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Difference Moving Frames

Variational problems and symmetry reduction

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Abstract

In this thesis we explore how moving frames can be applied to variational problems and symmetry reduction. First we consider the difference variational calculus. We show how the recently-developed difference prolongation space can be used to find a moving frame applicable to partial difference equations. This is used to develop the invariant difference calculus of variations for partial difference equations, which includes finding the Euler–Lagrange equations in an invariant form. Moreover, we use the infinitesimal and adjoint action to write the conversation laws for partial difference equations in terms of invariants and the adjoint action. Using difference forms, new formulas for the invariant Euler–Lagrange equations are found. Several different Lie group actions on the dependent variables are explored throughout.

This is extended from the standard rectangular mesh to include meshes constructed from non-rectangular tilings of the plane, looking particularly at the snub square tiling as a running example.

We define the differential-difference moving frame, using recent results on differential-difference structure. With this we develop the invariant differential-difference calculus of variations. This enables us to find the invariant formulation of differential-difference Euler–Lagrange equations for several different types of Lie group actions, including actions on an independent variable.

Finally, we expand the applicability of the moving frames symmetry reduction algorithm for ordinary difference equations. Currently, this does not address Lie group actions that depend on the independent variable, nor can it deal with partitioned difference equations. We give a framework in which these equations can be analysed and discuss differences and similarities between the canonical coordinates method and moving frames method.

Declaration

I declare that the work in this thesis is my own and has been carried out by myself with guidance from my supervisor. Any information from the literature is acknowledged in the text and given in the list of references. No part of this thesis has been presented for another award at this or any other institution.

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Contents

Chapter 1

Introduction

Since the modern development of moving frames by Fels and Olver [7, 8], finding different Lie group invariants such as differential invariants, joint invariants, and semi-differential invariants has become more algorithmic. In addition to this there has been the development of several practical applications, see [29] by Olver. For example in computer vision, moving frames have been used in a program that automatically assembles apictorial jigsaw puzzles. Also moving frames have been applied to practical image processing with a focus on medical imaging like, for example, an MRI scan. Additionally, moving frames are used for invariant numerical approximations within the study of symmetry-preserving numerical methods (geometric numerical integration/structure-preserving algorithms). This gears the motivation to consider moving frames for difference equations.

This thesis extends the theory of moving frames developed by Fels and Olver [7, 8] to variational problems and symmetry reduction. In particular, we use moving frames to determine the calculus of variations in terms of invariants of a Lie group action. This problem has been resolved for differential equations (with a Lie group action on the independent and/or dependent variables) by Kogan and Olver [19]. With $L^{\boldsymbol{\kappa}}$ the invariant form of the Lagrangian, they showed that the Euler–Lagrange equations of partial differential equations (PDEs) have the invariant form

$$\mathcal{A}^*\mathcal{E}\left(L^{\boldsymbol{\kappa}}\right)-\mathcal{B}^*\mathcal{H}\left(L^{\boldsymbol{\kappa}}\right)=0,$$

where $\mathcal{E}\left(L^{\boldsymbol{\kappa}}\right)$ is the invariantized Eulerian, $\mathcal{H}\left(L^{\boldsymbol{\kappa}}\right)$ a suitable invariantized Hamiltonian and $\mathcal{A}^*$, $\mathcal{B}^*$ certain invariant differential operators. Formulas for each of these components are given by Kogan and Olver [19]. The invariant calculus of variations for ordinary difference equations (OΔEs) has also been resolved by Mansfield *et al.* [26], who found that the original Euler–Lagrange equations in invariant form are equivalent to

$$\mathcal{H}^* \mathrm{E}_{\boldsymbol{\kappa}} \left(L^{\boldsymbol{\kappa}} \right) = 0.$$

Here $\mathcal{H}^*$ is a matrix of linear difference operators and $\mathrm{E}_{\boldsymbol{\kappa}}\left(L^{\boldsymbol{\kappa}}\right)$ is a column vector of the Euler operators of $L^{\boldsymbol{\kappa}}$ with respect to some generating invariants $\boldsymbol{\kappa}$. The method in both these papers starts by writing the Lagrangian in terms of the invariants of the Lie group action. This is done by using the moving frame and the replacement rule given by Fels and Olver [8].

Although the papers by Kogan and Olver [19], and Mansfield *et al.* [26] reached their desired outcomes, the methods used to attain their results are quite different. For PDEs, Kogan and Olver [19] introduce the invariant variational bicomplex, which is an invariant version of the variational bicomplex introduced by Anderson [1, 2]. For OΔEs, Mansfield *et al.* [26] use a method based on introducing an additional dummy variable, which was originally used in a textbook on invariant calculus by Mansfield [22]. In this book, it was shown, by letting the dependent variables depend on an additional variable, that the Euler–Lagrange equations could be written in terms of invariants of the Lie group action when the Lie group acts on the dependent variables.

Currently, there is no known Lie group invariant formulation of the Euler–Lagrange equations for partial difference equations (PΔEs) or for differential-difference equations (DΔEs). PΔEs with two independent variables typically lie on a regular square tiling. A generalization to other types of tilings has not yet been considered.

Several applications in the physical world can be modelled by PΔEs and DΔEs, see the difference equations book by Hydon [15] for some PΔEs examples. Therefore, it would be good to have an invariant formulation of the Euler–Lagrange equations for these different types of equations. So a natural question arises. "Is it possible to find an invariant formulation of the Euler–Lagrange equations for PΔEs and DΔEs?"

In this thesis, we derive the Lie group invariant Euler–Lagrange equations for PΔEs on a rectangular lattice and a non-rectangular tiling of the plane and indicate how this can be extended to various other tilings. For DΔEs, we construct the invariant calculus of variations for Lie group actions on the dependent variables. We also find the invariant Euler–Lagrange equation of a DΔE with a Lie group action on one independent variable only. Considering only one independent continuous variable and one dependent variable, the invariant Euler–Lagrange equation for a Lie group action on both continuous variables is found.

To achieve these results, we begin by looking at the PΔEs case using the method in the OΔEs paper [26], that is, using an additional dummy variable. Then we look at the same calculations using a difference version of the invariant variational bicomplex. This is constructed from the difference variational bicomplex introduced in a paper by Hydon and Mansfield [17]. From here the two methods can be compared showing many similarities between them. From this point, either method can be used to find the formula of the invariant Euler–Lagrange equations.

In addition to finding the invariant Euler–Lagrange equations for various types of equations, we also construct conservation laws of PΔEs by developing an equivariant form of Noether's Theorem. Mansfield [22] showed that for ODEs the conservation laws could be written in terms of the moving frame and the invariants of the Lie group. Similar methods for PDEs have been developed by Gonçalves and Mansfield in several recent papers [9, 10, 11], using extensions to the original method. Very recently, the method of introducing an additional variable to find the conservation laws has progressed from PDEs to OΔEs in two papers [25, 26], where both linear and semi-simple actions were explored.

At present, it is unknown whether or not a similar formula exists for writing the conservation laws of PΔEs in terms of invariants and the moving frame. Conservation laws are important for many reasons, including the fact that if a PΔE is the Euler–Lagrange equation for a variational problem then the conservation laws are linked with variational symmetries by Noether's Theorem for PΔEs (see

[15]). The goal is to find a similar formula for the conservation laws of PΔEs using the same approach as seen in the OΔEs paper by Mansfield *et al.* [26].

To finish this thesis we consider symmetry reduction. Symmetry reduction of ODEs was first described by Lie (see the texts by Olver [28], Bluman and Anco [5] and Hydon [16]). This approach is known as the canonical coordinates method, and it involves finding the invariant and equivariant components of a particular equation. This method has been extended to OΔEs in the difference equations book by Hydon [15]. Benson and Valiquette [4] recently showed that the moving frame can be used for symmetry reductions of OΔEs. It was also shown by Valiquette [34] that symmetry reduction can be done for ODEs by moving frames. At present, the OΔEs theory requires the Lie group actions not to depend on the independent variable. Additionally, the theory for partitioned OΔEs, as defined in the difference equations book by Hydon [15], has not been studied yet.

Many OΔEs have Lie group actions that depend on the independent variable so these must be considered. Therefore, we extend the moving frames method by Benson and Valiquette [4] to include these Lie group actions. Furthermore, we show how to apply this method for partitioned OΔEs. A thorough comparison of symmetry reductions of OΔEs by moving frames and canonical coordinates methods is also given.

Here is a summary of the topics covered in the rest of the thesis. In Section 2.1 the difference prolongation space is explained in detail; this space is the foundation of the difference moving frame theory. Section 2.2 introduces difference calculus of variations and gives some key results from the literature, including the difference case of Noether's theorem. Section 2.3 defines the Lie group action and gives the linearized symmetry condition, which is used to find the Lie group symmetries of an equation. Section 2.4 describes the current knowledge of moving frames and explores how we get from using a moving frame which is useful for the continuous case to one adapted to the discrete case. This will include the most important extension for us, the difference moving frame. Section 2.5 presents the main proposition of the chapter, which explains how the Euler–Lagrange equations can be calculated directly in terms of invariants of the Lie group action. Section 2.6 uses difference forms, which are comparable to differential forms, to complete the proof of the main proposition in Section 2.5. It also gives rise to two new formulae for the invariantized Euler–Lagrange equations. Indeed, this section amounts to the difference form version of what is shown by Kogan and Olver [19]. A running example is used throughout these sections to help with understanding these concepts. In Section 2.7, one of the new formulae for the invariantized Euler–Lagrange equations is used to revisit an example from the OΔE paper [25]. In Section 2.8 we look at the same Lagrangian and explore some of its different Lie subgroups. These include a one-parameter Lie group of translations, a two-parameter Lie group of translations, a 4-parameter Lie group of scalings and translations, and the projective SL(2) action which is a semi-simple group action. The Lie group action depends on the independent variables in both the two-parameter Lie group of translations example and the 4-parameter Lie group of scalings and translations example. When the Lie group action depends on the independent variables some additional complications can occur. For each example, the calculations to find the invariant Euler–Lagrange equations are shown. Also, we show how changing the normalization can change the difficulty of some of the computations. Section 2.9 discusses infinitesimal generators and the adjoint action, which is used to help formulate the conservation laws in Section 2.10.

In Chapter 3 the extension to non-rectangular tilings of the plane is considered. To help explore this topic the snub square tiling is used as a running example; the prolongation space for this is described in Section 3.1. (The prolongation for other regular and semi-regular tilings is outlined in Appendix B.) Then in Section 3.2 the difference variational calculus on non-rectangular tilings is discussed. Most of this section is an extension of the rectangular case with some subtle differences, so some details are omitted. Section 3.3 defines a difference moving frame on a non-rectangular mesh and explains how this is used to extend the definitions and propositions in Chapter 2. The last section in this chapter, Section 3.4, lays out the definition of difference forms on a non-rectangular mesh, proves the formula for the invariant Euler–Lagrange equations on non-rectangular mesh and provides a simple example to illustrate the theory.

Chapter 4 extends the theory by considering different variational problems for differential-difference equations (DΔEs). Section 4.1 explains the differential-difference structure developed very recently by Peng and Hydon [32]. This extends the difference prolongation space in Chapter 2 to include a differential structure. In Section 4.2 the difference moving frame is extended to include DΔEs. Then we consider the differential-difference calculus of variations in Section 4.3. Section 4.4 explains how to obtain the invariant differential-difference Euler–Lagrange equations, for several Lie groups.

Chapter 5 considers the use of moving frames to reduce and solve OΔEs. In Section 5.1 we introduce the inductive moving frame construction that is used for the moving frame reduction of OΔEs with a solvable symmetry group. Inductive moving frames have been developed across many papers [4, 18, 30, 33, 34] of which the most important for us is the OΔE reduction theory introduced by Benson and Valiquette [4]. Section 5.2 adapts this theory to the difference moving frame, enabling group actions to involve the independent variable. This section includes theory and examples for both one-parameter and multi-parameter solvable symmetry groups. Section 5.3 explores more examples, including a third order OΔE with a solvable symmetry group of dimension three. The theory of symmetry reductions using moving frames for systems of OΔEs is discussed, with an example, in Section 5.4. Section 5.5 gives the reduction method and an illustrative example for partitioned OΔEs. To conclude this chapter, Section 5.6 discusses some of the positives and negatives of the moving frame method compared to the canonical coordinates method. All of the examples in this chapter are OΔEs that have a Lie group that depends on the independent variable.

Finally, Chapter 6 gives some conclusions and potential future areas of research.

Throughout this thesis, propositions and lemmas represent new results found and theorems represent current results in the literature, as cited.

Chapter 2

Variational partial difference equations: rectangular mesh

This chapter extends the invariant calculus of variations theory from ordinary difference equations (OΔEs) to partial difference equations (PΔEs). This includes the difference moving frame, differential-difference invariants and syzygies, the invariant formulation of the Euler–Lagrange equations, and conservation laws. This newly extended theory is illustrated by several examples. The most novel material is in the section where the invariant formulation of the Euler–Lagrange equations is found using difference forms. This is then related to the existing method of finding the invariant formulation of Euler–Lagrange OΔEs.

2.1 Difference prolongation space

In the differential case, one can represent differential equations on a connected space using independent variables, dependent variables, and derivatives of the dependent variables (see [28]). As for the differential case, it is important to obtain a connected space to represent difference equations. The difference prolongation space achieves just this. We now outline the construction of the difference prolongation space for the case of one independent variable [26] and more than one independent variable [32].

The total space in which solutions of the difference equations lie is $\mathbb{Z}^p \times \mathbb{R}^q$, where the coordinates of the independent variables are $\mathbf{n} := \left(n^1, n^2, ..., n^p\right) \in \mathbb{Z}^p$,

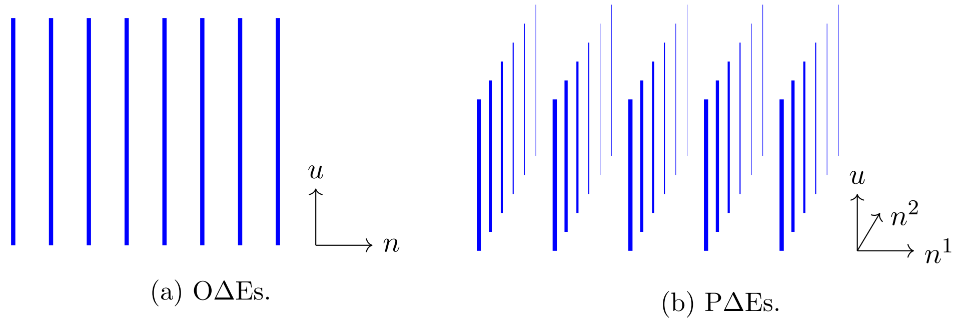


Figure 2.1: The total space for OΔEs (Figure 2.1a) is $\mathcal{T} = \mathbb{Z} \times \mathbb{R}$ and the total space for PΔEs with 2 independent variables (Figure 2.1b) is $\mathcal{T} = \mathbb{Z}^2 \times \mathbb{R}$.

and the dependent variables are $\mathbf{u} := \left(u^1, u^2, ..., u^q\right) \in \mathbb{R}^q$. As an example in Figure 2.1 a diagram of the OΔEs total space (with one dependent variable) (Figure 2.1a) and PΔEs total space (with one dependent and two independent variables) (Figure 2.1b) is shown.

We give here some definitions found in [15] that will help define a product lattice. For many difference equations, the independent variables do not take all possible values in $\mathbb{Z}^p$. When $p = 1$, the domain is a set of consecutive points in $\mathbb{Z}$; such a set is called a discrete interval, whether or not it is finite, that is, either the domain $D = \mathbb{Z}$ or some finite discrete interval $D = [n_0, n_1] \cap \mathbb{Z}$, where $n_0, n_1 \in \mathbb{Z}$ with $n_0 < n_1$. The simplest p-dimensional analogue of a discrete interval is the Cartesian product of p discrete intervals called a box. Now we introduce a valid lattice map, which is a map that does not shuffle points arbitrarily because the image of the lattice must be the Cartesian product of p copies of $\mathbb{Z}$. A valid lattice map is a bijective change of the independent variables that preserves the structure of the lattice. Then a product lattice is any set of points in $\mathbb{Z}^p$ that can be mapped to a box by a valid lattice map.

The total space is disconnected; therefore, finding a connected representation of this space, over each base point $\mathbf{m}$, is necessary to relate values of $\mathbf{u}$ on different fibres. Consider the horizontal translation

$$\mathrm{T}_{\mathbf{K}} : \mathbb{Z}^p \times \mathbb{R}^q \to \mathbb{Z}^p \times \mathbb{R}^q, \qquad \mathrm{T}_{\mathbf{K}} : (\mathbf{n}, \mathbf{u}) \mapsto (\mathbf{n} + \mathbf{K}, \mathbf{u}),$$

where $\mathbf{K} \in \mathbb{Z}^p$ is a multi-index. Let $u^\alpha_{\mathbf{J}}$ denote the value of u^α on the fibre $\mathbf{n} + \mathbf{J}$. The total prolongation space is $\mathbb{Z}^p \times P\left(\mathbb{R}^q\right)$, with coordinates $(\mathbf{n}, \mathbf{u}_{\mathbf{J}})$, where $u^\alpha_{\mathbf{J}} : P(\mathbb{R}^q) \mapsto \mathbb{R}$ are functions. Furthermore, we extend the translation operator to this space

$$\mathrm{T}_{\mathbf{K}} : (\mathbf{n}, (u^\alpha_{\mathbf{J}})) \mapsto (\mathbf{n} + \mathbf{K}, (u^\alpha_{\mathbf{J}})).$$

Thus, it is possible to move a prolonged dependent coordinate from one fibre to another using this same horizontal translation. It is also possible to construct a copy of the prolongation space $P(\mathbb{R}^q)$ over each $\mathbf{n}$; this prolongation space on $\mathbf{n}$, denoted by $P_{\mathbf{n}}(\mathbb{R}^q)$, contains the values of $\mathbf{u}$ on all fibres. The prolongation space allows us to represent values of $\mathbf{u}$ on different fibres onto a single fibre. To do this we introduce the pullback. Given f is a (locally smooth) function on $P(\mathbb{R}^q)$ with its restriction to $P_{\mathbf{n}}(\mathbb{R}^q)$ denoted

$$f_{\mathbf{n}}((u^\alpha_{\mathbf{J}})) = f(\mathbf{n}, (u^\alpha_{\mathbf{J}})),$$

the pullback $\mathrm{T}^*_{\mathbf{K}}$ of $f_{\mathbf{n}+\mathbf{K}}((u^\alpha_{\mathbf{J}}))$ to $P_{\mathbf{n}}(\mathbb{R}^q)$ is

$$\mathrm{T}^*_{\mathbf{K}} f_{\mathbf{n}+\mathbf{K}}((u^\alpha_{\mathbf{J}})) = f\left(\mathbf{n} + \mathbf{K}, \left(u^\alpha_{\mathbf{J}+\mathbf{K}}\right)\right).$$

Therefore, the pullback by $\mathrm{T}_{\mathbf{K}}$ takes $u^\alpha_{\mathbf{J}}$ on $\mathbf{n} + \mathbf{K}$ to the function $u^\alpha_{\mathbf{J}+\mathbf{K}}$ on $\mathbf{n}$. Finally, the shift operator is the operator on each continuous fibre $P_{\mathbf{n}}(\mathbb{R}^q)$ that mimics translation by $\mathbf{K}$, i.e.,

$$\mathrm{S}_{\mathbf{K}} f(\mathbf{n}, (u^\alpha_{\mathbf{J}})) = f\left(\mathbf{n} + \mathbf{K}, \left(u^\alpha_{\mathbf{J}+\mathbf{K}}\right)\right).$$

Therefore, the shift operator relates to the pullback as follows: $\mathrm{S}_{\mathbf{K}} f_{\mathbf{n}} := \mathrm{T}^*_{\mathbf{K}} f_{\mathbf{n}+\mathbf{K}}$. From now on the prolongation space over $\mathbf{n}$ is used to represent all values of $\mathbf{u}$ on different fibres onto a single fibre. Here the fixed $\mathbf{n}$ is called the base point.

Before we discuss difference variational problems, we need to define what is a regular domain D, [15]. To do this, let

$$\mathcal{A}(m, n, \{u_{i,j} : (i,j) \in \mathcal{S}\}) = 0$$

be the PΔE with $(m,n) \in D$ and $\mathcal{S}$ a non-empty finite set, called the stencil of the P∆E. Here we consider two independent variables but can extend this to more if required. For any P∆E the stencil at (m,n) is denoted by $\mathcal{S}(m,n)$ and is obtained by evaluating $\mathcal{A}=0$ at (m,n) and finding the stencil of the resulting expression. A border is a line in (i,j)-plane, $l_i(\mathbf{k})$, that includes at least two points of $\mathcal{S}$ and is not straddled by $\mathcal{S}$. A vertex is a point that belongs to two distinct borders, so each border contains exactly two vertices. Any point (m,n) at which at least one vertex of $\mathcal{S}$ does not belong to $\mathcal{S}(m,n)$ is called a singular point. A regular domain D for a given P∆E is a product lattice that has no singular points (for more details, see Chapter 4 of [15]).

For the rest of this thesis, we omit neighbourhoods of singularities and treat $\mathbf{n}$ as fixed, using the shift operator to represent structures over different base points $\mathbf{m}$. The difference prolongation space allows us to introduce key definitions and theorems for the difference calculus of variations. We use the Einstein summation convention to denote sums over all variables other than $\mathbf{n}$ where this is possible.

2.2 The difference variational calculus

The definitions and results in this section can be found for P∆Es in Chapter 6 of the difference equations book by Hydon [15] and for O∆Es in Section 3 of the paper by Mansfield *et al* [26]. We first write the Euler operators and Euler–Lagrange equations in terms of shifts and derivatives. A system of P∆Es is a given system of relations between the quantities $u^\alpha_{\mathbf{n}+\mathbf{J}}$, for each multi-index $\mathbf{J}=\left(j^1,...,j^p\right)$. For convenience we write $\mathbf{J}=j^k\mathbf{1}_k$, where $\mathbf{1}_k$ is the multi-index whose k^{th} entry is 1 and whose other entries are 0. The system will hold for all $\mathbf{n}$ in a given product lattice, which may or may not be finite, so we can suppress $\mathbf{n}$ and use the shorthand notation as before of $u^\alpha_{\mathbf{J}}$ for $u^\alpha(\mathbf{n}+\mathbf{J})$ and $\mathbf{u}_{\mathbf{J}}$ for $\mathbf{u}(\mathbf{n}+\mathbf{J})$. As noted in the last section $\mathbf{n}$ is taken to be the base point. From now we often drop the multi-index $\mathbf{0}$ when representing variables at $\mathbf{n}$; however, where the multi-index adds clarity it will be included. For any multi-index $\mathbf{J}$, the corresponding shift operator is $\mathrm{S}_{\mathbf{J}}=\mathrm{S}_1^{j_1}\cdots\mathrm{S}_p^{j_p}$, where $\mathrm{S}_i:=\mathrm{S}_{\mathbf{1}_i}$ denotes the forward shift with respect to n^i. Specifically, the forward shift operator in the direction n^k is

$$\mathrm{S}_k : (\mathbf{n}, f(\mathbf{n})) \mapsto (\mathbf{n}+\mathbf{1}_k, f(\mathbf{n}+\mathbf{1}_k)),$$

for all functions f whose domain includes $\mathbf{n}$ and $\mathbf{n}+\mathbf{1}_k$. Consequently,

$$\mathrm{S}_k : \mathbf{u} \mapsto \mathbf{u}_{\mathbf{1}_k}, \quad \mathrm{S}_k : \mathbf{u}_{\mathbf{J}} \mapsto \mathbf{u}_{\mathbf{J}+\mathbf{1}_k},$$

on any domain where these quantities are defined. The identity operator is

$$\mathrm{id} : \mathbf{n} \mapsto \mathbf{n}, \qquad \mathrm{id} : f(\mathbf{n}) \mapsto f(\mathbf{n}), \qquad \mathrm{id} : u^{\alpha}_{\mathbf{J}} \mapsto u^{\alpha}_{\mathbf{J}},$$

and the forward difference operator in the direction n^k is

$$\mathrm{D}_{n^k} = \mathrm{S}_k - \mathrm{id}.$$

A difference divergence is an expression of the form $\mathrm{Div}(F) = \mathrm{D}_{n^k} F^k$ for some $F := \left(F^1, \ldots, F^p\right)$. These are all the essential definitions for us to start looking at difference variational calculus, as introduced by Kupershmidt [20].

The basic variational problem is to find the extrema of a given functional

$$\mathcal{L}[\mathbf{u}] = \sum_{\mathbf{n}} \mathrm{L}(\mathbf{n}, [\mathbf{u}]), \tag{2.1}$$

with $[\mathbf{u}]$ representing finitely many shifts of the dependent variables. Throughout this thesis, $\mathrm{L} = \mathrm{L}(\mathbf{n}, [\mathbf{u}])$ is used to denote the Lagrangian in terms of the original variables $\mathbf{u}$ and their shifts. While $L^{\boldsymbol{\kappa}} = L^{\boldsymbol{\kappa}}(\mathbf{n}, [\boldsymbol{\kappa}])$ denotes the Lagrangian in terms of generating invariants $\boldsymbol{\kappa}$ and their shifts. The sum is over a regular domain D, which may be unbounded, if it is bounded $\mathbf{u}$ is prescribed at the boundaries of the domain. From here on, we work formally assuming the boundary terms do not contribute. The extrema can be found for variational PΔE problems by requiring that

$$\left\{ \frac{\mathrm{d}}{\mathrm{d}\epsilon} \mathcal{L}[\mathbf{u}+\epsilon \mathbf{w}] \right\}\Big|_{\epsilon=0} = 0,$$

for all $\mathbf{w} : \mathbb{Z}^p \to \mathbb{R}^q$ that vanish on the boundary (or in the appropriate limit, where the domain is unbounded). Consequently, the variation of $\mathcal{L}[\mathbf{u}]$ in the direction $\mathbf{w}$ is

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\mathcal{L}\left[\mathbf{u}+\epsilon\mathbf{w}\right] &= \sum_{\mathbf{n}}\left(\mathrm{S}_{\mathbf{J}}w^{\alpha}\frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}}\right)\\ &= \sum_{\mathbf{n}} w^{\alpha}\mathrm{S}_{-\mathbf{J}}\frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}} + \mathrm{Div}\left(A_{\mathbf{u}}\left(\mathbf{n},\mathbf{w}\right)\right)\\ &= \sum_{\mathbf{n}} w^{\alpha}\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right) + \mathrm{Div}\left(A_{\mathbf{u}}\left(\mathbf{n},\mathbf{w}\right)\right),\end{aligned} \tag{2.2}$$

where $\mathrm{S}_{-\mathbf{J}} = \mathrm{S}_1^{-j_1}\mathrm{S}_2^{-j_2}\cdots\mathrm{S}_p^{-j_p}$ and

$$\mathrm{E}_{u^{\alpha}} = \mathrm{S}_{-\mathbf{J}}\frac{\partial}{\partial u^{\alpha}_{\mathbf{J}}}$$

is the Euler operator with respect to u^{α}. This was derived by Kupershmidt in the book on discrete Lax equations and differential-difference calculus [20]. To derive (2.2), the summation by parts formula,

$$\left(\mathrm{S}_{\mathbf{J}}f\right)g = f\left(\mathrm{S}_{-\mathbf{J}}g\right) + \left(\mathrm{S}_{\mathbf{J}}-\mathrm{id}\right)f\left(\mathrm{S}_{-\mathbf{J}}g\right) \tag{2.3}$$

for each $\mathbf{J}$, is used. Hence, the last term in (2.2) is

$$\mathrm{Div}\left(A_{\mathbf{u}}\left(\mathbf{n},\mathbf{w}\right)\right) = \sum_{\mathbf{J}}\left(\mathrm{S}_{\mathbf{J}}-\mathrm{id}\right)w^{\alpha}\mathrm{S}_{-\mathbf{J}}\left(\frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}}\right).$$

This can be written into the form $\mathrm{D}_{n^i}\left(A^i_{\alpha}\left(\mathbf{n},[\mathbf{u}]\right)w^{\alpha}\right)$, with difference operators $A^i_{\alpha}\left(\mathbf{n},[\mathbf{u}]\right)$ for $i = 1,...,p$. It is known that the extrema satisfy the following system of Euler–Lagrange (difference) equations

$$\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right) = \mathrm{S}_{-\mathbf{J}}\left(\frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}}\right) = 0. \tag{2.4}$$

Each $\mathrm{E}_{u^{\alpha}}$ depends on only $\mathbf{n}$ and $[\mathbf{u}]$. Note if we let the dependent variable depend on an additional independent variable t one can achieve the same result using

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{(u^{\alpha})'=w^{\alpha}}\mathcal{L}\left[\mathbf{u}\right] = 0, \tag{2.5}$$

where $(u^{\alpha})' = \mathrm{d}u^{\alpha}/\mathrm{d}t$. The following sections will show how to find the invariantized version of the Euler–Lagrange equations using a similar method.

Certain variations $\mathbf{w}$ in (2.1) are important in relation to the Lagrangian. For a particular Lagrangian L, certain variations $\mathbf{Q}$ are linked with the symmetries of the Lagrangian. The variations $\mathbf{Q}$ linked with symmetries of the Lagrangian will leave the Lagrangian invariant up to a total difference term. Now we define the symmetry condition for a Lagrangian, $\mathrm{L} = 0$.

Definition 2.2.1. Suppose that a nonzero function $\mathbf{Q}$ satisfies

$$Q^{\alpha}_{\mathbf{J}}\left(\mathbf{n},\mathbf{u}\right)\frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}} = \mathrm{D}_{n^i} B^i\left(\mathbf{n},\mathbf{u}\right), \text{ where } Q^{\alpha}_{\mathbf{J}} = \mathrm{S}_{\mathbf{J}} Q^{\alpha}, \tag{2.6}$$

with potentially some or all $B^i\left(\mathbf{n},\mathbf{u}\right)$ being equal to zero. Then the Lagrangian L is said to have a variational symmetry (this is shorthand for a one-parameter local Lie group of variational symmetries, see [28]) with characteristic $\mathbf{Q}$. The Lagrangian is invariant under this symmetry if $B^i = 0$ for all i.

In this thesis, we only consider symmetries which leave the Lagrangian invariant. Section 9 of the OΔE paper [26] explains how to deal with variational symmetries for which not all B^i are zero. An extension of this method to PΔEs is trivial.

More details on the relationship between symmetries and characteristics are discussed in Section 2.3 and Section 2.9. The next theorem is Noether's Theorem for PΔEs, which outlines the importance of symmetries and their relationship to conservation laws.

Theorem 2.2.2 (Difference Noether's theorem)**.** Suppose that a Lagrangian L has a variational symmetry with characteristic $\mathbf{Q} \neq \mathbf{0}$. If $\overline{\mathbf{u}} = \mathbf{u}$ is a solution of the Euler–Lagrange system for L then

$$\left(\mathrm{D}_{n^i}\{A^i_{\alpha}\left(\mathbf{n},[\mathbf{u}]\right)Q^{\alpha} - B^i\left(\mathbf{n},\mathbf{u}\right)\}\right)|_{\mathbf{u}=\overline{\mathbf{u}}} = 0. \tag{2.7}$$

Proof. Substituting Q^{α} for w^{α} in (2.2) and using

$$\mathrm{Div}\left(A_{\mathbf{u}}\left(\mathbf{n},\mathbf{w}\right)\right) = \mathrm{D}_{n^i}\left(A^i_{\alpha}\left(\mathbf{n},[\mathbf{u}]\right)w^{\alpha}\right)$$

gives

$$Q^{\alpha}\left(\mathbf{n},\mathbf{u}\right)\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right)+\mathrm{D}_{n^{i}}\left(A_{\alpha}^{i}\left(\mathbf{n},\left[\mathbf{u}\right]\right)Q^{\alpha}\right)=Q_{\mathbf{J}}^{\alpha}\left(\mathbf{n},\mathbf{u}\right)\frac{\partial\mathrm{L}}{\partial u_{\mathbf{J}}^{\alpha}}=\mathrm{D}_{n^{i}}B^{i}\left(\mathbf{n},\mathbf{u}\right).$$

The result follows immediately. □

The expression in (2.7) is a conservation law for the Euler–Lagrange system.

Note for difference equations (and differential-difference equations) transformations in addition to Lie point symmetries are possible. For example, one can use a lattice transformation of the form $\widehat{\mathbf{n}} = A\mathbf{n} + \mathbf{n}_0$, where the matrix $A \in \mathrm{GL}_p\left(\mathbb{Z}\right)$ and $\mathbf{n}_0$ is a column vector with p rows, (see [15] for more details). However, only Lie point symmetries are considered here.

2.3 Lie group actions and symmetries

This section details some of the basic facts about Lie group actions and Lie symmetries, in particular, what they are and how to find them. To start we introduce the definition of a Lie group action using [35].

Definition 2.3.1. Let M be a manifold, and let G be a Lie group. A C^∞ map $G \times M \to M$ such that

$$\mu\left(g_1 g_2, z\right) = \mu\left(g_1, \mu\left(g_2, z\right)\right), \quad \mu\left(e, z\right) = z, \tag{2.8}$$

for all $g_1, g_2 \in G$, e the identity element of G and $z \in M$ is called an action of G on M on the left. If the map $\mu : G \times M \to M$ is an action of G on M on the left, then for a fixed $g \in G$ the map $z \mapsto \mu\left(g, z\right)$ is a diffeomorphism of M, that is, a smooth invertible map whose inverse is also smooth. Similarly, a C^∞ map $\mu : M \times G \to M$ such that

$$\mu\left(z, g_1 g_2\right) = \mu\left(\mu\left(z, g_1\right) g_2\right), \quad \mu\left(z, e\right) = z$$

for all $g_1, g_2 \in G$ and $z \in M$ is called an action of G on M on the right.

The choice of action can significantly affect the difficulty of computations. For simplicity, in this thesis, the left action is used throughout. Using $\mu\left(g, z\right) \mapsto g \cdot z$, the condition for a left action (2.8) above can be written as

$$(g_1 g_2) \cdot z = g_1 \cdot (g_2 \cdot z), \quad e \cdot z = z.$$

With this understanding of a Lie group action, one should now ask how to find the Lie group symmetries of an equation. As the majority of this thesis examines difference equations we restrict attention to the PΔEs case, which is described in the difference equations book by Hydon [15]. For the PDEs case, see the differential equations texts [5, 16, 28].

From Hydon [15] "A transformation of a differential or difference equation is a symmetry if every solution of the transformed equation is a solution of the original equation and vice versa. Thus the set of solutions is mapped invertibly to itself."

The type transformations we are interested in are diffeomorphisms. Given an equation (system) whose dependent variables are $\mathbf{u} = \left(u^1, \ldots, u^q\right)$, a point transformation is a locally defined diffeomorphism $\Gamma : \mathbf{u} \mapsto \widehat{\mathbf{u}}(\mathbf{u}; \epsilon)$; the term 'point' is used because $\widehat{\mathbf{u}}$ depends only on the point $\mathbf{u}$. A parameterized set of point transformations,

$$\Gamma_\epsilon : \mathbf{u} \mapsto \widehat{\mathbf{u}}(\mathbf{u}; \epsilon), \quad \epsilon \in (\epsilon_0, \epsilon_1),$$

where $\epsilon_0 < 0$ and $\epsilon_1 > 0$, is a one-parameter local Lie group if the following conditions are satisfied:

- Γ_0 is the identity map, so that $\widehat{\mathbf{u}} = \mathbf{u}$ when $\epsilon = 0$.
- $\Gamma_\delta \Gamma_\epsilon = \Gamma_{\delta+\epsilon}$ for every δ, ϵ sufficiently close to zero.
- Each $\widehat{u}^\alpha$ can be represented as a Taylor series in ϵ (in a neighbourhood of $\epsilon = 0$ that is determined by $\widehat{\mathbf{u}}$), and therefore

$$\widehat{u}^\alpha(\mathbf{u}; \epsilon) = u^\alpha + \epsilon Q^\alpha(\mathbf{n}, \mathbf{u}) + \mathcal{O}\left(\epsilon^2\right), \quad \alpha = 1, \ldots, q.$$

For further details see [15] by Hydon.

For this thesis, we only consider Lie point transformations that belong to a one-parameter local Lie group. Additionally, all the Lie symmetries are one-parameter local Lie group symmetries (more parameters may be used later). To find the Lie symmetries one needs to find each of the terms Q^α, more commonly known as the characteristics (see Section 2.10). One way to find the characteristic is by using the linearized symmetry condition (LSC) first stated in a paper by Maeda [21]. Given a system $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_s) = 0$ of s equations with q dependent variables and p independent variables, to find the LSC substitute

$$\widehat{\mathbf{n}} = \mathbf{n}, \quad \widehat{u}^\alpha_{\mathbf{K}} = u^\alpha_{\mathbf{K}} + \epsilon Q\left(\mathbf{n}+\mathbf{K}, u^\alpha_{\mathbf{K}}\right) + \mathcal{O}\left(\epsilon^2\right), \quad \mathbf{K} \in \mathbb{Z}^p,$$

into the P∆E, and use that $\mathcal{A} = 0$. This will leave an equation for the $\mathcal{O}(\epsilon)$ terms, which is the LSC. Then using an ansatz one can at least partially solve the LSC (see [15]).

Another method to solve the LSC is by differential elimination. For O∆Es there is a six-stage process to finding the characteristics Q by this method given in the difference equations book [15].

1. Write out the LSC for the O∆E using the method described above.

2. Differentiate with respect to the dependent variables u^α_k (and, if necessary, rearrange) the LSC repeatedly, so that at least one term involving a shift of $Q, Q', \ldots$ is eliminated. Continue doing this until an ODE is obtained.

3. Split the ODE into a system of ODEs whose coefficients involve only the arguments of the unknown function Q.

4. Simplify the system of ODEs (if possible).

5. Integrate the simplified ODEs, one step at a time, and substitute the results successively into the hierarchy of functional-differential equations that were constructed in stage 2. If possible, solve the linear O∆Es for coefficients of terms in Q, and solve these.

6. Finally, substitute Q into the LSC, collect and simplify any remaining O∆Es for coefficients of terms in Q, and solve these.

A similar process can be used for PΔEs (see [15] for examples). As an example of this method we show how to find the symmetries of the autonomous dpKdV equation. This example is given in [15] by Hydon.

Example 2.3.1. The autonomous dpKdV equation is

$$u_{1,1} = u + \frac{1}{u_{1,0} - u_{0,1}}\,. \tag{2.9}$$

For equations of the form $u_{1,1} = \omega\left(m, n, u, u_{1,0}, u_{0,1}\right)$ using the substitution

$$\widehat{u} = u + \epsilon Q\left(m, n, u\right) + \mathcal{O}\left(\epsilon^2\right)$$

yields

$$Q\left(m+1, n+1, \omega\right) = \\ \frac{\partial \omega}{\partial u} Q\left(m, n, u\right) + \frac{\partial \omega}{\partial u_{1,0}} Q\left(m+1, n, u_{1,0}\right) + \frac{\partial \omega}{\partial u_{0,1}} Q\left(m, n+1, u_{0,1}\right),$$

as the LSC. Thus, for the autonomous dpKdV equation (2.9) the LSC is

$$Q\left(m+1, n+1, \omega\right) = Q\left(m, n, u\right) + \frac{Q\left(m, n+1, u_{0,1}\right) - Q\left(m+1, n, u_{1,0}\right)}{\left(u_{1,0} - u_{0,1}\right)^2}\,. \tag{2.10}$$

Now applying the partial differential operator $\partial/\partial u_{1,0} + \partial/\partial u_{0,1}$ to (2.10) gives

$$Q'\left(m, n+1, u_{0,1}\right) - Q'\left(m+1, n, u_{1,0}\right) = 0. \tag{2.11}$$

Then applying the partial differential operator $-\partial/\partial u_{1,0}$ to (2.11) yields

$$Q''\left(m+1, n, u_{1,0}\right) = 0. \tag{2.12}$$

Shifting back (2.12) using S_1^{-1} and then integrating gives

$$Q'\left(m, n, u\right) = A\left(m, n\right).$$

Substituting shifts of this into (2.11) gives the condition

$$A\left(m+1,n\right)=A\left(m,n+1\right)$$

that leads to

$$Q'\left(m,n,u\right)=A\left(m+n\right).$$

Then integrating a second time yields

$$Q\left(m,n,u\right)=A\left(m+n\right)u+B\left(m,n\right). \tag{2.13}$$

Substituting this into (2.10) gives the following conditions

$$A\left(m+n+2\right)=-A\left(m+n+1\right)=A\left(m+n\right),$$
$$B\left(m+1,n+1\right)=B\left(m,n\right),\quad B\left(m+1,n\right)=B\left(m,n+1\right).$$

Therefore, the set of all characteristics of Lie point symmetries is spanned by

$$Q_1=1,\quad Q_2=\left(-1\right)^{m+n},\quad Q_3=\left(-1\right)^{m+n}u. \tag{2.14}$$

The last step we need to consider is how to get from the characteristics Q_i^α, for $i=1,\ldots,r$ and $\alpha=1,\ldots,q$ to the Lie group action on u^α. Let $\widehat{u}^\alpha\left(\mathbf{n},\mathbf{u};\epsilon\right)=g\cdot u^\alpha$ and $\widehat{Q}_i^\alpha=g\cdot Q_i^\alpha$, solving the system

$$\frac{\mathrm{d}\widehat{u}^\alpha}{\mathrm{d}\epsilon}=\widehat{Q}_i^\alpha\left(\mathbf{n},\widehat{\mathbf{u}};\epsilon\right),\qquad \widehat{u}^\alpha\left(\mathbf{n},\mathbf{u};0\right)=u^\alpha \tag{2.15}$$

yields $\widehat{u}^\alpha$. For an example we find the Lie group action on u of Q_3 in (2.14) above.

Example 2.3.2. Using (2.15) for Q_3 in (2.14) we obtain

$$\frac{\mathrm{d}\widehat{u}}{\mathrm{d}\epsilon}=\left(-1\right)^{m+n}\widehat{u},\quad \widehat{u}\left(m,n,u;0\right)=u.$$

Therefore, integrating this first-order differential equation gives

$$\ln\left|\widehat{u}\right|=\left(-1\right)^{m+n}\epsilon+c$$

using the condition $\widehat{u}\,(m,n,u,;\epsilon) = u$ gives $c = \ln|u|$ and so after simplification

$$\widehat{u} = e^{(-1)^{m+n}\epsilon}u.$$

2.4 Moving frames

Here we discuss the theory of moving frames, beginning by looking at these in general and then turning to their extension to discrete moving frames on a prolongation space. This allows us to introduce the important difference moving frame theory, where additional prolongation conditions apply. When discussing difference moving frames and related examples u is used as a coordinate. Otherwise when discussing moving frames in more general terms z is used. The results at the start of this section on general moving frames come from the seminal papers by Fels and Olver [7, 8], the book on invariant calculus by Mansfield [22] and are all included in the O∆E paper [26].

An important assumption for the moving frame theory (see [7, 8, 22]) is summed up in the following theorem.

Theorem 2.4.1. A moving frame exists in a neighbourhood of a point $z \in M$ if and only if the group G acts freely and regularly near z.

A group action is free if the only group element $g \in G$ which fixes every point in the neighbourhood is the identity. A group action is regular if the orbits form a regular foliation.

Therefore, to be free and regular in some domain $\Omega \subset M$ means, in effect, for every $z \in \Omega$ there is a neighborhood $\mathcal{U} \subset \Omega$ of z such that the following conditions are satisfied [22].

