(x\ell)}{\phi(x)} = \ell^{1/(2m)},$$

where ϕ is regularly varying at infinity with exponent of variation $1/2m$ (cf. [43]). Since it follows from (4.6) that

$$\lim_{n/N\to\ell}\sqrt{\beta_{n,N}(t;\lambda)} = \lim_{n/N\to\ell}\frac{\sqrt{\beta_n(t;\lambda)}}{\phi(n)}\frac{\phi(n)}{\phi(N)} = a\ell^{1/(2m)},$$

the recurrence coefficients $\beta_{n\cdot N}(t;\lambda)$ are said to be regularly varying at infinity with index $1/2m$ (cf. [42, Section 4.5]). From the property of regular variation, using [42, Theorem 1.4], it follows

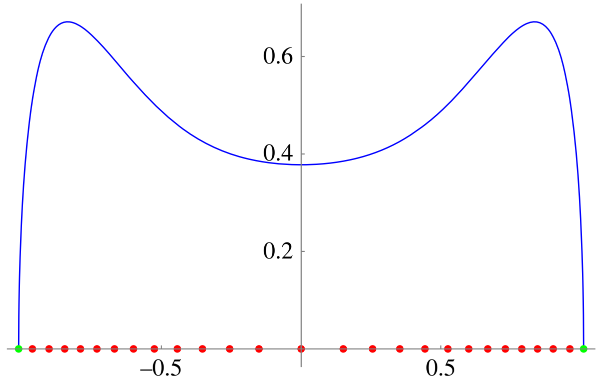

Figure 1. The zeros of $P_{n,N}(x)$ (red) for $\lambda = 0.5$, $t = 1$, $m = 3$, $n = N = 10$ and $\ell = 1$ with the corresponding limiting distribution (6.1) (blue) and endpoints $(-2a, 0)$ and $(2a, 0)$ (green).

that the asymptotic zero distribution has density

$$\begin{aligned}
&\frac{1}{\pi\ell}\int_0^{\ell} s^{-1/(2m)}\left(2a - xs^{-1/(2m)}\right)^{-1/2}\left(2a + xs^{-1/(2m)}\right)^{-1/2} \mathrm{d}s \\
&\quad = \frac{m}{a\pi\ell}\int_0^{\ell^{1/(2m)}} y^{2m-2}\left(1-\left(\frac{x}{2ay}\right)^2\right)^{-1/2} \mathrm{d}y \\
&\quad = \frac{m}{a\pi\ell}\int_0^{\ell^{1/(2m)}} y^{2m-2}\sum_{k=0}^{\infty}\frac{(1/2)_k}{k!}\left(\frac{x}{2ay}\right)^{2k} \mathrm{d}y \\
&\quad = \frac{m}{a\pi\ell}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{k!}\left(\frac{x}{2a}\right)^{2k}\int_0^{\ell^{1/(2m)}} y^{2m-2k-2}\,\mathrm{d}y \\
&\quad = \frac{m}{a\pi\ell^{1/(2m)}}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{k!}\frac{1}{2m-2k-1}\left(\frac{x}{2a\ell^{1/(2m)}}\right)^{2k} \\
&\quad = \frac{m}{a\pi\ell^{1/(2m)}(2m-1)}\sum_{k=0}^{\infty}\frac{(1/2)_k((1/2)-m)_k}{((3/2)-m)_k k!}\left(\frac{x}{2a\ell^{1/(2m)}}\right)^{2k} \\
&\quad = \frac{m}{a\pi(2m-1)\ell^{1/(2m)}}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}-m; \frac{3}{2}-m; \left(\frac{x}{2a\ell^{1/(2m)}}\right)^2\right).
\end{aligned}$$

■

Figure 1 shows the zeros and the asymptotic distribution according to theorem 6.1.

Figure 2 shows the asymptotic distribution of zeros according to theorem 6.1 for various values of ℓ.