- The group orbits all have the dimension of the group and foliate $\mathcal{U}$.
- There exists a submanifold $\mathcal{K} \subset \mathcal{U}$ that intersects the orbits of $\mathcal{U}$ transversely, and the intersection of $\mathcal{U}$ and $\mathcal{K}$ is a single point. The submanifold $\mathcal{K}$ is known as the cross-section and has dimension equal to $\dim(M) - \dim(G)$.
- If $\mathcal{O}(z)$ denotes the orbit through z, then the element $h \in G$ that takes $z \in \mathcal{U}$ to k, where $\{k\} = \mathcal{O}(z) \cap \mathcal{K}$, is unique.

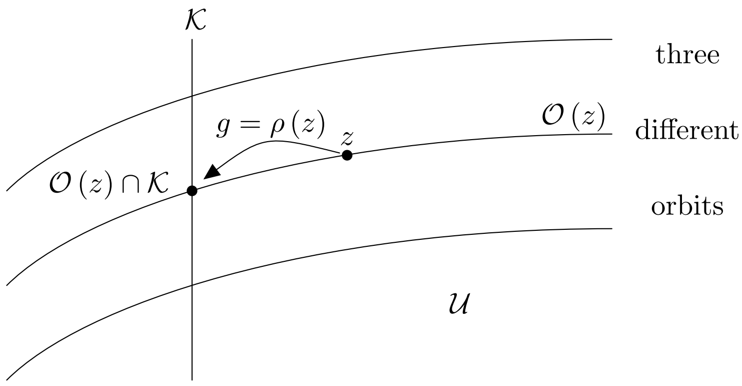


Figure 2.2: Moving frame defined by a cross-section.

If M is not free and regular replace it with a domain $\overline{M}$ on which the action is free and regular.

In Beffa and Mansfield [3] Remark 4.19 discusses research detailing conditions under which an action will become free and regular. For results on product actions, with a suffiently large number of products, see Boutin [6]. Using this we assume we have a free and regular group action.

Figure 2.2 shows an example for which there exists a cross-section $\mathcal{K} \subset M$ that is transverse to orbits $\mathcal{O}(z)$ for each $z \in M$. Furthermore, the set $\mathcal{K} \cap \mathcal{O}(z)$ has just one element which is the projection of z onto $\mathcal{K}$. The cross-section $\mathcal{K}$ is not unique, that is, there are different choices of the cross-section, some of which can vastly reduce the difficulty of computations.

Definition 2.4.2 (Moving frame)**.** Given a smooth Lie group action $G \times M \to M$, a moving frame is an equivariant map $\rho : \mathcal{U} \subset M \to G$. Here $\mathcal{U}$ is called the domain of the frame.

Note here we construct moving frames in a neighbourhood $\mathcal{U} \subset \Omega \subset M$ of any point z and the map ρ is a smooth equivariant map [22].

For a left action, a left equivariant map satisfies $\rho(g \cdot z) = g\rho(z)$ and a right equivariant map satisfies $\rho(g \cdot z) = \rho(z) g^{-1}$. The frame is called left or right accordingly. In theoretical developments, it is not necessary to choose the handedness of the frame, as to get a left frame from a right frame we take the inverse of the right frame. In order to find the frame let the cross-section $\mathcal{K}$ be a system of equations $\psi_r(z) = 0$, for $r = 1, \ldots, R$, where R is the dimension of the group G. Then by solving the so-called *normalization equations*

$$\psi_r\left(g \cdot z\right) = 0, \qquad r = 1, ..., R, \tag{2.16}$$

for g as a function of z, the solution is the group element $g = \rho\left(z\right)$ that maps z to its projection on $\mathcal{K}$. As a result, the frame $\rho\left(z\right)$ satisfies

$$\psi_r\left(\rho\left(z\right) \cdot z\right) = 0, \qquad r = 1, ..., R.$$

The conditions on the action above (i.e. freeness and regularity) are those for the implicit function theorem [12] to hold, so the solution $\rho\left(z\right)$ is unique. Therefore, the frame is right equivariant as both $\rho\left(g \cdot z\right)$ and $\rho\left(z\right) g^{-1}$ solve the equation

$$\psi_r\left(\rho\left(g \cdot z\right) \cdot \left(g \cdot z\right)\right) = 0.$$

A consequence of uniqueness is that

$$\rho\left(g \cdot z\right) = \rho\left(z\right) g^{-1}.$$

Solving the normalization equations produces a right frame. Throughout, this chapter examples will use a left action with a right frame. Knowing that the frames are equivariant will enable us to obtain invariants of a Lie group action.

Example 2.4.1. To illustrate the theory, we use the Lagrangian

$$\mathrm{L} = \frac{1}{2} \ln \left| \frac{\left(u_{2,0} - u_{1,1}\right)\left(u_{1,-1} - u_{0,0}\right)}{\left(u_{2,0} - u_{1,-1}\right)\left(u_{1,1} - u_{0,0}\right)} \right| \tag{2.17}$$

as a running example. This Lagrangian has six different symmetries which can be expressed by the infinitesimal generators

$$\begin{aligned} &\mathbf{v}_1 = u_{0,0}\,\partial_{u_{0,0}}, \qquad \mathbf{v}_2 = (-1)^{m+n}\,u_{0,0}\,\partial_{u_{0,0}}, \qquad \mathbf{v}_3 = \partial_{u_{0,0}}, \\ &\mathbf{v}_4 = (-1)^{m+n}\,\partial_{u_{0,0}}, \qquad \mathbf{v}_5 = u_{0,0}^2\,\partial_{u_{0,0}}, \qquad \mathbf{v}_6 = (-1)^{m+n}\,u_{0,0}^2\,\partial_{u_{0,0}}, \end{aligned} \tag{2.18}$$

where

$$\mathbf{v}_r = Q_r \partial_{u_{0,0}} = Q_r \frac{\partial}{\partial u_{0,0}}.$$

Additionally,

$$\mathbf{v}_r = (\mathrm{S}_{i,j} Q_r)\, \partial_{u_{i,j}} = (\mathrm{S}_{i,j} Q_r)\, \frac{\partial}{\partial u_{i,j}}, \tag{2.19}$$

see Chapter 6 of [15]. To show that the Lagrangian, L, is indeed invariant under these infinitesimal generators one can use the symmetry condition (Definition 2.2.1)

$$\mathbf{v}_r\,(\mathrm{L}) = 0 \quad \text{when} \quad \mathrm{L} = 0, \tag{2.20}$$

for each r (see [15]). As an example see Section A.1 in Appendix A for detailed calculations showing that L is invariant under $\mathbf{v}_1$. See Section 2.9 for more details on infinitesimal generators. The Lagrangian (2.17) is up to a divergence equivalent to the Lagrangian,

$$\mathrm{L}_0 = \ln \left| \frac{u_{1,0} - u_{0,1}}{u_{1,1} - u_{0,0}} \right|, \tag{2.21}$$

see Section A.2 in Appendix A for details. Divergence terms are in the kernel of the Euler operator, that is,

$$\mathrm{E}_{u^\alpha}\,(\mathrm{Div}\,(A)) = 0,$$

for any divergence term $\mathrm{Div}\,(A)$. Therefore, L has the same Euler–Lagrange equation as the Lagrangian L_0. To find the Euler–Lagrange equation of L (2.17) the key partial derivatives are

$$\begin{aligned} \frac{\partial \mathrm{L}}{\partial u_{0,0}} &= \frac{u_{1,-1} - u_{1,1}}{2\,(u_{0,0} - u_{1,-1})\,(u_{0,0} - u_{1,1})}, \quad & \frac{\partial \mathrm{L}}{\partial u_{2,0}} &= \frac{u_{1,1} - u_{1,-1}}{2\,(u_{1,1} - u_{2,0})\,(u_{1,-1} - u_{2,0})}, \\ \frac{\partial \mathrm{L}}{\partial u_{1,-1}} &= \frac{u_{0,0} - u_{2,0}}{2\,(u_{0,0} - u_{1,-1})\,(u_{2,0} - u_{1,-1})}, \quad & \frac{\partial \mathrm{L}}{\partial u_{1,1}} &= \frac{u_{0,0} - u_{2,0}}{2\,(u_{1,1} - u_{2,0})\,(u_{0,0} - u_{1,1})}, \end{aligned} \tag{2.22}$$

with key shifted partial derivatives

$$\begin{aligned} \mathrm{S}_{-2,0} \frac{\partial \mathrm{L}}{\partial u_{2,0}} &= \frac{u_{-1,1} - u_{-1,-1}}{2\,(u_{-1,1} - u_{0,0})\,(u_{-1,-1} - u_{0,0})}, \\ \mathrm{S}_{-1,1} \frac{\partial \mathrm{L}}{\partial u_{1,-1}} &= \frac{u_{-1,1} - u_{1,1}}{2\,(u_{-1,1} - u_{0,0})\,(u_{1,1} - u_{0,0})}, \\ \mathrm{S}_{-1,-1} \frac{\partial \mathrm{L}}{\partial u_{1,1}} &= \frac{u_{-1,-1} - u_{1,-1}}{2\,(u_{0,0} - u_{1,-1})\,(u_{-1,-1} - u_{0,0})}. \end{aligned}$$

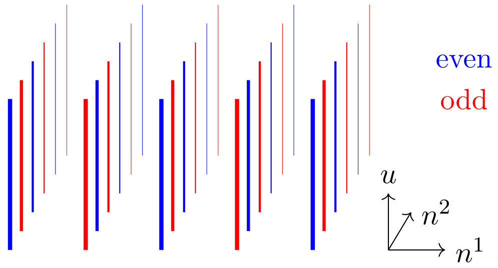


Figure 2.3: Partitioned PΔEs.

Using these and (2.4) gives

$$\begin{aligned}\mathrm{E}_u\left(\mathrm{L}\right) &= \frac{\partial \mathrm{L}}{\partial u_{0,0}} + \mathrm{S}_{-1,1}\frac{\partial \mathrm{L}}{\partial u_{1,-1}} + \mathrm{S}_{-1,-1}\frac{\partial \mathrm{L}}{\partial u_{1,1}} + \mathrm{S}_{-2,0}\frac{\partial \mathrm{L}}{\partial u_{2,0}} \\ &= \frac{1}{u_{1,1}-u_{0,0}} - \frac{1}{u_{-1,1}-u_{0,0}} - \frac{1}{u_{1,-1}-u_{0,0}} + \frac{1}{u_{-1,-1}-u_{0,0}} \\ &= 0,\end{aligned}$$

a Toda-type equation satisfied by all solutions of the autonomous dpKdV equation among others, (see [15] for details).

For the Toda-type equation, a phenomenon arises which can only arise for difference or differential-difference equations. As the total space is disconnected things can happen on one fibre without affecting another fibre. The Lagrangian, L, and Euler–Lagrange equation for the Toda-type equation contains only shifts of u with n^1+n^2 of even parity. Therefore, the odd and even parts of the lattice act entirely independent of one another (see Figure 2.3). Specifying initial conditions for the Euler–Lagrange equation on the odd (even) part of the lattice determines solutions on the odd (even) part only. However, it gives no information about the even (odd) part. When restricting to solutions of the equation the odd and even parts of the lattice are considered separately. The Lagrangian in (2.17) respects this partitioning whereas the Lagrangian (2.21) does not. Later this split in the lattice will manifest itself in other ways. For other Lagrangians it is possible that the lattice could be split up into something other than even and odd n^1+n^2.

As a running example, consider the moving frame method using the Lie group action derived from the infinitesimal generators $\mathbf{v}_1$ and $\mathbf{v}_3$. This group action is represented by

$$g : u_{i,j} \mapsto \widetilde{u}_{i,j} = au_{i,j} + b, \tag{2.23}$$

where $i, j \in \mathbb{Z}$, $a \in \mathbb{R}^+$ is a scaling and $b \in \mathbb{R}$ a translation. Choosing the normalization equations (2.16) to be

$$\psi_1([u]) = u_{0,0}, \quad \psi_2([u]) = u_{1,1} - 1, \tag{2.24}$$

with $[u]$ denoting shifts of u, yields

$$g \cdot u_{0,0} = au_{0,0} + b = 0, \quad g \cdot u_{1,1} = au_{1,1} + b = 1. \tag{2.25}$$

These normalization equations give the values of the parameters on the frame as

$$a = \frac{1}{u_{1,1} - u_{0,0}}, \quad b = \frac{-u_{0,0}}{u_{1,1} - u_{0,0}}. \tag{2.26}$$

Alternatively, this choice of normalization equations can be denoted by the coordinate cross-section

$$\mathcal{K} = \{u_{0,0} = 0, u_{1,1} = 1\}. \tag{2.27}$$

Remark 2.4.3. For the majority of applications and throughout this thesis, the normalization equations have the form

$$g \cdot z_1 = c_1, \quad g \cdot z_2 = c_2, \quad \dots \quad , g \cdot z_R = c_R, \tag{2.28}$$

where z_r are coordinates on M and c_r are constants, for $r = 1, \dots, R$. Typically, each of the constants c_r are either 0 or 1 for simplicity. From this point on instead of giving the functions $\psi_r(z)$, for $r = 1, \dots, R$, we write the normalization equations as in (2.28) or use a coordinate cross-section, that is,

$$\mathcal{K} = \{z_1 = c_1, z_2 = c_2, \dots, z_R = c_R\}. \tag{2.29}$$

Here there is a freedom in the choice of normalization equations, for example, we could also choose $g \cdot u_{0,0} = 0$ and $g \cdot u_{1,1} = 2$. Additionally, one could also use the normalization equations $g \cdot u_{\mathbf{I}} = 0$ and $g \cdot u_{\mathbf{J}} = 1$ for $\mathbf{I} \neq \mathbf{J}$ and $\mathbf{I}, \mathbf{J} \in \mathbb{Z}^2$. However, we cannot choose $g \cdot u_{0,0} = 0$ and $g \cdot u_{1,1} = 0$ as the resulting normalization equations cannot be solved in terms of the parameters. Therefore, an appropriate choice of normalization equations needs to be able to find the solutions for the parameters of the Lie group. Other nuances in the choice of a particular normalization equation will be discussed throughout this thesis.

Remark 2.4.4. This Lie group action is not free on the space $\mathbb{R}$ over $\mathbf{n}$, as this space only has the coordinate $u_{0,0}$ and is not a continuous space; therefore, to achieve freeness we need to work in a higher-dimensional continuous space. To do this we work on the prolongation space $P_{\mathbf{n}}(\mathbb{R})$, which includes the coordinate $u_{1,1}$.

Remark 2.4.5. As $a > 0$, this normalization is only valid throughout the half-space $\mathcal{U} = \{P_{\mathbf{n}}(\mathbb{R}) : u_{1,1} > u_{0,0}\}$. For the other half-space when $u_{1,1} < u_{0,0}$ an appropriate normalization is $g \cdot u_{0,0} = 0$ and $g \cdot u_{1,1} = -1$.

The following (three) definitions and (two) theorems come from Section 4 of the O∆E paper by Mansfield *et al.* [26], but analogous results can also be found in one of the papers by Fels and Olver [8].

Theorem 2.4.6 (Normalized invariants)**.** Given a left or right action $G \times M \to M$ and a right frame ρ, then

$$\iota(z) = \rho(z) \cdot z, \tag{2.30}$$

in the domain of the frame ρ, is invariant under the group action.

Proof. First, apply the group action to z; then, by definition,

$$\iota(g \cdot z) = \rho(g \cdot z) \cdot (g \cdot z) = \rho(z) \cdot g^{-1} g \cdot z = \rho(z) \cdot z = \iota(z),$$

so $\iota(z)$ is an invariant function. □

In general, let $\iota(u_{i,j}) = \rho([u]) \cdot u_{i,j}$, where $i, j \in \mathbb{Z}$. Additionally, the invariantization of $u^{\alpha}_{\mathbf{J}}$ is $\iota(u^{\alpha}_{\mathbf{J}}) = \rho([\mathbf{u}]) \cdot u^{\alpha}_{\mathbf{J}}$, for $\mathbf{J} \in \mathbb{Z}^p$.

Definition 2.4.7. The normalized invariants are the components of $\iota(z)$.

Definition 2.4.8. A set of invariants is said to be a generating, or complete, set for an algebra of invariants if any invariant in the algebra can be written as a function of elements of the generating set.

This definition will be important when finding the generating invariants of a group action, which enables us to write the Lagrangian in terms of invariants.

We now state the replacement rule and see that the normalized invariants provide the set of generating invariants.

Theorem 2.4.9 (Replacement rule)**.** If $F(z)$ is an invariant of a given Lie group action $G \times M \to M$ for a right moving frame ρ on M, then $F(z) = F(\iota(z))$.

Proof. As $F(z)$ is invariant it is clear that $F(z) = F(g \cdot z)$. Then taking $g = \rho(z)$ and using the definition of $\iota(z)$ gives the result. □

Definition 2.4.10 (Invariantization operator)**.** Given a right moving frame ρ, the map $z \mapsto \iota(z) = \rho(z) \cdot z$ is called the invariantization operator. This operator extends to functions as $f(z) \mapsto f(\iota(z))$, and

$$\iota(f(z)) = f(\iota(z)) \tag{2.31}$$

is called the invariantization of f.

If z has components z^α, then let $\iota(z^\alpha)$ denote the αth component of $\iota(z)$. The invariantization operator ι on z^α is also given by

$$\iota(z^\alpha) = g \cdot z^\alpha|_{g=\rho(z)},$$

where $|_{g=\rho(z)}$ takes the value of the parameters to their value on the frame. This form of the invariantization operator is important for when the invariantization operator is applied to different objects like $\mathrm{d}u^\alpha/\mathrm{d}t$ later in this section and $\mathrm{d}_v u^\alpha$ in Section 2.6.

Example 2.4.2. (Example 2.4.1 cont.) The action of the frame on $u_{i,j}$ (or invariantization of $u_{i,j}$) is

$$\iota(u_{i,j}) = \frac{u_{i,j} - u_{0,0}}{u_{1,1} - u_{0,0}}.$$

It is clear to see this is invariant under the group action.

Remark 2.4.11. It is possible to calculate the recurrence relations for these invariants and show that one can write all invariants in terms of two fundamental invariants for this group action. Note each shift of an invariant is an invariant itself; consequently, the recurrence relations can be found by simply using the replacement rule.

Let the two fundamental (or generating) invariants for the running example be

$$\kappa = \iota\left(u_{1,-1}\right) = \frac{u_{1,-1} - u_{0,0}}{u_{1,1} - u_{0,0}}, \qquad \lambda = \iota\left(u_{2,0}\right) = \frac{u_{2,0} - u_{0,0}}{u_{1,1} - u_{0,0}}. \tag{2.32}$$

As an example, we show how to find $\iota\left(u_{2,2}\right)$ in terms of these generating invariants and their shifts. Let $\kappa^{\beta}_{i,j}$ represent $\mathrm{S}_1^i\mathrm{S}_2^j\kappa^{\beta}$ for different generating invariants (this convention will be used throughout.) In general let $\kappa^{\beta}_{\mathbf{J}}$ represent $\mathrm{S}_{\mathbf{J}}\kappa^{\beta}$. Shift κ to involve $u_{2,2}$ and lower shifts of κ and λ:

$$\mathrm{S}_1\mathrm{S}_2\kappa = \kappa_{1,1} = \frac{u_{2,0} - u_{1,1}}{u_{2,2} - u_{1,1}}.$$

Then using the replacement rule to find the right-hand side of the shifted equation in terms of invariants gives

$$\kappa_{1,1} = \frac{\iota\left(u_{2,0}\right) - \iota\left(u_{1,1}\right)}{\iota\left(u_{2,2}\right) - \iota\left(u_{1,1}\right)},$$

thus, by substituting in $\iota\left(u_{2,0}\right) = \lambda$ and $\iota\left(u_{1,1}\right) = 1$, and rearranging gives the result

$$\iota\left(u_{2,2}\right) = \frac{\lambda - 1 + \kappa_{1,1}}{\kappa_{1,1}}.$$

Remark 2.4.12. Recurrence relations can also be constructed for different invariants $\iota\left(u_{i+r,j+s}\right)$ using shifts of $\iota\left(u_{i,j}\right)$. Indeed, a recurrence relation for $\iota\left(u_{i+1,j+1}\right)$ is possible by shifting $\iota\left(u_{i,j}\right)$ in both directions and invariantizing (2.31):

$$\iota\left(u_{i+1,j+1}\right) = \left(\frac{\lambda - 1}{\kappa_{1,1}}\right)\mathrm{S}_1\mathrm{S}_2\,\iota\left(u_{i,j}\right) + 1.$$

Note that taking $i = j = 1$ gives the same formula for $\iota\left(u_{2,2}\right)$ as before. Also if one calculates the recurrence relations for the positive shifts, that is, $\iota\left(u_{i,j}\right)$ to $\iota\left(u_{i+1,j}\right)$ and $\iota\left(u_{i,j}\right)$ to $\iota\left(u_{i,j+1}\right)$ then their inverses can be constructed. By changing the index from n^1 to $n^1 - 1$ and rearranging the formula for the new invariant $\iota\left(u_{i,j}\right)$ gives the recurrence formula for $\iota\left(u_{i,j}\right)$ to $\iota\left(u_{i-1,j}\right)$. Similarly, one can obtain the recurrence formula from $\iota(u_{i,j})$ to $\iota(u_{i,j-1})$. This enables one to find $\iota(u_{i,j})$ for all $i, j \in \mathbb{Z}$.

It is clear to see from the above example that the invariants $\iota(u_{\mathbf{J}})$ do not behave well under the shift map in the sense that $\mathrm{S}_{\mathbf{K}}\{\iota(u_{\mathbf{J}})\} \neq \iota(u_{\mathbf{J}+\mathbf{K}})$, in general. So, even though the shift operator takes invariants to other invariants there may be complicated expressions relating them.

2.4.1 Discrete moving frames

The discrete moving frame, as developed in papers by Mansfield, Beffa and Wang [23] and by Beffa and Mansfield [3] can be thought of as a moving frame adapted to discrete base points. The following definitions of the diagonal action of a group G on z, discrete moving frames and invariants for the discrete frame are given in the discrete moving frames papers [3, 23]. However, we follow closely the explanation provided in Section 4 of the OΔEs paper by Mansfield *et al.* [26]. The manifold on which G acts will be the Cartesian product manifold $\mathcal{M} = M^N$. The regularity and freeness of the action now refers to the diagonal action on the product; given a (left) action $(g, z_j) \mapsto g \cdot z_j$, for $z_j \in M$, the diagonal action of G on $z = (z_1, z_2, ..., z_N) \in \mathcal{M}$ is

$$g \cdot (z_1, z_2, ..., z_N) \mapsto (g \cdot z_1, g \cdot z_2, ..., g \cdot z_N) .$$

For discrete moving frames there are no assumptions made about any relationships between the elements $z_1, ..., z_N$.

Definition 2.4.13 (Discrete moving frames)**.** Let G^N denote the Cartesian product of N copies of the group G. A map

$$\rho : M^N \to G^N, \quad \rho(z) = (\rho_1(z), ..., \rho_N(z))$$

is a right discrete moving frame if

$$\rho_k(g \cdot z) = \rho_k(z)\, g^{-1}, \quad k = 1, ..., N,$$

and a left discrete moving frame if

$$\rho_k\left(g\cdot z\right)=g\rho_k\left(z\right), \quad k=1,...,N.$$

As with the continuous theory of moving frames, using the normalization equations (2.16) for discrete moving frames gives a right frame. So, the right moving frame component ρ_k is the unique element of the group G that takes z to the cross-section $\mathcal{K}_k$. The sequence of moving frames with a nontrivial intersection of domains (ρ_k) which makes up the discrete moving frame is, locally, uniquely determined by the cross-section $\mathcal{K}=(\mathcal{K}_1,...,\mathcal{K}_N)$ to the group orbit through z (see [23] for more details on discrete moving frames).

The invariants of the right (discrete) frame are

$$I_{k,j}:=\rho_k\left(z\right)\cdot z_j.$$

If M is q-dimensional then each z_j has q components $z_j^1,...,z_j^q$. So, there are q components of $I_{k,j}$ and these are

$$I_{k,j}^{\alpha}:=\rho_k\left(z\right)\cdot z_j^{\alpha}, \quad \alpha=1,...,q.$$

For the same reason as for normalized invariants of a general moving frame, each $I_{k,j}^{\alpha}$ is invariant.

The discrete moving frame applies to a wide variety of discrete domains. We now use the discrete moving frame and the difference prolongation space to develop the difference moving frame.

2.4.2 Difference moving frames

A difference moving frame is a particular type of discrete moving frame, so the definitions and theorems for discrete moving frames in [3, 23, 26] apply here. For PΔEs, the fibres lie on a regular lattice $\mathbb{Z}^p$. This gives us a geometric context which determines the additional structures on the manifold $\mathcal{M}$. Let $\mathcal{K}$ and $\rho\left([\mathbf{u}]\right)$ denote the cross-section and frame on $\mathbf{n}$, respectively. The cross-section on $\mathbf{n}$, denoted $\mathcal{K}$, is replicated for all the other base points $\mathbf{n}+\mathbf{J}$ if and only if the cross-section over $\mathbf{n}+\mathbf{J}$ is represented on $\mathcal{M}$ by $\mathrm{S}_{\mathbf{J}}\mathcal{K}$.

Definition 2.4.14. A difference moving frame is a discrete moving frame such that $\mathcal{M}$ is a prolongation space $P_{\mathbf{n}}(U)$ and the cross-section over $\mathbf{n}+\mathbf{J}$ is represented on $\mathcal{M}$ by $\mathrm{S}_{\mathbf{J}}\mathcal{K}$ for all necessary $\mathbf{J}$.

For difference moving frames, the invariants are

$$I_{\mathbf{K},\mathbf{J}} := \rho_{\mathbf{K}}([\mathbf{u}]) \cdot \mathbf{u}_{\mathbf{J}} = \left(\mathrm{S}_{\mathbf{K}}\rho([\mathbf{u}])\right) \cdot \left(\mathrm{S}_{\mathbf{J}}\mathbf{u}\right),$$

where $\mathbf{K}$ and $\mathbf{J}$ are multi-indices. Here the multi-index $\mathbf{K}$ relates to the number of shifts of the frame and the multi-index $\mathbf{J}$ relates to the number of shifts of $\mathbf{u}$. By definition $\mathrm{S}_i\left(I_{\mathbf{K},\mathbf{J}}\right) = I_{\mathbf{K}+\mathbf{1}_i,\mathbf{J}+\mathbf{1}_i}$. Hence, every invariant $I_{\mathbf{K},\mathbf{J}}$ can be expressed as a shift of $I_{\mathbf{0},\mathbf{J}-\mathbf{K}} = \left(\rho([\mathbf{u}])\right)\cdot\left(\mathrm{S}_{\mathbf{J}-\mathbf{K}}\mathbf{u}\right)$. Again we drop the multi-index $\mathbf{0}$ when talking about the moving frame on $\mathbf{n}$.

Definition 2.4.15 (Discrete Maurer–Cartan invariants)**.** Given a right discrete moving frame $\rho([\mathbf{u}])$ (often given in matrix form), the right discrete Maurer–Cartan group elements are

$$K_{(i)} = \left(\mathrm{S}_i\rho([\mathbf{u}])\right)\rho([\mathbf{u}])^{-1} = \iota\left(\mathrm{S}_i\rho([\mathbf{u}])\right), \qquad i = 1, ..., p.$$

To get multiple shifts in one or more different directions, use the fact that

$$\left(\mathrm{S}_i\mathrm{S}_j\rho([\mathbf{u}])\right)\rho([\mathbf{u}])^{-1} = \left(\mathrm{S}_j K_{(i)}\right) K_{(j)}.$$

It is possible to extend this formula further when necessary. The frame is equivariant, therefore, each $K_{(i)}$ is invariant under the group action of G. The *Maurer–Cartan invariants* are the components of $K_{(i)}$ and their shifts. The equality $\iota\left(\mathrm{S}_i\rho([\mathbf{u}])\right) = \left(\mathrm{S}_i\rho([\mathbf{u}])\right)\rho([\mathbf{u}])^{-1}$ is due to

$$\mathrm{S}_i\left(\rho(g\cdot[\mathbf{u}])\right) = \left(\mathrm{S}_i\rho([\mathbf{u}])\right)g^{-1},$$

by Definition 2.4.13 of a right moving frame. Therefore,

$$\iota\left(\mathrm{S}_i\rho([\mathbf{u}])\right) = \left(\mathrm{S}_i\rho(g\cdot[\mathbf{u}])\right)|_{g=\rho([\mathbf{u}])} = \left(\mathrm{S}_i\rho([\mathbf{u}])\right)\rho([\mathbf{u}])^{-1}. \tag{2.33}$$

Definition 2.4.16 (Syzygy)**.** A syzygy on a set of invariants is a relation between invariants that expresses functional dependency.

In other words, syzygies on a set of invariants are functions of invariants that, when expressed in terms of the underlying variables, are identically zero.

One way to obtain a set of generating invariants is to use the Maurer–Cartan elements; for example,

$$K_{(i)} \cdot I_{\mathbf{0},\mathbf{0}} = \left(\mathrm{S}_i \rho\left([\mathbf{u}]\right)\right) \rho\left([\mathbf{u}]\right)^{-1} \rho\left([\mathbf{u}]\right) \cdot \mathbf{u_0} = \left(\mathrm{S}_i \rho\left([\mathbf{u}]\right)\right) \cdot \mathbf{u_0} = I_{\mathbf{1_i},\mathbf{0}}.$$

It is possible to extend this to look at multiple shifts in different directions to achieve all invariants. The next proposition follows from Proposition 3.11 in the discrete moving frames paper by Mansfield *et al.* [23] and the fact that a difference moving frame is a particular type of discrete moving frame.

Proposition 2.4.17. Given a right discrete moving frame $\rho\left([\mathbf{u}]\right)$, the components of $K_{(i)}$, together with the set of all diagonal invariants, $I_{\mathbf{J},\mathbf{J}} = \rho_{\mathbf{J}}\left([\mathbf{u}]\right) \cdot \mathbf{u_J}$, generate all other invariants.

The difference identities, or syzygies, $K_{(i)} \cdot I_{\mathbf{0},\mathbf{0}} = I_{\mathbf{1_i},\mathbf{0}}$ and other extensions are also recurrence relations for the invariants. The next definition is from [26].

Definition 2.4.18. A set of invariants is a generating set for an algebra of difference invariants if any difference invariant in the algebra can be written as a function of elements of the generating set and their shifts.

Therefore, the right difference moving frame identity $I_{\mathbf{J},\mathbf{J}} = \mathrm{S}_{\mathbf{J}} I_{\mathbf{0},\mathbf{0}}$ together with $\mathrm{S}_{\mathbf{K}} K_{(i)}$, which can be written as shifts of the Maurer–Cartan components $K_{(j)}$, for $j = 1, ..., p$, gives us the following proposition.

Proposition 2.4.19. Given a right difference moving frame $\rho\left([\mathbf{u}]\right)$, the set of all invariants is generated by the set of components of $K_{(j)} = \left(\mathrm{S}_j \rho\left([\mathbf{u}]\right)\right) \rho\left([\mathbf{u}]\right)^{-1}$, where $j = 1, ..., p$, and $I_{\mathbf{0},\mathbf{0}} = \rho_{\mathbf{0}}\left([\mathbf{u}]\right) \cdot \mathbf{u_0}$.

2.4.3 Differential-difference invariants and syzygies

This subsection extends the results for OΔEs [26] to PΔEs and introduces another method of finding the first-order differential invariants. To enable us to write the Euler–Lagrange equations in terms of the invariants we need to find the so-called differential-difference syzygies between differential and difference invariants. Given any smooth path $t \mapsto z(t)$ in the space $\mathcal{M} = M^N$, consider the induced group action on the path and its tangent. The group action is extended to the dummy variable t so that t is invariant. The action extends to the first-order jet space of $\mathcal{M}$ as follows:

$$g \cdot \frac{\mathrm{d}z(t)}{\mathrm{d}t} = \frac{\mathrm{d}(g \cdot z(t))}{\mathrm{d}t}.$$

For a component $u_{\mathbf{0}}^{\alpha}$ we have

$$g \cdot \frac{\mathrm{d}u_{\mathbf{0}}^{\alpha}}{\mathrm{d}t} = \frac{\partial (g \cdot u_{\mathbf{0}}^{\alpha})}{\partial u_{\mathbf{0}}^{\delta}} \frac{\mathrm{d}u_{\mathbf{0}}^{\delta}}{\mathrm{d}t}. \tag{2.34}$$

If the action is free and regular on $\mathcal{M}$, it will remain so on the jet space and we may use the same frame to find the first-order differential invariants which are

$$\begin{aligned} I_{\mathbf{K},\mathbf{J};t}^{\alpha} :&= (\mathrm{S}_{\mathbf{K}}\rho([\mathbf{u}])) \cdot \frac{\mathrm{d}u_{\mathbf{J}}^{\alpha}(t)}{\mathrm{d}t} \\ &= g \cdot \frac{\mathrm{d}u_{\mathbf{J}}^{\alpha}(t)}{\mathrm{d}t}\bigg|_{g=\mathrm{S}_{\mathbf{K}}\rho([\mathbf{u}])}, \end{aligned} \tag{2.35}$$

for any multi-indexes $\mathbf{K}$ and $\mathbf{J}$ in the domain. The frame depends on $z(t)$, so in general,

$$I_{\mathbf{K},\mathbf{J};t}^{\alpha} \neq \frac{\mathrm{d}}{\mathrm{d}t}\left(I_{\mathbf{K},\mathbf{J}}^{\alpha}\right).$$

For the calculations of the invariantized Euler–Lagrange equations the important differential-difference syzygies are

$$\frac{\mathrm{d}\kappa^{\beta}}{\mathrm{d}t} = \mathcal{H}_{\alpha}^{\beta}\sigma^{\alpha}, \tag{2.36}$$

where κ^{β} represents the generating invariants and σ^{α} represents the generating first-order differential invariants. The terms $\mathcal{H}_{\alpha}^{\beta}$ are linear difference operators whose coefficients are functions of κ^{β} and their shifts. Relating this to the other notation, κ^{β} are components chosen from $I_{\mathbf{0},\mathbf{J}}^{\alpha}$ to give us a set which can generate all $I_{\mathbf{0},\mathbf{J}}^{\alpha}$ using shifts of κ^{β}. Therefore, the set of generating invariants κ^{β} are a subset of the set of invariants $I_{\mathbf{0},\mathbf{J}}^{\alpha}$. Typically, one needs at least $q \times p$ (the number of independent variables multiplied by the number of dependent variables) generating invariants κ^β, e.g., for the running example we need two (κ and λ). Additionally, $\sigma^\alpha = I_{\mathbf{0},\mathbf{0};t}$ are the generating first-order differential invariants, and in particular shifts of κ^β and σ^α are used to write each $I^\alpha_{\mathbf{0},\mathbf{K};t}$ for $\mathbf{K} \in \mathbb{Z}^p$. Here α represents $\alpha = 1, \ldots, q$, and β is used to represent different generating invariants.

One method to find these differential-difference syzygies is to write $\mathrm{d}\kappa^\beta/\mathrm{d}t$ in terms of the original variables and then apply the replacement rule (Theorem 2.4.9). This will be the method we adopt. The relations between the generating first-order differential invariants and other first-order differential invariants can be obtained by applying a shift to the generating first-order differential invariants in its original variables and then using the replacement rule. This is similar to the way of finding the recurrence relations. This implies that each of the first-order differential invariants can be written in terms of shifts of the generating invariants κ^β and the relevant shift of the generating first-order invariant σ^α. This method will be shown in Example 2.4.3, but first, we explain another way using matrices to achieve the same differential-difference syzygies.

The second method is derived from Theorem 4.6 in the discrete moving frames paper by Mansfield *et al.* [23]. It involves differentiating the Maurer–Cartan matrix with respect to t. The dependent variables u^α depend on t. Therefore, $\rho([\mathbf{u}])$ and $K_{(i)}$ are dependent on t. Given a matrix representation of $\rho([\mathbf{u}])$ and the product rule we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}K_{(i)} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\left(\mathrm{S}_i\rho([\mathbf{u}])\right)\rho([\mathbf{u}])^{-1}\Big)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{S}_i\rho([\mathbf{u}])\right)\Big(\mathrm{S}_i\rho([\mathbf{u}])^{-1}\Big)K_{(i)} - K_{(i)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho([\mathbf{u}])\right)\rho([\mathbf{u}])^{-1},\end{aligned} \tag{2.37}$$

for $i = 1, \ldots, p$.

Definition 2.4.20 (Curvature matrix)**.** The curvature matrix N is

$$N = \left(\frac{\mathrm{d}}{\mathrm{d}t}\rho([\mathbf{u}])\right)\rho([\mathbf{u}])^{-1} = \iota\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho([\mathbf{u}])\right)$$

when $\rho([\mathbf{u}])$ is in matrix form. Note that also

$$\mathrm{S}_iN = \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{S}_i\rho([\mathbf{u}])\right)\mathrm{S}_i\rho([\mathbf{u}])^{-1}$$

and it is clear to see that $\mathrm{S}_i\rho\left([\mathbf{u}]\right)^{-1} = \left(\mathrm{S}_i\rho\left([\mathbf{u}]\right)\right)^{-1}$ for difference moving frames.

The equality

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho\left([\mathbf{u}]\right)\right)\rho\left([\mathbf{u}]\right)^{-1} = \iota\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho\left([\mathbf{u}]\right)\right)$$

is as a result of N being invariant (allowing the use of the replacement rule Theorem 2.4.9) and

$$\iota\left(\rho\left([\mathbf{u}]\right)^{-1}\right) = \mathrm{Id},$$

(see (2.33)) where Id is some identity matrix.

It is easy to see that the curvature matrix N and any of its shifts are invariant matrices which involve first-order differential invariants. Using all these definitions, (2.37) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}K_{(i)} = \left(\mathrm{S}_i N\right)K_{(i)} - K_{(i)}N. \tag{2.38}$$

From these matrices we can solve componentwise for the differentials $\mathrm{d}\kappa^{\beta}/\mathrm{d}t$ to obtain the differential-difference syzygies in (2.36). It is clear there is some rearranging of the components to achieve the same results as in the first method, which uses the replacement rule; for this reason, we make use of the first method. Additionally, (2.38) leads to some redundant information (unlike the first method) because there are often more components of the matrix than generating invariants.

Example 2.4.3. (Example 2.4.1 cont.) We now find the differential invariants for the running example. Recall the group action is a scaling and translation on u (2.23). Writing $u_{i,j} = u_{i,j}\left(t\right)$, the action of the group on the derivative $u'_{i,j} = \mathrm{d}u_{i,j}/\mathrm{d}t$ is induced by the chain rule, as follows:

$$\begin{aligned} g \cdot u'_{i,j} &= \frac{\partial\left(g \cdot u_{i,j}\right)}{\partial u_{i,j}}u'_{i,j} \\ &= a u'_{i,j}. \end{aligned}$$

So, the Lie group action is not trivial on the derivative $u'_{i,j}$ as can be the case with some group actions. The invariantization (2.31) of $u'_{i,j}$ is given by

$$g \cdot \left.\frac{\mathrm{d}u_{i,j}}{\mathrm{d}t}\right|_{g=\rho_{0,0}([u])} = g \cdot u'_{i,j}|_{g=\rho_{0,0}([u])},$$

and is derived from (2.35). Therefore, with a given in (2.26) under the action of the frame

$$\iota\left(u'_{i,j}\right) = \frac{u'_{i,j}}{u_{1,1} - u_{0,0}},$$

which is a differential-difference invariant for $u'_{i,j}$. In particular, this is the differential-difference invariant for $u'_{i,j}$ derived from the invariantization operator (Definition 2.4.10).

With the generating differential-difference invariant

$$\sigma = \frac{u'_{0,0}}{u_{1,1} - u_{0,0}},$$

we can find the formula for the other differential-difference invariants by shifting σ and then applying the replacement rule (Theorem 2.4.9). As an example we work out $\iota\left(u'_{1,-1}\right)$. First, shift σ in the direction $\mathrm{S}_1\mathrm{S}_2^{-1}$:

$$\mathrm{S}_1\mathrm{S}_2^{-1}\sigma = \frac{u'_{1,-1}}{(u_{2,0} - u_{1,-1})}.$$

This gives us the differential $u'_{1,-1}$ we need. Then by using the replacement rule (Theorem 2.4.9), there is no change to the left-hand side of the equation, but the right-hand side now includes the differential invariant $\iota\left(u'_{1,-1}\right)$ and some additional invariants. Moving these additional invariants in the denominator to the other side achieves the result

$$\iota\left(u'_{1,-1}\right) = (\lambda - \kappa)\,\mathrm{S}_1\mathrm{S}_2^{-1}\sigma.$$

To find the differential-difference syzygies, we use the first method, involving the replacement rule (Theorem 2.4.9), that is described above. Firstly, find $\mathrm{d}\kappa/\mathrm{d}t$ and $\mathrm{d}\lambda/\mathrm{d}t$ in terms of the original variables. To do this let $\kappa := \kappa(t)$ and

$\lambda := \lambda(t)$, i.e.,

$$\kappa(t) = \frac{u_{1,-1}(t) - u_{0,0}(t)}{u_{1,1}(t) - u_{0,0}(t)}, \quad \lambda(t) = \frac{u_{2,0}(t) - u_{0,0}(t)}{u_{1,1}(t) - u_{0,0}(t)}.$$

Then the derivatives of the generating invariants are

$$\begin{aligned} \frac{\mathrm{d}\kappa}{\mathrm{d}t} &= \frac{u'_{1,-1} - u'_{0,0}}{u_{1,1} - u_{0,0}} - \frac{(u_{1,-1} - u_{0,0})\left(u'_{1,1} - u'_{0,0}\right)}{(u_{1,1} - u_{0,0})^2}, \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \frac{u'_{2,0} - u'_{0,0}}{u_{1,1} - u_{0,0}} - \frac{(u_{2,0} - u_{0,0})\left(u'_{1,1} - u'_{0,0}\right)}{(u_{1,1} - u_{0,0})^2}, \end{aligned} \tag{2.39}$$

(after dropping the variable t). Knowing that the generating invariants and t are invariant under the group action allows us to use the replacement rule (Theorem 2.4.9) to find the right-hand side of the equations above in terms of invariants. Therefore, the equations in (2.39) become

$$\begin{aligned} \frac{\mathrm{d}\kappa}{\mathrm{d}t} &= \iota\left(u'_{1,-1}\right) - \kappa\,\iota\left(u'_{1,1}\right) + (\kappa - 1)\,\iota\left(u'_{0,0}\right), \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \iota\left(u'_{2,0}\right) - \lambda\,\iota\left(u'_{1,1}\right) + (\lambda - 1)\,\iota\left(u'_{0,0}\right). \end{aligned} \tag{2.40}$$

Once each $\iota\left(u'_{i,j}\right)$ in (2.40) has been found, they can be substituted into (2.40) which gives

$$\begin{aligned} \frac{\mathrm{d}\kappa}{\mathrm{d}t} &= \left((\lambda - \kappa)\,\mathrm{S}_1\mathrm{S}_2^{-1} + \frac{(1-\lambda)\,\kappa}{\kappa_{1,1}}\mathrm{S}_1\mathrm{S}_2 + (\kappa - 1)\,\mathrm{id}\right)\sigma, \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \left(\frac{(1-\lambda)\,(\kappa_{1,1} - \lambda_{1,1})}{\kappa_{1,1}}\mathrm{S}_1^2 + \frac{\lambda\,(1-\lambda)}{\kappa_{1,1}}\mathrm{S}_1\mathrm{S}_2 + (\lambda - 1)\,\mathrm{id}\right)\sigma. \end{aligned} \tag{2.41}$$

Therefore, the differential-difference syzygy between the derivative of the generating difference invariants, $\mathrm{d}\kappa/\mathrm{d}t$ and $\mathrm{d}\lambda/\mathrm{d}t$, and the generating differential invariant, σ, can be put into the canonical form

$$\frac{\mathrm{d}\kappa}{\mathrm{d}t} = \mathcal{H}_\kappa\sigma, \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}t} = \mathcal{H}_\lambda\sigma,$$

where $\mathcal{H}_\kappa$ and $\mathcal{H}_\lambda$ are linear difference operators that depend only on the generating difference invariants and their shifts. Therefore, using the differential-difference syzygies in (2.41), the linear difference operators are

$$\begin{aligned}\mathcal{H}_{\kappa} &= (\lambda-\kappa)\,\mathrm{S}_1\mathrm{S}_2^{-1} + \frac{(1-\lambda)\,\kappa}{\kappa_{1,1}}\mathrm{S}_1\mathrm{S}_2 + (\kappa-1)\,\mathrm{id},\\ \mathcal{H}_{\lambda} &= \frac{(1-\lambda)\,(\kappa_{1,1}-\lambda_{1,1})}{\kappa_{1,1}}\mathrm{S}_1^2 + \frac{\lambda\,(1-\lambda)}{\kappa_{1,1}}\mathrm{S}_1\mathrm{S}_2 + (\lambda-1)\,\mathrm{id}.\end{aligned} \tag{2.42}$$

Remark 2.4.21. From the formula for the differential-difference syzygies (2.36), note that the number of generating first-order differential invariants is equal to the number of dependent variables. So the running example only requires one generating differential invariant, σ.

2.5 The invariant formulation of the Euler–Lagrange equations

Here we show how to calculate the Euler–Lagrange equations, in terms of invariants, for the Lie group invariant difference Lagrangian. The following definition is derived from Peng and Hydon [32].

Definition 2.5.1. Given a linear difference operator $\mathcal{H} = c^{\mathbf{J}}\mathrm{S}_{\mathbf{J}}$, the adjoint operator $\mathcal{H}^{\dagger}$ is defined by

$$\mathcal{H}^{\dagger}(F) = \mathrm{S}_{-\mathbf{J}}\left(c^{\mathbf{J}}F\right)$$

and the associated boundary term $A_{\mathcal{H}}$ is defined by

$$F\mathcal{H}(G) - \mathcal{H}^{\dagger}(F)\,G = \mathrm{Div}\left(A_{\mathcal{H}}(F,G)\right)$$

for all appropriate expressions F and G.

Now suppose we have a Lie group action $G \times M \to M$, and a difference frame for this action. Any group-invariant Lagrangian $\mathrm{L}(\mathbf{n},[\mathbf{u}])$ can be written, in terms of the generating invariants κ^{β} and their shifts $\kappa^{\beta}_{\mathbf{J}} = \mathrm{S}_{\mathbf{J}}\kappa^{\beta}$:

$$\mathrm{L}(\mathbf{n},[\mathbf{u}]) = L^{\boldsymbol{\kappa}}(\mathbf{n},[\boldsymbol{\kappa}]).$$

Proposition 2.5.2 (Invariant Euler–Lagrange equations)**.** Let $\mathcal{L}$ be a Lagrangian functional whose invariant Lagrangian is given in terms of the generating invariants,

$$\mathcal{L} = \sum_{\mathbf{n}} L^{\boldsymbol{\kappa}}\left(\mathbf{n}, [\boldsymbol{\kappa}]\right),$$

and suppose the differential-difference syzygies are

$$\frac{\mathrm{d}\kappa^{\beta}}{\mathrm{d}t} = \mathcal{H}^{\beta}_{\alpha}\sigma^{\alpha}.$$

Then,

$$\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right)\left(u^{\alpha}_{\mathbf{0}}\right)' = \left(\left(\mathcal{H}^{\beta}_{\alpha}\right)^{\dagger}\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right)\sigma^{\alpha}, \tag{2.43}$$

where

$$\mathrm{E}_{\kappa^{\beta}} = \mathrm{S}_{-\mathbf{J}}\frac{\partial}{\partial\kappa^{\beta}_{\mathbf{J}}} \tag{2.44}$$

is the difference Euler operator with respect to κ^{β}. Consequently, the invariantization (2.31) of the original Euler–Lagrange equations is,

$$\iota\left(\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right)\right) = \left(\mathcal{H}^{\beta}_{\alpha}\right)^{\dagger}\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right), \tag{2.45}$$

where $\alpha = 1, \ldots, q$. As a result, the original Euler–Lagrange equations, in invariant form, are equivalent to

$$\left(\mathcal{H}^{\beta}_{\alpha}\right)^{\dagger}\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right) = 0,$$

where $\alpha = 1, \ldots, q$.

In Section 2.6, this proposition will be proved using difference forms.

Example 2.5.1. (Example 2.4.1 cont.) For the Lagrangian (2.17) we continue to use the most convenient generating invariants $\kappa = \iota\left(u_{1,-1}\right)$ and $\lambda = \iota\left(u_{2,0}\right)$ which reduces the invariantized Lagrangian to

$$L^{\boldsymbol{\kappa}} = \frac{1}{2}\ln\left|\frac{\left(\lambda - 1\right)\kappa}{\lambda - \kappa}\right|. \tag{2.46}$$

It is often wise to choose the normalization equations (2.16) and generating invariants to fix as many of the original variables in the Lagrangian as possible. This can make the Euler operators with respect to the generating invariants easier to obtain because there are fewer shifts of the generating invariants. For the invariantized Lagrangian, the action of the Euler operators with respect to κ and λ gives

$$\begin{aligned} \mathrm{E}_{\kappa}\left(L^{\boldsymbol{\kappa}}\right) &= -\frac{\lambda}{2\kappa\left(\kappa-\lambda\right)}\,, \\ \mathrm{E}_{\lambda}\left(L^{\boldsymbol{\kappa}}\right) &= \frac{\kappa-1}{2\left(\lambda-1\right)\left(\kappa-\lambda\right)}\,. \end{aligned} \tag{2.47}$$

One can find the adjoint operators from the linear difference operators (2.42) by using Definition 2.5.1. Thus,

$$\begin{aligned} \mathcal{H}_{\kappa}^{\dagger} &= \left(\lambda_{-1,1}-\kappa_{-1,1}\right)\mathrm{S}_1^{-1}\mathrm{S}_2 + \frac{\left(1-\lambda_{-1,-1}\right)\kappa_{-1,-1}}{\kappa}\mathrm{S}_1^{-1}\mathrm{S}_2^{-1} \\ &\quad + \left(\kappa-1\right)\mathrm{id}, \\ \mathcal{H}_{\lambda}^{\dagger} &= \frac{\left(1-\lambda_{-2,0}\right)\left(\kappa_{-1,1}-\lambda_{-1,1}\right)}{\kappa_{-1,1}}\mathrm{S}_1^{-2} + \frac{\lambda_{-1,-1}\left(1-\lambda_{-1,-1}\right)}{\kappa}\mathrm{S}_1^{-1}\mathrm{S}_2^{-1} \\ &\quad + \left(\lambda-1\right)\mathrm{id}. \end{aligned} \tag{2.48}$$

By the formula in Proposition 2.5.2, the invariant Euler–Lagrange equation for the running example is

$$\mathcal{H}_{\lambda}^{\dagger}\mathrm{E}_{\lambda}\left(L^{\boldsymbol{\kappa}}\right) + \mathcal{H}_{\kappa}^{\dagger}\mathrm{E}_{\kappa}\left(L^{\boldsymbol{\kappa}}\right) = 0.$$

So substituting the adjoint linear difference operators and the Euler operators of $L^{\boldsymbol{\kappa}}$ with respect to the generating invariants into this equation and simplifying gives the invariant Euler–Lagrange equation

$$\begin{aligned} \mathcal{H}_{\lambda}^{\dagger}\mathrm{E}_{\lambda}\left(L^{\boldsymbol{\kappa}}\right) + \mathcal{H}_{\kappa}^{\dagger}\mathrm{E}_{\kappa}\left(L^{\boldsymbol{\kappa}}\right) &= \frac{\lambda_{-1,1}}{2\left(\kappa_{-1,1}\right)} + \frac{\kappa-\lambda_{-1,-1}-1}{2\kappa} \\ &\quad - \frac{\left(\kappa_{-2,0}-1\right)\left(\kappa_{-1,1}-\lambda_{-1,1}\right)}{2\left(\kappa_{-2,0}-\lambda_{-2,0}\right)\kappa_{-1,1}} \\ &= 0. \end{aligned} \tag{2.49}$$

It is possible to check the resulting invariantized Euler–Lagrange equation by changing it into the original variables $u_{i,j}$ and comparing this with the invariantization (2.31) of the original Euler–Lagrange equation, also given in terms of the original variables (see Section A.3 in Appendix A).

2.6 Invariant Euler–Lagrange equations by difference forms

This section explores a new way of finding the invariant formulation of the Euler–Lagrange equations and links it to the previous method.

2.6.1 The proof of the invariant Euler–Lagrange equations formula in Proposition 2.5.2 by difference forms

This subsection will prove the invariant Euler–Lagrange equations formula (2.45) in Proposition 2.5.2 and in doing so will give us two useful formulas for finding the Euler–Lagrange equations directly in terms of invariants. We use some basic features of the difference variational bicomplex, which was introduced in [17] and examined in detail in [31] and is analogous to the differential variational bicomplex [1, 2]. The difference structure is a consequence of the ordering of each independent variable. Hydon and Mansfield [17] introduced difference forms on $\mathbb{Z}^p$. These have the same algebraic properties as differential forms on $\mathbb{R}^p$, with the exterior algebra on p symbols, $\Delta^1, \ldots, \Delta^p$, replacing the exterior algebra on $\mathrm{d}x^1, \ldots, \mathrm{d}x^p$. For differential forms these algebraic properties include

$$\mathrm{d}x^i \wedge \mathrm{d}x^i = 0, \quad \mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i,$$

while for difference forms

$$\Delta^i \wedge \Delta^i = 0, \quad \Delta^i \wedge \Delta^j = -\Delta^j \wedge \Delta^i.$$

There are several other similarities between differential forms and difference forms. The symbols Δ^i at any two different points are related by (horizontal) translations, so that

$$\Delta^i|_{\mathbf{n}} = \mathrm{T}^*_{\mathbf{K}}\left(\Delta|_{\mathbf{n}+\mathbf{K}}\right) =: \mathrm{S}_{\mathbf{K}}\left(\Delta^i|_{\mathbf{n}}\right). \tag{2.50}$$

The standard exterior derivative on each fibre $P_{\mathbf{n}}\left(\mathbb{R}^q\right)$ is denoted d_v.

A (k, l)-form on $\mathbb{Z}^p \times P\left(\mathbb{R}^q\right)$ is a $(k + l)$-form w that can be written (without redundancies) as

$$w = f^{\mathbf{J}_1,\dots,\mathbf{J}_l}_{i_1,\dots,i_k;\alpha_1,\dots,\alpha_l}(\mathbf{n},[\mathbf{u}])\,\Delta^{i_1}\wedge\cdots\wedge\Delta^{i_k}\wedge \mathrm{d}_v u^{\alpha_1}_{\mathbf{J}_1}\wedge\cdots\wedge \mathrm{d}_v u^{\alpha_l}_{\mathbf{J}_l}; \qquad (2.51)$$

we denote the set of all such forms by $\Omega^{k,l}$. The exterior derivative is the mapping $\mathrm{d}_v : \Omega^{k,l} \mapsto \Omega^{k,l+1}$ whose action on (2.51) is

$$\mathrm{d}_v w = \frac{\partial}{\partial u^{\alpha}_{\mathbf{J}}}\{f^{\mathbf{J}_1,\dots,\mathbf{J}_l}_{i_1,\dots,i_k;\alpha_1,\dots,\alpha_l}(\mathbf{n},[\mathbf{u}])\}\mathrm{d}_v u^{\alpha}_{\mathbf{J}}\wedge\Delta^{i_1}\wedge\cdots\wedge\Delta^{i_k}\wedge \mathrm{d}_v u^{\alpha_1}_{\mathbf{J}_1}\wedge\cdots\wedge \mathrm{d}_v u^{\alpha_l}_{\mathbf{J}_l}. \qquad (2.52)$$

Additionally, the exterior difference operator is the mapping $\mathrm{d}^{\Delta}_h : \Omega^{k,l} \mapsto \Omega^{k+1,l}$ whose action on (2.51) is

$$\mathrm{d}^{\Delta}_h w = \mathrm{D}_{n^i}\{f^{\mathbf{J}_1,\dots,\mathbf{J}_l}_{i_1,\dots,i_k;\alpha_1,\dots,\alpha_l}(\mathbf{n},[\mathbf{u}])\}\Delta^{i}\wedge\Delta^{i_1}\wedge\cdots\wedge\Delta^{i_k}\wedge \mathrm{d}_v u^{\alpha_1}_{\mathbf{J}_1}\wedge\cdots\wedge \mathrm{d}_v u^{\alpha_l}_{\mathbf{J}_l}.$$

The notation d_v and d^{Δ}_h mirrors the notation used for the differential variational bicomplex where

$$\mathrm{d}_h = \mathrm{d}x^i \wedge D_i, \quad \mathrm{d}_v = \left(\mathrm{d}u^{\alpha}_{\mathbf{J}} - u^{\alpha}_{\mathbf{J}+\mathbf{1}_i}\mathrm{d}x^i\right)\wedge \frac{\partial}{\partial u^{\alpha}_{\mathbf{J}}},$$

are the horizontal and vertical derivatives with

$$D_i = \frac{\partial}{\partial x^i} + u_{\mathbf{J}+\mathbf{1}_i}\frac{\partial}{\partial u^{\alpha}_{\mathbf{J}}}.$$

Shifts of (2.51) are

$$\mathrm{S}_{\mathbf{K}} w = \mathrm{S}_{\mathbf{K}}\{f^{\mathbf{J}_1,\dots,\mathbf{J}_l}_{i_1,\dots,i_k;\alpha_1,\dots,\alpha_l}(\mathbf{n},[\mathbf{u}])\}\Delta^{i_1}\wedge\cdots\wedge\Delta^{i_k}\wedge \mathrm{d}_v u^{\alpha_1}_{\mathbf{J}_1+\mathbf{K}}\wedge\cdots\wedge \mathrm{d}_v u^{\alpha_l}_{\mathbf{J}_l+\mathbf{K}},$$

because (2.50) implies that $\mathrm{S}_{\mathbf{K}}\Delta^j = \Delta^j$. Denote the restriction of a difference form w on $\mathbb{Z}^p \times P(\mathbb{R}^q)$ to $P_{\mathbf{n}}(\mathbb{R}^q)$ by $w_{\mathbf{n}}$. Then the pullback of $\mathrm{T}_{\mathbf{K}}$ to $w_{\mathbf{n}}$ is

$$\mathrm{S}_{\mathbf{K}} w_{\mathbf{n}} = \mathrm{T}^{*}_{\mathbf{K}} w_{\mathbf{n}+\mathbf{K}}.$$

By the standard properties of the pullback, $\mathrm{S}_{\mathbf{K}}$ commutes with the wedge product and with the exterior derivative on the fibre $P(\mathbb{R}^q)$, so

$$\mathrm{S}_{\mathbf{K}}(w_1 \wedge w_2) = (\mathrm{S}_{\mathbf{K}}w_1) \wedge (\mathrm{S}_{\mathbf{K}}w_2), \qquad \mathrm{S}_{\mathbf{K}}(\mathrm{d}_v w) = \mathrm{d}_v(\mathrm{S}_{\mathbf{K}}w). \tag{2.53}$$

This gives us sufficient knowledge of difference forms to begin to explain the theory that will prove the formula (2.45) in Proposition 2.5.2.

Definition 2.6.1. A difference Lagrangian functional is the sum over $\mathbb{Z}^p$ of a $\Omega^{p,0}$ form,

$$\lambda^{\mathbf{u}} = \sum \mathrm{L}(\mathbf{n}, [\mathbf{u}])\, \Delta^1 \wedge \cdots \wedge \Delta^p \in \Omega^{p,0},$$

always assumed to be finite to avoid technical problems.

From this point on we use the notation vol for the volume form $\Delta^1 \wedge \cdots \wedge \Delta^p$. Therefore,

$$\lambda^{\mathbf{u}} = \sum \mathrm{L}(\mathbf{n}, [\mathbf{u}])\, \mathrm{vol}.$$

Additionally, when looking at forms there is no need to include $\mathbf{n}$ under the summation sign.

The exterior derivative of $\lambda^{\mathbf{u}}$ is

$$\mathrm{d}_v \lambda^{\mathbf{u}} = \sum \frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}} \mathrm{d}_v u^{\alpha}_{\mathbf{J}} \wedge \mathrm{vol}.$$

Using summation by parts gives the Euler operator of L with respect to u^{α}, as follows:

$$\begin{aligned} \mathrm{d}_v \lambda^{\mathbf{u}} &= \sum \mathrm{S}_{-\mathbf{J}} \frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}} \mathrm{d}_v u^{\alpha}_{\mathbf{0}} \wedge \mathrm{vol} \\ &= \sum \mathrm{E}_{u^{\alpha}}(\mathrm{L})\, \mathrm{d}_v u^{\alpha}_{\mathbf{0}} \wedge \mathrm{vol}. \end{aligned}$$

This can be adapted to finite domains using the discrete analogue of Stokes' theorem, see [24]. The divergence terms which arise disappear (for appropriate boundary conditions) because of the summation over the boundary. The exterior derivative is coordinate independent, therefore, if we have the same difference Lagrangian functional written in different variables,

$$\lambda^{\boldsymbol{\kappa}} = \sum L^{\boldsymbol{\kappa}}(\mathbf{n}, [\boldsymbol{\kappa}])\, \mathrm{vol} \in \Omega^{p,0},$$

then the exterior derivative of this is

$$\mathrm{d}_v \lambda^{\boldsymbol{\kappa}} = \sum \mathrm{E}_{\kappa^\beta} \left(L^{\boldsymbol{\kappa}}\right) \mathrm{d}_v \kappa^\beta \wedge \mathrm{vol},$$

where $\mathrm{E}_{\kappa^\beta}$ is the Euler operator with respect to κ^β.

Lemma 2.6.2. If $\mathrm{L}\left(\mathbf{n}, [\mathbf{u}]\right) = L^{\boldsymbol{\kappa}}\left(\mathbf{n}, [\boldsymbol{\kappa}]\right)$, with $\kappa^\beta = F^\beta\left([\mathbf{u}]\right)$ for some functions F^β, then the exterior derivatives $\mathrm{d}_v \mathrm{L}$ and $\mathrm{d}_v L^{\boldsymbol{\kappa}}$ are equal:

$$\sum \mathrm{E}_{u^\alpha}\left(\mathrm{L}\right) \mathrm{d}_v u^\alpha_{\mathbf{0}} \wedge \mathrm{vol} = \sum \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right) \mathrm{d}_v \kappa^\beta \wedge \mathrm{vol}. \tag{2.54}$$

This comes as no surprise as the exterior derivative d_v is coordinate independent as stated in Definition (2.6.1) above. However, the proof of this can be insightful for the approach to the proof of the formula (2.45) in Proposition 2.5.2 so it is shown here.

Proof. Starting with the right-hand side of (2.54),

$$\begin{aligned}
\mathrm{RHS} &= \sum \left(\mathrm{S}_{-\mathbf{J}} \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^\beta_{\mathbf{J}}}\right) \frac{\partial \kappa^\beta}{\partial u^\alpha_{\mathbf{K}}} \mathrm{d}_v u^\alpha_{\mathbf{K}} \wedge \mathrm{vol} \\
&= \sum \sum_{\mathbf{K}} \left(\mathrm{S}_{-\mathbf{J}} \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^\beta_{\mathbf{J}}}\right) \mathrm{S}_{\mathbf{K}} \left(\frac{\partial \kappa^\beta_{-\mathbf{K}}}{\partial u^\alpha_{\mathbf{0}}} \mathrm{d}_v u^\alpha_{\mathbf{0}}\right) \wedge \mathrm{vol} \\
&= \sum \sum_{\mathbf{K}} \left(\sum_{\mathbf{J}} \frac{\partial \left(\mathrm{S}_{-\mathbf{J}-\mathbf{K}} L^{\boldsymbol{\kappa}}\right)}{\partial \kappa^\beta_{-\mathbf{K}}}\right) \frac{\partial \kappa^\beta_{-\mathbf{K}}}{\partial u^\alpha_{\mathbf{0}}} \mathrm{d}_v u^\alpha_{\mathbf{0}} \wedge \mathrm{vol} \\
&= \sum \sum_{\mathbf{I}} \frac{\partial \left(\mathrm{S}_{-\mathbf{I}} L^{\boldsymbol{\kappa}}\right)}{\partial u^\alpha_{\mathbf{0}}} \mathrm{d}_v u^\alpha_{\mathbf{0}} \wedge \mathrm{vol} \quad (\text{where } \mathbf{I} = \mathbf{J} + \mathbf{K}) \\
&= \sum \left(\mathrm{S}_{-\mathbf{I}} \frac{\partial \mathrm{L}}{\partial u^\alpha_{\mathbf{I}}}\right) \mathrm{d}_v u^\alpha_{\mathbf{0}} \wedge \mathrm{vol} \quad (\text{using } \mathrm{L} = L^{\boldsymbol{\kappa}}) \\
&= \mathrm{LHS}.
\end{aligned}$$

□

Definition 2.6.3. Two difference functions are called equivalent if they differ by a divergence term.

Now we introduce an important lemma which uses the equivalence of functions.

Lemma 2.6.4. Let F, G be two difference functions and $\mathcal{P} = f^{\mathbf{J}}\mathrm{S}_{\mathbf{J}}$ a linear difference operator. Then,

$$\sum_{\mathbf{n}} F\mathcal{P}\left(G\right) = \sum_{\mathbf{n}} \mathcal{P}^{\dagger}\left(F\right)G,$$

where $\mathcal{P}^{\dagger}\left(F\right) = \mathrm{S}_{-\mathbf{J}}\left(f^{\mathbf{J}}F\right)$.

Proof. Using the summation by parts formula (2.3) and that the divergence terms disappear by Stokes' theorem,

$$\begin{aligned}\sum_{\mathbf{n}} F\mathcal{P}\left(G\right) &= \sum_{\mathbf{n}} F\left(f^{\mathbf{J}}\mathrm{S}_{\mathbf{J}}G\right)\\ &= \sum_{\mathbf{n}} \mathrm{S}_{-\mathbf{J}}\left(f^{\mathbf{J}}F\right)G\\ &= \sum_{\mathbf{n}} \mathcal{P}^{\dagger}\left(F\right)G.\end{aligned}$$

□

This idea works equally for difference forms.

We now introduce the matrix ϑ with components

$$\left(\vartheta\right)^{\alpha}_{\delta} = \left(\frac{\partial\left(g\cdot u^{\alpha}_{\mathbf{0}}\right)}{\partial u^{\delta}_{\mathbf{0}}}\right)\Bigg|_{g=\rho_{\mathbf{0}}([\mathbf{u}])}.$$

The inverse of this matrix is very important in the proof of (2.43) in Proposition 2.5.2 as it introduces a key equivariant component. Therefore, we need to show that ϑ is non-singular. The vertical derivative of $g\cdot u^{\alpha}_{\mathbf{0}}$ is

$$\mathrm{d}_v\left(g\cdot u^{\alpha}_{\mathbf{0}}\right) = \frac{\partial\left(g\cdot u^{\alpha}_{\mathbf{0}}\right)}{\partial u^{\delta}_{\mathbf{0}}}\mathrm{d}_v u^{\delta}_{\mathbf{0}}.$$

This difference form is not necessarily invariant under the group action; however, to achieve an invariant difference form one needs to introduce an equivariant component. Applying the action of the frame to this gives the invariant difference forms in terms of the original variables as

$$\begin{aligned}\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right) &= \left(g \cdot \mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right)|_{g=\rho_{\mathbf{0}}([\mathbf{u}])}\\ &= \left(\mathrm{d}_v \left(g \cdot u^{\alpha}_{\mathbf{0}}\right)\right)|_{g=\rho_{\mathbf{0}}([\mathbf{u}])}\\ &= (\vartheta)^{\alpha}_{\delta}\, \mathrm{d}_v u^{\delta}_{\mathbf{0}}.\end{aligned}$$

To prove that ϑ^{-1} exists, invariantize (2.31) this equation

$$\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right) = \iota\left((\vartheta)^{\alpha}_{\delta}\right)\iota\left(\mathrm{d}_v u^{\delta}_{\mathbf{0}}\right)$$

and note that the left-hand side is unchanged as it is already invariant

$$\iota\left(\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right)\right) = \iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right).$$

Not only this but it is clear to see that

$$\iota\left((\vartheta)^{\alpha}_{\delta}\right) = \begin{cases} 1 & \text{if} \quad \alpha = \delta, \\ 0 & \text{if} \quad \alpha \neq \delta. \end{cases}$$

Consequently, $\iota(\vartheta) = \mathrm{Id_q}$, where $\mathrm{Id_q}$ is the $q \times q$ identity matrix, and so its determinant is 1. Using the property

$$\iota\left(\det(\vartheta)\right) = \det\left(\iota(\vartheta)\right)$$

of invariantization (2.31) means it is obvious to see that

$$\iota\left(\det(\vartheta)\right) = 1 \quad \text{implies} \quad \det(\vartheta) \neq 0,$$

and so the inverse ϑ^{-1} exists. From here onwards we are more interested in the components of the inverse matrix ϑ^{-1}, so we introduce $\theta_{\mathbf{0}} = \vartheta^{-1}$. The shifts of $\theta_{\mathbf{0}}$ are

$$\theta_{\mathbf{J}} = \mathrm{S}_{\mathbf{J}}\theta_{\mathbf{0}}.$$

In all that follows, $\mathrm{E}_{u^{\alpha}}$ represents the Euler operator with respect to the original variables, u^{α}, and $\mathrm{E}_{\kappa^{\beta}}$ represents the Euler operator with respect to the generating invariants, κ^β.

Proposition 2.6.5. The invariantization of the original Euler–Lagrange equations is

$$\iota\left(\mathrm{E}_{u^\alpha}\left(\mathrm{L}\right)\right) = \left(\mathcal{H}_\alpha^\beta\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right), \tag{2.55}$$

where

$$\left(\mathcal{H}_\alpha^\beta\right)^\dagger = \sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}} \left(\iota\left(\frac{\partial \kappa^\beta}{\partial u_{\mathbf{J}}^\delta}\right)\iota\left(\left(\theta_{\mathbf{J}}\right)_\alpha^\delta\right)\right)\mathrm{S}_{-\mathbf{J}}. \tag{2.56}$$

Proof. The exterior derivative of the generating invariant κ^β is

$$\mathrm{d}_v \kappa^\beta = \frac{\partial \kappa^\beta}{\partial u_{\mathbf{J}}^\delta}\mathrm{d}_v u_{\mathbf{J}}^\delta; \tag{2.57}$$

this was used without introducing it in the proof of Lemma 2.6.2. It is stated here because is central to the calculations. We show how to construct invariant difference forms; developing the theory of an invariant difference variational bicomplex, that is comparable to developments for the differential case in the PDEs paper by Kogan and Olver [19]. For the difference case the horizontal forms Δ^i are all invariant, as the lattice remains unchanged. In the differential case or differential-difference it is possible that the independent variables can be changed by a Lie group action. The invariant difference forms have already been defined and are

$$\iota\left(\mathrm{d}_v u_{\mathbf{0}}^\alpha\right) = \left(\vartheta\right)_\delta^\alpha \mathrm{d}_v u_{\mathbf{0}}^\delta.$$

Inverting the matrix on the right-hand side gives

$$\left(\theta_{\mathbf{0}}\right)_\alpha^\delta \iota\left(\mathrm{d}_v u_{\mathbf{0}}^\alpha\right) = \mathrm{d}_v u_{\mathbf{0}}^\delta,$$

which can be shifted to get

$$\left(\theta_{\mathbf{J}}\right)_\alpha^\delta \mathrm{S}_{\mathbf{J}} \iota\left(\mathrm{d}_v u_{\mathbf{0}}^\alpha\right) = \mathrm{d}_v u_{\mathbf{J}}^\delta.$$

The key identity is

$$\iota\left(\left(\theta_{\mathbf{J}}\right)_\alpha^\delta\right) \mathrm{S}_{\mathbf{J}} \iota\left(\mathrm{d}_v u_{\mathbf{0}}^\alpha\right) = \iota\left(\mathrm{d}_v u_{\mathbf{J}}^\delta\right), \tag{2.58}$$

where $\iota\left(\mathrm{S}_{\mathbf{J}}\iota\left(\mathrm{d}_v u_{\mathbf{0}}^\alpha\right)\right) = \mathrm{S}_{\mathbf{J}}\iota\left(\mathrm{d}_v u_{\mathbf{0}}^\alpha\right)$ as the invariantization (2.31) of an invariant is unchanged. So, by invariantizing (2.57), we obtain

$$\iota\left(\mathrm{d}_v \kappa^{\beta}\right) = \iota\left(\frac{\partial \kappa^{\beta}}{\partial u^{\delta}_{\mathbf{J}}}\right) \iota\left(\mathrm{d}_v u^{\delta}_{\mathbf{J}}\right),$$

and hence substituting in (2.58) yields

$$\iota\left(\mathrm{d}_v \kappa^{\beta}\right) = \sum_{\mathbf{J}} \iota\left(\frac{\partial \kappa^{\beta}}{\partial u^{\delta}_{\mathbf{J}}}\right) \iota\left(\left(\theta_{\mathbf{J}}\right)^{\delta}_{\alpha}\right) \mathrm{S}_{\mathbf{J}} \iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right). \tag{2.59}$$

Let

$$\mathcal{H}^{\beta}_{\alpha} = \sum_{\mathbf{J}} \iota\left(\frac{\partial \kappa^{\beta}}{\partial u^{\delta}_{\mathbf{J}}}\right) \iota\left(\left(\theta_{\mathbf{J}}\right)^{\delta}_{\alpha}\right) \mathrm{S}_{\mathbf{J}}, \tag{2.60}$$

so that (2.59) is

$$\iota\left(\mathrm{d}_v \kappa^{\beta}\right) = \mathcal{H}^{\beta}_{\alpha}\, \iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right). \tag{2.61}$$

Lemma 2.6.2 implies $\mathrm{d}_v \lambda^{\mathbf{u}} = \mathrm{d}_v \lambda^{\boldsymbol{\kappa}}$. By invariantizing (2.31) both sides of this equation, we obtain

$$\sum \iota\left(\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right)\right) \iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right) \wedge \mathrm{vol} = \sum \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right) \iota\left(\mathrm{d}_v \kappa^{\beta}\right) \wedge \mathrm{vol}, \tag{2.62}$$

where $\iota\left(\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right) = \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)$, as $\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)$ is invariant. Also $\iota\left(\mathrm{vol}\right) = \mathrm{vol}$ as vol does not depend on the Lie group action. From (2.61),

$$\sum \iota\left(\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right)\right) \iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right) \wedge \mathrm{vol} = \sum \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right) \mathcal{H}^{\beta}_{\alpha}\, \iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right) \wedge \mathrm{vol}.$$

Then Lemma 2.6.4 gives

$$\sum \iota\left(\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right)\right) \iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right) \wedge \mathrm{vol} = \sum \left(\mathcal{H}^{\beta}_{\alpha}\right)^{\dagger} \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\, \iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0}}\right) \wedge \mathrm{vol}.$$

Therefore, pulling out the coefficients of the invariant difference forms,

$$\iota\left(\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right)\right) = \left(\mathcal{H}^{\beta}_{\alpha}\right)^{\dagger} \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right),$$

where

$$\left(\mathcal{H}^{\beta}_{\alpha}\right)^{\dagger} = \sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}} \left(\iota\left(\frac{\partial \kappa^{\beta}}{\partial u^{\delta}_{\mathbf{J}}}\right) \iota\left(\left(\theta_{\mathbf{J}}\right)^{\delta}_{\alpha}\right)\right) \mathrm{S}_{-\mathbf{J}}.$$

Accordingly, the invariantized original Euler–Lagrange equations are

$$\left(\mathcal{H}_{\alpha}^{\beta}\right)^{\dagger} \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)=0. \tag{2.63}$$

□

Remark 2.6.6. The adjoint operators (2.56) can be worked out independent of the form of the invariant Lagrangian. For the adjoint operators only the Lie group action, normalization equations (2.16) and generating invariants are required.

Example 2.6.1. The running example has only one dependent variable, so we adapt notation for $\mathcal{H}^{\dagger}$, to include the generating invariant as a subscript, i.e., $\mathcal{H}_{\lambda}^{\dagger}$ and $\mathcal{H}_{\kappa}^{\dagger}$.

The difference operator $\mathcal{H}_{\kappa}^{\dagger}$ for this example is

$$\mathcal{H}_{\kappa}^{\dagger}=\sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}}\left[\iota\left(\frac{\partial \kappa}{\partial u_{\mathbf{J}}}\right) \iota\left(\theta_{\mathbf{J}}\right)\right] \mathrm{S}_{-\mathbf{J}}. \tag{2.64}$$

We differentiate κ with respect to $u_{0,0}$, $u_{1,1}$, and $u_{1,-1}$, as these are the only shifts of the dependent variable in the definition of the generating invariant κ. Invariantizing (2.31) these derivatives gives

$$\iota\left(\frac{\partial \kappa}{\partial u_{0,0}}\right)=\kappa-1, \qquad \iota\left(\frac{\partial \kappa}{\partial u_{1,1}}\right)=-\kappa, \qquad \iota\left(\frac{\partial \kappa}{\partial u_{1,-1}}\right)=1.$$

As the action (2.23) is a translation and scaling, $\mathrm{d}\left(g \cdot u_{0,0}\right) / \mathrm{d} u_{0,0}=a$, where a comes from the scaling part of the group action. Taking the inverse of this and applying the value of the parameter on the moving frame (2.26) gives

$$\theta_{0,0}=u_{1,1}-u_{0,0}.$$

To find all relevant $\iota\left(\theta_{\mathbf{J}}\right)$, shift the above equation by all relevant values $\mathbf{J}$ and invariantize (2.31):

$$\iota\left(\theta_{0,0}\right)=1, \qquad \iota\left(\theta_{1,1}\right)=\frac{\lambda-1}{\kappa_{1,1}}, \qquad \iota\left(\theta_{1,-1}\right)=\lambda-\kappa.$$

Substituting these results into the formula (2.64) gives

$$\mathcal{H}_\kappa^\dagger = (\lambda_{-1,1} - \kappa_{-1,1})\,\mathrm{S}_1^{-1}\mathrm{S}_2 + \frac{(1-\lambda_{-1,-1})\,\kappa_{-1,-1}}{\kappa}\mathrm{S}_1^{-1}\mathrm{S}_2^{-1} + (\kappa - 1)\,\mathrm{id}.$$

Next, we go through similar calculations for

$$\mathcal{H}_\lambda^\dagger = \sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}} \left[\iota\left(\frac{\partial\lambda}{\partial u_{\mathbf{J}}}\right)\iota\left(\theta_{\mathbf{J}}\right)\right]\mathrm{S}_{-\mathbf{J}}. \tag{2.65}$$

The components are

$$\iota\left(\frac{\partial\lambda}{\partial u_{0,0}}\right) = \lambda - 1, \qquad \iota\left(\frac{\partial\lambda}{\partial u_{1,1}}\right) = -\lambda, \qquad \iota\left(\frac{\partial\lambda}{\partial u_{2,0}}\right) = 1,$$

and

$$\iota\left(\theta_{0,0}\right) = 1, \qquad \iota\left(\theta_{1,1}\right) = \frac{\lambda-1}{\kappa_{1,1}}, \qquad \iota\left(\theta_{2,0}\right) = \frac{(1-\lambda)\left(\kappa_{1,1}-\lambda_{1,1}\right)}{\kappa_{1,1}}.$$

Substituting these into (2.65) gives

$$\mathcal{H}_\lambda^\dagger = \frac{(1-\lambda_{-2,0})\left(\kappa_{-1,1}-\lambda_{-1,1}\right)}{\kappa_{-1,1}}\mathrm{S}_1^{-2} + \frac{\lambda_{-1,-1}\left(1-\lambda_{-1,-1}\right)}{\kappa}\mathrm{S}_1^{-1}\mathrm{S}_2^{-1} + (\lambda-1)\,\mathrm{id}.$$

These two adjoint linear difference operators are the same as (2.48) found in Example 2.4.3. Using $\mathrm{E}_\kappa\left(L^{\boldsymbol{\kappa}}\right)$ and $\mathrm{E}_\lambda\left(L^{\boldsymbol{\kappa}}\right)$ from before one can then find the invariant formulation of the Euler–Lagrange equations using Proposition 2.6.5.

There is a second way to get the invariant formulation of the Euler–Lagrange equations. This produces a different formula for the adjoint linear difference operators (2.56) compared to the one found in Proposition 2.6.5. However, it achieves the same results as the adjoint operators (2.56) in Proposition 2.6.5, and more interestingly it demonstrates a link between the order of invariantizing and shifting.

Lemma 2.6.7. The invariantization of the original Euler–Lagrange equations is

$$\iota\left(\mathrm{E}_{u^\alpha}\left(\mathrm{L}\right)\right) = \left(\mathcal{H}_\alpha^\beta\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right), \tag{2.66}$$

where $\left(\mathcal{H}_{\alpha}^{\beta}\right)^{\dagger}$ is a difference operator given by

$$\left(\mathcal{H}_{\alpha}^{\beta}\right)^{\dagger} = \sum_{\mathbf{J}} \iota\left(\frac{\partial \kappa_{-\mathbf{J}}^{\beta}}{\partial u_{\mathbf{0}}^{\alpha}}\right) \mathrm{S}_{-\mathbf{J}}. \tag{2.67}$$

Proof. Knowing that the exterior derivative is coordinate independent allows us to manipulate the right-hand side of (2.54), using (2.57) and Lemma 2.6.4, so that the difference form which remains is $\mathrm{d}_v u_{\mathbf{0}}^{\alpha}$, as follows:

$$\begin{aligned}
\text{RHS} &= \sum \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right) \frac{\partial \kappa^{\beta}}{\partial u_{\mathbf{J}}^{\alpha}} \mathrm{d}_v u_{\mathbf{J}}^{\alpha} \wedge \mathrm{vol} \\
&= \sum \sum_{\mathbf{J}} \frac{\partial \kappa_{-\mathbf{J}}^{\beta}}{\partial u_{\mathbf{0}}^{\alpha}} \mathrm{S}_{-\mathbf{J}}\left(\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right) \mathrm{d}_v u_{\mathbf{0}}^{\alpha} \wedge \mathrm{vol}. \qquad (2.68)
\end{aligned}$$

Therefore, invariantizing (2.31) the left-hand side of (2.54) and the right-hand side of (2.68) gives, after dropping the summation over $\mathbf{n}$ and extracting components of difference forms,

$$\iota\left(\mathrm{E}_{u^{\alpha}}(\mathrm{L})\right) = \sum_{\mathbf{J}} \iota\left(\frac{\partial \kappa_{-\mathbf{J}}^{\beta}}{\partial u_{\mathbf{0}}^{\alpha}}\right) \mathrm{S}_{-\mathbf{J}}\left(\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right).$$

where $\iota\left(\mathrm{S}_{-\mathbf{J}}\left(\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right)\right) = \mathrm{S}_{-\mathbf{J}}\left(\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right)$ as this is already invariant. Writing

$$\left(\mathcal{H}_{\alpha}^{\beta}\right)^{\dagger} = \sum_{\mathbf{J}} \iota\left(\frac{\partial \kappa_{-\mathbf{J}}^{\beta}}{\partial u_{\mathbf{0}}^{\alpha}}\right) \mathrm{S}_{-\mathbf{J}},$$

gives the result. □

Corollary 2.6.8. As a consequence of Proposition 2.6.5 and Lemma 2.6.7 the two representations of the linear difference operators $\left(\mathcal{H}_{\alpha}^{\beta}\right)^{\dagger}$ are equivalent when acting on $\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)$. Therefore,

$$\sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}}\left[\iota\left(\frac{\partial \kappa^{\beta}}{\partial u_{\mathbf{J}}^{\delta}}\right) \iota\left(\left(\theta_{\mathbf{J}}\right)_{\alpha}^{\delta}\right) \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right] = \sum_{\mathbf{J}} \iota\left(\frac{\partial \kappa_{-\mathbf{J}}^{\beta}}{\partial u_{\mathbf{0}}^{\alpha}}\right)\left[\mathrm{S}_{-\mathbf{J}} \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right]. \tag{2.69}$$

This is a new identity that gives a relationship between invariantization that is done before and after a shift.

2.6.2 How the difference form theory relates to using the additional variable trick

In Section 2.4 we considered the first-order differential invariants

$$I_{\mathbf{K},\mathbf{J};t} = \rho_{\mathbf{K}}\left([\mathbf{u}]\right) \cdot \frac{\mathrm{d}\mathbf{u}_{\mathbf{J}}}{\mathrm{d}t}.$$

The components of these first-order differential invariants are

$$I^{\alpha}_{\mathbf{K},\mathbf{J};t} = \rho_{\mathbf{K}}\left([\mathbf{u}]\right) \cdot \frac{\mathrm{d}u^{\alpha}_{\mathbf{J}}}{\mathrm{d}t},$$

so

$$\begin{aligned} I^{\alpha}_{\mathbf{K},\mathbf{J};t} &= \left(\frac{\partial\left(g \cdot u^{\alpha}_{\mathbf{J}}\right)}{\partial u^{\delta}_{\mathbf{J}}}\right)\Bigg|_{g=\rho_{\mathbf{K}}([\mathbf{u}])} \frac{\mathrm{d}u^{\delta}_{\mathbf{J}}}{\mathrm{d}t} \\ &= \left(\frac{\partial\left(g \cdot u^{\alpha}_{\mathbf{J}}\right)}{\partial u^{\delta}_{\mathbf{J}}}\right)\Bigg|_{g=\rho_{\mathbf{K}}([\mathbf{u}])} \left(u^{\delta}_{\mathbf{J}}\right)'. \end{aligned}$$

For $\mathbf{J} = \mathbf{K} = \mathbf{0}$, this gives the first-order generating invariants

$$I^{\alpha}_{\mathbf{0},\mathbf{0};t} = \left(\frac{\partial\left(g \cdot u^{\alpha}_{\mathbf{0}}\right)}{\partial u^{\delta}_{\mathbf{0}}}\right)\Bigg|_{g=\rho_{\mathbf{0}}([\mathbf{u}])} \left(u^{\delta}_{\mathbf{0}}\right)' = (\vartheta)^{\alpha}_{\delta}\left(u^{\delta}_{\mathbf{0}}\right)' = \sigma^{\alpha}, \qquad (2.70)$$

and so

$$\left(u^{\delta}_{\mathbf{0}}\right)' = (\theta_{\mathbf{0}})^{\delta}_{\alpha}\,\sigma^{\alpha}.$$

To find the invariants $\iota\left(\left(u^{\delta}_{\mathbf{J}}\right)'\right)$ one shifts the above formula by $\mathbf{J}$

$$\left(u^{\delta}_{\mathbf{J}}\right)' = (\theta_{\mathbf{J}})^{\alpha}_{\delta}\,\mathrm{S}_{\mathbf{J}}\sigma^{\alpha},$$

and then invariantizes (2.31):

$$\iota\left(\left(u^{\delta}_{\mathbf{J}}\right)'\right) = \iota\left((\theta_{\mathbf{J}})^{\delta}_{\alpha}\right)\mathrm{S}_{\mathbf{J}}\sigma^{\alpha}. \qquad (2.71)$$

Here the invariantization in the last line leaves $\mathrm{S}_{\mathbf{J}}\sigma^{\alpha}$ unchanged, as it is already invariant.