Remark 6.2. Note that the formula on [42, p. 189, line 22] should be $(1/t)\int_0^t (1/s^{\lambda})\omega'_{[b-2a,b+2a]}(xs^{-\lambda})\,\mathrm{d}s$.

(b) Bounds for the extreme zeros

From the three-term recurrence relation (4.4), we obtain bounds for the extreme zeros of monic generalized higher-order Freud polynomials.

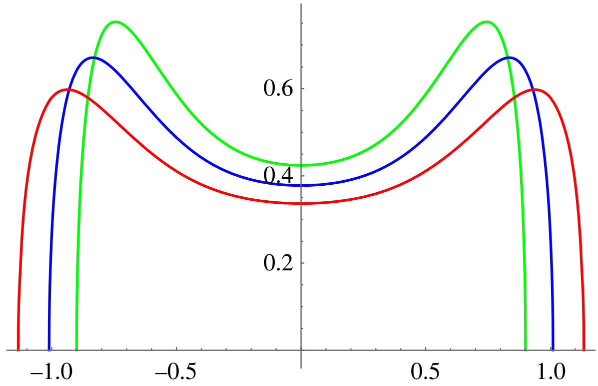

Figure 2. The limiting distribution of the zeros $a_3(\ell)$ for $\ell = 0.5$ (green), $\ell = 1$ (blue) and $\ell = 2$ (red).

Theorem 6.3. *For each $n = 2, 3, \ldots$, the largest zero, $x_{1,n}$, of monic generalized higher-order Freud polynomials $P_n(x)$ orthogonal with respect to the weight (1.1), satisfies*

$$0 < x_{1,n} < \max_{1\le k\le n-1} \sqrt{c_n \beta_k(t;\lambda)},$$

where $c_n = 4\cos^2\left(\dfrac{\pi}{n+1}\right) + \varepsilon$, $\varepsilon > 0$.

Proof. The upper bound for the largest zero $x_{1,n}$ follows by applying [44, Theorem 2 and 3], based on the Wall–Wetzel Theorem to the three-term recurrence relation (4.4). ■

(c) Monotonicity of the zeros

Theorem 6.4. *Consider $0 < x_{\lfloor n/2\rfloor,n} < \cdots < x_{2,n} < x_{1,n}$, the positive zeros of monic orthogonal polynomials $P_n(x)$ with respect to the generalized higher-order Freud weight (1.1), where $\lfloor k\rfloor$ denotes the largest integer less than or equal to k. Then, for $\lambda > -1$, $t \in \mathbb{R}$ and for a fixed value of ν, $\nu \in \{1, 2, \ldots, \lfloor n/2\rfloor\}$, the ν-th zero $x_{n,\nu}$ increases when (i) λ increases; and (ii) t increases.*

Proof. This follows from [2, Lemma 4.5], taking $C(x) = x$, $D(x) = x^2$, $\rho = 2\lambda + 1$ and $\omega_0(x) = \exp(-x^{2m})$. ■

(d) Interlacing of the zeros

Next, for fixed $\lambda > -1$, $t \in \mathbb{R}$ and $k \in (0, 1]$, we consider the relative positioning of the zeros of the monic generalized higher-order Freud polynomials $\{P_n(x;\lambda)\}$ orthogonal with respect to the weight (1.1), and the zeros of $\{P_n(x;\lambda + k)$, $k \in (0, 1]$, orthogonal with respect to the weight

$$\omega(x; t, \lambda) = |x|^{2\lambda+2k+1} \exp(tx^2 - x^{2m}), \quad m = 2, 3, \ldots .$$

The zeros of monic generalized higher-order Freud polynomials $\{P_n(x;\lambda)\}$ orthogonal with respect to the symmetric weight (1.1) are symmetric around the origin. We denote the positive zeros of $P_{2n}(x;\lambda)$ by

$$0 < x^{\lambda}_{n,2n} < x^{\lambda}_{n-1,2n} < \cdots < x^{\lambda}_{2,2n} < x^{\lambda}_{1,2n},$$

and the positive zeros of $P_{2n+1}(x;\lambda)$ by

$$0 < x^{\lambda}_{n,2n+1} < x^{\lambda}_{n-1,2n+1} < \cdots < x^{\lambda}_{2,2n+1} < x^{\lambda}_{1,2n+1},$$

noting that $x_{n+1,2n+1} = 0$.

royalsocietypublishing.org/journal/rspa Proc. R. Soc. A 479: 20220788

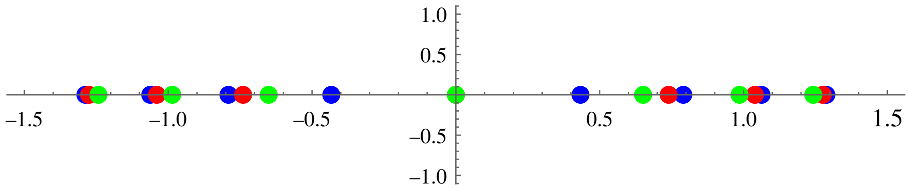

Figure 3. The zeros of $P_7(x;\lambda)$ (green), $P_7(x;\lambda+1)$ (red) and $P_8(x;\lambda)$ (blue) for $\lambda = 0.5$ and $t = 1$.

Theorem 6.5. *Let $\lambda > -1$ and $t \in \mathbb{R}$. Let $\{P_n(x;\lambda)\}$ be the monic generalized higher-order Freud polynomials orthogonal with respect to the weight (1.1). Then, for $\ell \in \{1,\dots,n-1\}$ and $k \in (0,1)$, we have*

$$x^{\lambda}_{\ell+1,2n} < x^{\lambda}_{\ell,2n-1} < x^{\lambda+k}_{\ell,2n-1} < x^{\lambda+1}_{\ell,2n-1} < x^{\lambda}_{\ell,2n}, \tag{6.2}$$

and, for $\ell \in \{1,\dots,n\}$,

$$x^{\lambda}_{\ell+1,2n+1} < x^{\lambda}_{\ell,2n} < x^{\lambda+k}_{\ell,2n} < x^{\lambda+1}_{\ell,2n} = x^{\lambda}_{\ell,2n+1}. \tag{6.3}$$

Proof. The zeros of two consecutive polynomials in the sequence of generalized higher-order Freud orthogonal polynomials are interlacing, i.e.

$$0 < x^{\lambda}_{n,2n} < x^{\lambda}_{n-1,2n-1} < x^{\lambda}_{n-1,2n} < \cdots < x^{\lambda}_{2,2n} < x^{\lambda}_{1,2n-1} < x^{\lambda}_{1,2n} \tag{6.4}$$

and

$$0 < x^{\lambda}_{n,2n} < x^{\lambda}_{n,2n+1} < x^{\lambda}_{n-1,2n} < \cdots < x^{\lambda}_{2,2n+1} < x^{\lambda}_{1,2n} < x^{\lambda}_{1,2n+1}. \tag{6.5}$$

On the other hand, we proved in theorem 6.4 that the positive zeros of generalized higher-order Freud polynomials monotonically increase as the parameterλ increases. This implies that, for each fixed $\ell \in \{1,2,\dots,n\}$ and $k \in (0,1)$,

$$x^{\lambda}_{\ell,2n} < x^{\lambda+k}_{\ell,2n} < x^{\lambda+1}_{\ell,2n} \tag{6.6}$$

and

$$x^{\lambda}_{\ell,2n-1} < x^{\lambda+k}_{\ell,2n-1} < x^{\lambda+1}_{\ell,2n-1}. \tag{6.7}$$

Next, we prove that the zeros of $P_{2n}(x;\lambda)$ interlace with those of $P_{2n-1}(x;\lambda+1)$. From (5.1*b*),

$$P_{2n-1}(x;\lambda+1) = \frac{xP_{2n}(x;\lambda) - (\beta_{2n}(\lambda) + P'_{2n+1}(0;\lambda)/P'_{2n-1}(0;\lambda))P_{2n-1}(x;\lambda)}{x^2}. \tag{6.8}$$