We now inspect the differential-difference invariants $\mathrm{d}\kappa^{\beta}/\mathrm{d}t$. The aim is to show by using (2.71) that we can achieve the same differential-difference syzygies

(2.36) with linear difference operators (2.60). Thus, by direct computation

$$\begin{aligned}\frac{\mathrm{d}\kappa^\beta}{\mathrm{d}t} &= \frac{\partial \kappa^\beta}{\partial u^\delta_{\mathbf{J}}}\left(u^\delta_{\mathbf{J}}\right)' \\ &= \iota\left(\frac{\partial \kappa^\beta}{\partial u^\delta_{\mathbf{J}}}\right)\iota\left(\left(u^\delta_{\mathbf{J}}\right)'\right) \\ &= \sum_{\mathbf{J}}\iota\left(\frac{\partial \kappa^\beta}{\partial u^\delta_{\mathbf{J}}}\right)\iota\left(\left(\theta_{\mathbf{J}}\right)^\delta_\alpha\right)\mathrm{S}_{\mathbf{J}}\sigma^\alpha \\ &= \mathcal{H}^\beta_\alpha\sigma^\alpha.\end{aligned}$$

This shows that the method of using the additional variable t gives the same linear difference operators as in the difference forms theory. The generating first-order invariants σ^α in effect play the same role as $\iota\left(\mathrm{d}_v u^\alpha_{\mathbf{0}}\right)$ does in the difference form theory. Likewise, the first-order invariants $\iota\left(\left(u^\delta_{\mathbf{J}}\right)'\right)$ play the role of $\iota\left(\mathrm{d}_v u^\delta_{\mathbf{J}}\right)$ in the difference forms theory and $\mathrm{d}_v u^\alpha_{\mathbf{0}}$ plays a similar role to $\left(u^\alpha_{\mathbf{0}}\right)'$. Using these similarities we now prove several propositions for both the additional variable and difference form methods.

Proposition 2.6.9. The following identity holds

$$\mathrm{E}_{u^\alpha}\left(\mathrm{L}\right)\mathrm{d}_v u^\alpha_{\mathbf{0}}\wedge\mathrm{vol} = \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\iota\left(\mathrm{d}_v u^\alpha_{\mathbf{0}}\right)\wedge\mathrm{vol}. \tag{2.72}$$

Proof. From the proof of Proposition 2.6.5 the identity

$$\iota\left(\mathrm{E}_{u^\alpha}\left(\mathrm{L}\right)\right)\iota\left(\mathrm{d}_v u^\alpha_{\mathbf{0}}\right)\wedge\mathrm{vol} = \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\iota\left(\mathrm{d}_v u^\alpha_{\mathbf{0}}\right)\wedge\mathrm{vol},$$

implies

$$\mathrm{E}_{u^\alpha}\left(\mathrm{L}\right)\mathrm{d}_v u^\alpha_{\mathbf{0}}\wedge\mathrm{vol} - \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\iota\left(\mathrm{d}_v u^\alpha_{\mathbf{0}}\right)\wedge\mathrm{vol} \in \mathrm{Ker}\left(\iota\right),$$

where this is invariant under the group action. The only $(p,1)$-form which exists with $\iota\left(\Omega\right)=\Omega$ and $\Omega\in\mathrm{Ker}\left(\iota\right)$ is $\Omega = 0$. Therefore, this proves the identity. □

Next, we prove identity (2.43) in Proposition 2.5.2 by implementing a similar approach.

Proposition 2.6.10. The following identity holds

$$\mathrm{E}_{u^\alpha}(\mathrm{L})\left(u^\alpha_{\mathbf{0}}\right)' = \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\sigma^\alpha. \tag{2.73}$$

Proof. Multiply the main result in Proposition 2.6.5 by σ^α to obtain

$$\iota\left(\mathrm{E}_{u^\alpha}(\mathrm{L})\right)\sigma^\alpha = \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\sigma^\alpha.$$

Consequently,

$$\mathrm{E}_{u^\alpha}(\mathrm{L})\left(u^\alpha_{\mathbf{0}}\right)' - \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\sigma^\alpha \in \operatorname{Ker}(\iota),$$

where this is invariant under the group action. Similar to the proof of the previous proposition, the only function $F\left(u^\alpha_{\mathbf{0}}\right)'$ which is invariant under the group action and is an element of $\operatorname{Ker}(\iota)$ is $F = 0$, which proves the identity. □

This proves the formula (2.43) in Proposition 2.5.2 which completes the extension to PΔEs of the main theorem in the paper [26] by Mansfield *et al.* Now we use this identity to prove another result for the divergence terms. Note the divergence terms first appeared in the calculation of the Euler–Lagrange equations in (2.2). Also recall the additional independent variable t method (2.5).

Proposition 2.6.11. The divergence terms

$$\begin{aligned}
\operatorname{Div}\left(A_{\mathbf{u}}\right) &= \frac{\partial \mathrm{L}}{\partial u^\alpha_{\mathbf{J}}}\frac{\mathrm{d}u^\alpha_{\mathbf{J}}}{\mathrm{d}t} - \left(\mathrm{S}_{-\mathbf{J}}\frac{\partial \mathrm{L}}{\partial u^\alpha_{\mathbf{J}}}\right)\frac{\mathrm{d}u^\alpha}{\mathrm{d}t},\\
\operatorname{Div}\left(A_{\boldsymbol{\kappa}}\right) &= \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^\beta_{\mathbf{J}}}\frac{\mathrm{d}\kappa^\beta_{\mathbf{J}}}{\mathrm{d}t} - \left(\mathrm{S}_{-\mathbf{J}}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^\beta_{\mathbf{J}}}\right)\frac{\mathrm{d}\kappa^\beta}{\mathrm{d}t},\\
\operatorname{Div}\left(A_{\mathcal{H}}\right) &= \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\mathcal{H}^\beta_\alpha\sigma^\alpha - \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\sigma^\alpha,
\end{aligned}$$

have the following relationship

$$\operatorname{Div}\left(A_{\mathbf{u}}\right) = \operatorname{Div}\left(A_{\boldsymbol{\kappa}}\right) + \operatorname{Div}\left(A_{\mathcal{H}}\right). \tag{2.74}$$

Proof. In order to effect the variation, we set $u^\alpha = u^\alpha(t)$ where the group action acts trivially on t and compare

$$\frac{\mathrm{dL}}{\mathrm{d}t} = \mathrm{E}_{u^\alpha}\left(\mathrm{L}\right)\left(u^\alpha_{\mathbf{0}}\right)' + \mathrm{Div}\left(A_{\mathbf{u}}\right), \tag{2.75}$$

with the same calculation in terms of the invariants. (This is the same result as (2.2) but replacing w^α with $(u^\alpha)'$.) Note that $\mathrm{dL}/\mathrm{d}t = \mathrm{d}L^{\boldsymbol{\kappa}}/\mathrm{d}t$, so

$$\begin{aligned}
\frac{\mathrm{d}L^{\boldsymbol{\kappa}}}{\mathrm{d}t} &= \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^\beta_{\mathbf{J}}}\frac{\mathrm{d}\kappa^\beta_{\mathbf{J}}}{\mathrm{d}t}\\
&= \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^\beta_{\mathbf{J}}}\mathrm{S}_{\mathbf{J}}\frac{\mathrm{d}\kappa^\beta}{\mathrm{d}t}\\
&= \left(\mathrm{S}_{-\mathbf{J}}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^\beta_{\mathbf{J}}}\right)\frac{\mathrm{d}\kappa^\beta}{\mathrm{d}t} + \mathrm{Div}\left(A_{\boldsymbol{\kappa}}\right)\\
&= \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\frac{\mathrm{d}\kappa^\beta}{\mathrm{d}t} + \mathrm{Div}\left(A_{\boldsymbol{\kappa}}\right)\\
&= \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\left(\mathcal{H}^\beta_\alpha\sigma^\alpha\right) + \mathrm{Div}\left(A_{\boldsymbol{\kappa}}\right)\\
&= \left(\left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\right)\sigma^\alpha + \mathrm{Div}\left(A_{\boldsymbol{\kappa}}\right) + \mathrm{Div}\left(A_{\mathcal{H}}\right).
\end{aligned} \tag{2.76}$$

Thus, using the identity in Proposition 2.6.10 and comparing (2.75) to (2.76) gives the result. □

Let the divergence terms be equal to

$$\begin{aligned}
\mathrm{Div}\left(A_{\mathbf{u}}\right) &= \frac{\partial \mathrm{L}}{\partial u^\alpha_{\mathbf{J}}}\frac{\mathrm{d}u^\alpha_{\mathbf{J}}}{\mathrm{d}t} - \left(\mathrm{S}_{-\mathbf{J}}\frac{\partial \mathrm{L}}{\partial u^\alpha_{\mathbf{J}}}\right)\frac{\mathrm{d}u^\alpha}{\mathrm{d}t} = \mathrm{D}_{n^i}\left(A^i_\alpha\left(\mathbf{n},[\mathbf{u}]\right)\left(u^\alpha\right)'\right),\\
\mathrm{Div}\left(A_{\boldsymbol{\kappa}}\right) &= \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^\beta_{\mathbf{J}}}\frac{\mathrm{d}\kappa^\beta_{\mathbf{J}}}{\mathrm{d}t} - \left(\mathrm{S}_{-\mathbf{J}}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^\beta_{\mathbf{J}}}\right)\frac{\mathrm{d}\kappa^\beta}{\mathrm{d}t} = \mathrm{D}_{n^i}\left(F^i_\beta\left(\mathbf{n},[\boldsymbol{\kappa}]\right)\left(\kappa^\beta\right)'\right),\\
\mathrm{Div}\left(A_{\mathcal{H}}\right) &= \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\mathcal{H}^\beta_\alpha\sigma^\alpha - \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\sigma^\alpha = \mathrm{D}_{n^i}\left(H^i_\alpha\left(\mathbf{n},[\boldsymbol{\kappa}]\right)\sigma^\alpha\right),
\end{aligned}$$

for some difference operators A^i_α, F^i_β and H^i_α. In Section 2.10, these divergence terms will be vital for calculating the conservation laws.

Remark 2.6.12. The divergence $\mathrm{Div}\left(A_{\boldsymbol{\kappa}}\right)$ is linear in $\mathrm{d}\kappa^\beta/\mathrm{d}t$ and their shifts, while $\mathrm{Div}\left(A_{\mathcal{H}}\right)$ is linear in σ^α and their shifts.

2.7 A linear action of SL(2) on the plane

In the previous example of finding the invariantized Euler–Lagrange equation, the matrices $\iota(\theta_{\mathbf{J}})$ were trivial 1×1 matrices. In the following OΔE example of a linear action of SL(2) on the plane, the matrices $\iota(\theta_{\mathbf{J}})$ are not 1×1 or diagonal. Therefore, it is a more representative example of the general theory. This example is taken from the OΔE paper [25], which also explores other SL(2) actions. By using the same normalization and generating invariants here the results can be compared with those found in [25]. As this is a OΔE example replace the multi-index $\mathbf{J}$ by j and use S_j to denote j shifts.

Example 2.7.1. Consider the following action of SL(2) on the prolongation space $P_n\left(\mathbb{R}^2\right)$, which has coordinates $\left(u^1_j, u^2_j\right)$. The infinitesimal generators are

$$\mathbf{v}_a = u^1\partial_{u^1} - u^2\partial_{u^2}, \quad \mathbf{v}_b = u^2\partial_{u^1}, \quad \mathbf{v}_c = u^1\partial_{u^2},$$

and the action is

$$g : \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix} = \begin{pmatrix} \widetilde{u}^1_0 \\ \widetilde{u}^2_0 \end{pmatrix}, \quad ad - bc = 1.$$

A Lagrangian has an SL(2) symmetry if it satisfies the symmetry condition,

$$\mathbf{v}_r\left(\mathrm{L}\right) = 0 \quad \text{when} \quad \mathrm{L} = 0$$

for $r = a$, b and c. In this example we assume we have a general Lagrangian that satisfies this condition.

The normalization used in [25] is

$$g \cdot u^1_0 = 1, \quad g \cdot u^1_1 = 0, \quad g \cdot u^2_0 = 0,$$

which gives the values of the parameters on the frame as

$$a = \frac{u^2_1}{u^1_0 u^2_1 - u^1_1 u^2_0}, \quad b = -\frac{u^1_1}{u^1_0 u^2_1 - u^1_1 u^2_0}, \quad c = -u^2_0, \quad d = u^1_0.$$

Also, the generating invariants are

$$\kappa^1 = \frac{\iota\left(u_2^2\right)}{\mathrm{S}\left(\iota\left(u_1^2\right)\right)} = \frac{u_0^1 u_2^2 - u_2^1 u_0^2}{u_1^1 u_2^2 - u_2^1 u_1^2}, \qquad \kappa^2 = \iota\left(u_1^2\right) = u_0^1 u_1^2 - u_1^1 u_0^2.$$

Any Lagrangian L with an SL(2) symmetry can be written in the invariant form $L^{\boldsymbol{\kappa}} = L^{\boldsymbol{\kappa}}\left(\left[\kappa^1\right], \left[\kappa^2\right]\right)$. To use (2.60), we need

$$\begin{aligned}
&\iota\left(\frac{\partial \kappa^1}{\partial u_0^1}\right) = \kappa^1, \quad \iota\left(\frac{\partial \kappa^1}{\partial u_1^1}\right) = -\left(\kappa^1\right)^2, \quad \iota\left(\frac{\partial \kappa^1}{\partial u_2^1}\right) = \frac{\kappa^1 \kappa^2}{\kappa_1^2},\\
&\iota\left(\frac{\partial \kappa^1}{\partial u_0^2}\right) = \frac{1}{\kappa^2}, \quad \iota\left(\frac{\partial \kappa^1}{\partial u_1^2}\right) = -\frac{\kappa^1}{\kappa^2}, \quad \iota\left(\frac{\partial \kappa^1}{\partial u_2^2}\right) = \frac{1}{\kappa_1^2},\\
&\iota\left(\frac{\partial \kappa^2}{\partial u_0^1}\right) = \kappa^2, \quad \iota\left(\frac{\partial \kappa^2}{\partial u_1^1}\right) = 0, \quad \iota\left(\frac{\partial \kappa^2}{\partial u_0^2}\right) = 0, \quad \iota\left(\frac{\partial \kappa^2}{\partial u_1^2}\right) = 1.
\end{aligned}$$

Furthermore,

$$\theta_0 = \begin{pmatrix} u_0^1 & \frac{u_1^1}{u_0^1 u_1^2 - u_1^1 u_0^2} \\ u_0^2 & \frac{u_1^2}{u_0^1 u_1^2 - u_1^1 u_0^2} \end{pmatrix},$$

which can be shifted to find the matrices θ_j for all j. The formula for the invariantized Euler–Lagrange equations requires the values of the invariantized (2.31) components of each θ_j for $j = 0, 1, 2$:

$$\begin{aligned}
\iota\left(\theta_0\right) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\\
\iota\left(\theta_1\right) &= \begin{pmatrix} 0 & -\frac{1}{\kappa^2} \\ \kappa^2 & \kappa^1 \end{pmatrix},\\
\iota\left(\theta_2\right) &= \begin{pmatrix} -\frac{\kappa_1^2}{\kappa^2} & -\frac{\kappa_1^1}{\kappa^2} \\ \kappa^1 \kappa_1^2 & \frac{\kappa_1^1 \kappa^1 \kappa_1^2 - \kappa^2}{\kappa_1^2} \end{pmatrix}.
\end{aligned}$$

The components of the inverse matrices together with the invariantized derivatives of the generating invariants give the values of the linear difference operators of the differential-difference syzygies by using (2.60):

$$\begin{aligned}
\mathcal{H}^1_1 &= \sum_j \iota\left(\frac{\partial \kappa^1}{\partial u^1_j}\right)\iota\left(\left(\theta_j\right)^1_1\right)\mathrm{S}_j + \iota\left(\frac{\partial \kappa^1}{\partial u^2_j}\right)\iota\left(\left(\theta_j\right)^2_1\right)\mathrm{S}_j \\
&= \kappa^1\left(\mathrm{id}-\mathrm{S}\right), \\
\mathcal{H}^1_2 &= \sum_j \iota\left(\frac{\partial \kappa^1}{\partial u^1_j}\right)\iota\left(\left(\theta_j\right)^1_2\right)\mathrm{S}_j + \iota\left(\frac{\partial \kappa^1}{\partial u^2_j}\right)\iota\left(\left(\theta_j\right)^2_2\right)\mathrm{S}_j \\
&= \frac{1}{\kappa^2}\mathrm{id} - \frac{\kappa^2}{\left(\kappa^2_1\right)^2}\mathrm{S}_2. \\
\mathcal{H}^2_1 &= \sum_j \iota\left(\frac{\partial \kappa^2}{\partial u^1_j}\right)\iota\left(\left(\theta_j\right)^1_1\right)\mathrm{S}_j + \iota\left(\frac{\partial \kappa^2}{\partial u^2_j}\right)\iota\left(\left(\theta_j\right)^2_1\right)\mathrm{S}_j \\
&= \kappa^2\left(\mathrm{S}+\mathrm{id}\right), \\
\mathcal{H}^2_2 &= \sum_j \iota\left(\frac{\partial \kappa^2}{\partial u^1_j}\right)\iota\left(\left(\theta_j\right)^1_2\right)\mathrm{S}_j + \iota\left(\frac{\partial \kappa^2}{\partial u^2_j}\right)\iota\left(\left(\theta_j\right)^2_2\right)\mathrm{S}_j \\
&= \kappa^1\mathrm{S}.
\end{aligned}$$

The adjoint of these can be found by using (2.56). Consequently, by (2.63), the invariantized Euler–Lagrange equations are

$$\begin{aligned}
\left(\mathrm{id}-\mathrm{S}_{-1}\right)\kappa^1\mathrm{E}_{\kappa^1}\left(L^{\boldsymbol{\kappa}}\right) + \left(\mathrm{id}+\mathrm{S}_{-1}\right)\kappa^2\mathrm{E}_{\kappa^2}\left(L^{\boldsymbol{\kappa}}\right) &= 0, \\
-\mathrm{S}_{-2}\left(\frac{\kappa^2}{\left(\kappa^2_1\right)^2}\mathrm{E}_{\kappa^1}\left(L^{\boldsymbol{\kappa}}\right)\right) + \frac{1}{\kappa^2}\mathrm{E}_{\kappa^1}\left(L^{\boldsymbol{\kappa}}\right) + \mathrm{S}_{-1}\left(\kappa^1\mathrm{E}_{\kappa^2}\left(L^{\boldsymbol{\kappa}}\right)\right) &= 0.
\end{aligned} \tag{2.77}$$

When comparing (2.77) with Equation 2.11 found in [25] one will see the results are the same. However, this is a simpler method of obtaining the result found in [25].

Remark 2.7.1. The matrix $\iota\left(\theta_1\right)$, in this example, is the inverse of the Maurer–Cartan matrix $K_0 = \rho_1\left([\mathbf{u}]\right)\rho_0\left([\mathbf{u}]\right)^{-1}$ given in the paper [25] on OΔEs. This occurs when the group action on the dependent variables is of the linear homogeneous form

$$g \cdot u^\alpha_{\mathbf{0}} = a^\alpha_1 u^1_{\mathbf{0}} + \cdots + a^\alpha_q u^q_{\mathbf{0}}, \quad \alpha = 1, \ldots, q.$$

When this happens, $\iota\left(\theta_{\mathbf{J}}\right) = \left(\rho_{\mathbf{J}}\left([\mathbf{u}]\right)\rho_{\mathbf{0}}\left([\mathbf{u}]\right)^{-1}\right)^{-1}$ in general, which can be written as shifts of concatenating Maurer–Cartan matrices $K_{(i)}$.

2.8 Examples of different group actions

The action can have a significant impact on the difficulty of calculations in the moving frame method for finding the invariant Euler–Lagrange equations. To illustrate this consider the same Lagrangian (2.17) with different Lie group actions. In this section we use the additional variable t method, used throughout the running example, to find the invariant Euler–Lagrange equations. Alternatively, one could use the formulas for the invariant Euler–Lagrange equations found in Section 2.6.

2.8.1 One-parameter group of translations

Example 2.8.1. We start with the most simple example of the moving frame method, that is, a single translation, $\mathbf{v}_3$ in the infinitesimal generators (2.18). The action of this translation on the original variables is

$$g : u_{i,j} \mapsto \widetilde{u}_{i,j} = u_{i,j} + a. \tag{2.78}$$

Now using the normalization equation (2.16)

$$g \cdot u_{0,0} = 0$$

the value of the parameter on the frame is

$$a = -u_{0,0}.$$

Consequently, the invariantization (2.31) of $u_{i,j}$ is

$$\iota\left(u_{i,j}\right) = u_{i,j} - u_{0,0},$$

and therefore, let the two generating invariants be

$$\kappa = \iota\left(u_{1,1}\right) = u_{1,1} - u_{0,0}, \quad \lambda = \iota\left(u_{2,0}\right) = u_{2,0} - u_{0,0}.$$

Now to write the Lagrangian (2.17) in terms of the generating invariants we need $\iota(u_{1,-1})$ in terms of shifts of λ and κ. This is achieved by looking at

$$\begin{aligned}\kappa_{1,-1} &= u_{2,0} - u_{1,-1}\\ &= \lambda - \iota(u_{1,-1}),\end{aligned}$$

which gives

$$\iota(u_{1,-1}) = \lambda - \kappa_{1,-1}.$$

Now using the replacement rule (Theorem 2.4.9) on the Lagrangian (2.17) gives

$$L^{\kappa} = \frac{1}{2}\ln\left|\frac{(\lambda-\kappa)(\lambda-\kappa_{1,-1})}{(\kappa_{1,-1})\kappa}\right|.$$

Then the Euler operators (2.44) of L^{κ} with respect to the generating invariants are

$$\begin{aligned}\mathrm{E}_{\kappa}(L^{\kappa}) &= \frac{(\lambda-\lambda_{-1,1})\kappa - 2\lambda(\lambda_{-1,1})}{2\kappa(\kappa-\lambda)(\kappa-\lambda_{-1,1})},\\ \mathrm{E}_{\lambda}(L^{\kappa}) &= \frac{-2\lambda+\kappa_{1,-1}+\kappa}{2(\kappa-\lambda)(\lambda-\kappa_{1,-1})}.\end{aligned}$$

Next to find the differential-difference syzygies (2.36) extend the action (2.78) to the first-order jet space

$$g\cdot\frac{\mathrm{d}u_{i,j}(t)}{\mathrm{d}t} = \frac{\mathrm{d}u_{i,j}(t)}{\mathrm{d}t}.$$

Therefore, the Lie group action is trivial on the derivative $u'_{i,j}$. Consequently, take $\sigma = u'_{0,0}$ to be the first-order generating invariant then each $u'_{i,j}$ can be written as a direct shift of σ, or more precisely $u'_{i,j} = \mathrm{S}_1^i\mathrm{S}_2^j\sigma$. As a result, $\iota\left(u'_{i,j}\right) = \mathrm{S}_1^i\mathrm{S}_2^j\sigma$. Now the first derivative of the generating invariants with respect to the additional variable t are

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}t} &= u'_{1,1} - u'_{0,0} = \iota\left(u'_{1,1}\right) - \iota\left(u'_{0,0}\right) = (\mathrm{S}_1\mathrm{S}_2 - \mathrm{id})\,\sigma = \mathcal{H}_{\kappa}\sigma,\\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= u'_{2,0} - u'_{0,0} = \iota\left(u'_{2,0}\right) - \iota\left(u'_{0,0}\right) = \left(\mathrm{S}_1^2 - \mathrm{id}\right)\sigma = \mathcal{H}_{\lambda}\sigma.\end{aligned}$$

Consequently, the two linear difference operators (2.60) are

$$\mathcal{H}_\kappa = \mathrm{S}_1\mathrm{S}_2 - \mathrm{id}, \quad \mathcal{H}_\lambda = \mathrm{S}_1^2 - \mathrm{id},$$

and their adjoint operators (2.56) are

$$\mathcal{H}_\kappa^\dagger = \mathrm{S}_1^{-1}\mathrm{S}_2^{-1} - \mathrm{id}, \quad \mathcal{H}_\lambda^\dagger = \mathrm{S}_1^{-2} - \mathrm{id}.$$

Then using the formula in Proposition 2.5.2, the invariantized Euler–Lagrange equation is

$$\begin{aligned}\mathcal{H}_\lambda^\dagger \mathrm{E}_\lambda\left(L^{\boldsymbol{\kappa}}\right) + \mathcal{H}_\kappa^\dagger \mathrm{E}_\kappa\left(L^{\boldsymbol{\kappa}}\right) = {} & \frac{-2\lambda_{-2,0} + \kappa_{-1,-1} + \kappa_{-2,0}}{2\left(\kappa_{-2,0} - \lambda_{-2,0}\right)\left(\lambda_{-2,0} - \kappa_{-1,-1}\right)} \\ & - \frac{-2\lambda + \kappa_{1,-1} + \kappa}{2\left(\kappa - \lambda\right)\left(\lambda - \kappa_{1,-1}\right)} - \frac{\left(\lambda - \lambda_{-1,1}\right)\kappa - 2\lambda\left(\lambda_{-1,1}\right)}{2\kappa\left(\kappa - \lambda\right)\left(\kappa - \lambda_{-1,1}\right)} \\ & + \frac{\left(\lambda_{-1,-1} + \lambda_{-2,0}\right)\kappa_{-1,-1} - 2\lambda_{-1,-1}\left(\lambda_{-2,0}\right)}{2\kappa_{-1,-1}\left(\kappa_{-1,-1} - \lambda_{-1,-1}\right)\left(\kappa_{-1,-1} - \lambda_{-2,0}\right)}.\end{aligned}$$

2.8.2 Two-parameter group of translations

Example 2.8.2. Another easy example of the moving frame method involves looking at the action of the two translations. The infinitesimal generators (2.18) for these two translations are $\mathbf{v}_3$ and $\mathbf{v}_4$. The action of the translations on the original variables is

$$g : u_{i,j} \mapsto \widetilde{u}_{i,j} = u_{i,j} + a + b(-1)^{m+n+i+j}. \tag{2.79}$$

Now using the normalization equations (2.16)

$$g \cdot u_{0,0} = 0, \quad g \cdot u_{1,0} = 0,$$

the values of the parameters on the frame are

$$a = \frac{-\left(u_{0,0} + u_{1,0}\right)}{2}, \qquad b = \frac{(-1)^{n+m}\left(u_{1,0} - u_{0,0}\right)}{2}.$$

Consequently, there are two different invariantizations (2.31) of $u_{i,j}$ depending on the parity of $i+j$. For $i+j$ even the invariantization is

$$\iota\left(u_{i,j}\right) = u_{i,j} - u_{0,0},$$

and for $i+j$ odd the invariantization is

$$\iota\left(u_{i,j}\right) = u_{i,j} - u_{1,0}.$$

Now we concentrate on writing the Lagrangian (2.17) in terms of invariants of this new Lie group action. Take the two generating invariants to be

$$\lambda = \iota\left(u_{2,0}\right) = u_{2,0} - u_{0,0}, \quad \kappa = \iota\left(u_{0,1}\right) = u_{0,1} - u_{1,0}.$$

Unlike the running example, not all components of the invariantized Lagrangian have been worked out in terms of these generating invariants and their shifts. Here we still require $\iota\left(u_{1,-1}\right)$ and $\iota\left(u_{1,1}\right)$ in terms of the generating invariants and their shifts. To find $\iota\left(u_{1,1}\right)$ look at the shifted invariant

$$\kappa_{1,0} = u_{1,1} - u_{2.0} = \iota\left(u_{1,1}\right) - \lambda,$$

which after rearranging gives $\iota\left(u_{1,1}\right) = \kappa_{1,0} + \lambda$ and in a similar manner the other invariant is $\iota\left(u_{1,-1}\right) = -\kappa_{0,-1}$. Using these identities along with $\iota\left(u_{0,0}\right) = 0$, from the normalization, and $\iota\left(u_{2,0}\right) = \lambda$, by definition, the Lagrangian (2.17) is written in terms of these invariants as

$$L^{\boldsymbol{\kappa}} = \frac{1}{2}\ln\left|\frac{\left(\kappa_{1,0}\right)\left(\kappa_{0,-1}\right)}{\left(\lambda+\kappa_{0,-1}\right)\left(\kappa_{1,0}+\lambda\right)}\right|.$$

Then the Euler operators (2.44) of $L^{\boldsymbol{\kappa}}$ with respect to κ and λ are

$$\begin{aligned}
\mathrm{E}_{\kappa}\left(L^{\boldsymbol{\kappa}}\right) &= \frac{\kappa\left(\lambda_{-1,0}+\lambda_{0,1}\right)+2\left(\lambda_{-1,0}\right)\left(\lambda_{0,1}\right)}{2\kappa\left(\kappa+\lambda_{-1,0}\right)\left(\lambda_{0,1}+\kappa\right)}, \\
\mathrm{E}_{\lambda}\left(L^{\boldsymbol{\kappa}}\right) &= \frac{-\kappa_{1,0}-2\lambda-\kappa_{0,-1}}{2\left(\lambda+\kappa_{0,-1}\right)\left(\kappa_{1,0}+\lambda\right)}.
\end{aligned}$$

Next to find the differential-difference syzygies (2.36) extend the action (2.79) to the first-order jet space, which gives

$$g \cdot \frac{\mathrm{d}u_{i,j}\left(t\right)}{\mathrm{d}t} = \frac{\mathrm{d}u_{i,j}\left(t\right)}{\mathrm{d}t}.$$

Similar to the one-parameter group of translations example (Example 2.8.1), the Lie group action is trivial on the derivative $u'_{i,j}$. Consequently, with $\sigma = u'_{0,0}$, each $u'_{i,j}$ can be written as a direct shift of σ, or $u'_{i,j} = \mathrm{S}_1^i \mathrm{S}_2^j \sigma$. As a result, $\iota\left(u'_{i,j}\right) = \mathrm{S}_1^i \mathrm{S}_2^j \sigma$. Now the first derivative of the generating invariants with respect to the additional variable t are

$$\begin{aligned}
\frac{\mathrm{d}\kappa}{\mathrm{d}t} &= u'_{0,1} - u'_{1,0} = \iota\left(u'_{0,1}\right) - \iota\left(u'_{1,0}\right) = \left(\mathrm{S}_2 - \mathrm{S}_1\right)\sigma = \mathcal{H}_\kappa \sigma, \\
\frac{\mathrm{d}\lambda}{\mathrm{d}t} &= u'_{2,0} - u'_{0,0} = \iota\left(u'_{2,0}\right) - \iota\left(u'_{0,0}\right) = \left(\mathrm{S}_1^2 - \mathrm{id}\right)\sigma = \mathcal{H}_\lambda \sigma.
\end{aligned}$$

Consequently, the two linear difference operators (2.60) are

$$\mathcal{H}_\kappa = \mathrm{S}_2 - \mathrm{S}_1, \quad \mathcal{H}_\lambda = \mathrm{S}_1^2 - \mathrm{id},$$

and their adjoint operators (2.56) are

$$\mathcal{H}_\kappa^\dagger = \mathrm{S}_2^{-1} - \mathrm{S}_1^{-1}, \quad \mathcal{H}_\lambda^\dagger = \mathrm{S}_1^{-2} - \mathrm{id}.$$

Then using the formula in Proposition 2.5.2, the invariantized Euler–Lagrange equation is

$$\begin{aligned}
\mathcal{H}_\lambda^\dagger \mathrm{E}_\lambda\left(L^\kappa\right) + \mathcal{H}_\kappa^\dagger \mathrm{E}_\kappa\left(L^\kappa\right) &= \frac{\kappa_{1,0} + 2\lambda + \kappa_{0,-1}}{2\left(\lambda + \kappa_{0,-1}\right)\left(\kappa_{1,0} + \lambda\right)} \\
&- \frac{\kappa_{-1,0} + 2\lambda_{-2,0} + \kappa_{-2,-1}}{2\left(\lambda_{-2,0} + \kappa_{-2,-1}\right)\left(\kappa_{-1,0} + \lambda_{-2,0}\right)} + \frac{\kappa_{0,-1}\left(\lambda_{-1,-1} + \lambda\right) + 2\left(\lambda_{-1,-1}\right)\left(\lambda\right)}{2\kappa_{0,-1}\left(\kappa_{0,-1} + \lambda_{-1,-1}\right)\left(\lambda + \kappa_{0,-1}\right)} \\
&- \frac{\kappa_{-1,0}\left(\lambda_{-2,0} + \lambda_{-1,1}\right) + 2\left(\lambda_{-2,0}\right)\left(\lambda_{-1,1}\right)}{2\kappa_{-1,0}\left(\kappa_{-1,0} + \lambda_{-2,0}\right)\left(\lambda_{-1,1} + \kappa_{-1,0}\right)} \\
&= 0.
\end{aligned}$$

This is slightly more complex when compared to the invariant Euler–Lagrange equation (2.49) attained using the scaling and translation Lie group action.

In the following examples, the calculations of the different invariantized Euler–

Lagrange equations get unwieldy. So, unless otherwise stated we show the results for the linear difference operators (2.60), their adjoints (2.56) and the Euler operators (2.44) of the Lagrangian with respect to the relevant generating invariants. Of course, the formula in Proposition 2.5.2 can then be used to find the invariantized Euler–Lagrange equation for each example.

2.8.3 Dependence on the independent variables

An example of a Lie group that depends on the independent variables has already been given in Example 2.8.2. Nevertheless, it is appropriate to show a more substantial example which uses the same Lagrangian (2.17).

Example 2.8.3. Consider the 4-parameter Lie group that has the infinitesimal generators $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ from the possible infinitesimal generators (2.18). The action of the group on $u_{i,j}$ can be written as

$$g : u_{i,j} \mapsto \widetilde{u}_{i,j} = \exp\left(a + b\,(-1)^{m+n+i+j}\right) u_{i,j} + c + d\,(-1)^{m+n+i+j}. \qquad (2.80)$$

However, a more practical representation of the action is

$$\widetilde{u}_{i,j} = \exp(a)\left(\cosh(b) + (-1)^{m+n+i+j}\sinh(b)\right) u_{i,j} + c + d\,(-1)^{m+n+i+j}.$$

One possible choice of normalization equations (2.16) for this group action is

$$g \cdot u_{0,0} = 0, \quad g \cdot u_{1,1} = 1, \quad g \cdot u_{1,0} = 1, \quad g \cdot u_{0,1} = 0,$$

where the value of the parameters, for this normalization, are

$$\begin{aligned}
\exp(a)\cosh(b) &= \frac{1}{u_{1,1} - u_{0,0}} + \frac{1}{u_{1,0} - u_{0,1}},\\
\exp(a)\sinh(b) &= \frac{(-1)^{m+n}}{2}\left(\frac{1}{u_{1,1} - u_{0,0}} - \frac{1}{u_{1,0} - u_{0,1}}\right),\\
c &= -\frac{1}{2}\left(\frac{u_{0,0}}{u_{1,1} - u_{0,0}} + \frac{u_{0,1}}{u_{1,0} - u_{0,1}}\right),\\
d &= -\frac{(-1)^{m+n}}{2}\left(\frac{u_{0,0}}{u_{1,1} - u_{0,0}} - \frac{u_{0,1}}{u_{1,0} - u_{0,1}}\right).
\end{aligned}$$

This gives a similar split in the invariantization (2.31) of $u_{i,j}$ as the 2-parameter

Lie group of translations has in Example 2.8.2. This is because the 4-parameter Lie group depends on the parity of $i+j$ again. The invariantization of $u_{i,j}$ is

$$\iota\left(u_{i,j}\right)=\frac{u_{i,j}-u_{0,0}}{u_{1,1}-u_{0,0}}, \quad \text{for } i+j \text{ even,}$$
$$\iota\left(u_{i,j}\right)=\frac{u_{i,j}-u_{0,1}}{u_{1,0}-u_{0,1}}, \quad \text{for } i+j \text{ odd.}$$

In this example take $\kappa=\iota\left(u_{0,-1}\right)$ and $\lambda=\iota\left(u_{2,0}\right)$ as the generating invariants. It follows that $\iota\left(u_{1,-1}\right)=(\lambda-1)\,\kappa_{1,0}+1$ is the other relation needed to write the invariantized Lagrangian in terms of the generating invariants and their shifts:

$$L^{\boldsymbol{\kappa}}=\frac{1}{2}\ln\left|\frac{\kappa_{1,0}\,(1-\lambda)-1}{\kappa_{1,0}-1}\right|.$$

Accordingly, the Euler operators (2.44) of $L^{\boldsymbol{\kappa}}$ with respect to κ and λ are

$$\mathrm{E}_{\kappa}\left(L^{\boldsymbol{\kappa}}\right)=-\frac{\lambda_{-1,0}}{2\,(\kappa-1)\left(\left(\lambda_{-1,0}-1\right)\kappa+1\right)},$$
$$\mathrm{E}_{\lambda}\left(L^{\boldsymbol{\kappa}}\right)=\frac{\kappa_{1,0}}{2+(2\lambda-2)\,\kappa_{1,0}}.$$

We now find the differential-difference syzygies. Firstly, the induced action is

$$g\cdot u'_{i,j}=\exp\left(a+b\,(-1)^{m+n+i+j}\right)u'_{i,j}, \tag{2.81}$$

and applying the parameters on the frame to this gives the first-order invariants

$$\iota\left(u'_{i,j}\right)=\frac{1}{2}\left(\left(1+(-1)^{i+j}\right)\frac{1}{u_{1,1}-u_{0,0}}+\left(1-(-1)^{i+j}\right)\frac{1}{u_{1,0}-u_{0,1}}\right)u'_{i,j},$$

for each i and j. Consequently, there are two different invariantizations (2.31) of $u'_{i,j}$, depending on the parity of $i+j$, specifically

$$\iota\left(u'_{i,j}\right)=\frac{u'_{i,j}}{u_{1,1}-u_{0,0}}, \quad \text{for } i+j \text{ even,}$$
$$\iota\left(u'_{i,j}\right)=\frac{u'_{i,j}}{u_{1,0}-u_{0,1}}, \quad \text{for } i+j \text{ odd.}$$

The first-order generating invariant is then

$$\sigma=\frac{u_{0,0}}{u_{1,1}-u_{0,0}}.$$

Looking at the derivative of the generating invariants with respect to t,

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}t} &= \frac{u'_{0,-1} - u'_{0,1}}{u_{1,0} - u_{0,1}} - \frac{\left(u_{0,-1} - u_{0,1}\right)\left(u'_{1,0} - u'_{0,1}\right)}{\left(u_{1,0} - u_{0,1}\right)^2}\,, \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \frac{u'_{2,0} - u'_{0,0}}{u_{1,1} - u_{0,0}} - \frac{\left(u_{2,0} - u_{0,0}\right)\left(u'_{1,1} - u'_{0,0}\right)}{\left(u_{1,1} - u_{0,0}\right)^2}\,,\end{aligned} \tag{2.82}$$

and using the replacement rule (Theorem 2.4.9) to invariantize (2.31) the right-hand side of these equations gives

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}t} &= \iota\left(u'_{0,-1}\right) - \kappa\,\iota\left(u'_{1,0}\right) + (\kappa - 1)\,\iota\left(u'_{0,1}\right), \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \iota\left(u'_{2,0}\right) - \lambda\,\iota\left(u'_{1,1}\right) + (\lambda - 1)\,\iota\left(u'_{0,0}\right).\end{aligned} \tag{2.83}$$

A small complication arises here when the values $\iota\left(u'_{i,j}\right)$ need to be found in terms of the generating invariants and $\mathrm{S}_1^i\mathrm{S}_2^j\sigma$. To find even $\iota\left(u'_{i,j}\right)$ is fairly straightforward and we continue as before by shifting σ in the original variables by $\mathrm{S}_1^i\mathrm{S}_2^j$ and then using the replacement rule. To find odd $\iota\left(u'_{i,j}\right)$ one can simplify the calculations by shifting $\iota\left(u'_{0,1}\right)$ in the original variables by $\mathrm{S}_1^i\mathrm{S}_2^{j-1}$ and then using the replacement rule and the identity

$$\iota\left(u'_{0,1}\right) = \frac{\mathrm{S}_2\sigma}{1 - \kappa_{1,1}\left(1 - \lambda_{0,1}\right)}\,.$$

Once the values for all $\iota\left(u'_{i,j}\right)$ are found in terms of the generating invariants and $\mathrm{S}_1^i\mathrm{S}_2^j\sigma$, they can be substituted into (2.83) giving the differential-difference syzygies (2.36)

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}t} &= \frac{(1-\kappa)\left(1 + \left(\lambda_{0,-1} - 1\right)\kappa_{1,-1}\right)}{1 - \kappa_{1,-1}\left(1 - \lambda_{0,-1}\right)}\mathrm{S}_2^{-1}\sigma - \frac{\kappa\left(\lambda_{0,-1} - 1\right)(1-\kappa)}{1 - \left(\kappa_{2,0}\right)\left(1 - \lambda_{1,0}\right)}\mathrm{S}_1\sigma \\ &\quad + \frac{\kappa - 1}{1 - \left(\kappa_{1,1}\right)\left(1 - \lambda_{0,1}\right)}\mathrm{S}_2\sigma, \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \frac{-\left((\lambda - 1)\kappa_{0,1} - \lambda - \lambda_{1,1} + 1\right)\left(\lambda_{0,2} - 1\right)\left(\kappa_{1,2}\right) - \left(\lambda_{0,2}\right)\left(\lambda_{1,1}\right)}{\left(1 + \left(\lambda_{0,2} - 1\right)\kappa_{1,2}\right)\left(\kappa_{0,1} - 1\right)}\mathrm{S}_1^2\sigma \\ &\quad + \frac{\kappa_{0,1}(1-\lambda) + \lambda + \lambda_{1,1} - 1}{\left(1 + \left(\lambda_{0,2} - 1\right)\kappa_{1,2}\right)\left(\kappa_{0,1} - 1\right)}\mathrm{S}_1^2\sigma - \frac{\lambda\left(\kappa_{1,2} - 1\right)\left(\lambda_{0,2} - 1\right)}{\left(\kappa_{0,1} - 1\right)\left(1 + \kappa_{1,2}\left(\lambda_{0,2} - 1\right)\right)}\mathrm{S}_1\mathrm{S}_2\sigma \\ &\quad + (\lambda - 1)\,\sigma.\end{aligned}$$

Therefore, the linear difference operators (2.60) are

$$
\begin{aligned}
\mathcal{H}_\kappa =& \frac{(1-\kappa)\left(1+(\lambda_{0,-1}-1)\,\kappa_{1,-1}\right)}{1-\kappa_{1,-1}\left(1-\lambda_{0,-1}\right)}\mathrm{S}_2^{-1} - \frac{\kappa\left(\lambda_{0,-1}-1\right)(1-\kappa)}{1-(\kappa_{2,0})\left(1-\lambda_{1,0}\right)}\mathrm{S}_1 \\
&+ \frac{\kappa-1}{1-(\kappa_{1,1})\left(1-\lambda_{0,1}\right)}\mathrm{S}_2, \\
\mathcal{H}_\lambda =& \frac{-\left((\lambda-1)\,\kappa_{0,1}-\lambda-\lambda_{1,1}+1\right)(\lambda_{0,2}-1)\,(\kappa_{1,2})-(\lambda_{0,2})\,(\lambda_{1,1})}{\left(1+(\lambda_{0,2}-1)\,\kappa_{1,2}\right)(\kappa_{0,1}-1)}\mathrm{S}_1^2 \\
&+ \frac{\kappa_{0,1}\,(1-\lambda)+\lambda+\lambda_{1,1}-1}{\left(1+(\lambda_{0,2}-1)\,\kappa_{1,2}\right)(\kappa_{0,1}-1)}\mathrm{S}_1^2 - \frac{\lambda\,(\kappa_{1,2}-1)\,(\lambda_{0,2}-1)}{(\kappa_{0,1}-1)\left(1+\kappa_{1,2}\,(\lambda_{0,2}-1)\right)}\mathrm{S}_1\mathrm{S}_2 \\
&+ (\lambda-1)\,\mathrm{id},
\end{aligned}
$$

and their adjoints (2.56)

$$
\begin{aligned}
\mathcal{H}_\kappa^\dagger =& \frac{(1-\kappa_{0,1})\left(1+(\lambda-1)\,\kappa_{1,0}\right)}{1-\kappa_{1,0}\,(1-\lambda)}\mathrm{S}_2 - \frac{\kappa_{-1,0}\left(\lambda_{-1,-1}-1\right)(1-\kappa_{-1,0})}{1-(\kappa_{1,0})\,(1-\lambda)}\mathrm{S}_1^{-1} \\
&+ \frac{\kappa_{0,-1}-1}{1-(\kappa_{1,0})\,(1-\lambda)}\mathrm{S}_2^{-1}, \\
\mathcal{H}_\lambda^\dagger =& \frac{-\left((\lambda_{-2,0}-1)\,\kappa_{-2,1}-\lambda_{-2,0}-\lambda_{-1,1}+1\right)(\lambda_{-2,2}-1)\,(\kappa_{-1,2})}{\left(1+(\lambda_{-2,2}-1)\,\kappa_{-1,2}\right)(\kappa_{-2,1}-1)}\mathrm{S}_1^{-2} \\
&+ \frac{-(\lambda_{-2,2})\,(\lambda_{-1,1})+\kappa_{-2,1}\,(1-\lambda_{-2,0})+\lambda_{-2,0}+\lambda_{-1,1}-1}{\left(1+(\lambda_{-2,2}-1)\,\kappa_{-1,2}\right)(\kappa_{-2,1}-1)}\mathrm{S}_1^{-2} \\
&- \frac{\lambda_{-1,-1}\,(\kappa_{0,1}-1)\,(\lambda_{-1,1}-1)}{(\kappa_{-1,0}-1)\left(1+\kappa_{0,1}\,(\lambda_{-1,1}-1)\right)}\mathrm{S}_1^{-1}\mathrm{S}_2^{-1} + (\lambda-1)\,\mathrm{id}.
\end{aligned}
$$

The next example will explore a different normalization of the same group action in Example 2.8.3 to show some of the subtle differences this can have on the results. The new choice of normalization shows some computations can be made simpler than those in Example 2.8.3.