By evaluating (6.8) at consecutive zeros $x_\ell = x^{(\lambda)}_{\ell,n}$ and $x_{\ell+1} = x^{(\lambda)}_{\ell+1,n}$, $\ell = 1,2,\dots,n-1$, of $P_{2n}(x;\lambda)$, we obtain

$$\begin{aligned} &P_{2n-1}(x_\ell;\lambda+1)P_{2n-1}(x_{\ell+1};\lambda+1) \\ &\quad = \frac{(\beta_{2n}(\lambda) + P'_{2n+1}(0;\lambda)/P'_{2n-1}(0;\lambda))^2 P_{2n-1}(x_\ell;\lambda)P_{2n-1}(x_{\ell+1};\lambda)}{x_\ell^2 x_{\ell+1}^2} < 0, \end{aligned}$$

since the zeros of $P_{2n}(x;\lambda)$ and $P_{2n-1}(x;\lambda)$ separate each other. So there is at least one positive zero of $P_{2n}(x;\lambda+1)$ in the interval $(x_\ell, x_\ell+1)$ for each $\ell = 1,2,\dots,n-1$ since there are exactly n positive zeros, and this implies that

$$0 < x^{\lambda}_{n,2n} < x^{\lambda+1}_{n-1,2n-1} < x^{\lambda}_{n-1,2n} < x^{\lambda+1}_{n-2,2n-1} < \cdots < x^{\lambda+1}_{2,2n-1} < x^{\lambda}_{2,2n} < x^{\lambda+1}_{1,2n-1} < x^{\lambda}_{1,2n}. \tag{6.9}$$

Equations (6.4), (6.7) and (6.9) yield (6.2). To prove (6.3), we note that by (5.1*a*), the n positive zeros of $P_{2n}(x,\lambda+1)$ and $P_{2n+1}(x;\lambda)$ coincide, i.e. $x^{\lambda+1}_{\ell,2n} = x^{\lambda}_{\ell,2n+1}$ for $\ell \in \{1,2,\dots,n\}$, and the result follows using (6.5) and (6.6). ■

Figure 3 shows the interlacing of the zeros of polynomials orthogonal with respect to the generalized higher-order Freud weight (1.1) for $m = 3$ as described in (6.4) of theorem 6.5.

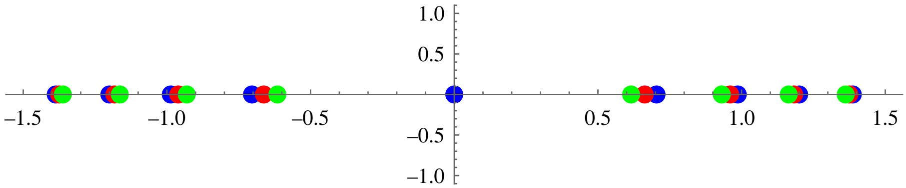

Figure 4. The zeros of $P_8(x;\lambda)$ (green), $P_8(x;\lambda+0.5)$ (red), $xP_8(x;\lambda+1)$ (blue) and $P_9(x;\lambda)$ (blue) for $\lambda=1.5$ and $t=2.3$.

Figure 4 illustrates the interlacing of the zeros of polynomials orthogonal with respect to the generalized higher-order Freud weight (1.1) for $m=3$ as described in (6.5) of theorem 6.5.

7. Quadratic decomposition of the generalized higher-order Freud weight

We apply known results [45, Chapter 1, theorem 9.1] on the quadratic decomposition of any symmetric polynomials to this particular case of generalized higher-order Freud weights. Precisely, if

$$P_{2n}(x;t,\lambda)=B_n(x^2;t,\lambda) \quad \text{and} \quad P_{2n+1}(x;t,\lambda)=xR_n(x^2;t,\lambda), \quad \text{for all } n\geq 0,$$