Example 2.8.4. To start the new normalization equations (2.16) are

$$
g\cdot u_{0,0}=0, \quad g\cdot u_{1,1}=1, \quad g\cdot u_{1,0}=1, \quad g\cdot u_{0,-1}=0.
$$

The only change from the previous normalization is the substitution of the normalization equation $g\cdot u_{0,-1}=0$ for $g\cdot u_{0,1}=0$. This changes the parameters on the frame to

$$
\begin{aligned}
\exp(a)\cosh(b) &= \frac{1}{u_{1,1}-u_{0,0}} + \frac{1}{u_{1,0}-u_{0,-1}}\,,\\
\exp(a)\sinh(b) &= \frac{(-1)^{m+n}}{2}\left(\frac{1}{u_{1,1}-u_{0,0}} - \frac{1}{u_{1,0}-u_{0,-1}}\right),\\
c &= -\frac{1}{2}\left(\frac{u_{0,0}}{u_{1,1}-u_{0,0}} + \frac{u_{0,-1}}{u_{1,0}-u_{0,-1}}\right),\\
d &= -\frac{(-1)^{m+n}}{2}\left(\frac{u_{0,0}}{u_{1,1}-u_{0,0}} - \frac{u_{0,-1}}{u_{1,0}-u_{0,-1}}\right).
\end{aligned}
$$

Likewise, the invariantization (2.31) of $u_{i,j}$ changes to

$$
\begin{aligned}
\iota(u_{i,j}) &= \frac{u_{i,j}-u_{0,0}}{u_{1,1}-u_{0,0}}\,, \quad \text{for } i+j \text{ even},\\
\iota(u_{i,j}) &= \frac{u_{i,j}-u_{0,-1}}{u_{1,0}-u_{0,-1}}\,, \quad \text{for } i+j \text{ odd}.
\end{aligned}
$$

For this case let the two generating invariants be $\lambda = \iota(u_{2,0})$ and $\kappa = \iota(u_{0,1})$, consequently, the invariant Lagrangian is

$$
L^{\boldsymbol{\kappa}} = \frac{1}{2}\ln|1-(\kappa_{1,0})\,\lambda|,
$$

and so the Euler operators (2.44) of $L^{\boldsymbol{\kappa}}$ with respect to κ and λ are

$$
\begin{aligned}
\mathrm{E}_{\kappa}(L^{\boldsymbol{\kappa}}) &= \frac{\lambda_{-1,0}}{2(\lambda_{-1,0})\,\kappa-2}\,,\\
\mathrm{E}_{\lambda}(L^{\boldsymbol{\kappa}}) &= \frac{\kappa_{1,0}}{2(\kappa_{1,0})\,\lambda-2}\,.
\end{aligned}
$$

Now to find the differential-difference syzygies again we apply the values of the parameters on the frame to the induced action (2.81), which gives the invariant first-order derivatives

$$
\iota\left(u'_{i,j}\right) = \frac{1}{2}\left(\left(1+(-1)^{i+j}\right)\frac{1}{u_{1,1}-u_{0,0}} + \left(1-(-1)^{i+j}\right)\frac{1}{u_{1,0}-u_{0,-1}}\right)u'_{i,j},
$$

for each i, j. As in Example 2.8.3 there are two different invariantizations (2.31) of $u'_{i,j}$, depending on the parity of $i+j$, specifically

$$\iota\left(u'_{i,j}\right) = \frac{u'_{i,j}}{u_{1,1}-u_{0,0}}\,, \quad \text{for } i+j \text{ even,}$$
$$\iota\left(u'_{i,j}\right) = \frac{u'_{i,j}}{u_{1,0}-u_{0,-1}}\,, \quad \text{for } i+j \text{ odd.}$$

The most advantageous part of this current normalization is its ease of moving from the odd to even $\iota\left(u'_{i,j}\right)$. For example, with $\sigma = \iota\left(u_{0,0}\right)$ it is clear to see that

$$\mathrm{S}_2^{-1}\sigma = \frac{u'_{0,-1}}{u_{1,0}-u_{0,-1}} = \iota\left(u'_{0,-1}\right).$$

Here

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}t} &= \frac{u'_{0,1}-u'_{0,-1}}{u_{1,0}-u_{0,-1}} - \frac{\left(u_{0,1}-u_{0,-1}\right)\left(u'_{1,0}-u'_{0,-1}\right)}{\left(u_{1,0}-u_{0,-1}\right)^2}\,,\\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \frac{u'_{2,0}-u'_{0,0}}{u_{1,1}-u_{0,0}} - \frac{\left(u_{2,0}-u_{0,0}\right)\left(u'_{1,1}-u'_{0,0}\right)}{\left(u_{1,1}-u_{0,0}\right)^2}\,,\end{aligned} \tag{2.84}$$

so

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}t} &= \iota\left(u'_{0,1}\right) - \kappa\,\iota\left(u'_{1,0}\right) + (\kappa-1)\,\iota\left(u'_{0,-1}\right),\\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \iota\left(u'_{2,0}\right) - \lambda\,\iota\left(u'_{1,1}\right) + (\lambda-1)\,\iota\left(u'_{0,0}\right).\end{aligned} \tag{2.85}$$

After working out the values of relevant $\iota\left(u_{i,j}\right)$, they can be substituted into (2.85), giving the differential-difference syzygies (2.36)

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}t} &= -\frac{\left(\kappa_{1,1}-1\right)(\kappa-1)}{\left(\lambda_{0,1}\right)\left(\kappa_{1,1}\right)-1}\mathrm{S}_2\sigma - \frac{\kappa\left(\lambda_{0,1}-1\right)(\kappa-1)}{\left(\lambda_{0,1}\right)\left(\kappa_{1,1}\right)-1}\mathrm{S}_1\sigma + (\kappa-1)\,\mathrm{S}_2^{-1}\sigma,\\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} &= -\frac{\left(\lambda_{1,1}-1\right)(\lambda-1)}{\left(\kappa_{2,1}\right)\left(\lambda_{1,1}\right)-1}\mathrm{S}_1^2\sigma - \frac{\lambda\left(\kappa_{2,1}-1\right)(\lambda-1)}{\left(\kappa_{2,1}\right)\left(\lambda_{1,1}\right)-1}\mathrm{S}_1\mathrm{S}_2\sigma + (\lambda-1)\,\sigma.\end{aligned}$$

Therefore, the linear difference operators (2.60) are

$$\begin{aligned}\mathcal{H}_\kappa &= -\frac{\left(\kappa_{1,1}-1\right)(\kappa-1)}{\left(\lambda_{0,1}\right)\left(\kappa_{1,1}\right)-1}\mathrm{S}_2 - \frac{\kappa\left(\lambda_{0,1}-1\right)(\kappa-1)}{\left(\lambda_{0,1}\right)\left(\kappa_{1,1}\right)-1}\mathrm{S}_1 + (\kappa-1)\,\mathrm{S}_2^{-1},\\ \mathcal{H}_\lambda &= -\frac{\left(\lambda_{1,1}-1\right)(\lambda-1)}{\left(\kappa_{2,1}\right)\left(\lambda_{1,1}\right)-1}\mathrm{S}_1^2 - \frac{\lambda\left(\kappa_{2,1}-1\right)(\lambda-1)}{\left(\kappa_{2,1}\right)\left(\lambda_{1,1}\right)-1}\mathrm{S}_1\mathrm{S}_2 + (\lambda-1)\,\mathrm{id},\end{aligned}$$

and the adjoint (2.56) of these are

$$\begin{aligned}\mathcal{H}_\kappa^\dagger &= -\frac{(\kappa_{1,0}-1)(\kappa_{0,-1}-1)}{\lambda(\kappa_{1,0})-1}\mathrm{S}_2^{-1} - \frac{\kappa_{-1,0}(\lambda_{-1,1}-1)(\kappa_{-1,0}-1)}{(\lambda_{-1,1})(\kappa_{0,1})-1}\mathrm{S}_1^{-1} \\ &\quad + (\kappa_{0,1}-1)\,\mathrm{S}_2, \\ \mathcal{H}_\lambda^\dagger &= -\frac{(\lambda_{-1,1}-1)(\lambda_{-2,0}-1)}{(\kappa_{0,1})(\lambda_{-1,1})-1}\mathrm{S}_1^{-2} - \frac{\lambda_{-1,-1}(\kappa_{1,0}-1)(\lambda_{-1,-1}-1)}{(\kappa_{1,0})(\lambda)-1}\mathrm{S}_1^{-1}\mathrm{S}_2^{-1} \\ &\quad + (\lambda-1)\,\mathrm{id}.\end{aligned}$$

Remark 2.8.1. An interesting question arises from this example "can we find a better choice of normalization equations and generating invariants than those in Example 2.8.4 to give a less complex invariant Euler–Lagrange equation?" The answer to that question is yes. If we choose two of the normalization equations (2.16) to be those in the running example (2.25), that is,

$$g \cdot u_{0,0} = 0, \quad g \cdot u_{1,1} = 1,$$

with the other two say,

$$g \cdot u_{0,-1} = 0, \quad g \cdot u_{1,0} = 1.$$

Then using the same choice of generating invariants as in the running example, $\kappa = \iota(u_{1,-1})$ and $\lambda = \iota(u_{2,0})$, we get the same invariant Euler–Lagrange equations as in (2.49). Now why is this the case? Well for this choice of normalization the invariants $\iota(u_{i,j})$ with even $i+j$ are the same as those in the running example. Therefore, the generating invariants κ and λ, and the recurrence relations for even $\iota(u_{i,j})$ are the same as those in (2.32). So, the Lagrangian (2.17) reduces to the same as before (2.46) and as a consequence has the same Euler operators (2.44) with respect to the generating invariants (2.47). Additionally, as the relationship between the generating invariants and $\iota(u_{i,j})$ for even $i+j$ remains the same, the adjoint operators also remain the same (2.48) meaning the invariant Euler–Lagrange equations are the same as those in (2.49). Note that the choice of normalization equations for $i+j$ odd does not matter in this example. We could have used

$$g \cdot u_{2,-1} = 2 \quad g \cdot u_{1,2} = 0,$$

and the result would remain the same.

The same situation would happen in Example 2.8.2 if we had used the normalization equation $g \cdot u_{0,0} = 0$ and generating invariants as in Example 2.8.1. In this case Example 2.8.2 would have the same invariant Euler–Lagrange equations as in Example 2.8.1.

Therefore, for Lagrangians like L in (2.17) with symmetries that depend on $(-1)^{m+n}$ it might be worth considering only those symmetries without $(-1)^{m+n}$ and taking the generating invariants to be of the same parity as the Lagrangian. As this seems to find the best representation of the invariant Euler–Lagrange equations for these particular types of group actions.

2.8.4 A semi-simple group action SL(2)

Example 2.8.5. Now consider the semi-simple group action of SL(2). The infinitesimal generators (2.18) of this action are $\mathbf{v}_1$, $\mathbf{v}_3$ and $\mathbf{v}_5$. However, by taking the group action to be

$$g : u_{i,j} \mapsto \widetilde{u}_{i,j} = \frac{au_{i,j} + b}{cu_{i,j} + d}, \quad \text{where} \quad ad - bc = 1,$$

the adapted infinitesimal generators are

$$\mathbf{v}_a = 2\mathbf{v}_1, \quad \mathbf{v}_b = \mathbf{v}_3, \quad \mathbf{v}_c = -\mathbf{v}_5.$$

Let the normalization equations (2.16) be

$$g \cdot u_{1,1} = 1/2, \quad g \cdot u_{0,0} = 0, \quad g \cdot u_{1,-1} = -1/2.$$

For this normalization the value of the parameters are

$$
\begin{aligned}
a &= \sqrt{-\frac{u_{1,1}-u_{1,-1}}{4u_{0,0}^2-4u_{0,0}u_{1,1}-4u_{1,-1}u_{0,0}+4u_{1,1}u_{1,-1}}},\\
b &= -\sqrt{-\frac{u_{1,1}-u_{1,-1}}{4u_{0,0}^2-4u_{0,0}u_{1,1}-4u_{1,-1}u_{0,0}+4u_{1,1}u_{1,-1}}}\,u_{0,0},\\
c &= \frac{2u_{0,0}-u_{1,1}-u_{1,-1}}{2\left(u_{0,0}-u_{1,-1}\right)\left(u_{0,0}-u_{1,1}\right)\sqrt{-\frac{u_{1,1}-u_{1,-1}}{4u_{0,0}^2-4u_{0,0}u_{1,1}-4u_{1,-1}u_{0,0}+4u_{1,1}u_{1,-1}}}},\\
d &= \frac{\left(-u_{1,1}-u_{1,-1}\right)u_{0,0}+2u_{1,1}u_{1,-1}}{2\left(u_{0,0}-u_{1,-1}\right)\left(u_{0,0}-u_{1,1}\right)\sqrt{-\frac{u_{1,1}-u_{1,-1}}{4u_{0,0}^2-4u_{0,0}u_{1,1}-4u_{1,-1}u_{0,0}+4u_{1,1}u_{1,-1}}}},
\end{aligned}
$$

consequently, the invariantization (2.31) of $u_{i,j}$ is

$$
\iota\left(u_{i,j}\right) = -\frac{\left(u_{0,0}-u_{i,j}\right)\left(u_{1,1}-u_{1,-1}\right)}{\left(2u_{0,0}-4u_{1,1}+2u_{i,j}\right)u_{1,-1}+\left(2u_{1,1}-4u_{i,j}\right)u_{0,0}+2u_{i,j}u_{1,1}}.
$$

This invariant can be hard to use, so instead let the generating invariants be

$$
\begin{aligned}
\kappa &= \frac{\left(u_{0,0}-u_{1,1}\right)\left(u_{0,2}-u_{1,-1}\right)}{\left(u_{0,0}-u_{1,-1}\right)\left(u_{0,2}-u_{1,1}\right)} = \frac{1+2\iota\left(u_{0,2}\right)}{1-2\iota\left(u_{0,2}\right)},\\
\lambda &= \frac{\left(u_{0,0}-u_{1,1}\right)\left(u_{2,0}-u_{1,-1}\right)}{\left(u_{0,0}-u_{1,-1}\right)\left(u_{2,0}-u_{1,1}\right)} = \frac{1+2\iota\left(u_{2,0}\right)}{1-2\iota\left(u_{2,0}\right)}.
\end{aligned}
$$

A similar trick for the generating invariants is used in the OΔE paper, [25], for the SL(2) projective action. Accordingly, the invariants $\iota\left(u_{0,2}\right)$ and $\iota\left(u_{2,0}\right)$ can be written in terms of generating invariants as

$$
\iota\left(u_{0,2}\right) = \frac{1}{2}\left(\frac{\kappa-1}{\kappa+1}\right), \qquad \iota\left(u_{2,0}\right) = \frac{1}{2}\left(\frac{\lambda-1}{\lambda+1}\right),
$$

and so the invariantized Lagrangian in terms of the generating invariants is

$$
L^{\boldsymbol{\kappa}} = \frac{1}{2}\ln\left|\frac{1}{\lambda}\right|.
$$

Taking this normalization simplifies the invariant Lagrangian considerably. Consequently, the Euler operators (2.44) of $L^{\boldsymbol{\kappa}}$ with respect to κ and λ are

$$
\mathrm{E}_{\kappa}\left(L^{\boldsymbol{\kappa}}\right) = 0, \qquad \mathrm{E}_{\lambda}\left(L^{\boldsymbol{\kappa}}\right) = -\frac{1}{2\lambda}.
$$

As $\mathrm{E}_\kappa\left(L^{\boldsymbol{\kappa}}\right) = 0$, we only need to look at the derivative $\mathrm{d}\lambda/\mathrm{d}t$, which when invariantized (2.31) is

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -4\lambda\iota\left(u'_{0,0}\right) - (\lambda-1)\,\lambda\,\iota\left(u'_{1,1}\right) + (\lambda-1)\,\iota\left(u'_{1,-1}\right) + (\lambda+1)^2\,\iota\left(u'_{2,0}\right). \quad (2.86)$$

To find the form of the first-order differential invariants use the induced group action on $u'_{i,j}$,

$$\frac{\mathrm{d}\left(g\cdot u_{i,j}\right)}{\mathrm{d}t} = \frac{u'_{i,j}}{\left(cu_{i,j}+d\right)^2},$$

and then substitute the value of the parameters on the frame:

$$\iota\left(u'_{i,j}\right) = -\frac{u'_{i,j}\left(u_{1,1}-u_{1,-1}\right)}{4\left(u_{0,0}-u_{1,-1}\right)\left(u_{0,0}-u_{1,1}\right)}.$$

Here

$$\sigma = \iota\left(u'_{0,0}\right) = -\frac{u'_{0,0}\left(u_{1,1}-u_{1,-1}\right)}{4\left(u_{0,0}-u_{1,-1}\right)\left(u_{0,0}-u_{1,1}\right)},$$

is the generating first-order differential invariant. This leads to the differential-difference syzygy (2.36)

$$\begin{aligned}\frac{\mathrm{d}\lambda}{\mathrm{d}t} &= -4\lambda\sigma + \frac{4\lambda\left(\lambda-1\right)\left(\kappa_{1,1}-\lambda_{0,2}\right)\mathrm{S}_1\mathrm{S}_2\sigma}{\left(\kappa_{1,1}-1\right)\left(\kappa-\lambda\right)} - \frac{4\left(\lambda-1\right)\lambda\mathrm{S}_1\mathrm{S}_2^{-1}\sigma}{\kappa_{1,-1}-1}\\ &+ \frac{4\left(\lambda-1\right)^2\left(\kappa_{1,-1}-1\right)\lambda\,\lambda_{1,1}\left(\kappa_{1,1}-1\right)\left(\kappa-\lambda\right)\mathrm{S}_1^2\sigma}{\tau\left(\left[\kappa\right],\left[\lambda\right]\right)},\end{aligned}$$

where

$$\begin{aligned}\tau\left(\left[\kappa\right],\left[\lambda\right]\right) = {} & \left(\kappa\right)\left(\lambda_{1,1}\right)\left(\kappa_{1,-1}-\lambda_{1,-1}\right)\left(\kappa_{1,1}-1\right)\\ & - \left(\kappa_{1,-1}-1\right)\left(\kappa_{1,1}-\lambda_{0,2}\right)\left(\lambda\right)\\ & + \left[\left(\lambda_{1,-1}-1\right)\kappa_{1,1} + \left(1-\lambda_{0,2}\right)\kappa_{1,-1} + \lambda_{0,2} - \lambda_{1,-1}\right]\left(\lambda_{1,1}\right)\left(\lambda\right).\end{aligned}$$

Thus, the linear difference operator (2.60) is

$$\begin{aligned}\mathcal{H}_\lambda &= -4\lambda\,\mathrm{id} + \frac{4\lambda\left(\lambda-1\right)\left(\kappa_{1,1}-\lambda_{0,2}\right)\mathrm{S}_1\mathrm{S}_2}{\left(\kappa_{1,1}-1\right)\left(\kappa-\lambda\right)} - \frac{4\left(\lambda-1\right)\lambda\mathrm{S}_1\mathrm{S}_2^{-1}}{\kappa_{1,-1}-1}\\ &+ \frac{4\left(\lambda-1\right)^2\left(\kappa_{1,-1}-1\right)\lambda\,\lambda_{1,1}\left(\kappa_{1,1}-1\right)\left(\kappa-\lambda\right)\mathrm{S}_1^2}{\tau\left(\left[\kappa\right],\left[\lambda\right]\right)},\end{aligned}$$

and the corresponding adjoint operator (2.56) is

$$\begin{aligned}\mathcal{H}_{\lambda}^{\dagger} &= -4\lambda\,\mathrm{id} + \frac{4\left(\lambda_{-1,-1}\right)\left(\lambda_{-1,-1}-1\right)\left(\kappa-\lambda_{-1,1}\right)\mathrm{S}_1^{-1}\mathrm{S}_2^{-1}}{\left(\kappa-1\right)\left(\kappa_{-1,-1}-\lambda_{-1,-1}\right)}\\
&- \frac{4\left(\lambda_{-1,1}-1\right)\left(\lambda_{-1,1}\right)\mathrm{S}_1^{-1}\mathrm{S}_2}{\kappa-1}\\
&+ \frac{4\left(\kappa_{-1,-1}-1\right)\left(\lambda_{-2,0}\right)\,\left(\lambda_{-1,1}\right)\left(\kappa_{-1,1}-1\right)\left(\kappa_{-2,0}-\lambda_{-2,0}\right)\mathrm{S}_1^{-2}}{\tau^{\dagger}\left(\left[\kappa\right],\left[\lambda\right]\right)},\end{aligned}$$

with

$$\begin{aligned}\tau^{\dagger}\left(\left[\kappa\right],\left[\lambda\right]\right) &= \left(\kappa_{-2,0}\right)\left(\lambda_{-1,1}\right)\left(\kappa_{-1,-1}-\lambda_{-1,-1}\right)\left(\kappa_{-1,1}-1\right)\\
&- \left(\kappa_{-1,-1}-1\right)\left(\kappa_{-1,1}-\lambda_{-2,2}\right)\left(\lambda_{-2,0}\right)\\
&+ \left[\left(\lambda_{-1,-1}-1\right)\kappa_{-1,1}+\left(1-\lambda_{-2,2}\right)\kappa_{-1,-1}+\lambda_{-2,2}-\lambda_{-1,-1}\right]\left(\lambda_{-1,1}\right)\left(\lambda_{-2,0}\right).\end{aligned}$$

Then as $\mathrm{E}_{\kappa}\left(L^{\boldsymbol{\kappa}}\right)=0$, the invariantized Euler–Lagrange equation is

$$\mathcal{H}_{\lambda}^{\dagger}\mathrm{E}_{\lambda}\left(L^{\boldsymbol{\kappa}}\right)=0.$$

As is shown in the examples the type of Lie subgroup used can have a significant impact on the resulting invariantized Lagrangian and invariantized Euler–Lagrange equations. The calculations involved in the moving frame formulation are also significantly more difficult for some Lie subgroups. For the Lagrangian (2.17), possibly the best Lie subgroup, normalization equations and generating invariants are given in the running example. (Here a 2-parameter Lie group of scalings and translations is used.) The main advantage is this reduces the Lagrangian and Euler–Lagrange equation down to something quite simple. This is something which the other Lie subgroups struggle to do. The more parameters used, (i.e., the larger the Lie subgroup) the further one can reduce the Lagrangian down. For example, the SL(2) action reduces the Lagrangian (2.17) to one generating invariant. However, this comes with the added difficulty of a more complex group action and relation between the invariants resulting in a complex expression for the adjoint linear difference operator.

Using a Lie subgroup with fewer parameters, like in Example 2.8.1, results in nice expressions for the adjoint linear difference operators but does not simplify the Lagrangian significantly resulting in difficult expressions for the Euler operators of the Lagrangian with respect to the generating invariants. Using Lie subgroups which involve the independent variable (and partitioning) comes with its own set of complications. The normalization equations need to be chosen so that the invariantization (2.31) of $u_{i,j}$ on the odd (even) part of the lattice has only variables $u_{i,j}$ from the odd (even) part.

It is currently not yet fully understood why using a particular Lie subgroup will give better reductions of the invariant Euler–Lagrange equations. It appears using the entire Lie group may in fact be the least optimal choice for some examples. As a result this topic deserves further investigation.

Different Lagrangians invariant under a particular Lie group will need a different choice of Lie subgroup, normalization equations and generating invariants. Remark 2.6.6 helps find the best choice of these by allowing one to calculate the adjoint operators (2.56) individually for different Lie subgroups, normalization equations and generating invariants to find the most suitable choice for a particular invariant Lagrangian and its Euler operators. Importantly for PΔEs (with two generating invariants) there exists a syzygy between the two generating invariants meaning if one has a solution to the Euler–Lagrange equation for either generating invariant the solution for the other can be found (if one can solve the resulting PΔE from the syzygy). One can find the solution to the original Euler–Lagrange equation from either generating invariant, (however, for PΔEs this is difficult.) For OΔEs the number of dependent variables is equal to the number of generating invariants so there is no syzygy. Solving the Euler–Lagrange OΔEs is discussed in Section 8 of paper by Mansfield *et al.* [26].

2.9 On infinitesimal and adjoint action

This section extends the ideas in the OΔEs paper by Mansfield *et al.* [26] to PΔEs. For us to state the results for conservation laws, it is essential to introduce the action of infinitesimal generators of a Lie group on a manifold, including the adjoint action of the Lie group. Infinitesimal generators and some of their properties have already been discussed in Section 2.4. Here we start with a more rigorous definition of the infinitesimal generator.

Definition 2.9.1. Let $G \times U \to U$ be a smooth local Lie group action. If $\gamma(t)$ is a path in G with $\gamma(0) = e$, the identity element in G, then

$$\mathbf{v} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t) \cdot \mathbf{u} \tag{2.87}$$

is called the infinitesimal generator of the group action at $\mathbf{u} \in U$, in the direction $\gamma'(0) \in T_eG$, where T_eG is the tangent space to G at e. In coordinates, the components of the infinitesimal generator are $Q^\alpha = \mathbf{v}(u^\alpha)$ so

$$\mathbf{v} = Q^\alpha \frac{\partial}{\partial u^\alpha}\,.$$

The infinitesimal generator is extended to the prolongation space $P_{\mathbf{n}}(\mathbb{R}^q)$ by the prolongation formula

$$\mathbf{v}(u^\alpha) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t) \cdot u^\alpha_{\mathbf{J}} = Q^\alpha_{\mathbf{J}} = \mathrm{S}_{\mathbf{J}} Q^\alpha,$$

for all $\mathbf{J}$ in the domain (see Chapter 6 in [15]).

Therefore, in coordinates, the prolonged infinitesimal generator is

$$\mathbf{v} = Q^\alpha_{\mathbf{J}} \frac{\partial}{\partial u^\alpha_{\mathbf{J}}}\,, \tag{2.88}$$

this is a generalization of (2.19).

Lemma 2.9.2. If a Lagrangian $\mathrm{L}(\mathbf{n}, [\mathbf{u}])$ is invariant under the Lie group action $G \times \mathcal{M} \to \mathcal{M}$, the components of the infinitesimal generator of the group action from Definition 2.9.1 form the characteristic of a variational symmetry of $\mathrm{L}(\mathbf{n}, [\mathbf{u}])$, as defined in Definition 2.2.1.

Proof. The Lagrangian L is invariant so

$$\mathrm{L}(\mathbf{n}, [\mathbf{u}]) = \mathrm{L}(\mathbf{n}, g \cdot [\mathbf{u}])$$

for all g, where $g \cdot [\mathbf{u}]$ represents the action of the group on the dependent variables and finitely many of their shifts. Thus,

$$0=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathrm{L}\left(\mathbf{n},\gamma\left(t\right)\cdot\left[\mathbf{u}\right]\right)=\mathbf{v}\left(\mathrm{L}\right)=Q^{\alpha}_{\mathbf{J}}\frac{\partial\mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}}.$$

By Definition 2.2.1, the components Q^{α} of the infinitesimal generator are the components of the characteristic of a variational symmetry of L. □

By the same reasoning as in the OΔEs paper [26] the elements $\gamma'\left(0\right)\in T_eG$ determine each of the infinitesimal generators, with the remainder of the path in G being immaterial. A property of the tangent space T_eG is that it is isomorphic to the Lie algebra $\mathfrak{g}$, where $\mathfrak{g}$ is the set of right-invariant vector fields on G. The right-invariance yields a Lie algebra homomorphism from $\mathfrak{g}$ to the set $\mathcal{X}$ of infinitesimal generators of symmetries (see [28]). Further, if the group action is faithful, that is, there are no group elements g (except the identity element) such that $g\cdot u=u$ for all $u\in U$, then this is an isomorphism.

Now we provide details on how to obtain the adjoint representation of g from [26]. Let G be an R-dimensional Lie group. The Lie group G can be parameterized by $\mathbf{a}=\left(a^1,...,a^R\right)$ in a neighbourhood of the identity, e, where the general group element is $\Gamma\left(\mathbf{a}\right)$ and the identity element of the group is $\Gamma\left(\mathbf{0}\right)=e$. Let the action of G on the local coordinates, $\mathbf{u}=\left(u^1,...,u^q\right)$ on U, be $\widehat{\mathbf{u}}=\Gamma\left(a\right)\cdot\mathbf{u}$. Varying the parameters a^r one by one in the process above yields R infinitesimal generators,

$$\mathbf{v}_r=Q^{\alpha}_r\left(\mathbf{n},\mathbf{u}\right)\partial_{u^{\alpha}},\quad \text{where}\quad Q^{\alpha}_r=\left.\frac{\partial\widehat{u}^{\alpha}}{\partial a^r}\right|_{\mathbf{a}=\mathbf{0}}. \tag{2.89}$$

These form a basis for $\mathcal{X}$. As the set of infinitesimal generators of symmetries, $\mathcal{X}$, is homomorphic to the Lie algebra, $\mathfrak{g}$, the adjoint representation of G on $\mathfrak{g}$ gives rise to the adjoint representation of G on $\mathcal{X}$. Given $g\in G$, the adjoint representation Ad_g is the tangent map on $\mathfrak{g}$ induced by the conjugation $h\mapsto ghg^{-1}$. The corresponding adjoint representation on $\mathcal{X}$ is expressed by a matrix, $\mathcal{A}d\left(g\right)=\left(a^s_r\right)$, which is obtained in the following way.

First calculate a basis for $\mathcal{X}$,

$$\mathbf{v}_r=Q^{\alpha}_r\left(\mathbf{n},\mathbf{u}\right)\partial_{u^{\alpha}},\quad r=1,...,R.$$

Then let $\widetilde{\mathbf{u}} = g \cdot \mathbf{u}$ and define

$$\widetilde{\mathbf{v}}_r = Q_r^\alpha \left(\mathbf{n}, \widetilde{\mathbf{u}}\right) \partial_{\widetilde{u}^\alpha}, \quad r = 1, ..., R.$$

Now to determine $\mathcal{A}d\left(g\right)$, express each $\mathbf{v}_r$ in terms of $\widetilde{\mathbf{v}}_1, ..., \widetilde{\mathbf{v}}_R$ and use the identity

$$\mathbf{v}_r = a_r^s \widetilde{\mathbf{v}}_s, \quad \text{where} \quad \widetilde{\mathbf{v}}_s := \mathbf{v}_s|_{z \mapsto g \cdot z = \widetilde{z}}\,. \tag{2.90}$$

As $\mathbf{v}_r$ and $\widetilde{\mathbf{v}}_r$ are known for $r = 1, \ldots, R$, one can read off the components a_r^s of the adjoint representation of g. Additionally, by regarding the infinitesimal generators as differential operators and applying the left-hand side of the identity (2.90) to each $\widetilde{u}^\alpha$ in turn, one obtains via the chain rule

$$\mathbf{v}_r \left(\widetilde{u}^\alpha\right) = \left(\frac{\partial \widetilde{u}^\alpha}{\partial u^\beta}\right) Q_r^\beta \left(\mathbf{n}, \mathbf{u}\right), \tag{2.91}$$

where $\partial \widetilde{u}^\alpha / \partial u^\beta$ denotes the α row and β column component of the Jacobian matrix. Similarly, applying the right-hand side of (2.90) to each $\widetilde{u}^\alpha$ in turn gives

$$a_r^s \widetilde{\mathbf{v}}_s \left(\widetilde{u}^\alpha\right) = a_r^s Q_s^\alpha \left(\mathbf{n}, \widetilde{\mathbf{u}}\right). \tag{2.92}$$

Combining the results (2.91) and (2.92) yields

$$\left(\frac{\partial \widetilde{u}^\alpha}{\partial u^\beta}\right) Q_r^\beta \left(\mathbf{n}, \mathbf{u}\right) = a_r^s Q_s^\alpha \left(\mathbf{n}, \widetilde{\mathbf{u}}\right). \tag{2.93}$$

This identity is extended on to the prolongation space $P_{\mathbf{n}}\left(\mathbb{R}^q\right)$ where points in that space include coordinates of the form $\mathbf{u}_{\mathbf{K}}$ which lie in this domain.

The infinitesimal generators $\mathbf{v}_r$, prolonged to all variables $\mathbf{u}_{\mathbf{K}}$ for all $\mathbf{K}$ in the domain, satisfy (2.90). Applying this identity to $\mathbf{u}_{\mathbf{J}}$ gives the formula

$$\left(\frac{\partial \widetilde{u}_{\mathbf{J}}^\alpha}{\partial u_{\mathbf{J}}^\beta}\right) Q_r^\beta \left(\mathbf{n} + \mathbf{J}, \mathbf{u}_{\mathbf{J}}\right) = a_r^s Q_s^\alpha \left(\mathbf{n} + \mathbf{J}, \widetilde{\mathbf{u}_{\mathbf{J}}}\right). \tag{2.94}$$

2.10 Conservation laws

For ODEs [9, 11, 22], PDEs [10] and OΔEs [25, 26] it has been shown that in general, the conservation laws are not invariant. However, they are equivariant and they can be written in terms of the invariants and the moving frame. For PΔEs, the key result is that the R conservation laws can be written in the form

$$\mathrm{D}_{n^i}\{V_s^i a_r^s|_{g=\rho_{\mathbf{0}}([\mathbf{u}])}\} = 0, \quad r = 1, \ldots, R, \tag{2.95}$$

where $a_r^s|_{g=\rho_{\mathbf{0}}([\mathbf{u}])}$ are the components of the adjoint representation of $\rho_{\mathbf{0}}\left([\mathbf{u}]\right)$ and V_s^i are invariants for each i.

We use the same reasoning as is given in the OΔEs paper by Mansfield *et al.* in the calculation of the Euler–Lagrange equations and boundary terms. In terms of the original variables, suppose that the dummy variable t effecting the variation is a group parameter for G, under which the Lagrangian is invariant. Then the resulting boundary terms yield conservation laws; this is the difference version of Noether's theorem (Theorem 2.2.2). So it is useful to identify t with a group parameter by considering the following path in G:

$$t \mapsto \gamma_r\left(t\right) = \Gamma\left(a^1\left(t\right), \ldots, a^R\left(t\right)\right), \text{ where } a^r\left(t\right) = t \text{ and } a^l\left(t\right) = 0,\ l \neq r. \tag{2.96}$$

We know that $\mathbf{a} \mapsto \Gamma\left(\mathbf{a}\right)$ expresses the general group element in terms of the coordinates $\mathbf{a}$. On this path, each $(u_0^\alpha)'$ at $t = 0$ is an infinitesimal generator, from (2.89).

For the invariantized calculation, we follow essentially the same route to the result, identifying the dummy variable effecting the variation with each group parameter in turn. Remember that $L^{\boldsymbol{\kappa}}$ is a function of $\boldsymbol{\kappa}$ with each κ^β a function of $\mathbf{u}$ and their shifts. The dependent variables u^α depend on t, so $L^{\boldsymbol{\kappa}}$ depends on t. The identity

$$\begin{aligned}\frac{\mathrm{d}L^{\boldsymbol{\kappa}}}{\mathrm{d}t} &= \left(\mathcal{H}_\alpha^\beta\right)^\dagger \mathrm{E}_{k^\beta}\left(L^{\boldsymbol{\kappa}}\right)\sigma^\alpha + \mathrm{Div}\left(A_{\boldsymbol{\kappa}}\right) + \mathrm{Div}\left(A_{\mathcal{H}}\right),\\ &= \left(\mathcal{H}_\alpha^\beta\right)^\dagger \mathrm{E}_{k^\beta}\left(L^{\boldsymbol{\kappa}}\right)\sigma^\alpha + \mathrm{D}_{n^i}\left(F_\beta^i\left(\mathbf{n},[\boldsymbol{\kappa}]\right)\left(\kappa^\beta\right)'\right) + \mathrm{D}_{n^i}\left(H_\alpha^i\left(\mathbf{n},[\boldsymbol{\kappa}]\right)\sigma^\alpha\right),\end{aligned} \tag{2.97}$$

is given in the proof of Proposition 2.6.11 and is important here. Recall Remark 2.6.12 which states that $\mathrm{Div}\left(A_{\boldsymbol{\kappa}}\right)$ is linear in $\left(\kappa^\beta\right)'$ and their shifts, while $\mathrm{Div}\left(A_{\mathcal{H}}\right)$ is linear in σ^α and their shifts. As t is a group parameter and each κ^β is invariant,

$\left(\kappa^{\beta}\right)' = 0$. As a consequence, (2.97) reduces to

$$\left(\mathcal{H}_{\alpha}^{\beta}\right)^{\dagger} \mathrm{E}_{k^{\beta}}\left(L^{\boldsymbol{\kappa}}\right) \sigma^{\alpha} + \mathrm{D}_{n^{i}}\left(H_{\alpha}^{i}\left(\mathbf{n},[\boldsymbol{\kappa}]\right) \sigma^{\alpha}\right) = 0, \tag{2.98}$$

so $\mathrm{D}_{n^{i}}\left(H_{\alpha}^{i}\left(\mathbf{n},[\boldsymbol{\kappa}]\right) \sigma^{\alpha}\right) = 0$ on all solutions of the invariantized Euler–Lagrange equations, i.e., when

$$\left(\mathcal{H}_{\alpha}^{\beta}\right)^{\dagger} \mathrm{E}_{k^{\beta}}\left(L^{\boldsymbol{\kappa}}\right) = 0. \tag{2.99}$$

We now derive the conservation laws from this condition.

Proposition 2.10.1. Suppose that the conditions of Proposition 2.5.2 hold. Write

$$H_{\alpha}^{i}\left(\mathbf{n},[\boldsymbol{\kappa}]\right) \sigma^{\alpha} = \mathcal{C}_{\alpha}^{i,\mathbf{J}} \mathrm{S}_{\mathbf{J}}\left(\sigma^{\alpha}\right) \tag{2.100}$$

where the $H_{\alpha}^{i}\left(\mathbf{n},[\boldsymbol{\kappa}]\right)$ come from $\mathrm{D}_{n^{i}}\left(H_{\alpha}^{i}\left(\mathbf{n},[\boldsymbol{\kappa}]\right) \sigma^{\alpha}\right)$ and each $\mathcal{C}_{\alpha}^{i,\mathbf{J}}$ depends only on $\mathbf{n}$, $\boldsymbol{\kappa}$ and its shifts. Let Q_{s}^{α} be the component of the matrix of characteristics corresponding to the dependent variable $u_{\mathbf{0}}^{\alpha}$ and the group parameter a^{s}. Then the R conservation laws amount to

$$\mathrm{D}_{n^{i}}\left(\mathcal{C}_{\alpha}^{i,\mathbf{J}} \mathrm{S}_{\mathbf{J}}\{\iota\left(Q_{s}^{\alpha}\right) a_{r}^{s}|_{g=\rho_{\mathbf{0}}([\mathbf{u}])}\}\right) = 0. \tag{2.101}$$

That is, to obtain the conservation laws, it is sufficient to make the replacement

$$\sigma^{\alpha} \mapsto \{\widetilde{Q}_{s}^{\alpha} a_{r}^{s}\}|_{g=\rho_{\mathbf{0}}([\mathbf{u}])} = \iota\left(Q_{s}^{\alpha}\right) a_{r}^{s}|_{g=\rho_{\mathbf{0}}([\mathbf{u}])} \tag{2.102}$$

in each $\mathrm{D}_{n^{i}}\left(H_{\alpha}^{i}\left(\mathbf{n},[\boldsymbol{\kappa}]\right) \sigma^{\alpha}\right)$, with $\widetilde{Q}_{s}^{\alpha} = g \cdot Q_{s}^{\alpha}\left(\mathbf{n},\mathbf{u}\right) = Q_{s}^{\alpha}\left(\mathbf{n},\widetilde{\mathbf{u}}\right)$.

Proof. Recall that

$$\sigma^{\alpha} = \rho_{\mathbf{0}}\left([\mathbf{u}]\right) \cdot \left(u_{\mathbf{0}}^{\alpha}\right)' = \left.\left(\frac{\mathrm{d}\left(g \cdot u_{\mathbf{0}}^{\alpha}\right)}{\mathrm{d}t}\right)\right|_{g=\rho_{\mathbf{0}}([\mathbf{u}])}. \tag{2.103}$$

To obtain the conservation laws, conflate t with the group parameter a^{r}, making the replacement

$$\rho_{\mathbf{0}}\left([\mathbf{u}]\right) \cdot \left(u_{\mathbf{0}}^{\alpha}\right)' \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho_{\mathbf{0}}\left([\mathbf{u}]\right) \cdot \gamma_{r}\left(t\right) \cdot u_{\mathbf{0}}^{\alpha} \tag{2.104}$$

in $\mathrm{D}_{n^i}\left(H^i_\alpha\left(\mathbf{n},[\boldsymbol{\kappa}]\right)\sigma^\alpha\right)$, where $\gamma_r\left(t\right)$ is the path defined in (2.96). For any $g\in G$

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(g\cdot\gamma_r\left(t\right)\cdot u^\alpha_{\mathbf{0}}\right) &= \left(\frac{\partial\left(g\cdot\gamma_r\left(t\right)\cdot u^\alpha_{\mathbf{0}}\right)}{\partial\left(\gamma_r\left(t\right)\cdot u^\beta_{\mathbf{0}}\right)}\right)\Bigg|_{t=0}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\gamma_r\left(t\right)\cdot u^\beta_{\mathbf{0}}\right)\\ &= \frac{\partial\left(g\cdot u^\alpha_{\mathbf{0}}\right)}{\partial u^\beta_{\mathbf{0}}}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\gamma_r\left(t\right)\cdot u^\beta_{\mathbf{0}}\right).\end{aligned}\tag{2.105}$$

Using (2.94) yields

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(g\cdot\gamma_r\left(t\right)\cdot u^\alpha_{\mathbf{0}}\right) = \frac{\partial\left(g\cdot u^\alpha_{\mathbf{0}}\right)}{\partial u^\beta_{\mathbf{0}}}Q^\beta_r = \widetilde{Q}^\alpha_s a^s_r.$$

As a result, setting $g=\rho_{\mathbf{0}}\left([\mathbf{u}]\right)$, the required replacement is

$$\sigma^\alpha\mapsto\iota\left(Q^\alpha_s\right)a^s_r|_{g=\rho_{\mathbf{0}}([\mathbf{u}])}.\tag{2.106}$$

□

By the prolongation formula $\mathrm{S}_{\mathbf{J}}\left(\rho_{\mathbf{0}}\left([\mathbf{u}]\right)\right)=\rho_{\mathbf{J}}\left([\mathbf{u}]\right)$, the conservation laws amount to

$$\mathrm{D}_{n^i}\left(\mathcal{C}^{i,\mathbf{J}}_\alpha\left(\mathrm{S}_{\mathbf{J}}\iota\left(Q^\alpha_s\right)\right)a^s_r|_{g=\rho_{\mathbf{J}}([\mathbf{u}])}\right)=0.\tag{2.107}$$

As the adjoint is a group representation

$$a^l_s|_{g=\rho_{\mathbf{J}}([\mathbf{u}])}\cdot\left(a^{-1}\right)^s_r|_{g=\rho_{\mathbf{0}}([\mathbf{u}])}=a^l_r|_{g=\rho_{\mathbf{J}}([\mathbf{u}])\rho_{\mathbf{0}}([\mathbf{u}])^{-1}}$$

(with a^{-1} the inverse adjoint matrix) is invariant, this leads to the following corollary.

Corollary 2.10.2. The conservation laws for a difference frame may be written in the form

$$\mathrm{D}_{n^i}\{V^i_s a^s_r|_{g=\rho_{\mathbf{0}}([\mathbf{u}])}\}=0,\tag{2.108}$$

where V^i_s are invariant components, specifically

$$V^i_s=\mathcal{C}^{i,\mathbf{J}}_\alpha\left(\mathrm{S}_{\mathbf{J}}\left(\iota\left(Q^\alpha_l\right)\right)\right)\left(a^l_s|_{g=\rho_{\mathbf{J}}([\mathbf{u}])\rho_{\mathbf{0}}([\mathbf{u}])^{-1}}\right).\tag{2.109}$$

As the conservation laws depend only on the terms arising from $\mathrm{Div}\left(A_{\mathcal{H}}\right)$, the laws can be calculated for all Lagrangian in the relevant invariant class, in terms of $\mathrm{E}_{\kappa^\alpha}(L^{\boldsymbol{\kappa}})$, independently of the precise form that $L^{\boldsymbol{\kappa}} = L^{\boldsymbol{\kappa}}(\mathbf{n}, [\boldsymbol{\kappa}])$ takes.

Now using Proposition 2.6.5 we can derive a useful result for the divergence term $\mathrm{Div}(A_{\mathcal{H}})$.

Corollary 2.10.3. The divergence term $\mathrm{Div}(A_{\mathcal{H}})$ is given by

$$\mathrm{Div}(A_{\mathcal{H}}) = \sum_{\mathbf{J}} (\mathrm{S}_{\mathbf{J}} - \mathrm{id}) \left[\mathrm{S}_{-\mathbf{J}} \left(\iota \left(\frac{\partial \kappa^\beta}{\partial u^\alpha_{\mathbf{J}}} \right) \iota \left((\theta_{\mathbf{J}})^\delta_\alpha \right) \mathrm{E}_{\kappa^\beta}(L^{\boldsymbol{\kappa}}) \right) \sigma^\alpha \right].$$

Proof. The divergence term is

$$\mathrm{Div}(A_{\mathcal{H}}) = \mathrm{E}_{\kappa^\beta}(L^{\boldsymbol{\kappa}}) \left(\mathcal{H}^\beta_\alpha \sigma^\alpha \right) - \left(\left(\mathcal{H}^\beta_\alpha \right)^\dagger \mathrm{E}_{\kappa^\beta}(L^{\boldsymbol{\kappa}}) \right) \sigma^\alpha. \tag{2.110}$$

The formulas for $\mathcal{H}^\beta_\alpha$ and $\left(\mathcal{H}^\beta_\alpha\right)^\dagger$ in Proposition 2.6.5 give

$$\mathrm{E}_{\kappa^\beta}(L^{\boldsymbol{\kappa}}) \left(\mathcal{H}^\beta_\alpha \sigma^\alpha \right) = \sum_{\mathbf{J}} \mathrm{E}_{\kappa^\beta}(L^{\boldsymbol{\kappa}})\, \iota \left(\frac{\partial \kappa^\beta}{\partial u^\alpha_{\mathbf{J}}} \right) \iota \left((\theta_{\mathbf{J}})^\delta_\alpha \right) \mathrm{S}_{\mathbf{J}} \sigma^\alpha,$$
$$\left(\mathcal{H}^\beta_\alpha \right)^\dagger \mathrm{E}_{\kappa^\beta}(L^{\boldsymbol{\kappa}})\, \sigma^\alpha = \sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}} \left(\mathrm{E}_{\kappa^\beta}(L^{\boldsymbol{\kappa}})\, \iota \left(\frac{\partial \kappa^\beta}{\partial u^\alpha_{\mathbf{J}}} \right) \iota \left((\theta_{\mathbf{J}})^\delta_\alpha \right) \right) \sigma^\alpha,$$

Thus, substituting these into (2.110) gives the result. □

Remark 2.10.4. Using the formulas for $\mathcal{H}^\beta_\alpha$ and $\left(\mathcal{H}^\beta_\alpha\right)^\dagger$ in Lemma 2.6.7 would give

$$\mathrm{Div}(A_{\mathcal{H}}) = \sum_{\mathbf{J}} (\mathrm{S}_{\mathbf{J}} - \mathrm{id}) \left[\iota \left(\frac{\partial \kappa^\beta_{-\mathbf{J}}}{\partial u^\alpha_{\mathbf{0}}} \right) \left(\mathrm{S}_{-\mathbf{J}} \mathrm{E}_{\kappa^\beta}(L^{\boldsymbol{\kappa}}) \right) \sigma^\alpha \right].$$

From this point we can use identities to write different $(\mathrm{S}_{\mathbf{J}} - \mathrm{id})\, F$ in terms of $\mathrm{D}_{n^i} F^i$, that is,

$$(\mathrm{S}_{\mathbf{J}} - \mathrm{id})\, F = \mathrm{D}_{n^1} F^1 + \cdots + \mathrm{D}_{n^p} F^p,$$

for each F^i with $i = 1, \ldots, p$. As a consequence, the terms H^i_α and $\mathcal{C}^{i,\mathbf{J}}_\alpha$ in (2.100) can be found.

Example 2.10.1. Here we find the conservation laws of the Lagrangian (2.17) when considering the group action in the running example of a scaling and translation (2.23). First, we calculate the divergence term $\mathrm{Div}(A_{\mathcal{H}})$, using the Euler operators of $L^{\boldsymbol{\kappa}}$ with respect to the generating invariants (2.47), linear difference operators (2.42) and adjoint linear difference operators (2.48). Therefore,

$$\begin{aligned}
\mathrm{E}_\kappa\left(L^{\boldsymbol{\kappa}}\right)\left(\mathcal{H}_\kappa\sigma\right) &= \frac{\lambda}{2\kappa}\mathrm{S}_1\mathrm{S}_2^{-1}\sigma - \frac{\lambda\left(1-\lambda\right)}{2\kappa_{1,1}\left(\kappa-\lambda\right)}\mathrm{S}_1\mathrm{S}_2\sigma - \frac{\lambda\left(\kappa-1\right)}{2\kappa\left(\kappa-\lambda\right)}\sigma,\\
\mathrm{E}_\lambda\left(L^{\boldsymbol{\kappa}}\right)\left(\mathcal{H}_\lambda\sigma\right) &= -\frac{\left(\kappa-1\right)\left(\kappa_{1,1}-\lambda_{1,1}\right)}{2\kappa_{1,1}\left(\kappa-\lambda\right)}\mathrm{S}_1^2\sigma - \frac{\lambda\left(\kappa-1\right)}{2\kappa_{1,1}\left(\kappa-\lambda\right)}\mathrm{S}_1\mathrm{S}_2\sigma\\
&\quad + \frac{\kappa-1}{2\left(\kappa-\lambda\right)}\sigma,\\
\left(\left(\mathcal{H}_\kappa^\dagger\right)\mathrm{E}_\kappa\left(L^{\boldsymbol{\kappa}}\right)\right) &= \frac{\lambda_{-1,1}}{2\kappa_{-1,1}}\sigma - \frac{\lambda_{-1,-1}\left(1-\lambda_{-1,-1}\right)}{2\kappa\left(\kappa_{-1,-1}-\lambda_{-1,-1}\right)}\sigma - \frac{\lambda\left(\kappa-1\right)}{2\kappa\left(\kappa-\lambda\right)}\sigma,\\
\left(\left(\mathcal{H}_\lambda^\dagger\right)\mathrm{E}_\lambda\left(L^{\boldsymbol{\kappa}}\right)\right) &= -\frac{\left(\kappa_{-2,0}-1\right)\left(\kappa_{-1,1}-\lambda_{-1,1}\right)}{2\kappa_{-1,1}\left(\kappa_{-2,0}-\lambda_{-2,0}\right)}\sigma - \frac{\lambda_{-1,-1}\left(\kappa_{-1,-1}-1\right)}{2\kappa\left(\kappa_{-1,-1}-\lambda_{-1,-1}\right)}\sigma\\
&\quad + \frac{\kappa-1}{2\left(\kappa-\lambda\right)}\sigma.
\end{aligned}$$

Using (2.110) the divergence term is

$$\begin{aligned}
\mathrm{Div}\left(A_{\mathcal{H}}\right) &= \mathrm{E}_\kappa\left(L^{\boldsymbol{\kappa}}\right)\left(\mathcal{H}_\kappa\sigma\right) - \left(\left(\mathcal{H}_\kappa^\dagger\right)\mathrm{E}_\kappa\left(L^{\boldsymbol{\kappa}}\right)\right)\\
&\quad + \mathrm{E}_\lambda\left(L^{\boldsymbol{\kappa}}\right)\left(\mathcal{H}_\lambda\sigma\right) - \left(\left(\mathcal{H}_\lambda^\dagger\right)\mathrm{E}_\lambda\left(L^{\boldsymbol{\kappa}}\right)\right)\\
&= \left(\mathrm{S}_1\mathrm{S}_2^{-1}-\mathrm{id}\right)\left(\frac{\lambda_{-1,1}}{2\kappa_{-1,1}}\sigma\right) + \left(\mathrm{S}_1\mathrm{S}_2-\mathrm{id}\right)\left(-\frac{\lambda_{-1,-1}}{2\kappa}\sigma\right)\\
&\quad + \left(\mathrm{S}_1^2-\mathrm{id}\right)\left(-\frac{\left(\kappa_{-2,0}-1\right)\left(\kappa_{-1,1}-\lambda_{-1,1}\right)}{2\kappa_{-1,1}\left(\kappa_{-2,0}-\lambda_{-2,0}\right)}\sigma\right).
\end{aligned}$$

Now we need to write this in the form

$$\mathrm{Div}\left(A_{\mathcal{H}}\right) = \left(\mathrm{S}_1-\mathrm{id}\right)G^1 + \left(\mathrm{S}_2-\mathrm{id}\right)G^2.$$

To do this we use the following identities

$$\begin{aligned}
\left(\mathrm{S}_1\mathrm{S}_2^{-1}-\mathrm{id}\right)F^1 &= \mathrm{S}_1\mathrm{S}_2^{-1}F^1 - F^1 = \left(\mathrm{S}_1-\mathrm{id}\right)\left(\mathrm{S}_2^{-1}F^1\right) + \left(\mathrm{S}_2-\mathrm{id}\right)\left(-\mathrm{S}_2^{-1}F^1\right),\\
\left(\mathrm{S}_1^2-\mathrm{id}\right)F^2 &= \mathrm{S}_1^2F^2 - F^2 = \left(\mathrm{S}_1-\mathrm{id}\right)\left(\mathrm{S}_1F^2\right) + \left(\mathrm{S}_1-\mathrm{id}\right)F^2,\\
\left(\mathrm{S}_1\mathrm{S}_2-\mathrm{id}\right)F^3 &= \mathrm{S}_1\mathrm{S}_2F^3 - F^3 = \left(\mathrm{S}_1-\mathrm{id}\right)\left(\mathrm{S}_2F^3\right) + \left(\mathrm{S}_2-\mathrm{id}\right)F^3.
\end{aligned}$$

Thus, the key divergence term is

$$\begin{aligned}
\operatorname{Div}(A_{\mathcal{H}}) &= (\mathrm{S}_1 - \mathrm{id})\left(\frac{\lambda_{-1,0}}{2\kappa_{-1,0}}\mathrm{S}_2^{-1}\sigma\right) + (\mathrm{S}_2 - \mathrm{id})\left(-\frac{\lambda_{-1,0}}{2\kappa_{-1,0}}\mathrm{S}_2^{-1}\sigma\right)\\
&+ (\mathrm{S}_1 - \mathrm{id})\left(-\frac{(\kappa_{-1,0}-1)(\kappa_{0,1}-\lambda_{0,1})}{2\kappa_{0,1}(\kappa_{-1,0}-\lambda_{-1,0})}\mathrm{S}_1\sigma\right)\\
&+ (\mathrm{S}_1 - \mathrm{id})\left(-\frac{(\kappa_{-2,0}-1)(\kappa_{-1,1}-\lambda_{-1,1})}{2\kappa_{-1,1}(\kappa_{-2,0}-\lambda_{-2,0})}\sigma\right)\\
&+ (\mathrm{S}_1 - \mathrm{id})\left(-\frac{\lambda_{-1,0}}{2\kappa_{0,1}}\mathrm{S}_1\sigma\right) + (\mathrm{S}_2 - \mathrm{id})\left(-\frac{\lambda_{-1,-1}}{2\kappa}\sigma\right),
\end{aligned}$$

from which we find the corresponding values of $\mathcal{C}_\alpha^{i,\mathbf{J}}$ (where $\alpha = 1$ as we take $u = u^1$) are the following

$$\begin{aligned}
\mathcal{C}_1^{1,0,0} &= -\frac{(\kappa_{-2,0}-1)(\kappa_{-1,1}-\lambda_{-1,1})}{2\kappa_{-1,1}(\kappa_{-2,0}-\lambda_{-2,0})},\\
\mathcal{C}_1^{1,0,1} &= -\frac{\lambda_{-1,0}}{2\,\kappa_{0,1}},\\
\mathcal{C}_1^{1,1,0} &= -\frac{(\kappa_{-1,0}-1)(\kappa_{0,1}-\lambda_{0,1})}{2\kappa_{0,1}(\kappa_{-1,0}-\lambda_{-1,0})},\\
\mathcal{C}_1^{1,0,-1} &= \frac{\lambda_{-1,0}}{2\,\kappa_{-1,0}},\\
\mathcal{C}_1^{2,0,-1} &= -\frac{\lambda_{-1,0}}{2\,\kappa_{-1,0}},\\
\mathcal{C}_1^{2,0,0} &= -\frac{\lambda_{-1,-1}}{2\kappa}.
\end{aligned} \tag{2.111}$$

Note for PΔEs the values of $\mathcal{C}_\alpha^{i;\mathbf{J}}$ are not unique. To obtain the conservation laws we need the invariantized (2.31) form of infinitesimals restricted to the variable u

$$\iota\left(Q_1^1\right) = 0, \qquad \iota\left(Q_2^1\right) = 1,$$

and the components of the adjoint matrix on the frame:

$$\begin{aligned}
a_1^1|_{g=\rho_{0,0}([u])} &= 1, \quad a_2^1|_{g=\rho_{0,0}([u])} = 0,\\
a_1^2|_{g=\rho_{0,0}([u])} &= \frac{u_{0,0}}{u_{1,1}-u_{0,0}}, \quad a_2^2|_{g=\rho_{0,0}([u])} = \frac{1}{u_{1,1}-u_{0,0}}.
\end{aligned}$$

Using the formula (2.101) gives the conservation laws

$$
\begin{aligned}
&\mathrm{D}_{n^1}\left\{\mathcal{C}_1^{1,0,0}\left(a_j^2|_{g=\rho_{0,0}([u])}\right)\right\}+\mathrm{D}_{n^1}\left\{\mathcal{C}_1^{1,0,1}\left(a_j^2|_{g=\rho_{0,1}([u])}\right)\right\}+\\
&\mathrm{D}_{n^1}\left\{\mathcal{C}_1^{1,1,0}\left(a_j^2|_{g=\rho_{1,0}([u])}\right)\right\}+\mathrm{D}_{n^1}\left\{\mathcal{C}_1^{1,0,-1}\left(a_j^2|_{g=\rho_{0,-1}([u])}\right)\right\}+\\
&\mathrm{D}_{n^2}\left\{\mathcal{C}_1^{2,0,-1}\left(a_j^2|_{g=\rho_{0,-1}([u])}\right)\right\}+\mathrm{D}_{n^2}\left\{\mathcal{C}_1^{2,0,0}\left(a_j^2|_{g=\rho_{0,0}([u])}\right)\right\}=0,
\end{aligned}
$$

for $j = 1, 2$. These expressions are complicated, so we omit writing them explicitly.

Chapter 3

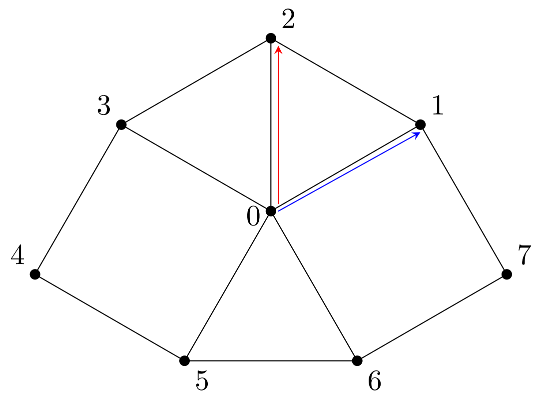
Extension to a non-rectangular mesh

In this chapter, the results on variational problems for the rectangular mesh are extended to a non-rectangular mesh. In particular, the snub square tiling, a type of semi-regular tiling, is explored. We build on a new prolongation space for this tiling due to Hydon [14], which is summarized in Section 3.1. This enables us to explore the calculus of variations and its invariantization.

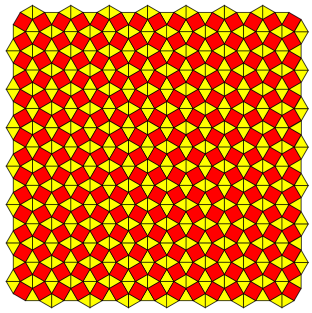
3.1 The prolongation space of the snub square tiling

Most of the current literature considers regular tilings of the plane, that is, tilings using regular polygons (squares, equilateral triangles and regular hexagons). The snub square tiling is a semi-regular tiling, that is, its tiling uses more than one regular polygon. Considering such tilings aids the development towards looking at completely free mesh, which would have possible applications in numerical approximations. The material in this section is due to Hydon [14]. The base points in the snub square mesh (see Figure 3.1) are the vertices of the snub square tiling (see Figure 3.1b). Given an arbitrary base point, which is labelled 0, the tiles touching 0 form the following arrangement, called the standard template (see Figure 3.1a).

Definition 3.1.1. Neighbouring points (vertices) are connected via edges on the graphs.



(a) Standard template.



(b) Tiling[1].

Figure 3.1: Snub square standard template and tiling.

Here we introduce steps which allow one to move along edges between different points in a tiling.

Definition 3.1.2. A step, T, maps a tiling $\mathcal{T}$ of the plane to itself, i.e.,

$$\mathrm{T} : \mathcal{T} \to \mathcal{T}.$$

Steps can be applied more than once, r steps of T being denoted by T^r. Two steps T_1 and T_2 are independent if and only if

$$\mathrm{T}_1^r(\mathbf{J}) \neq \mathrm{T}_2(\mathbf{J}), \quad \mathrm{T}_2^s(\mathbf{J}) \neq \mathrm{T}_1(\mathbf{J})$$

for all $r, s \in \mathbb{Z}$ and where $\mathbf{J}$ represents a particular point in the tiling. We consider tilings of the plane with at most two independent steps.

Steps we consider consist of a translation between neighbouring points, possibly followed by a rotation. In the regular square domain, the two independent steps are the horizontal and vertical translations with no rotations for either. We now define two independent steps for the snub square tiling. For a brief outline of other regular and semi-regular tilings, see Appendix B.

The group of symmetries of the mesh include all translations from one point to another (with a rotation following the translation to map the standard template on the source point 0, to the standard template on the target.) The rotations

[1] 1-uniform_n9 by Tomruen, retrieved from "`https://commons.wikimedia.org/wiki/File:1-uniform_n9.svg`" is licensed under CC BY-SA 4.0. This image has been rotated and some shapes have been added.

are integer multiples of $\pi/2$. There are also reflections, but these are not used to describe the prolongation space over 0.

Let T_1 denote the translation from 0 to 1 followed by a rotation of $\pi/2$ clockwise, and let T_2 denote the translation from 0 to 2 followed by a rotation of π. On Figure 3.1a the step T_1 is denoted by the blue arrow and T_2 by the red arrow. So $1 = \mathrm{T}_1(0)$ and $2 = \mathrm{T}_2(0)$. Moreover,

$$3 = \mathrm{T}_1\mathrm{T}_2(0),$$
$$4 = \mathrm{T}_1\mathrm{T}_2\mathrm{T}_1\mathrm{T}_2(0),$$
$$5 = \mathrm{T}_1\mathrm{T}_2\mathrm{T}_1\mathrm{T}_2\mathrm{T}_1\mathrm{T}_2(0),$$
$$6 = \mathrm{T}_1^3(0),$$
$$7 = \mathrm{T}_1^2(0),$$

and

$$\mathrm{T}_1^4 = (\mathrm{T}_1\mathrm{T}_2)^4 = \mathrm{T}_2^2 = \mathrm{id}.$$

So applying T_1 repeatedly to 0 gives the cycle

$$0 \mapsto 1 \mapsto 7 \mapsto 6 \mapsto 0,$$

clockwise around the right-hand square in the standard template, and similarly applying $\mathrm{T}_1\mathrm{T}_2$ repeatedly cycles anticlockwise around the left-hand square:

$$0 \mapsto 3 \mapsto 4 \mapsto 5 \mapsto 0.$$

Additionally, applying T_2 repeatedly cycles between 0 and 2.

A path between 0 and any other point $\mathbf{J}$ which may be anywhere in the mesh, is obtained by applying T_1^k, $k \in \{1,2,3\}$ and T_2 successively to translate along the edges between the tiles; this may produce a very long expression for the path. Note that the order of the steps for the snub square tiling is important. The steps for the snub square tiling do not commute, as $\mathrm{T}_1\mathrm{T}_2(0) \neq \mathrm{T}_2\mathrm{T}_1(0)$. Two different paths between 0 and $\mathbf{J}$, P_1 and P_2 say, differ by a trivial path: $P_2(0) = P_2\left(P_2^{-1}P_1(0)\right)$, so $P_2^{-1}P_1$ is a trivial path (either the identity or a cycle, i.e., a closed loop). For instance, if one goes from 0 to 6 in the standard template via 5, the path is not $P_1 = \mathrm{T}_1^3(0)$, but could be $P_2 = \mathrm{T}_2(\mathrm{T}_1\mathrm{T}_2\mathrm{T}_1\mathrm{T}_2\mathrm{T}_1\mathrm{T}_2)(0)$.

Then $P_2^{-1}P_1(0)$ is the trivial path

$$0 \mapsto 1 \mapsto 7 \mapsto 6 \mapsto 5 \mapsto 4 \mapsto 3 \mapsto 0,$$

which is a cycle (note some of this path goes off the standard template). The identities $\mathrm{T}_1^4 = (\mathrm{T}_1\mathrm{T}_2)^4 = \mathrm{T}_2^2 = \mathrm{id}$, enable us to factor out such trivial paths: two paths P_1 and P_2 are equivalent if and only if $P_2^{-1}P_1 = \mathrm{id}$ (modulo the identities). This means they produce a path between 0 and some point $\mathbf{J}$ in the tiling. In the example above, $P_2 = \mathrm{T}_1^{-1}(\mathrm{T}_1\mathrm{T}_2)^4$, where T_1^{-1} is equivalent to T_1^3. Consequently, $P_2 \equiv P_1$.

Using equivalence of paths enables one to write complicated paths in a shortest possible form. Even then, the notation can be cumbersome.

For a function u (or, more generally, a difference form – yet to be defined) on the mesh, the prolongation space over 0, $P_{\mathbf{n}}(U)$, is the product space giving the values of u at all points in the mesh. Here, we are not thinking about any particular function, but rather the space of all functions so $u_{\mathbf{J}}$ can take any value in $\mathbb{R}$ (or $\mathbb{C}$ if needed). Here $u_{\mathbf{J}}$ is the value of u at the point $\mathbf{J}$, that is, the pullback of u by any translation from 0 to $\mathbf{J}$. We talk about the path from 0 to $\mathbf{J}$ henceforth, as equivalent paths are counted as the same. We think of the equivalence class of all paths which take 0 to $\mathbf{J}$. It is important for the calculations later that we use a single representation of a path from 0 to $\mathbf{J}$.

The structure of the mesh (and hence the ability to distinguish difference forms from arbitrary discrete forms) is built into this construction, as every path consists of translations between successive adjacent points (and its points can be labelled by a set of consecutive integers).

For the snub square tiling, T_2 only occurs (in paths) raised to the first power, whereas T_1 can be raised to the power 1, 2 or 3. As in Chapter 2, $\mathrm{T}_{\mathbf{K}}^*$ denotes the pullback by $\mathrm{T}_{\mathbf{K}}$. To shorten the notation, write $\mathbf{J}$ in $u_{\mathbf{J}}$ as a sequence of colons (:) representing T_2^* and digits $k \in \{1, 2, 3\}$, representing $\left(\mathrm{T}_1^k\right)^*$. In this chapter we mainly focus on the case of one dependent variable only. The extension to more dependent variables is trivial.

For instance, $\mathbf{J} = P_{\mathbf{J}}(0)$, where $P_{\mathbf{J}} = \mathrm{T}_2\mathrm{T}_1^3\mathrm{T}_2\mathrm{T}_1\mathrm{T}_2\mathrm{T}_1$, gives

$$u_{\mathbf{J}} = P_{\mathbf{J}}^* u = \mathrm{T}_1^*\mathrm{T}_2^*\mathrm{T}_1^*\mathrm{T}_2^*\left(\mathrm{T}_1^3\right)^*\mathrm{T}_2^* u,$$

so $u_{\mathbf{J}} = u_{1:1:3:}$. Bear in mind that the indices used in going from 0 to $\mathbf{J}$ are read from left to right in the pullback. Therefore, the pullback of $P_{\mathbf{K}}P_{\mathbf{J}}$ is $P_{\mathbf{J}}^*P_{\mathbf{K}}^*$, and so $P_{\mathbf{J}}^*P_{\mathbf{K}}^*u = u_{\mathbf{JK}}$, where $\mathbf{JK}$ represents the concatenation of indices, simplified by the relations

$$\begin{aligned} :: &= 0, \\ 1:1:1:1: &= 0, \\ :1:1:1:1 &= 0, \\ 3:3:3:3: &= 0, \\ :3:3:3:3 &= 0, \\ n\,m &= n+m \,(\mathrm{mod}\,4)\,; \end{aligned}$$

any zeros that occur are deleted and simplification continues until no further simplification is possible.

One can also use the identities

$$\begin{aligned} 1:1:1 &= :3:\,, \\ 3:3:3 &= :1:\,, \\ :1:1: &= 3:3\,, \\ :3:3: &= 1:1\,, \end{aligned}$$

to keep the representation as short as possible.

In calculations, because we represent the paths from 0 to $\mathbf{J}$ with one representation the pullback also only has one representation.

3.2 The non-rectangular difference variational calculus

Much of the theory for PΔEs in Chapter 2 extends to the non-rectangular case, with a few subtle differences. The main difference to consider is that the multi-index $\mathbf{J}$ for the PΔEs is now described using a set of consecutive integers. As much of the theory remains the same, a proof will be given only when required.

The equations hold for all $\mathbf{n}$ in $\mathbb{Z}^2$. Assume all equations are regular over $\mathbb{Z}^2$ [15]. Therefore, $\mathbf{n}$ is suppressed and $u_{\mathbf{J}}$ is used to represent u on the fibre $\mathbf{nJ}$. The $\mathbf{J}$ here is the set of integers in the pullback of the path $P_{\mathbf{J}}$ (from 0 to $\mathbf{J}$). Taking $\mathbf{n}$ to be the base point, we must find the Euler operators and Euler–Lagrange equations in terms of shifts and derivatives. The identity operator remains the same, that is,

$$\mathrm{id} : \mathbf{n} \mapsto \mathbf{n}, \quad \mathrm{id} : f(\mathbf{n}) \mapsto f(\mathbf{n}), \quad \mathrm{id} : u_{\mathbf{J}} \mapsto u_{\mathbf{J}},$$

but the shift operator becomes

$$\mathrm{S}_{\mathbf{K}} : \mathbf{n} \mapsto \mathbf{nK}, \quad \mathrm{S}_{\mathbf{K}} : f(\mathbf{n}) \mapsto f(\mathbf{nK}), \quad \mathrm{S}_{\mathbf{K}} : u_{\mathbf{J}} \mapsto u_{\mathbf{JK}},$$

with $\mathbf{K}$ the set of integers in the pullback of the path by $P_{\mathbf{K}}$. If $f_{\mathbf{nJ}}$ is a function on the fibre $\mathbf{nJ}$ then the pullback is related to the shift operator by $\mathrm{S}_{\mathbf{K}} f_{\mathbf{n}} := \mathrm{T}^*_{\mathbf{K}} f_{\mathbf{nK}}$.

The variational problem is to find the extrema of a given functional

$$\mathcal{L}[\mathbf{u}] = \sum_{\mathbf{n}} \mathrm{L}(\mathbf{n}, [\mathbf{u}]),$$

where the sum is over $\mathbb{Z}^2$. The extrema are found using

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathcal{L}[\mathbf{u} + \epsilon \mathbf{w}] = 0,$$

for all $\mathbf{w}$ that vanish on the boundary (or in the appropriate limit, where the domain is unbounded). It is simple to check that the variation of $\mathcal{L}(\mathbf{u})$ in the direction $\mathbf{w}$ is

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathcal{L}\left[\mathbf{u}+\epsilon\mathbf{w}\right] = \sum_{\mathbf{n}} w^{\alpha} \mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right) + \mathrm{Div}\left(A_{\mathbf{u}}\left(\mathbf{n},\mathbf{w}\right)\right). \tag{3.1}$$

Here

$$\mathrm{E}_{u^{\alpha}} = \mathrm{S}_{-\mathbf{J}} \frac{\partial}{\partial u^{\alpha}_{\mathbf{J}}}, \quad \mathrm{Div}\left(A_{\mathbf{u}}\left(\mathbf{n},\mathbf{w}\right)\right) = \sum_{\mathbf{J}} \left(\mathrm{S}_{\mathbf{J}} - \mathrm{id}\right) w^{\alpha} \mathrm{S}_{-\mathbf{J}} \left(\frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}}\right),$$

with $-\mathbf{J}$ the set of integers in the pullback of the inverse path $P_{-\mathbf{J}}$ (path from $\mathbf{J}$ to 0). The divergence terms $\mathrm{Div}\left(A_{\mathbf{u}}\right)$ summed over the domain will only give boundary terms which are assumed to disappear. Therefore, the extrema satisfy the system of Euler–Lagrange equations

$$\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right) = \sum_{\mathbf{n}} \mathrm{S}_{-\mathbf{J}} \frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}} = \sum_{\mathbf{n}} \sum_{\mathbf{J}} \frac{\partial \left(\mathrm{S}_{-\mathbf{J}}\mathrm{L}\right)}{\partial u^{\alpha}_{\mathbf{0}}} = 0,$$

and the Euler operator with respect to the original variable u^{α} is

$$\mathrm{E}_{u^{\alpha}} = \mathrm{S}_{-\mathbf{J}} \frac{\partial}{\partial u^{\alpha}_{\mathbf{J}}}.$$

Divergence terms are in the kernel of the Euler operator, that is,

$$\mathrm{E}_{u^{\alpha}}\left(\mathrm{Div}\left(A\right)\right) = 0,$$

for any divergence term $\mathrm{Div}\left(A\right)$. The systems of Euler–Lagrange equations over $\mathbf{n}$ are

$$\mathrm{E}_{u^{\alpha}}\left(\mathrm{L}\right) = \mathrm{S}_{-\mathbf{J}} \frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J}}} = 0, \quad \alpha = 1, \ldots, q. \tag{3.2}$$

Example 3.2.1. For an example consider the Lagrangian,

$$\mathrm{L} = \ln\left(\frac{u_{:} - u_0}{u_1 - u_0}\right), \tag{3.3}$$

and find the Euler–Lagrange equation. Using (3.2) the Euler–Lagrange equation is

$$\mathrm{E}_{u}\left(\mathrm{L}\right) = \frac{u_{:} - u_1}{\left(u_{:} - u_0\right)\left(u_1 - u_0\right)} + \mathrm{S}_{-(1)}\left(\frac{1}{u_0 - u_1}\right) + \mathrm{S}_{-(:)}\left(-\frac{1}{u_0 - u_{:}}\right) = 0$$

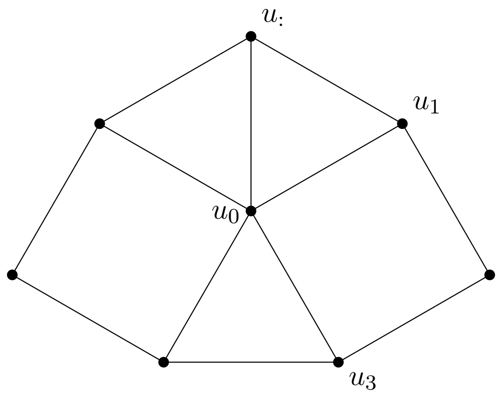


Figure 3.2: Relevant points for $\mathrm{E}_u\left(\mathrm{L}\right)$.

where $\mathrm{S}_{-(1)} = \mathrm{S}_3$ and $\mathrm{S}_{-(:)} = \mathrm{S}_:$, so

$$\mathrm{E}_u\left(\mathrm{L}\right) = \frac{u_: - u_1}{\left(u_: - u_0\right)\left(u_1 - u_0\right)} + \frac{1}{u_3 - u_0} - \frac{1}{u_: - u_0} = 0. \tag{3.4}$$

Figure 3.2 shows all the points that are involved in the Euler–Lagrange equation $\mathrm{E}_u\left(\mathrm{L}\right) = 0$.

Remark 3.2.1. The Euler–Lagrange equations can also be found using an additional independent variable t:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{(u^\alpha)' = w^\alpha} \mathcal{L}\left[\mathbf{u}\right] = 0.$$

with $(u^\alpha)' = \mathrm{d}u^\alpha/\mathrm{d}t$. Alternatively, one could also use difference forms and a vertical derivative, as described in Chapter 2.

3.3 Difference moving frames for a non-rectangular mesh

The geometric setting for functions on the snub square tiling is represented by its prolongation space. Coupling this prolongation space with the discrete moving frame (Definition 2.4.13) leads to a difference moving frame. Therefore, let $\mathcal{K}$ and $\rho\left(\left[\mathbf{u}\right]\right)$ denote the cross-section and frame on $\mathbf{n}$, respectively.

Definition 3.3.1. A difference moving frame for the snub square tiling is a discrete moving frame on the prolongation space $P_{\mathbf{n}}\left(U\right)$ where the cross-section over $\mathbf{nJ}$ is represented by $\mathrm{S}_{\mathbf{J}}\mathcal{K}$ and this holds for all $\mathbf{J}$.

Remark 3.3.2. This definition also applies to other tilings with two independent steps (see Appendix B), not just the snub square tiling.

From this definition, a large amount of the content in Chapter 2 can be replicated for the non-rectangular case. This includes all the definitions and propositions in Section 2.4 that follow after Definition 2.4.13 of the discrete moving frame. The only change to make for these definitions and propositions is that the multi-index $\mathbf{J}$ now represents the concatenation of indices in the pullback of the path, $P_{\mathbf{J}}$, from 0 to $\mathbf{J}$. Therefore, the invariants $I^{\alpha}_{\mathbf{0},\mathbf{0}} = \rho_{\mathbf{0}}([\mathbf{u}]) \cdot u^{\alpha}_{\mathbf{0}}$ along with the Maurer–Cartan invariants $K_{(i)} = (\mathrm{S}_i\rho([\mathbf{u}]))\,\rho([\mathbf{u}])^{-1}$ with $i = 1, :$, and their shifts, provide the set of all invariants. From here if one is using the additional dummy variable method, the important differential-difference syzygies are still

$$\frac{\mathrm{d}\kappa^{\beta}}{\mathrm{d}t} = \mathcal{H}^{\beta}_{\alpha}\sigma^{\alpha}.$$

These are then found using either method described in Section 2.4. Using difference forms, as defined in the next section, the linear difference operator

$$\mathcal{H}^{\beta}_{\alpha} = \sum_{\mathbf{J}} \iota\left(\frac{\partial \kappa^{\beta}}{\partial u^{\delta}_{\mathbf{J}}}\right) \iota\left(\left(\theta_{\mathbf{J}}\right)^{\delta}_{\alpha}\right) \mathrm{S}_{\mathbf{J}}$$

from Section 2.6 can be used. The adjoint of a linear difference operator is defined in the same way as in Section 2.5. Therefore, the adjoint linear operator can be found and is equivalent to

$$\left(\mathcal{H}^{\beta}_{\alpha}\right)^{\dagger} = \sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}} \left(\iota\left(\frac{\partial \kappa^{\beta}}{\partial u^{\delta}_{\mathbf{J}}}\right) \iota\left(\left(\theta_{\mathbf{J}}\right)^{\delta}_{\alpha}\right)\right) \mathrm{S}_{-\mathbf{J}},$$

found in Chapter 2.

3.4 Non-rectangular invariant Euler–Lagrange equations

To see how Proposition 2.6.5 and its proof carry over to the non-rectangular case, all that remains is to understand how difference forms can be defined on a non-rectangular tiling. The following definition of difference forms is a standard application of homology theory, developed by Hydon [13].

Let $\mathcal{T}$ be an edge-to-edge tiling of $\mathbb{R}^2$; this consists of points, edges and tiles (which need not be the same shape as one another). The tiling splits $\mathbb{R}^2$ into disjoint interiors of tiles, interiors of edges, and points. The points are the vertices of the polygonal tiles, and the edges link neighbouring points.

The standard (anticlockwise) orientation on $\mathbb{R}^2$ leads to a compatible orientation on each tile, which induces an orientation on the edges that bound the tile; neighbouring tiles that share an edge induce opposite orientations on their common edge.

Consider what can be deduced directly from the given tiling $\mathcal{T}$. In the following, $\mathbf{I}$ is any index.

Let $c^{(0)}_{\mathbf{I}}$ be the point labelled $\mathbf{I}$; the set of all fundamental 0-chains is the set of all $c^{(0)}_{\mathbf{I}}$. A 0-chain is a linear combination of fundamental 0-chains.

Let $c^{(1)}_{\mathbf{I}}$ be the edge labelled $\mathbf{I}$, and let $c^{(2)}_{\mathbf{I}}$ be the tile labelled $\mathbf{I}$; the set of all fundamental r-chains is the set of all $c^{(r)}_{\mathbf{I}}$, and an r-chain is a linear combination of fundamental r-chains. The vector space of all r-chains is denoted by $\Lambda^r(\mathcal{T})$

Example 3.4.1. Consider the 2-tile system shown in Figure 3.3; taking orientation into account, label the fundamental 2-tiles as $c^{(2)}_{1245}$ and $c^{(2)}_{234}$. The fundamental 1-chains are the edges $c^{(1)}_{12}$, $c^{(1)}_{15}$, $c^{(1)}_{23}$, $c^{(1)}_{24}$, $c^{(1)}_{34}$ and $c^{(1)}_{45}$. The fundamental 0-chains are $c^{(0)}_{1}$, $c^{(0)}_{2}$, $c^{(0)}_{3}$, $c^{(0)}_{4}$ and $c^{(0)}_{5}$. The index used is a matter of convenience; Cartesian coordinates would be just as well. The boundary of $c^{(2)}_{1245}$ is

$$\partial c^{(2)}_{1245} = c^{(1)}_{12} + c^{(1)}_{24} + c^{(1)}_{45} - c^{(1)}_{15}.$$

Here the assumed orientation of the fundamental 1-chains is from the smallest index to the largest; if we had chosen to label the edge between points 1 and 5 as $c^{(1)}_{51}$, with the orientation from the first index to the second, we would have written $c^{(1)}_{51}$ in place of $-c^{(1)}_{15}$. Similarly $\partial c^{(2)}_{234} = c^{(1)}_{23} + c^{(1)}_{34} - c^{(1)}_{24}$. The boundary operator ∂ acts on the fundamental 1-forms $c^{(1)}_{ij}$ as follows: $\partial c^{(1)}_{ij} = c^{(0)}_{j} - c^{(0)}_{i}$.

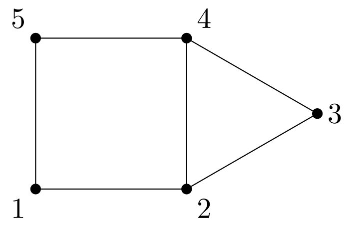


Figure 3.3: Two tiles of the snub square tiling.

Note that

$$\begin{aligned}\partial\left(\partial c^{(2)}_{234}\right) &= \partial\left(c^{(1)}_{23}+c^{(1)}_{34}-c^{(1)}_{24}\right)\\ &= \left(c^{(0)}_{3}-c^{(0)}_{2}\right)+\left(c^{(0)}_{4}-c^{(0)}_{3}\right)-\left(c^{(0)}_{4}-c^{(0)}_{2}\right)=0.\end{aligned}$$

Similarly, $\partial^2 c^{(2)}_{1245}=0$.