then from the recurrence relation (2.6), we have

$$B_{n+1}(x;t,\lambda)=R_{n+1}(x;t,\lambda)+\beta_{2n+2}R_n(x;t,\lambda)$$

and

$$xR_n(x;t,\lambda)=B_{n+1}(x;t,\lambda)+\beta_{2n+1}B_n(x;t,\lambda),$$

and this gives second-order recurrence relations for both $\{B_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ as follows:

$$\begin{cases} B_{n+1}(x;t,\lambda)=(x-\beta_{2n}-\beta_{2n+1})B_n(x;t,\lambda)-\beta_{2n-1}\beta_{2n}B_{n-1}(x;t,\lambda), & n\geq 1,\\ B_1(x;t,\lambda)=x-\beta_1, \quad B_0(x;t,\lambda)=1, \end{cases}$$

and

$$\begin{cases} R_{n+1}(x;t,\lambda)=(x-\beta_{2n+2}-\beta_{2n+1})R_n(x;t,\lambda)-\beta_{2n+1}\beta_{2n}R_{n-1}(x;t,\lambda),\\ R_1(x;t,\lambda)=x-\beta_1-\beta_2, \quad R_0(x;t,\lambda)=1. \end{cases}$$

Furthermore, $\{B_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ satisfy the orthogonality relations:

$$\int_0^\infty B_k(x;t,\lambda)B_n(x;t,\lambda)x^\lambda \exp(tx-x^m)\,dx=h_{2n}(t,\lambda)\delta_{n,k}$$

and

$$\int_0^\infty R_k(x;t,\lambda)R_n(x;t,\lambda)x^{\lambda+1} \exp(tx-x^m)\,dx=h_{2n+1}(t,\lambda)\delta_{n,k}, \quad n,k\geq 0.$$

Data accessibility. This article has no additional data.

Authors' contributions. P.A.C.: funding acquisition, investigation, methodology, writing—review and editing; K.J.: funding acquisition, investigation, methodology, writing—original draft, writing—review and editing; A.L.: investigation, methodology, writing—original draft, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

Funding. P.A.C. and K.J. gratefully acknowledge the support of a Royal Society Newton Advanced Fellowship NAF\R2\180669.

Acknowledgements. We also thank the reviewers for helpful suggestions and additional references.

References

1. Clarkson PA, Jordaan K. 2018 Properties of generalized Freud weight. *J. Approx. Theory* **25**, 148–175. (doi:10.1016/j.jat.2017.10.001)
2. Clarkson PA, Jordaan K. 2021 Generalised Airy polynomials. *J. Phys. A* **54**, 185202. (doi:10.1088/1751-8121/abf019)
3. Clarkson PA, Jordaan K. 2021 A generalized sextic Freud weight. *Integral Transforms Spec. Funct.* **32**, 458–482. (doi:10.1080/10652469.2020.1798949)
4. Clarkson PA, Jordaan K, Kelil A. 2016 A generalized Freud weight. *Stud. Appl. Math.* **136**, 288–320. (doi:10.1111/sapm.12105)
5. Magnus AP. 1985 A proof of Freud's conjecture about the orthogonal polynomials related to $|x|^\rho \exp(-x^2m)$, for integer m. In *Orthogonal polynomials and applications* (eds C Brezinski, A Draux, AP Magnus, P Maroni A Ronveaux). Lecture Notes in Mathematics, vol. 1171, pp. 362–372. Berlin, Germany: Springer-Verlag.
6. Magnus AP. 1986 On Freud's equations for exponential weights. *J. Approx. Theory* **46**, 65–99. (doi:10.1016/0021-9045(86)90088-2)
7. Shohat J. 1939 A differential equation for orthogonal polynomials. *Duke Math. J.* **5**, 401–417. (doi:10.1215/S0012-7094-39-00534-X)
8. Maroni P. 1987 Prolégomènes à l'étude des polynômes orthogonaux semi-classiques. *Ann. Mat. Pura Appl. (4)* **149**, 165–184. (doi:10.1007/BF01773932)
9. Maroni P. 1991 Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques. In *Orthogonal polynomials and their applications* (eds C Brezinski *et al.*). IMACS Annals of Computational and Applied Mathematics, vol. 9, pp. 95–130. Basel, Switzerland: Baltzer.
10. Hendriksen E, van Rossum H. 1985 Semi-classical orthogonal polynomials. In *Orthogonal polynomials and applications* (eds C Brezinski, A Draux, AP Magnus, P Maroni, A Ronveaux). Lecture Notes in Mathematics, vol. 1171, pp. 354–361. Berlin, Germany: Springer-Verlag.
11. Olver FWJ *et al.* (eds.) 2022 NIST Digital Library of Mathematical Functions. See http://dlmf.nist.gov/ Release 1.1.8 of 2022-12-15.
12. Muldoon ME. 1977/78 Higher monotonicity properties of certain Sturm-Liouville functions. V. *Proc. R. Soc. Edinb. Sect. A* **77**, 23–37. (doi:10.1017/S0308210500018011)
13. Vein PR, Dale P. 1999 *Determinants and their applications in mathematical physics*. New York, NY: Springer-Verlag.
14. Van Assche W. 2018 *Orthogonal polynomials and painlevé equations*. Australian Mathematical Society Lecture Series. Cambridge, UK: Cambridge University Press.
15. Maroni P. 1999 Semi-classical character and finite-type relations between polynomial sequences. *Appl. Numer. Math.* **31**, 295–330. (doi:10.1016/S0168-9274(98)00137-8)
16. Benassi C, Moro A. 2020 Thermodynamic limit and dispersive regularization in matrix models. *Phys. Rev. E* **101**, 052118. (doi:10.1103/PhysRevE.101.052118)
17. Bonora M, Martellini L, Xiong CS. 1992 Integrable discrete linear systems and one-matrix random models. *Nucl. Phys. B* **375**, 453–477. (doi:10.1016/0550-3213(92)90040-I)
18. Freud G. 1976 On the coefficients in the recursion formulae of orthogonal polynomials. *Proc. R. Ir. Acad. A Math. Phys. Sci.* **76**, 1–6.
19. Laguerre E. 1885 Sur la réduction en fractions continues d?une fraction qui satisfait à une ï£¡ï£¡quation différentielle linéaire du premier ordre dont les coefficients sont rationnels. *J. Math. Pures Appl.* **1**, 135–165.
20. Belmehdi S, Ronveaux A. 1994 Laguerre-Freud's equations for the recurrence coefficients of semi-classical orthogonal polynomials. *J. Approx. Theory* **76**, 351–368. (doi:10.1006/jath.1994.1022)
21. Hounkonnou MN, Hounga C, Ronveaux A. 2000 Discrete semi-classical orthogonal polynomials: generalized Charlier. *J. Comput. Appl. Math.* **114**, 361–366. (doi:10.1016/S0377-0427(99)00275-7)
22. Magnus AP. 1995 Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials. *J. Comput. Appl. Math.* **57**, 215–237. (doi:10.1016/0377-0427(93)E0247-J)
23. Magnus AP. 1999 Freud's equations for orthogonal polynomials as discrete Painlevé equations. In *Symmetries and integrability of difference equations* (eds PA Clarkson, FW Nijhoff). London Mathematical Society Lecture Note Series, vol. 255, pp. 228–243. Cambridge, UK: Cambridge University Press.