More generally, the boundary operator ∂ is defined as the linear map taking each (oriented) r-chain to its boundary. This is a slight adaptation of what happens for a triangular tiling (where every tile is a triangle). In that case, the orientation of the triangle is shown by the ordering of its vertices $c^{(2)}_{ijk}$, and the boundary operator gives

$$\partial c^{(2)}_{ijk}=c^{(1)}_{ij}+c^{(1)}_{jk}-c^{(1)}_{ik}.$$

Similarly, $\partial c^{(1)}_{rs}=c^{(0)}_{s}-c^{(0)}_{r}$, so that $\partial^2 c^{(2)}_{ijk}=0$.

One can regard $c^{(2)}_{1245}$ in the example above as a chain consisting of $c^{(2)}_{124}+c^{(2)}_{145}$, so that the boundary is

$$\begin{aligned}\partial\left(c^{(2)}_{124}+c^{(2)}_{145}\right) &= \partial c^{(2)}_{124}+\partial c^{(2)}_{145}\\ &= \left(c^{(1)}_{12}+c^{(1)}_{24}-c^{(1)}_{14}\right)+\left(c^{(1)}_{14}+c^{(1)}_{45}-c^{(1)}_{15}\right)\\ &= c^{(1)}_{12}+c^{(1)}_{24}+c^{(1)}_{45}-c^{(1)}_{15},\end{aligned}$$

as expected. The internal 'edge', $c^{(1)}_{14}$, which is not part of the tiling, is cancelled out by the boundary operator.

The point is that $\partial:\Lambda_r(\mathcal{T})\mapsto\Lambda_{r-1}(\mathcal{T})$, $r\geq 1$, is a familiar object, behaving in exactly the same way as the boundary operator for simplicial homology. The coboundary operator for simplicial complexes is defined by $(\delta\omega)(c) = (\omega)(\partial c)$, where ω is an r-form and c is an $(r+1)$-chain; brackets are used to indicate the pairing between forms and chains. Note that if ω is an r-form and c is an $(r+2)$-chain,

$$\left(\delta^2\omega\right)(c) = (\delta\omega)(\partial c) = \omega\left(\partial^2 c\right) = 0,$$

so $\delta^2 = 0$.

The construction of the coboundary operator in the simplicial case suggests how to deal with difference forms on a general tiling. Define the fundamental r-forms $\Delta^{\mathbf{I}}_{(r)}$ dual to the fundamental r-chains. With the pairing indicated by an integral sign,

$$\int_{c^{(r)}_{\mathbf{J}}} \Delta^{\mathbf{I}}_{(r)} = \delta^{\mathbf{I}}_{\mathbf{J}} = \begin{cases} 1, & \mathbf{I} = \mathbf{J}, \\ 0, & \mathbf{I} \neq \mathbf{J}, \end{cases}$$

for all fundamental r-chains $c^{(r)}_{\mathbf{J}}$.

Then an r-cochain is a linear combination of the fundamental r-cochains, with coefficients in the same (abelian) group as the r-chains. So if $c^{(r)} = a^{\mathbf{J}} c^{(r)}_{\mathbf{J}}$, and $\omega = b_{\mathbf{I}} \Delta^{\mathbf{I}}_{(r)}$ is an r-cochain,

$$\int_{c^{(r)}} \omega = a^{\mathbf{J}} b_{\mathbf{I}} \int_{c^{(r)}_{\mathbf{J}}} \Delta^{\mathbf{I}}_{(r)} = a^{\mathbf{J}} b_{\mathbf{I}} \delta^{\mathbf{I}}_{\mathbf{J}} = a^{\mathbf{J}} b_{\mathbf{J}}.$$

The coboundary operator Δ is defined by

$$\int_{c^{(r+1)}} \Delta\omega = \int_{\partial c^{(r+1)}} \omega,$$

for all $c^{(r+1)} \in \Lambda_{r+1}$, which looks like Stokes' Theorem, but merely expresses a definition that is analogous to that for the simplicial coboundary operator. Let $\Lambda^r(\mathcal{T})$ denote the vector space of r-cochains; then $\Delta : \Delta^r(\mathcal{T}) \mapsto \Delta^{r+1}(\mathcal{T})$ and $\Delta^2 = 0$.

For now, assume that the coefficients of chains and cochains are arbitrary real numbers. A function f takes values on the vertices, so it can be regarded as a mapping $f : \Lambda_0 \to \mathbb{R}$. So $f = f_i \Delta^i_{(0)}$, where $f_i \in \mathbb{R}$, and therefore

$$\int_{c_j^{(0)}} f = f_i \delta^i_j = f_j.$$

Similarly, a 1-form λ can be written as

$$\lambda = \lambda_{ij} \Delta^{ij}_{(1)}$$

(as every edge can be indexed by the points it joins). Thus

$$\int_{c_{kl}^{(1)}} \lambda = \lambda_{ij} \delta^{ij}_{kl} = \lambda_{ij} \delta^i_k \delta^j_l = \lambda_{kl}.$$

By definition,

$$\int_{c_{kl}^{(1)}} \Delta f = \int_{\partial c_{kl}^{(1)}} f = \int_{c_l^{(0)}} f - \int_{c_k^{(0)}} f = f_l - f_k,$$

and therefore $\Delta f = (f_l - f_k)\, \Delta^{kl}_{(1)}$. Note that as $\Delta^{lk}_{(1)} = -\Delta^{kl}_{(1)}$, this would have been written as $\Delta f = (f_k - f_l)\, \Delta^{lk}_{(1)}$ if we had chosen the indexing differently.

Similarly, for a given 2-form, $\eta = \eta_{\mathbf{I}} \Delta^{\mathbf{I}}_{(2)}$, where $\eta_{\mathbf{I}} \in \mathbb{R}$,

$$\int_{c_{\mathbf{J}}^{(2)}} \eta = \eta_{\mathbf{J}},$$

and

$$\int_{c_{\mathbf{J}}^{(2)}} \Delta\lambda = \int_{\partial c_{\mathbf{J}}^{(2)}} \lambda$$

defines $\Delta\lambda$ in terms of $\Delta^{\mathbf{I}}_{(2)}$. Given that some tiles may not be triangles, this needs to be calculated case-by-case.

Example 3.4.2. (Example 3.4.1 Cont.)

For the 2-tile in Figure 3.3, the 2-forms are

$$\eta = \eta_{1245} \Delta^{1245}_{(2)} + \eta_{234} \Delta^{234}_{(2)}.$$

In particular,

$$\int_{c^{(2)}_{1245}} \Delta\lambda = \int_{c^{(1)}_{12}} \lambda + \int_{c^{(1)}_{24}} \lambda + \int_{c^{(1)}_{45}} \lambda - \int_{c^{(1)}_{15}} \lambda = \lambda_{12} + \lambda_{24} + \lambda_{45} - \lambda_{15},$$

$$\int_{c^{(2)}_{234}} \Delta\lambda = \int_{c^{(1)}_{23}} \lambda + \int_{c^{(1)}_{34}} \lambda - \int_{c^{(1)}_{24}} \lambda = \lambda_{23} + \lambda_{34} - \lambda_{24},$$

so

$$\Delta\lambda = (\lambda_{12} + \lambda_{24} + \lambda_{45} - \lambda_{15})\,\Delta^{1245}_{(2)} + (\lambda_{23} + \lambda_{34} - \lambda_{24})\,\Delta^{234}_{(2)}$$

Note that if

$$\begin{aligned}\lambda &= \Delta f\\ &= (f_2 - f_1)\,\Delta^{12}_{(1)} + (f_5 - f_1)\,\Delta^{15}_{(1)} + (f_3 - f_2)\,\Delta^{23}_{(1)}\\ &\quad + (f_4 - f_2)\,\Delta^{24}_{(1)} + (f_4 - f_3)\,\Delta^{34}_{(1)} + (f_5 - f_4)\,\Delta^{45}_{(1)}\end{aligned}$$

then we obtain $\Delta^2 f = 0$, as required.

Everything that has been done so far is coordinate-free. Functions are paired with points, 1-forms are paired with edges, and 2-forms are paired with tiles. The homology and cohomology groups will be exactly those that would be obtained using a simplicial approach for any triangulation that uses the same set of vertices.

The above definition of difference forms enables us to find the invariantized Euler–Lagrange equations for a Lie group invariant Lagrangian. The setup is the same as before: suppose that we have a Lie group action $G \times M \to M$, and a difference frame for this action. Any group-invariant Lagrangian $\mathrm{L}\,(\mathbf{n}, [\mathbf{u}])$ can be written, in terms of the generating invariants κ^β and their shifts $\kappa^\beta_{\mathbf{J}} = \mathrm{S}_{\mathbf{J}}\kappa^\beta$, as $L^{\boldsymbol{\kappa}}\,(\mathbf{n}, [\boldsymbol{\kappa}])$, for some finite number of shifts of the generating difference invariants. Let the difference Euler operator with respect to κ^β be the same as before, namely

$$\mathrm{E}_{\kappa^\beta} = \mathrm{S}_{-\mathbf{J}}\,\frac{\partial}{\partial \kappa^\beta_{\mathbf{J}}}\,,$$

where $\mathbf{J}$ $(-\mathbf{J})$ is the concatenation of indices in the pullback of the path, $P_{\mathbf{J}}$ $(P_{-\mathbf{J}})$, from 0 to $\mathbf{J}$ ($\mathbf{J}$ to 0).

Proposition 3.4.1. The invariantization of the original Euler–Lagrange equations is

$$\iota\left(\mathrm{E}_{u^\alpha}\left(\mathrm{L}\right)\right) = \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right), \tag{3.5}$$

where

$$\left(\mathcal{H}^\beta_\alpha\right)^\dagger = \sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}} \left(\iota\left(\frac{\partial \kappa^\beta}{\partial u^\delta_{\mathbf{J}}}\right) \iota\left(\left(\theta_{\mathbf{J}}\right)^\delta_\alpha\right)\right) \mathrm{S}_{-\mathbf{J}}. \tag{3.6}$$

Proof. The proof remains the same as the one in Proposition 2.6.5, however, the difference forms are defined as above. Assuming an anti-clockwise orientation on all the tiles means that tiles which share an edge will have that edge summed both positively and negatively. Therefore, any divergence terms will vanish leaving only boundary terms which are assumed to disappear. □

Remark 3.4.2. In a similar manner to the rectangular case for the non-rectangular case, the invariant Euler–Lagrange equations can also be written as

$$\iota\left(\mathrm{E}_{u^\alpha}\left(\mathrm{L}\right)\right) = \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right),$$

with

$$\left(\mathcal{H}^\beta_\alpha\right)^\dagger = \sum_{\mathbf{J}} \iota\left(\frac{\partial \kappa^\beta_{-\mathbf{J}}}{\partial u^\alpha_{\mathbf{0}}}\right) \mathrm{S}_{-\mathbf{J}}.$$

Therefore, as before there are two representations of the invariant Euler–Lagrange equations as

$$\sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}} \left[\iota\left(\frac{\partial \kappa^\beta}{\partial u^\delta_{\mathbf{J}}}\right) \iota\left(\left(\theta_{\mathbf{J}}\right)^\delta_\alpha\right) \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\right] = \sum_{\mathbf{J}} \iota\left(\frac{\partial \kappa^\beta_{-\mathbf{J}}}{\partial u^\alpha_{\mathbf{0}}}\right) \left[\mathrm{S}_{-\mathbf{J}} \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right)\right].$$

Example 3.4.3. Here we show how to find the invariant Euler–Lagrange equation of the Lagrangian

$$\mathrm{L} = \ln\left(\frac{u_{:} - u_0}{u_1 - u_0}\right). \tag{3.7}$$

This has several Lie symmetries including scaling and translations, for which the action is

$$g \cdot u = au + b.$$

For the construction of the moving frame let the cross-section be given by the normalization equations (2.16)

$$g \cdot u_0 = 0, \quad g \cdot u_1 = 1$$

which gives the value of the parameters

$$a = \frac{1}{u_1 - u_0}, \quad b = \frac{-u_0}{u_1 - u_0}. \tag{3.8}$$

In this example, take the two generating invariants to be

$$\kappa^1 = \iota\left(u_{:}\right) = \frac{u_{:} - u_0}{u_1 - u_0}, \quad \kappa^2 = \iota\left(u_2\right) = \frac{u_2 - u_0}{u_1 - u_0}.$$

In a similar manner to Remark 2.4.12 one can find recurrence relations that determine all $\iota\left(u_{\mathbf{J}}\right)$ in terms of shifts of the generating invariants.

Thus, the invariant form of the Lagrangian (3.7) is

$$L^{\boldsymbol{\kappa}} = \ln\left(\kappa^1\right). \tag{3.9}$$

Therefore,

$$\mathrm{E}_{\kappa^1}\left(L^{\boldsymbol{\kappa}}\right) = \frac{1}{\kappa^1}, \qquad \mathrm{E}_{\kappa^2}\left(L^{\boldsymbol{\kappa}}\right) = 0.$$

To find the invariant Euler–Lagrange equation, use the formula in Proposition 3.4.1. For this example, the relevant adjoint linear operator is

$$\left(\mathcal{H}_1^1\right)^\dagger = \sum_{\mathbf{J}} \mathrm{S}_{-\mathbf{J}} \left(\iota\left(\frac{\partial \kappa^1}{\partial u_{\mathbf{J}}^1}\right) \iota\left(\left(\theta_{\mathbf{J}}\right)_1^1\right)\right) \mathrm{S}_{-\mathbf{J}},$$

where $u^1 = u$. The invariantization (2.31) of the nonzero partial derivatives gives

$$\iota\left(\frac{\partial \kappa^1}{\partial u_0}\right) = \kappa^1 - 1, \quad \iota\left(\frac{\partial \kappa^1}{\partial u_1}\right) = -\kappa^1, \quad \iota\left(\frac{\partial \kappa^1}{\partial u_{:}}\right) = 1. \tag{3.10}$$

The values of the components $\iota(\theta_{\mathbf{J}})$ for $\mathbf{J} = 0, 1, :$ are

$$\begin{aligned}
\theta_0 &= u_1 - u_0, \\
\theta_1 &= u_2 - u_1, \\
\theta_{:} &= u_{:1} - u_{:},
\end{aligned}$$

and the invariantization (2.31) of these are

$$\begin{aligned}
\iota(\theta_0) &= 1, \\
\iota(\theta_1) &= \kappa^2 - 1, \\
\iota(\theta_{:}) &= \iota(u_{:1}) - \kappa^1 = -\frac{\kappa^1}{\kappa^1_{:}} .
\end{aligned}$$

Therefore, the invariant form of the Euler–Lagrange equation is

$$\left(\mathcal{H}^1_1\right)^{\dagger} \mathrm{E}_{\kappa^1}\left(L^{\boldsymbol{\kappa}}\right) = \frac{\kappa^1 - 1}{\kappa^1} - \left(\kappa^2_3 - 1\right) - \frac{1}{\kappa^1} , \tag{3.11}$$

which can be checked by comparing with the invariantization of the original Euler–Lagrange equation.

Chapter 4

Differential-difference variational problems

This chapter introduces and develops the invariant calculus of variations for differential-difference equations (DΔEs). The first DΔE we consider has one independent continuous variable, one independent discrete variable, one dependent variable and a Lie group action on the dependent variable only. This is then extended to DΔEs with more of each of these variables with the group action still only on the dependent variables. Finally, the case where the group action is on the independent variable or both variables is discussed for DΔEs with one of each type of variable.

4.1 Differential-difference structure

This section summarizes the geometric setting for DΔEs by Peng and Hydon [32]. This paper resolves the extent to which a continuous symmetry can depend on the discrete independent variables and dependent variables. DΔEs, like PDEs and PΔEs, have independent and dependent variables. The distinction between these variables leads to a geometric structure, the prolongation space, whose preservation determines the class of transformations and, in particular, symmetries. The prolongation structure of PDEs and PΔEs is well known, (see [26], [28]). A combination of these structures gives rise to the structure for DΔEs. Restricting attention to DΔEs on $\mathbb{R}^p \times \mathbb{Z}^m$ (for other domains, treat these variables as local

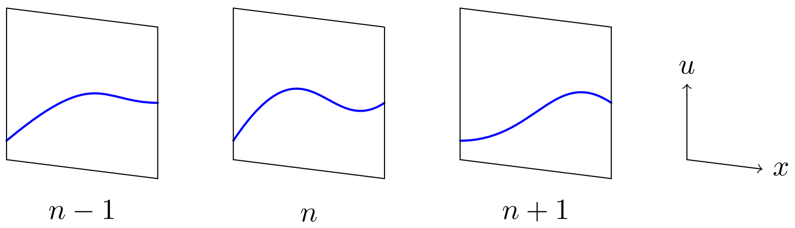


Figure 4.1: The total prolongation space for a DΔE with one independent continuous variable, one independent discrete variable and one dependent variable is $\mathcal{T} = \mathbb{R} \times \mathbb{Z} \times \mathbb{R}$. A representation is shown in this figure.

coordinates), the continuous independent variables are $\mathbf{x} = \left(x^1, \ldots, x^p\right) \in \mathbb{R}^p$, the discrete independent variables are $\mathbf{n} = \left(n^1, \ldots, n^m\right) \in \mathbb{Z}^m$, and the dependent variables are $\mathbf{u} = \left(u^1, \ldots, u^q\right) \in \mathbb{R}^q$. The total space is $\mathcal{T} = \mathbb{R}^q \times \mathbb{Z}^m \times \mathbb{R}^q$. As an example Figure 4.1 shows a graph of the total space for one of each variable.

To avoid discussion of technicalities associated with singularities and other discontinuities, from this point onwards all functions are assumed to be locally smooth in each of their continuous arguments, for each $\mathbf{n}$.

4.1.1 Differential structure

Given $\mathbf{n} \in \mathbb{Z}^m$, a slice $\mathcal{T}_{\mathbf{n}} = \mathbb{R}^p \times \{\mathbf{n}\} \times \mathbb{R}^q$ is a continuous space whose functions $\mathbf{u} = \mathbf{f}\left(\mathbf{x}, \mathbf{n}\right)$, restricted to this slice, can be prolonged by differentiation as many times as is needed. Hence one can define the infinite jet space over $\mathbf{n}$, denoted by $J^\infty\left(\mathcal{T}_{\mathbf{n}}\right)$. The coordinates $u^\alpha_{\mathbf{J};\mathbf{0}}$ with $\mathbf{J} = \left(j^1, \ldots, j^p\right)$ represent the values of the derivatives of the dependent variables, where each j^i denotes the number of derivatives with respect to x^i. In particular, $u^\alpha_{\mathbf{0};\mathbf{0}} = u^\alpha$ and the first derivatives of $u^\alpha = f^\alpha\left(\mathbf{x}, \mathbf{n}\right)$ are represented by the coordinate values

$$u_{\mathbf{1}_i;\mathbf{0}} = \frac{\partial f^\alpha\left(\mathbf{x}, \mathbf{n}\right)}{\partial x^i}, \quad i = 1, \ldots p,$$

where $\mathbf{1}_i$ has 1 in the i-th entry and zero elsewhere. More generally, the action of the first derivative with respect to x^i on any differentiable function defined on $J^\infty\left(\mathcal{T}_{\mathbf{n}}\right)$ is given by the operator

$$D_i|_{J^\infty(\mathcal{T}_{\mathbf{n}})} := \frac{\partial}{\partial x^i} + u^\alpha_{\mathbf{J}+\mathbf{1}_i;\mathbf{0}} \frac{\partial}{\partial u^\alpha_{\mathbf{J};\mathbf{0}}}.$$

It is enough to consider the jet space over a single (arbitrary) slice $\mathcal{T}_{\mathbf{n}}$ as a copy of this can be produced over every $\mathbf{n}$. This is due to $\mathbf{x}$ and $\mathbf{n}$ being mutually independent; consequently, the total jet space is

$$J^{\infty}\left(\mathcal{T}\right) \cong \mathbb{Z}^m \times J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right).$$

4.1.2 Difference structure

The majority of the difference structure remains the same as in Section 2.1. However, there are some important extensions to make for DΔEs. The total space $\mathcal{T}$ is preserved by all translations

$$\mathrm{T}_{\mathbf{K}} : \mathcal{T} \to \mathcal{T}, \qquad \mathrm{T}_{\mathbf{K}} : (\mathbf{x}, \mathbf{n}, \mathbf{u}) \mapsto (\mathbf{x}, \mathbf{n} + \mathbf{K}, \mathbf{u}),$$

with $\mathrm{T}_{\mathbf{L}} \circ \mathrm{T}_{\mathbf{K}} = \mathrm{T}_{\mathbf{K}+\mathbf{L}}$ for all $\mathbf{K}, \mathbf{L} \in \mathbb{Z}^m$. The total space is disconnected; however, a connected representation is possible over each $\mathbf{n}$. Using the pullback $\mathrm{T}^*_{\mathbf{K}}$, for all $\mathbf{K}$, each slice can be prolonged to include coordinates on other slices. The prolongation space, denoted $P\left(\mathcal{T}_{\mathbf{n}}\right)$ includes all coordinates

$$u^{\alpha}_{\mathbf{0};\mathbf{K}} = \mathrm{T}^*_{\mathbf{K}}\left(u^{\alpha}_{\mathbf{0};\mathbf{0}}|_{\mathbf{n}+\mathbf{K}}\right).$$

This difference prolongation structure must be preserved by every transformation.

To combine the differential and difference structure first extend the translation operator $\mathrm{T}_{\mathbf{K}}$ to the total jet space:

$$\mathrm{T}_{\mathbf{K}} : J^{\infty}\left(\mathcal{T}\right) \to J^{\infty}\left(\mathcal{T}\right), \quad \mathrm{T}_{\mathbf{K}} : \left(\mathbf{n}, \mathbf{x}, \ldots, u^{\alpha}_{\mathbf{J};\mathbf{0}}, \ldots\right) \mapsto \left(\mathbf{n} + \mathbf{K}, \mathbf{x}, \ldots, u^{\alpha}_{\mathbf{J};\mathbf{0}}, \ldots\right).$$

Then the coordinates on the jet space over $\mathbf{n} + \mathbf{K}$ can be pulled back to the prolongation space over $\mathbf{n}$ in a similar way to the difference case. Therefore, the space $P\left(J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right)\right)$ has coordinates

$$u^{\alpha}_{\mathbf{J};\mathbf{K}} = \mathrm{T}^*_{\mathbf{K}}\left(u^{\alpha}_{\mathbf{J};\mathbf{0}}|_{\mathbf{n}+\mathbf{K}}\right)$$

and is a connected component on $\mathbf{n}$ of the total prolongation space denoted

$P\left(J^{\infty}\left(\mathcal{T}\right)\right) \cong \mathbb{Z}^m \times P\left(J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right)\right)$. The composition rule for translations gives

$$u^{\alpha}_{\mathbf{J};\mathbf{K}+\mathbf{L}} = \mathrm{T}^{*}_{\mathbf{K}}\left(u^{\alpha}_{\mathbf{J};\mathbf{L}}|_{\mathbf{n}+\mathbf{K}}\right).$$

In general, if f is a (locally smooth) function on $P\left(J^{\infty}\left(\mathcal{T}\right)\right)$ with its restriction to $P\left(J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right)\right)$ denoted by

$$f_{\mathbf{n}}\left(\mathbf{x}, \ldots, u^{\alpha}_{\mathbf{J};\mathbf{L}}, \ldots\right) := f\left(\mathbf{x}, \mathbf{n}, \ldots, u^{\alpha}_{\mathbf{J};\mathbf{L}}, \ldots\right),$$

then the pullback of $f_{\mathbf{n}+\mathbf{K}}\left(\mathbf{x}, \ldots, u^{\alpha}_{\mathbf{J};\mathbf{L}}, \ldots\right)$ to $P\left(J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right)\right)$ is

$$\mathrm{T}^{*}_{\mathbf{K}} f_{\mathbf{n}+\mathbf{K}} = f\left(\mathbf{x}, \mathbf{n}+\mathbf{K}, \ldots, u^{\alpha}_{\mathbf{J};\mathbf{K}+\mathbf{L}}, \ldots\right).$$

The shift operator,

$$\mathrm{S}_{\mathbf{K}} : f\left(\mathbf{x}, \mathbf{n}, \ldots, u^{\alpha}_{\mathbf{J};\mathbf{L}}, \ldots\right) \mapsto f\left(\mathbf{x}, \mathbf{n}+\mathbf{K}, \ldots, u^{\alpha}_{\mathbf{J};\mathbf{L}+\mathbf{K}}, \ldots\right),$$

mimics the action of the translation $\mathrm{T}_{\mathbf{K}}$ on $P\left(J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right)\right)$, with the same relation $\mathrm{S}_{\mathbf{K}} f_{\mathbf{n}} = \mathrm{T}^{*}_{\mathbf{K}} f_{\mathbf{n}+\mathbf{K}}$ as before. From here we use shifts to denote points on different slices, that is, we use $\mathrm{S}_{\mathbf{K}}\left(u^{\alpha}_{\mathbf{0};\mathbf{0}}|_{\mathbf{n}}\right)$ or $\mathrm{S}_{\mathbf{K}} u^{\alpha}_{\mathbf{0};\mathbf{0}}$ for $\mathrm{T}^{*}_{\mathbf{K}}\left(u^{\alpha}_{\mathbf{0};\mathbf{0}}|_{\mathbf{n}+\mathbf{K}}\right)$. Consequently, the derivative with respect to x^i on $J^{\infty}\left(\mathcal{T}\right)$ is represented on $P\left(J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right)\right)$ by the total derivative

$$D_i = \frac{\partial}{\partial x^i} + u^{\alpha}_{\mathbf{J}+\mathbf{1}_i;\mathbf{K}} \frac{\partial}{\partial u^{\alpha}_{\mathbf{J};\mathbf{K}}}.$$

Crucially, all the total derivatives and shift operators commute:

$$D_i D_j = D_j D_i, \qquad D_i \mathrm{S}_{\mathbf{K}} = \mathrm{S}_{\mathbf{K}} D_i, \qquad \mathrm{S}_{\mathbf{K}} \mathrm{S}_{\mathbf{L}} = \mathrm{S}_{\mathbf{L}} \mathrm{S}_{\mathbf{K}}.$$

It is convenient to use the following shorthand notation for a product of total derivatives:

$$D_{\mathbf{J}} = D_1^{j^1} \cdots D_p^{j^p}, \qquad \text{where} \quad \mathbf{J} = \left(j^1, \ldots, j^p\right).$$

The difference operators on the continuous space $P\left(J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right)\right)$ arise from the ordering of each n^i. For any index $\mathbf{K} = \left(k^1, \ldots, k^m\right)$, the corresponding shift operator is $\mathrm{S}_{\mathbf{J}} = \mathrm{S}_1^{k_1} \cdots \mathrm{S}_m^{k_m}$, where $\mathrm{S}_i := \mathrm{S}_{\mathbf{1}_i}$ denotes the forward shift with respect to n^i. Then the forward difference in the n^i-direction is represented on $P\left(J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right)\right)$ by the operator

$$\mathrm{D}_{n^i} := \mathrm{S}_i - \mathrm{id},$$

where id is the identity mapping. A differential-difference divergence is an expression $\mathcal{C}$ of the form

$$\mathcal{C} = D_i F^i + \mathrm{D}_{n^i} G^i.$$

The formal adjoint of a linear operator $\mathcal{H}$ is the unique operator $\mathcal{H}^\dagger$ such that

$$f\mathcal{H}g - \left(\mathcal{H}^\dagger f\right) g = \mathrm{Div}\left(A_{\mathcal{H}}\right),$$

where $\mathrm{Div}\left(A_{\mathcal{H}}\right)$ is a (differential-difference) divergence for all functions f and g defined on the prolongation space $P\left(J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right)\right)$. Here the adjoint of the ith shift operator, the ith differential operator and the identity operator are

$$\mathrm{S}_i^\dagger = \mathrm{S}_i^{-1}, \quad D_i^\dagger = -D_i, \quad \mathrm{id}^\dagger = \mathrm{id},$$

respectively. The composition rule $\left(\mathcal{H}_1\mathcal{H}_2\right)^\dagger = \mathcal{H}_2^\dagger\mathcal{H}_1^\dagger$ determines the adjoint of a product of linear operators. Thus, as the shift operators and differential operators commute,

$$\mathrm{S}_{\mathbf{K}}^\dagger = \mathrm{S}_{-\mathbf{K}}, \qquad D_{\mathbf{J}}^\dagger = (-D)_{\mathbf{J}} := (-1)^{j^1+\cdots+j^p} D_{\mathbf{J}}.$$

4.1.3 Lie point transformations for differential-difference equations

This subsection identifies the constraints that must be satisfied by transformations of DΔEs. Using the passive viewpoint, that is, viewing a transformation as a change of coordinates, a point transformation is a transformation of the total space

$$\Gamma : \mathcal{T} \to \mathcal{T}, \quad \Gamma : (\mathbf{x}, \mathbf{n}, \mathbf{u}) \mapsto \left(\hat{\mathbf{x}}, \hat{\mathbf{n}}, \hat{\mathbf{u}}\right).$$

Here we do not consider lattice transformations (see [15]), so $\hat{\mathbf{n}} = \mathbf{n}$. Every one-parameter Lie group from the total space $\mathcal{T}$ to itself can be expressed in the form

$$\hat{x}^i = x^i + \epsilon\xi^i(\mathbf{x},\mathbf{n},\mathbf{u}) + \mathcal{O}\left(\epsilon^2\right), \quad \hat{u}^\alpha = u^\alpha + \epsilon\eta^\alpha(\mathbf{x},\mathbf{n},\mathbf{u}) + \mathcal{O}\left(\epsilon^2\right). \tag{4.1}$$

As such mappings change only $\mathbf{x}$ and $\mathbf{u}$, they are represented on $P\left(J^\infty\left(\mathcal{T}_\mathbf{n}\right)\right)$ by using the same shift operator $\mathrm{S}_\mathbf{K}$ in both the original and transformed coordinates. Now we introduce an important theorem from Peng and Hydon [32] about the allowable variables for each ξ^i.

Theorem 4.1.1. A one-parameter Lie group of mappings (4.1) prolongs to a transformation group for $P\left(J^\infty\left(\mathcal{T}_\mathbf{n}\right)\right)$ if and only if each ξ^i is independent of $\mathbf{n}$ and $\mathbf{u}$.

Therefore, from Theorem 4.1.1, every one-parameter Lie group from $\mathcal{T}$ to itself that preserves the prolongation structure is necessarily of the form

$$\hat{x}^i = x^i + \epsilon\xi^i(\mathbf{x}) + \mathcal{O}\left(\epsilon^2\right), \quad \hat{n}^i = n^i, \quad \hat{u}^\alpha = u^\alpha + \epsilon\eta^\alpha(\mathbf{x},\mathbf{n},\mathbf{u}) + \mathcal{O}\left(\epsilon^2\right). \tag{4.2}$$

As both continuous independent variables and dependent variables can be acted on by a Lie point symmetry group, there is an increase in the potential choice of normalization equations (2.16) in comparison to the PΔEs case. However, to make the calculations slightly simpler, we restrict the possible normalization equations when a group action on an independent variable exists.

Definition 4.1.2. A projectable normalization is a normalization for which the group parameters affecting the independent variables are written only in terms of the independent variables.

A sufficient condition for this to happen is that the group action is free on the space of independent continuous variables.

4.2 Differential-difference moving frames

The construction of a differential-difference moving frame is similar to that of the difference moving frame. For a differential-difference moving frame, $\mathcal{K}$ and $\rho(\mathbf{x},[\mathbf{u}])$ denote the cross-section and frame on $\mathbf{n}$, respectively.

Definition 4.2.1. A differential-difference moving frame is a discrete moving frame on the prolongation space $P\left(J^{\infty}\left(\mathcal{T}_{\mathbf{n}}\right)\right)$ and the cross-section over $\mathbf{n}+\mathbf{K}$ is represented by $\mathrm{S}_{\mathbf{K}}\mathcal{K}$ and this holds for all necessary $\mathbf{K}$.

Remark 4.2.2. In the differential case, it is often ideal to use the dependent variables with the least number of derivatives for the normalization equations. This advice extends to choosing the normalization equations of a cross-section in the differential-difference case.

Remark 4.2.3. As stated in Chapter 2, discussion of research detailing conditions under which an action will become free and regular is given in [3]. Results for product actions are given by Boutin [6]. Additionally, results for a jet bundle can be found in [8] by Fels and Olver, under a sufficiently large prolongation (considering higher-order derivatives). Here we conjecture that similar results hold for actions in the differential–difference space.

4.3 The differential-difference calculus of variations

The majority of DΔEs of interest have one independent continuous variable and one independent discrete variable. Therefore, we initially consider DΔEs of this type and also assume for simplicity there is only one dependent variable. If one wants a formula for multiple dependent variables just replace u by u^{α} in the equations below. The Lagrangian in this case is $\mathrm{L}=\mathrm{L}\left(x,n,[u]\right)$, where $[u]$ denotes finitely many derivatives and shifts of the original dependent variable u. To find the Euler–Lagrange equations for the difference case one can use difference forms or the additional dummy variable trick. The difference forms method leads to the explicit formula of the invariant formulation of the Euler–Lagrange equations in a more natural way. Therefore, the important horizontal differential-difference forms are

$$\omega=f_{j_1,\ldots,j_m;i_1,\ldots,i_k}\left(\mathbf{x},\mathbf{n},[\mathbf{u}]\right)\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_m}\wedge\Delta^{i_1}\wedge\cdots\wedge\Delta^{i_k}.$$

From this point on, we use D_j (S_j) to denote j derivatives (shifts) for the case with one independent continuous (discrete) variable. This is done so we can use the Einstein summation convention and reduce the number of summation signs. The

Euler–Lagrange equation in the original variables is obtained as follows (using the exterior derivative (2.52))

$$
\begin{aligned}
\mathrm{d}_v \sum \int \mathrm{L}\, \mathrm{d}x \wedge \Delta &= \sum \int \mathrm{d}_v \mathrm{L} \wedge \mathrm{d}x \wedge \Delta \\
&= \sum \int \frac{\partial \mathrm{L}}{\partial u_{j;k}} \mathrm{d}_v u_{j;k} \wedge \mathrm{d}x \wedge \Delta \\
&= \sum \int \left[\left(\mathrm{S}_{-k} \frac{\partial \mathrm{L}}{\partial u_{j;k}} \right) \mathrm{d}_v u_{j;0} + \mathrm{Div}\left(A_{\mathrm{S}}\right) \right] \wedge \mathrm{d}x \wedge \Delta \\
&= \sum \int \left[\left(D_j^{\dagger} \mathrm{S}_{-k} \frac{\partial \mathrm{L}}{\partial u_{j;k}} \right) \mathrm{d}_v u_{0;0} + \mathrm{Div}\left(A_D\right) + \mathrm{Div}\left(A_{\mathrm{S}}\right) \right] \wedge \mathrm{d}x \wedge \Delta \\
&= \sum \int \mathrm{E}_u\left(\mathrm{L}\right) \mathrm{d}_v u_{0;0} \wedge \mathrm{d}x \wedge \Delta.
\end{aligned}
\tag{4.3}
$$

The divergence terms arising from the summation and integration by parts are $\mathrm{Div}\left(A_{\mathrm{S}}\right)$ and $\mathrm{Div}\left(A_{\mathrm{D}}\right)$, respectively. Hence the Euler–Lagrange equation is

$$
\mathrm{E}_u\left(\mathrm{L}\right) = D_j^{\dagger} \mathrm{S}_{-k} \frac{\partial \mathrm{L}}{\partial u_{j;k}} = 0, \tag{4.4}
$$

with $D_j^{\dagger} = \left(-1\right)^j D_j$.

Unlike in the difference case, it is important to consider the Lagrangian functional and not just the Lagrangian itself. This is because Lie point symmetries can act on the independent continuous variable meaning the differential form component $\mathrm{d}x$ is not necessarily invariant. If there is a group action on the independent continuous variable then the invariant differential form is $\iota\left(\mathrm{d}x\right) = J\mathrm{d}x$ with

$$
J = \left(\frac{\partial (g \cdot x)}{\partial x} \right) \Bigg|_{g=\rho_{;k}(x,[u])}, \tag{4.5}
$$

for all $k \in \mathbb{Z}$ when a projectable normalization (Definition 4.1.2) is used. Here $\rho_{;k}\left(x, [u]\right)$ denotes k shifts of the moving frame $\rho_{;0}\left(x, [u]\right)$. Therefore, the Lagrangian functional

$$
\sum \int \mathrm{L}\left(x, n, [u]\right) \mathrm{d}x \wedge \Delta
$$

can also be written as

$$
\sum \int \overline{\mathrm{L}}\left(x, n, [u]\right) \iota\left(\mathrm{d}x\right) \wedge \Delta \tag{4.6}
$$

with $\overline{\mathrm{L}} = J^{-1}\mathrm{L}$. This form of the Lagrangian functional is helpful to find the invariant formulation of the Euler–Lagrange equation when using a group action on the independent variable. The Euler operator of $\overline{L}$ with respect to u can be achieved using differential-difference forms

$$\begin{aligned}
\mathrm{d}_v \sum \int \overline{\mathrm{L}}\iota\left(\mathrm{d}x\right) \wedge \Delta &= \sum \int \mathrm{d}_v \overline{\mathrm{L}} \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
&= \sum \int \frac{\partial \overline{\mathrm{L}}}{\partial u_{j;k}} \mathrm{d}_v u_{j;k} \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
&= \sum \int \left[\left(\mathrm{S}_{-k}\frac{\partial \overline{\mathrm{L}}}{\partial u_{j;k}}\right) \mathrm{d}_v u_{j;0} + \operatorname{Div}\left(\overline{A_{\mathrm{S}}}\right)\right] \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
&= \sum \int \left[\left(D_j^{\dagger}\mathrm{S}_{-k}\frac{\partial \overline{\mathrm{L}}}{\partial u_{j;k}}\right) \mathrm{d}_v u_{0;0} + \operatorname{Div}\left(\overline{A_D}\right)\right] \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
&= \sum \int \mathrm{E}_u\left(\overline{\mathrm{L}}\right) \mathrm{d}_v u_{0;0} \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta.
\end{aligned} \tag{4.7}$$

This is not the Euler–Lagrange equation of L with respect to u. To find this, use the identity

$$\frac{\partial \overline{\mathrm{L}}}{\partial u_{j;k}} = \frac{\partial\left(J^{-1}\mathrm{L}\right)}{\partial u_{j;k}} = J^{-1}\frac{\partial \mathrm{L}}{\partial u_{j;k}} + \mathrm{L}\underbrace{\frac{\partial J^{-1}}{\partial u_{j;k}}}_{=0} = J^{-1}\frac{\partial \mathrm{L}}{\partial u_{j;k}},$$

and the following relation from the Euler operator of $\overline{\mathrm{L}}$ with respect to u

$$\begin{aligned}
\mathrm{E}_u\left(\overline{\mathrm{L}}\right) &= \sum_j (-1)^j D_j \mathrm{S}_{-k}\left(J^{-1}\frac{\partial \mathrm{L}}{\partial u_{j;k}}\right) \\
&= \sum_j (-1)^j D_j\left(J^{-1}\mathrm{S}_{-k}\frac{\partial \mathrm{L}}{\partial u_{j;k}}\right) \\
&= \sum_j \sum_{l=0}^{j} (-1)^j \binom{j}{l}\left(D_{j-l}\left(J^{-1}\right)\right)\left(D_l \mathrm{S}_{-k}\frac{\partial \mathrm{L}}{\partial u_{j;k}}\right) \\
&= \sum_j \left((-1)^j\left(J^{-1}\right) D_j \mathrm{S}_{-k}\frac{\partial \mathrm{L}}{\partial u_{j;k}}\right) \\
&\qquad + \sum_{j\geq 1}\sum_{l=0}^{j-1} (-1)^j \binom{j}{l}\left(D_{j-l}\left(J^{-1}\right)\right)\left(D_l \mathrm{S}_{-k}\frac{\partial \mathrm{L}}{\partial u_{j;k}}\right) \\
&= J^{-1}\mathrm{E}_u\left(\mathrm{L}\right) \\
&\qquad + \sum_{j\geq 1}\sum_{l=0}^{j-1} (-1)^j \binom{j}{l}\left(D_{j-l}\left(J^{-1}\right)\right)\left(D_l \mathrm{S}_{-k}\frac{\partial \mathrm{L}}{\partial u_{j;k}}\right).
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
\mathrm{E}_u\left(\mathrm{L}\right) &= J\mathrm{E}_u\left(\overline{\mathrm{L}}\right) - J\sum_{j\geq 1}\sum_{l=0}^{j-1}(-1)^j\binom{j}{l}\left(D_{j-l}\left(J^{-1}\right)\right)\left(D_l\mathrm{S}_{-k}\frac{\partial \mathrm{L}}{\partial u_{j;k}}\right)\\
&= J\mathrm{E}_u\left(\overline{\mathrm{L}}\right) - P, \qquad (4.8)
\end{aligned}$$

with

$$P = J\sum_{j\geq 1}\sum_{l=0}^{j-1}(-1)^j\binom{j}{l}\left(D_{j-l}\left(J^{-1}\right)\right)\left(D_l\mathrm{S}_{-k}\frac{\partial \mathrm{L}}{\partial u_{j;k}}\right).$$

This will be used later to find the invariant formulation of the Euler–Lagrange equations of L when a group action is on the independent variable.

The differential-difference form method is similar to the additional variable t trick with $\mathrm{d}u_{j;k}/\mathrm{d}t$ replacing $\mathrm{d}_v u_{j;k}$. Just as in the difference form theory, the terms $\iota\left(\mathrm{d}_v u_{0;0}\right)$ and $\sigma = \iota\left(\mathrm{d}u_{0;0}/\mathrm{d}t\right)$ are equivalent in the different methods. Having looked at the simple case, we now look at having more than one of each variable.

For a DΔE with q dependent variables, p continuous independent variables and m discrete independent variables, assume that the Lie group action on all independent continuous variables is trivial, i.e., $g \cdot x^i = x^i$, for all $i = 1, \ldots, p$. Consequently $\iota\left(\mathrm{d}x^i\right) = \mathrm{d}x^i$, for all $i = 1, \ldots, p$ and $\mathrm{L} = \overline{\mathrm{L}}$. Similar to Chapter 2 we use the notation (vol) for the volume form, that is,

$$\mathrm{vol} = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p \wedge \Delta^1 \wedge \cdots \wedge \Delta^m.$$

We again use differential-difference forms to find the Euler–Lagrange equations in terms of the original variables:

$$
\begin{aligned}
\mathrm{d}_v \sum \int & \mathrm{L}\,\mathrm{vol} \\
&= \sum \int \mathrm{d}_v \mathrm{L} \wedge \mathrm{vol} \\
&= \sum \int \frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J};\mathbf{K}}} \mathrm{d}_v u^{\alpha}_{\mathbf{J};\mathbf{K}} \wedge \mathrm{vol} \\
&= \sum \int \left(\mathrm{S}_{-\mathbf{K}} \frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J};\mathbf{K}}} \right) \mathrm{d}_v u^{\alpha}_{\mathbf{J};\mathbf{0}} \wedge \mathrm{vol} \\
&= \sum \int \left(D^{\dagger}_{\mathbf{J}} \mathrm{S}_{-\mathbf{K}} \frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J};\mathbf{K}}} \right) \mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}} \wedge \mathrm{vol} \\
&= \sum \int \mathrm{E}_{u^{\alpha}} (\mathrm{L})\, \mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}} \wedge \mathrm{vol},
\end{aligned}
$$

where $D^{\dagger}_{\mathbf{J}} = (-1)^{|\mathbf{J}|} D_{\mathbf{J}}$ and $|\mathbf{J}| = j^1 + \cdots + j^p$. Therefore, the Euler–Lagrange equations are

$$
\mathrm{E}_{u^{\alpha}} (\mathrm{L}) = (-1)^{|\mathbf{J}|} D_{\mathbf{J}} \mathrm{S}_{-\mathbf{K}} \frac{\partial \mathrm{L}}{\partial u^{\alpha}_{\mathbf{J};\mathbf{K}}} = 0.
$$

If one needs to look at a DΔEs with a Lie group action on more than one independent continuous variable then the invariant differential forms and invariant derivatives change slightly. So the invariant forms are

$$
\iota \left(\mathrm{d}x^i \right) = \left(\left. \frac{\partial \left(g \cdot x^i \right)}{\partial x^j} \right|_{g = \rho_{;\mathbf{K}}(\mathbf{x}, [\mathbf{u}])} \right) \mathrm{d}x^j = J^i_j \mathrm{d}x^j
$$

for all multi-indices $\mathbf{K}$ and $i, j = 1, \ldots, p$. Accordingly, the invariant derivatives are

$$
\mathcal{D}_i = \mathcal{J}^j_i D_j,
$$

where $\mathcal{J}^j_i$ are the components of the inverse matrix J^{-1}. Additionally, for group actions on the independent variable $\iota\,(\mathrm{vol}) \neq \mathrm{vol}$ as

$$
\iota\,(\mathrm{vol}) = \iota \left(\mathrm{d}x^1 \right) \wedge \cdots \wedge \iota \left(\mathrm{d}x^p \right) \wedge \Delta^1 \wedge \cdots \wedge \Delta^m.
$$

In this case, the invariant calculations will become quite complex, so for simplicity we focus on the first three cases and find the Euler–Lagrange equations in terms of their invariants.