24. Cresswell C, Joshi N. 1999 The discrete first, second and thirty-fourth Painlevé hierarchies. *J. Phys. A* **32**, 655–669. (doi:10.1088/0305-4470/32/4/009)
25. Cresswell C, Joshi N. 1999 The discrete Painlevé I hierarchy. In *Symmetries and integrability of difference equations* (eds PA Clarkson, FW Nijhoff). London Mathematical Society Lecture Note Series, vol. 255, pp. 197–205. Cambridge, UK: Cambridge University Press.
26. Douglas MR, Shenker SH. 1990 Strings in less than one dimension. *Nucl. Phys.* **B335**, 635–654. (doi:10.1016/0550-3213(90)90522-F)
27. Gross DJ, Migdal AA. 1990 Nonperturbative two-dimensional quantum gravity. *Phys. Rev. Lett.* **64**, 127–130. (doi:10.1103/PhysRevLett.64.127)
28. Fokas AS, Its AR, Kitaev AV. 1991 Discrete Painlevé equations and their appearance in quantum-gravity. *Commun. Math. Phys.* **142**, 313–344. (doi:10.1007/BF02102066)
29. Fokas AS, Its AR, Kitaev AV. 1992 The isomonodromy approach to matrix models in 2D quantum-gravity. *Commun. Math. Phys.* **147**, 395–430. (doi:10.1007/BF02096594)
30. Kharchev S, Marshakov A, Mironov A, Orlov A, Zabrodin A. 1991 Matrix models among integrable theories: forced hierarchies and operator formalism. *Nucl. Phys. B* **366**, 569–601. (doi:10.1016/0550-3213(91)90030-2)
31. Periwal V, Shevitz D. 1990 Unitary-matrix models as exactly solvable string theories. *Phys. Rev. Lett.* **64**, 1326–1329. (doi:10.1103/PhysRevLett.64.1326)
32. Nevai P. 1986 Géza Freud, orthogonal polynomials and Christoffel functions. A case study. *J. Approx. Theory* **48**, 3–167. (doi:10.1016/0021-9045(86)90016-X)
33. Lubinsky D, Mhaskar H, Saff EB. 1988 A proof of Freud's conjecture for exponential weights. *Constr. Approx.* **4**, 65–83. (doi:10.1007/BF02075448)
34. Lubinsky D, Mhaskar H, Saff EB. 1986 Freud's conjecture for exponential weights. *Bull. Am. Math. Soc.* **15**, 217–221. (doi:10.1090/S0273-0979-1986-15480-7)
35. Lubinsky DS, Saff EB. 1988 Uniform and mean approximation by certain weighted polynomials, with applications. *Constr. Approx.* **4**, 21–64. (doi:10.1007/BF02075447)
36. Szegő G. 1975 *Orthogonal polynomials*. **23**. AMS Colloquium Publications, vol. Providence, RI: Amer. Math. Soc.
37. Ismail MEH. 2005 *Classical and quantum orthogonal polynomials in one variable*. Cambridge, UK: Cambridge University Press.
38. García-Ardila JC, Marcellán F, Villamil-Hernández PH. 2022 Associated orthogonal polynomials of the first kind and Darboux transformations. *J. Math. Anal. Appl.* **508**, 25883. (doi:10.1016/j.jmaa.2021.125883)
39. Bueno MI, Marcellán F. 2004 Darboux transformation and perturbation of linear functionals. *Linear Algebra Appl.* **384**, 215–242. (doi:10.1016/j.laa.2004.02.004)
40. Gautschi W. 1982 An algorithmic implementation of the generalized Christoffel theorem. In *Numerical integration* (ed. G Hämmerlin). International Series of Numerical Mathematics, vol. 57, pp. 89–106. Basel, Switzerland: Birkhäuser.
41. Gautschi W. 1996 Orthogonal polynomials: applications and computation. *Acta Numer.* **5**, 45–119. (doi:10.1017/S0962492900002622)
42. Kuijlaars ABJ, Van Assche W. 1999 The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients. *J. Approx. Theory* **99**, 167–197. (doi:10.1006/jath.1999.3316)
43. Van Assche W, Geronimo JS. 1989 Asymptotics for orthogonal polynomials with regularly varying recurrence coefficients. *Rocky Mt. J. Math.* **19**, 39–49.
44. Ismail MEH, Li X. 1992 Bounds for extreme zeros of orthogonal polynomials. *Proc. Am. Math. Soc.* **115**, 131–140. (doi:10.1090/S0002-9939-1992-1079891-5)
45. Chihara TS. 1978 *An introduction to orthogonal polynomials*. New York, NY: Gordon and Breach. (Reprinted by Dover Publications, 2011).