4.4 Invariant formulation of the differential-difference Euler-Lagrange equations

This section shows the calculations for the invariant Euler–Lagrange equations for several different types of Lie group actions. First, we consider a group action on the dependent variable only, and compare this with the difference case. The invariant formulation of the Euler–Lagrange equations when there is a group action on the independent continuous variable only is discussed, restricting attention to one of each type of variable for simplicity. Finally, we consider a Lie group action on both the independent continuous variable and dependent variable. Again, we only consider the case with one of each type of variable.

4.4.1 A group action on the dependent variable only

In this case, $g \cdot x = x$, that is, the group action on the continuous independent variables is trivial. Consequently, $\iota(\mathrm{d}x) = \mathrm{d}x$ and $\mathrm{L} = \overline{\mathrm{L}}$. Therefore, let $L^{\boldsymbol{\kappa}}(x, n, [\boldsymbol{\kappa}]) = L^{\boldsymbol{\kappa}}$ be the invariant Lagrangian written in terms of invariants, with $[\boldsymbol{\kappa}]$ representing finitely many derivatives and shifts of κ^{β}. The Euler operator with respect to κ^{β} is

$$\mathrm{E}_{\kappa^{\beta}} = D_j^{\dagger}\mathrm{S}_{-k}\frac{\partial}{\partial \kappa^{\beta}_{j;k}} \tag{4.9}$$

when the group action is trivial on the independent variable. The components

$$\theta_{j;k} = D_j \mathrm{S}_k \theta_{0;0} = D_j \mathrm{S}_k \left(\frac{1}{\left.\frac{\partial(g \cdot u_{0;0})}{\partial u_{0;0}}\right|_{g=\rho_{;0}(x,[u])}} \right)$$

are important for the calculations (similar to the difference case in Chapter 2).

Proposition 4.4.1. The invariantization of the original Euler–Lagrange equation with a group action on the dependent variable only is

$$\iota\left(\mathrm{E}_u(\mathrm{L})\right) = \left(\mathcal{H}^{\beta}\right)^{\dagger} \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)$$

where

$$\left(\mathcal{H}^{\beta}\right)^{\dagger}(f)=\sum_{j}\sum_{k}\sum_{i=0}^{j}(-1)^{i}\binom{j}{i}D_{i}\mathrm{S}_{-k}\left(\iota\left(\frac{\partial\kappa^{\beta}}{\partial u_{i;j}}\right)\iota\left(\theta_{j-i;k}\right)f\right).$$

Proof. The differential-difference forms approach is used; it is similar to the method for the difference case in Chapter 2. The change of variables formula,

$$\mathrm{d}_{v}\kappa^{\beta}=\frac{\partial\kappa^{\beta}}{\partial u_{i;j}}\mathrm{d}_{v}u_{i;j},$$

is invariantized (2.31) to give

$$\iota\left(\mathrm{d}_{v}\kappa^{\beta}\right)=\iota\left(\frac{\partial\kappa^{\beta}}{\partial u_{i;j}}\right)\iota\left(\mathrm{d}_{v}u_{i;j}\right). \tag{4.10}$$

Then we need to write $\iota\left(\mathrm{d}_{v}u_{i;j}\right)$ in terms of $\iota\left(\mathrm{d}_{v}u_{0;0}\right)$. As in the difference case, the invariant differential form for $i=j=0$ is

$$\iota\left(\mathrm{d}_{v}u_{0;0}\right)=\vartheta\mathrm{d}_{v}u_{0;0},$$

where

$$\vartheta=\left.\frac{\partial\left(g\cdot u_{0;0}\right)}{\partial u_{0;0}}\right|_{g=\rho_{;0}(x,[u])}.$$

Let the reciprocal of ϑ be $\theta_{0;0}$ giving

$$\mathrm{d}_{v}u_{0;0}=\theta_{0;0}\iota\left(\mathrm{d}_{v}u_{0;0}\right),$$

similar to the difference formula found in Section 2.6.

The general Leibniz rule gives

$$\begin{aligned}\mathrm{d}_{v}u_{j;0}=D_{j}\mathrm{d}_{v}u_{0;0}&=D_{j}\left[\theta_{0;0}\iota\left(\mathrm{d}_{v}u_{0;0}\right)\right]\\&=\sum_{i=0}^{j}\binom{j}{i}\left(D_{j-i}\theta_{0;0}\right)\left(D_{i}\iota\left(\mathrm{d}_{v}u_{0;0}\right)\right)\\&=\sum_{i=0}^{j}\binom{j}{i}\theta_{j-i;0}\left(D_{i}\iota\left(\mathrm{d}_{v}u_{0;0}\right)\right).\end{aligned}$$

Applying k shifts to this gives

$$\begin{aligned}
\mathrm{d}_v u_{j;k} = \mathrm{S}_k \mathrm{d}_v u_{j;0} &= \mathrm{S}_k \left[\sum_{i=0}^{j} \binom{j}{i} \theta_{j-i;0} \left(D_i \iota \left(\mathrm{d}_v u_{0;0} \right) \right) \right] \\
&= \left[\sum_{i=0}^{j} \binom{j}{i} \left(\theta_{j-i;k} \right) \left(\mathrm{S}_k D_i \iota \left(\mathrm{d}_v u_{0;0} \right) \right) \right],
\end{aligned}$$

and invariantizing (2.31) this gives

$$\begin{aligned}
\iota \left(\mathrm{d}_v u_{j;k} \right) &= \sum_{i=0}^{j} \binom{j}{i} \iota \left(\theta_{j-i;k} \right) \iota \left(\mathrm{S}_k D_i \iota \left(\mathrm{d}_v u_{0;0} \right) \right) \\
&= \sum_{i=0}^{j} \binom{j}{i} \iota \left(\theta_{j-i;k} \right) \mathrm{S}_k D_i \iota \left(\mathrm{d}_v u_{0;0} \right).
\end{aligned}$$

Therefore, the invariantized change of variables formula (4.10) can be written as

$$\begin{aligned}
\iota \left(\mathrm{d}_v \kappa^{\beta} \right) &= \sum_k \sum_j \iota \left(\frac{\partial \kappa^{\beta}}{\partial u_{j;k}} \right) \sum_{i=0}^{j} \binom{j}{i} \iota \left(\theta_{j-i;k} \right) \mathrm{S}_k D_i \iota \left(\mathrm{d}_v u_{0;0} \right) \\
&= \sum_k \sum_j \sum_{i=0}^{j} \binom{j}{i} \iota \left(\frac{\partial \kappa^{\beta}}{\partial u_{j;k}} \right) \iota \left(\theta_{j-i;k} \right) \mathrm{S}_k D_i \iota \left(\mathrm{d}_v u_{0;0} \right) \\
&= \mathcal{H}^{\beta} \iota \left(\mathrm{d}_v u_{0;0} \right).
\end{aligned}$$

The adjoint of this linear differential-difference operator acting on f is

$$\left(\mathcal{H}^{\beta} \right)^{\dagger} (f) = \sum_j \sum_k \sum_{i=0}^{j} \binom{j}{i} D_i^{\dagger} \mathrm{S}_k^{\dagger} \left(\iota \left(\frac{\partial \kappa^{\beta}}{\partial u_{i;j}} \right) \iota \left(\theta_{j-i;k} \right) f \right).$$

Consequently,

$$\begin{aligned}
\mathrm{d}_v \sum \int L^{\boldsymbol{\kappa}} \iota\left(\mathrm{d}x\right) \wedge \Delta &= \sum \int \mathrm{d}_v L^{\boldsymbol{\kappa}} \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
&= \sum \int \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^{\beta}_{j;k}} \mathrm{d}_v \kappa^{\beta}_{j;k} \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
&= \sum \int \left[\left(\mathrm{S}_{-k} \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^{\beta}_{j;k}}\right) \mathrm{d}_v \kappa^{\beta}_{j;0} + \mathrm{Div}\left(B_{\mathrm{S}}\right)\right] \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
&= \sum \int \left[\left(D_j^{\dagger} \mathrm{S}_{-k} \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^{\beta}_{j;k}}\right) \mathrm{d}_v \kappa^{\beta}_{0;0} + \mathrm{Div}\left(B_D\right)\right] \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
&= \sum \int \left[\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right) \mathcal{H}^{\beta} \iota\left(\mathrm{d}_v u_{0;0}\right)\right] \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
&= \sum \int \left[\left(\left(\mathcal{H}^{\beta}\right)^{\dagger} \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right) \iota\left(\mathrm{d}_v u_{0;0}\right) + \mathrm{Div}\left(B_{\mathcal{H}}\right)\right] \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
&= \sum \int \left(\left(\mathcal{H}^{\beta}\right)^{\dagger} \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right) \iota\left(\mathrm{d}_v u_{0;0}\right) \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta.
\end{aligned} \tag{4.11}$$

The divergence terms $\mathrm{Div}\left(B_{\mathrm{S}}\right)$, $\mathrm{Div}\left(B_{\mathrm{D}}\right)$ and $\mathrm{Div}\left(B_{\mathcal{H}}\right)$ all become boundary terms which we assume disappear. As the vertical derivative is coordinate independent,

$$\begin{aligned}
\sum \int \iota\left(\mathrm{E}_u\left(\mathrm{L}\right)\right) \iota\left(\mathrm{d}_v u_{0;0}\right) \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta \\
= \sum \int \left(\mathcal{H}^{\beta}\right)^{\dagger} \left(\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right) \iota\left(\mathrm{d}_v u_{0;0}\right) \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta
\end{aligned}$$

and therefore

$$\iota\left(\mathrm{E}_u\left(\mathrm{L}\right)\right) = \left(\mathcal{H}^{\beta}\right)^{\dagger} \left(\mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right)$$

As a result, the invariant Euler–Lagrange equation is

$$\begin{aligned}
&\left(\mathcal{H}^{\beta}\right)^{\dagger} \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right) = \\
&\qquad \sum_j \sum_k \sum_{i=0}^{j} (-1)^i \binom{j}{i} \left(D_i\right) \mathrm{S}_{-k} \left(\iota\left(\frac{\partial \kappa^{\beta}}{\partial u_{i;j}}\right) \iota\left(\theta_{j-i;k}\right) \mathrm{E}_{\kappa^{\beta}}\left(L^{\boldsymbol{\kappa}}\right)\right) = 0.
\end{aligned}$$

□

Example 4.4.1. Consider the Lagrangian

$$\mathrm{L} = \frac{u_{1;0}}{u_{0;1}},$$

with the Lie group action

$$g \cdot u_{j;k} = a u_{j;k}, \qquad g \cdot x = x.$$

Using the normalization equation (2.16) $g \cdot u_{0;0} = 1$ gives

$$\iota\left(u_{j;k}\right) = \frac{u_{j;k}}{u_{0;0}}.$$

Let the two generating differential-difference invariants be

$$\kappa^1 = \frac{u_{0;1}}{u_{0;0}}, \qquad \kappa^2 = \frac{u_{1;0}}{u_{0;0}}.$$

The Lagrangian in terms of these generating invariants is

$$L^{\boldsymbol{\kappa}} = \frac{\kappa^2}{\kappa^1}.$$

The partial derivatives of the generating invariants are

$$\frac{\partial \kappa^1}{\partial u_{0;0}} = -\frac{u_{0;1}}{\left(u_{0;0}\right)^2}, \quad \frac{\partial \kappa^1}{\partial u_{0;1}} = \frac{1}{u_{0;0}}, \quad \frac{\partial \kappa^2}{\partial u_{0;0}} = -\frac{u_{1;0}}{\left(u_{0;0}\right)^2}, \quad \frac{\partial \kappa^2}{\partial u_{1;0}} = \frac{1}{u_{0;0}},$$

with invariantizations (2.31)

$$\iota\left(\frac{\partial \kappa^1}{\partial u_{0;0}}\right) = -\kappa^1, \quad \iota\left(\frac{\partial \kappa^1}{\partial u_{0;1}}\right) = 1, \quad \iota\left(\frac{\partial \kappa^2}{\partial u_{0;0}}\right) = -\kappa^2, \quad \iota\left(\frac{\partial \kappa^2}{\partial u_{1;0}}\right) = 1.$$

The component

$$\vartheta = \frac{1}{u_{0;0}},$$

has the inverse

$$\theta_{0;0} = u_{0;0}.$$

As a consequence,

$$\theta_{1;0} = u_{1;0}, \qquad \theta_{0;1} = u_{0;1},$$

and the invariantizations (2.31) of the θ components are

$$\iota\left(\theta_{0;0}\right) = 1, \qquad \iota\left(\theta_{1;0}\right) = \kappa^2, \qquad \iota\left(\theta_{0;1}\right) = \kappa^1.$$

Finally, the Euler operators (4.9) of $L^{\boldsymbol{\kappa}}$ with respect to the generating invariants are

$$\mathrm{E}_{\kappa^1}\left(L^{\boldsymbol{\kappa}}\right)=-\frac{\kappa^2}{\left(\kappa^1\right)^2}, \qquad \mathrm{E}_{\kappa^2}\left(L^{\boldsymbol{\kappa}}\right)=\frac{1}{\kappa^1},$$

and so using Proposition 4.4.5 yields

$$\begin{aligned}
\left(\mathcal{H}^1\right)^{\dagger}\mathrm{E}_{\kappa^1}\left(L^{\boldsymbol{\kappa}}\right) &= \iota\left(\frac{\partial\kappa^1}{\partial u_{0;0}}\right)\iota\left(\theta_{0;0}\right)\mathrm{E}_{\kappa^1}\left(L^{\boldsymbol{\kappa}}\right)+\mathrm{S}^{-1}\left(\iota\left(\frac{\partial\kappa^1}{\partial u_{0;1}}\right)\iota\left(\theta_{0;1}\right)\mathrm{E}_{\kappa^1}\left(L^{\boldsymbol{\kappa}}\right)\right)\\
&= \frac{\kappa^2}{\kappa^1}-\frac{\kappa^2_{0;-1}}{\kappa^1_{0;-1}},\\
\left(\mathcal{H}^2\right)^{\dagger}\mathrm{E}_{\kappa^2}\left(L^{\boldsymbol{\kappa}}\right) &= \iota\left(\frac{\partial\kappa^2}{\partial u_{0;0}}\right)\iota\left(\theta_{0;0}\right)\mathrm{E}_{\kappa^2}\left(L^{\boldsymbol{\kappa}}\right)+\iota\left(\frac{\partial\kappa^2}{\partial u_{1;0}}\right)\iota\left(\theta_{1;0}\right)\mathrm{E}_{\kappa^2}\left(L^{\boldsymbol{\kappa}}\right)\\
&\quad - D\left(\iota\left(\frac{\partial\kappa^2}{\partial u_{1;0}}\right)\iota\left(\theta_{0;0}\right)\mathrm{E}_{\kappa^2}\left(L^{\boldsymbol{\kappa}}\right)\right)\\
&= -\frac{\kappa^2}{\kappa^1}+\frac{\kappa^2_{0;1}}{\kappa^1}.
\end{aligned}$$

Consequently, the invariantization of the Euler–Lagrange equation is

$$\left(\mathcal{H}^1\right)^{\dagger}\mathrm{E}_{\kappa^1}\left(L^{\boldsymbol{\kappa}}\right)+\left(\mathcal{H}^2\right)^{\dagger}\mathrm{E}_{\kappa^2}\left(L^{\boldsymbol{\kappa}}\right)=\frac{\kappa^2_{0;1}}{\kappa^1}-\frac{\kappa^2_{0;-1}}{\kappa^1_{0;-1}}=0.$$

For comparison, the original Euler–Lagrange equation (4.4) is

$$\begin{aligned}
\mathrm{E}_u\left(\mathrm{L}\right) &= D_j^{\dagger}\mathrm{S}_{-k}\frac{\partial\mathrm{L}}{\partial u_{j;k}}\\
&= -D\left(\frac{\partial\mathrm{L}}{\partial u_{1;0}}\right)+\mathrm{S}_{-1}\left(\frac{\partial\mathrm{L}}{\partial u_{0;1}}\right)\\
&= \frac{u_{1,1}}{\left(u_{0;1}\right)^2}-\frac{u_{1;-1}}{\left(u_{0;0}\right)^2}.
\end{aligned}$$

Therefore, the denominator has been reduced from a square term to a first-order term in the invariantized Euler–Lagrange equation.

4.4.2 Multi-variable case with a group action on the dependent variable only

Now consider the case with more than one of each type of variable, where the group action on all the independent variables is trivial. Therefore, $\iota\left(\mathrm{d}x^i\right)=\mathrm{d}x^i$

and the Lagrangian functional can be written as

$$\sum \int \mathrm{L}\left(\mathbf{x}, \mathbf{n}, [\mathbf{u}]\right) \mathrm{vol} = \sum \int L^{\boldsymbol{\kappa}}\left(\mathbf{x}, \mathbf{n}, [\boldsymbol{\kappa}]\right) \mathrm{vol}.$$

To help simplify the formula, we use the notation

$$\sum_{\mathbf{J} \geq \mathbf{I}} \binom{\mathbf{J}}{\mathbf{I}} = \sum_{i^1}^{j^1} \cdots \sum_{i^p}^{j^p} \binom{j^1}{i^1} \cdots \binom{j^p}{i^p},$$

where $\mathbf{J} \geq \mathbf{I}$ means that $j^l \geq i^l$ for $l = 1, \ldots, p$. The Euler operator with respect to κ^β is

$$\mathrm{E}_{\kappa^\beta} = D_{\mathbf{J}}^{\dagger} \mathrm{S}_{-\mathbf{J}} \frac{\partial}{\partial \kappa^\beta_{\mathbf{J};\mathbf{K}}}.$$

Similarly, if

$$(\vartheta)^\alpha_\delta = \left.\frac{\partial \left(g \cdot u^\alpha_{\mathbf{0};\mathbf{0}}\right)}{\partial u^\delta_{\mathbf{0};\mathbf{0}}}\right|_{g = \rho_{;\mathbf{0}}(\mathbf{x}, [\mathbf{u}])},$$

then $\theta_{\mathbf{0};\mathbf{0}} = \vartheta^{-1}$ and $\left(\theta_{\mathbf{J};\mathbf{K}}\right)^\delta_\alpha = D_{\mathbf{J}} \mathrm{S}_{\mathbf{K}} \left(\vartheta^{-1}\right)^\delta_\alpha$. This allows us to introduce the proposition.

Proposition 4.4.2. The invariantization of the original Euler–Lagrange equations with a group action on the dependent variables only is

$$\iota\left(\mathrm{E}_{u^\alpha}\left(\mathrm{L}\right)\right) = \left(\mathcal{H}^\beta_\alpha\right)^\dagger \mathrm{E}_{\kappa^\beta}\left(L^{\boldsymbol{\kappa}}\right),$$

where

$$\left(\mathcal{H}^\beta_\alpha\right)^\dagger (f) = \sum_{\mathbf{K}} \sum_{\mathbf{J}} \sum_{\mathbf{J} \geq \mathbf{I}} \binom{\mathbf{J}}{\mathbf{I}} D_{\mathbf{I}}^{\dagger} \mathrm{S}_{-\mathbf{K}} \left[\iota\left(\frac{\partial \kappa^\beta}{\partial u^\delta_{\mathbf{J};\mathbf{K}}}\right) \iota\left(\theta_{\mathbf{J}-\mathbf{I};\mathbf{K}}\right)^\delta_\alpha f\right].$$

Proof. Using differential-difference forms again, the change of variables formula is

$$\mathrm{d}_v \kappa^\beta = \frac{\partial \kappa^\beta}{\partial u^\delta_{\mathbf{J};\mathbf{K}}} \mathrm{d}_v u^\delta_{\mathbf{J};\mathbf{K}}$$

with invariantization (2.31)

$$\iota\left(\mathrm{d}_v \kappa^\beta\right) = \iota\left(\frac{\partial \kappa^\beta}{\partial u^\delta_{\mathbf{J};\mathbf{K}}}\right) \iota\left(\mathrm{d}_v u^\delta_{\mathbf{J};\mathbf{K}}\right).$$

Then the fundamental invariant differential-difference forms are

$$\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right) = (\vartheta)^{\alpha}_{\delta}\,\mathrm{d}_v u^{\delta}_{\mathbf{0};\mathbf{0}},$$

so

$$\mathrm{d}_v u^{\delta}_{\mathbf{0};\mathbf{0}} = \left(\theta_{\mathbf{0};\mathbf{0}}\right)^{\delta}_{\alpha}\,\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right).$$

Using the general Leibniz rule gives

$$\begin{aligned}
\mathrm{d}_v u^{\delta}_{\mathbf{J};\mathbf{0}} &= D_{\mathbf{J}}\mathrm{d}_v u^{\delta}_{\mathbf{0};\mathbf{0}} = D_{\mathbf{J}}\left[\left(\theta_{\mathbf{0};\mathbf{0}}\right)^{\delta}_{\alpha}\,\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right)\right]\\
&= D_{0,j^2,\ldots,j^p}\sum_{i^1}^{j^1}\binom{j^1}{i^1}\left(\theta_{j^1-i^1,0,\ldots,0;\mathbf{0}}\right)^{\delta}_{\alpha} D_{i^1}\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right)\\
&= \ldots\\
&= \sum_{i^1}^{j^1}\cdots\sum_{i^p}^{j^p}\binom{j^1}{i^1}\cdots\binom{j^p}{i^p}\left(\theta_{j^1-i^1,\ldots,j^p-i^p;\mathbf{0}}\right)^{\delta}_{\alpha} D_{i^p}\ldots D_{i^1}\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right)\\
&= \sum_{\mathbf{J}\geq\mathbf{I}}\binom{\mathbf{J}}{\mathbf{I}}\left(\theta_{\mathbf{J}-\mathbf{I};\mathbf{0}}\right)^{\delta}_{\alpha} D_{\mathbf{I}}\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right).
\end{aligned}$$

Applying $\mathbf{K}$ shifts to this gives

$$\begin{aligned}
\mathrm{d}_v u^{\delta}_{\mathbf{J};\mathbf{K}} &= \mathrm{S}_{\mathbf{K}}\left[\sum_{\mathbf{J}\geq\mathbf{I}}\binom{\mathbf{J}}{\mathbf{I}}\left(\theta_{\mathbf{J}-\mathbf{I};\mathbf{0}}\right)^{\delta}_{\alpha} D_{\mathbf{I}}\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right)\right]\\
&= \sum_{\mathbf{J}\geq\mathbf{I}}\binom{\mathbf{J}}{\mathbf{I}}\left(\theta_{\mathbf{J}-\mathbf{I};\mathbf{K}}\right)^{\delta}_{\alpha}\mathrm{S}_{\mathbf{K}} D_{\mathbf{I}}\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right),
\end{aligned}$$

and therefore

$$\begin{aligned}
\iota\left(\mathrm{d}_v u^{\delta}_{\mathbf{J};\mathbf{K}}\right) &= \sum_{\mathbf{J}\geq\mathbf{I}}\binom{\mathbf{J}}{\mathbf{I}}\iota\left(\theta_{\mathbf{J}-\mathbf{I};\mathbf{K}}\right)^{\delta}_{\alpha}\,\iota\left(\mathrm{S}_{\mathbf{K}} D_{\mathbf{I}}\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right)\right)\\
&= \sum_{\mathbf{J}\geq\mathbf{I}}\binom{\mathbf{J}}{\mathbf{I}}\iota\left(\theta_{\mathbf{J}-\mathbf{I};\mathbf{K}}\right)^{\delta}_{\alpha}\mathrm{S}_{\mathbf{K}} D_{\mathbf{I}}\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iota\left(\mathrm{d}_v \kappa^{\beta}\right) &= \sum_{\mathbf{K}}\sum_{\mathbf{J}}\sum_{\mathbf{J}\geq\mathbf{I}}\binom{\mathbf{J}}{\mathbf{I}}\iota\left(\frac{\partial\kappa^{\beta}}{\partial u^{\delta}_{\mathbf{J};\mathbf{K}}}\right)\iota\left(\theta_{\mathbf{J}-\mathbf{I};\mathbf{K}}\right)^{\delta}_{\alpha}\mathrm{S}_{\mathbf{K}} D_{\mathbf{I}}\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right)\\
&= \mathcal{H}^{\beta}_{\alpha}\iota\left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right),
\end{aligned}$$

and so for any smooth function f,

$$\left(\mathcal{H}_{\alpha}^{\beta}\right)^{\dagger} f = \sum_{\mathbf{K}} \sum_{\mathbf{J}} \sum_{\mathbf{J} \geq \mathbf{I}} (-1)^{|\mathbf{I}|} \binom{\mathbf{J}}{\mathbf{I}} D_{\mathbf{I}} \mathrm{S}_{-\mathbf{K}} \left[\iota \left(\frac{\partial \kappa^{\beta}}{\partial u^{\delta}_{\mathbf{J};\mathbf{K}}} \right) \iota \left(\theta_{\mathbf{J}-\mathbf{I};\mathbf{K}}\right)^{\delta}_{\alpha} f \right].$$

Consequently,

$$\begin{aligned}
&\mathrm{d}_v \sum \int L^{\boldsymbol{\kappa}} \,\mathrm{vol} \\
&= \sum \int \mathrm{d}_v L^{\boldsymbol{\kappa}} \wedge \mathrm{vol} \\
&= \sum \int \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^{\beta}_{\mathbf{J};\mathbf{K}}} \mathrm{d}_v \kappa^{\beta}_{\mathbf{J};\mathbf{K}} \wedge \mathrm{vol} \\
&= \sum \int \left(D^{\dagger}_{\mathbf{J}} \mathrm{S}_{-\mathbf{K}} \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^{\beta}_{\mathbf{J};\mathbf{K}}} \right) \mathrm{d}_v \kappa^{\beta}_{\mathbf{0};\mathbf{0}} \wedge \mathrm{vol} \\
&= \sum \int \mathrm{E}_{\kappa^{\beta}} \left(L^{\boldsymbol{\kappa}}\right) \mathcal{H}^{\beta}_{\alpha} \iota \left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right) \wedge \mathrm{vol} \\
&= \left(\sum \int \left(\mathcal{H}^{\beta}_{\alpha}\right)^{\dagger} \mathrm{E}_{\kappa^{\beta}} \left(L^{\boldsymbol{\kappa}}\right) \right) \iota \left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right) \wedge \mathrm{vol}.
\end{aligned}$$

As the vertical derivative is coordinate independent,

$$\sum \int \iota \left(\mathrm{E}_{u^{\alpha}} (\mathrm{L})\right) \iota \left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right) \wedge \mathrm{vol} = \sum \int \left(\mathcal{H}^{\beta}_{\alpha}\right)^{\dagger} \mathrm{E}_{\kappa^{\beta}} \left(L^{\boldsymbol{\kappa}}\right) \iota \left(\mathrm{d}_v u^{\alpha}_{\mathbf{0};\mathbf{0}}\right) \wedge \mathrm{vol}.$$

So the adjoint of this linear differential-difference operator acting on $\mathrm{E}_{\kappa^{\beta}} \left(L^{\boldsymbol{\kappa}}\right)$ gives the formula for the invariant Euler–Lagrange equations

$$\begin{aligned}
&\left(\mathcal{H}^{\beta}_{\alpha}\right)^{\dagger} \mathrm{E}_{\kappa^{\beta}} \left(L^{\boldsymbol{\kappa}}\right) = \\
&\qquad \sum_{\mathbf{K}} \sum_{\mathbf{J}} \sum_{\mathbf{J} \geq \mathbf{I}} (-1)^{|\mathbf{I}|} \binom{\mathbf{J}}{\mathbf{I}} D_{\mathbf{I}} \mathrm{S}_{-\mathbf{K}} \left[\iota \left(\frac{\partial \kappa^{\beta}}{\partial u^{\delta}_{\mathbf{J};\mathbf{K}}} \right) \iota \left(\theta_{\mathbf{J}-\mathbf{I};\mathbf{K}}\right)^{\delta}_{\alpha} \mathrm{E}_{\kappa^{\beta}} \left(L^{\boldsymbol{\kappa}}\right) \right] = 0.
\end{aligned}$$

□

This looks somewhat similar to the difference case in Proposition 2.6.5. Indeed, if there are no derivatives this formula reduces to the one found in Proposition 2.6.5.

4.4.3 A group action on the independent variable only

Now assume that $g \cdot u_{0;0} = u_{0;0}$, so the group action on the dependent variable is trivial. As now we consider a group action on the independent variable alone, the invariant differential form is

$$\iota\left(\mathrm{d}x\right) = J\mathrm{d}x,$$

where a projectable normalization is used and

$$J = \left(\frac{\partial\left(g \cdot x\right)}{\partial x}\right)\Bigg|_{g=\rho_{;k}(x,[u])}.$$

The Lagrangian functional can also be changed from

$$\sum \int \mathrm{L}\left(x, n, [u]\right) \mathrm{d}x \wedge \Delta$$

to its invariant form

$$\sum \int \overline{\mathrm{L}}\left(x, n, [u]\right) \iota\left(\mathrm{d}x\right) \wedge \Delta$$

with $\overline{\mathrm{L}} = \mathcal{J}\mathrm{L}$ where $\mathcal{J} = J^{-1}$. To find the invariant formulation of the Euler–Lagrange equation of L, use the relation (4.8). Its invariantization (2.31),

$$\iota\left(\mathrm{E}_u\left(\mathrm{L}\right)\right) = \iota\left(\mathrm{E}_u\left(\overline{\mathrm{L}}\right)\right) - \iota\left(P\right), \tag{4.12}$$

is needed.

Proposition 4.4.3. The invariantization of the original Euler–Lagrange equation with a group action on the independent variable is

$$\iota\left(\mathrm{E}_u\left(\mathrm{L}\right)\right) = \sum_j \iota\left(H_j^j\right) - \mathcal{P}, \tag{4.13}$$

where

$$H_j^j = (-1)^j \underbrace{D\big[\dots\big[D}_{j \text{ times}}\big[H_0^j \underbrace{\mathcal{J}\big]\dots\big]\mathcal{J}\big]}_{j \text{ times}}, \quad H_0^j = \mathrm{S}_{-k}\frac{\partial L^{\kappa}}{\partial \kappa_{j;k}} \tag{4.14}$$

and the correction term is

$$\mathcal{P} = \sum_{j\geq 1}\sum_{l=0}^{j-1}(-1)^j\,\iota\left(D_{j-l}\left(\mathcal{J}\right)\right)\sum_{s=0}^{l}\binom{l}{s}\iota\left(D_s J\right)\iota\left(D_{l-s}\mathrm{S}_{-k}\left(\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^{\beta}_{i;r}}\frac{\partial \kappa^{\beta}_{i;r}}{\partial u_{j;k}}\right)\right).$$

Proof. To show this, we use (4.12) to split the calculation into two parts. First, we find $\iota\left(\mathrm{E}_u\left(\overline{\mathrm{L}}\right)\right)$ in terms of invariants, then turn to finding $\iota\left(P\right)$ in terms of invariants. The normalization equation (2.16) must include the only independent continuous variable x, so the Lagrangian will be of the form $L^{\boldsymbol{\kappa}} = L^{\boldsymbol{\kappa}}\left(n, [\kappa]\right)$, with the functional

$$\sum\int \overline{\mathrm{L}}\iota\left(\mathrm{d}x\right)\wedge\Delta = \sum\int L^{\boldsymbol{\kappa}}\iota\left(\mathrm{d}x\right)\wedge\Delta. \tag{4.15}$$

As there is an action on the independent continuous variables the derivatives themselves are not necessarily invariant. The invariant differential operator is $\mathcal{D} = \mathcal{J}D$. Therefore, by definition $\kappa_{1;0} = \mathcal{D}\kappa_{0;0}$. The invariant differential operator, like the differential operator, commutes with shifts (because the normalization is projectable). (For more than one continuous independent variable, the invariant differential operators do not necessarily commute; however, we do not consider this case.) Using differential-difference forms, summation by parts, and the first iteration of integration by parts one moves from the vertical derivative of the Lagrangian functional to a formula involving the differential form $\mathrm{d}_v\kappa_{j-1;0}$ in the following way:

$$\begin{aligned}
\sum\int \mathrm{d}_v L^{\boldsymbol{\kappa}}\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta &= \sum\int \frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa_{j;k}}\mathrm{d}_v\kappa_{j;k}\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta\\
&= \sum\int\left(\mathrm{S}_{-k}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa_{j;k}}\right)\mathrm{d}_v\kappa_{j;0}\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta\\
&= \sum\int\left(\mathrm{S}_{-k}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa_{j;k}}\right)\mathcal{D}\mathrm{d}_v\kappa_{j-1;0}\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta\\
&= \sum\int\left(\mathrm{S}_{-k}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa_{j;k}}\right)\mathcal{J}\left(D\mathrm{d}_v\kappa_{j-1;0}\right)\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta\\
&= \sum\int\left[D^{\dagger}\left(\left(\mathrm{S}_{-k}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa_{j;k}}\right)\mathcal{J}\right)\right]\mathrm{d}_v\kappa_{j-1;0}\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta.
\end{aligned}$$

As we only consider projectable normalizations the vertical derivative of $\iota\left(\mathrm{d}x\right)$ is zero. Now introducing the components

$$H_0^j = \mathrm{S}_{-k}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{j;k}}, \quad H_1^j = D^{\dagger}\left[H_0^j \mathcal{J}\right] = -D\left[H_0^j \mathcal{J}\right],$$
$$H_2^j = D^{\dagger}\left[H_1^j \mathcal{J}\right] = D^{\dagger}\left[D^{\dagger}\left[H_0^j \mathcal{J}\right]\mathcal{J}\right] = D\left[D\left[H_0^j \mathcal{J}\right]\mathcal{J}\right],$$

and more generally,

$$H_j^j = \underbrace{D^{\dagger}[\dots[D^{\dagger}}_{j\text{ times}}[H_0^j \underbrace{\mathcal{J}]\dots]\mathcal{J}]}_{j\text{ times}} = (-1)^j \underbrace{D[\dots[D}_{j\text{ times}}[H_0^j \underbrace{\mathcal{J}]\dots]\mathcal{J}]}_{j\text{ times}}.$$

After the second iteration of integration by parts the vertical derivative of the Lagrangian functional becomes

$$\sum\int \mathrm{d}_v L^{\boldsymbol{\kappa}} \wedge \iota(\mathrm{d}x) \wedge \Delta = \sum\int\sum_j H_2^j \mathrm{d}_v \kappa_{j-2;0} \wedge \iota(\mathrm{d}x) \wedge \Delta$$

and after j iterations of integration by parts the formula becomes

$$\sum\int \mathrm{d}_v L^{\boldsymbol{\kappa}} \wedge \iota(\mathrm{d}x) \wedge \Delta = \sum\int\sum_j H_j^j \mathrm{d}_v \kappa_{0;0} \wedge \iota(\mathrm{d}x) \wedge \Delta.$$

Thus,

$$\sum\int \iota\left(\mathrm{d}_v L^{\boldsymbol{\kappa}}\right) \wedge \iota(\mathrm{d}x) \wedge \Delta = \sum\int\sum_j \iota\left(H_j^j\right)\iota\left(\mathrm{d}_v u_{0;0}\right) \wedge \iota(\mathrm{d}x) \wedge \Delta. \quad (4.16)$$

This gives the equivalence

$$\sum\int \iota\left(\mathrm{E}_u\left(\overline{\mathrm{L}}\right)\right)\iota\left(\mathrm{d}_v u_{0;0}\right) \wedge \iota(\mathrm{d}x) \wedge \Delta = \sum\int\sum_j \iota\left(H_j^j\right)\iota\left(\mathrm{d}_v u_{0;0}\right) \wedge \iota(\mathrm{d}x) \wedge \Delta, \quad (4.17)$$

which gives the first term $\iota\left(\mathrm{E}_u\left(\overline{\mathrm{L}}\right)\right)$ in the relation (4.12). The correction term in the original variables is

$$P = J\sum_{j\geq 1}\sum_{l=0}^{j-1}(-1)^j\binom{j}{l}\left(D_{j-l}(\mathcal{J})\right)\left(D_l \mathrm{S}_{-k}\frac{\partial \mathrm{L}}{\partial u_{j;k}}\right).$$

We now write the invariantization (2.31) of the last term,

$$\iota\left(D_l \mathrm{S}_{-k} \frac{\partial \mathrm{L}}{\partial u_{j;k}}\right),$$

in terms of the invariant Lagrangian. The original Lagrangian can be written as $\mathrm{L} = JL^{\boldsymbol{\kappa}}$, so

$$\iota\left(D_l \mathrm{S}_{-k} \frac{\partial \mathrm{L}}{\partial u_{j;k}}\right) = \iota\left(D_l \mathrm{S}_{-k} \frac{\partial \left(JL^{\boldsymbol{\kappa}}\right)}{\partial \kappa_{i;r}} \frac{\partial \kappa_{i;r}}{\partial u_{j;k}}\right),$$

and by the chain rule,

$$\frac{\partial \left(JL^{\boldsymbol{\kappa}}\right)}{\partial \kappa_{i;r}} = J \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{i;r}} + \underbrace{\frac{\partial J}{\partial \kappa_{i;r}}}_{=0} L^{\boldsymbol{\kappa}} = J \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{i;r}}$$

(as we use a projectable normalization). Therefore,

$$\begin{aligned} \iota\left(D_l \mathrm{S}_{-k} \frac{\partial \mathrm{L}}{\partial u_{j;k}}\right) &= \iota\left(D_l \mathrm{S}_{-k}\left(J \frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{i;r}} \frac{\partial \kappa_{i;r}}{\partial u_{j;k}}\right)\right) \\ &= \iota\left(D_l\left(J \mathrm{S}_{-k}\left(\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{i;r}} \frac{\partial \kappa_{i;r}}{\partial u_{j;k}}\right)\right)\right) \end{aligned}$$

and using the general Leibniz rule again gives

$$\begin{aligned} \iota\left(D_l \mathrm{S}_{-k} \frac{\partial \mathrm{L}}{\partial u_{j;k}}\right) &= \iota\left(\sum_{s=0}^{l} \binom{l}{s} \left(D_s J\right) \left(D_{l-s} \mathrm{S}_{-k}\left(\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{i;r}} \frac{\partial \kappa_{i;r}}{\partial u_{j;k}}\right)\right)\right) \\ &= \sum_{s=0}^{l} \binom{l}{s} \iota\left(D_s J\right) \iota\left(D_{l-s} \mathrm{S}_{-k}\left(\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{i;r}} \frac{\partial \kappa_{i;r}}{\partial u_{j;k}}\right)\right). \end{aligned}$$

Thus, the invariantized (2.31) correction term $\mathcal{P} = \iota\left(P\right)$ amounts to

$$\mathcal{P} = \sum_{j \geq 1} \sum_{l=0}^{j-1} (-1)^j \binom{j}{l} \iota\left(D_{j-l} \mathcal{J}\right) \sum_{s=0}^{l} \binom{l}{s} \iota\left(D_s J\right) \iota\left(D_{l-s} \mathrm{S}_{-k}\left(\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{i;r}} \frac{\partial \kappa_{i;r}}{\partial u_{j;k}}\right)\right),$$

with $\iota\left(J\right) = 1$ for a similar reason to $\iota\left(\vartheta\right) = 1$. Finally, using the relation (4.12) for the Euler–Lagrange equation of L, the invariant formulation of the Euler–Lagrange equation is

$$\iota\left(\mathrm{E}_u\left(\mathrm{L}\right)\right) = \sum_j \iota\left(H_j^j\right) - \mathcal{P}, \tag{4.18}$$

with H_j^j, for each $j \in \mathbb{Z}^+$, and $\mathcal{P}$ as defined above. □

Example 4.4.2. Consider the Lagrangian,

$$\mathrm{L} = \frac{u_{1;0}}{u_{0;1}},$$

with Lagrangian functional

$$\sum \int \mathrm{L}\, \mathrm{d}x \wedge \Delta.$$

Using the group action

$$g \cdot x = ax, \qquad g \cdot u_{0;0} = u_{0;0},$$

and the normalization equation (2.16) $g \cdot x = 1$, the Lagrangian and Lagrangian functional become

$$\overline{\mathrm{L}} = \frac{x u_{1;0}}{u_{0;1}} \qquad \text{and} \qquad \sum \int \overline{\mathrm{L}}\, \iota\left(\mathrm{d}x\right) \wedge \Delta$$

respectively, as

$$\iota\left(\mathrm{d}x\right) = \left(\left. \frac{\partial \left(g \cdot x\right)}{\partial x} \right|_{g=\rho_{;0}(x,[u])} \right) \mathrm{d}x = \frac{\mathrm{d}x}{x}.$$

The invariantization (2.31) of $u_{j;k}$ is

$$\iota\left(u_{j;k}\right) = x^j u_{j;k},$$

and the generating invariant is

$$\kappa = \iota\left(u_{0;0}\right) = u_{0;0}.$$

For the invariant Euler–Lagrange equation, we need the terms H_1^1 and H_0^0 in Proposition 4.4.3. The Lagrangian in terms of shifts and derivatives of the generating invariant is

$$L^{\kappa} = \frac{\kappa_{1;0}}{\kappa_{0;1}}$$

and therefore

$$
\begin{aligned}
H_0^0 &= \mathrm{S}_{-1}\left(\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{0;1}}\right) = -\frac{\kappa_{1;-1}}{\left(\kappa_{0;0}\right)^2}\,,\\
H_1^1 &= D^{\dagger}\left[H_0^1 \mathcal{J}\right] = -D\left[\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{1;0}} x\right] = -\frac{1}{\kappa_{0;1}} + \frac{\kappa_{1;1}}{\left(\kappa_{0;1}\right)^2}\,.
\end{aligned}
$$

The invariantization (2.31) of these leaves them unchanged so

$$
\iota\left(H_0^0\right) = -\frac{\kappa_{1;-1}}{\left(\kappa_{0;0}\right)^2}\,, \qquad \iota\left(H_1^1\right) = -\frac{1}{\kappa_{0;1}} + \frac{\kappa_{1;1}}{\left(\kappa_{0;1}\right)^2}\,.
$$

The correction term for this example is

$$
\begin{aligned}
\mathcal{P} &= (-1)\binom{1}{0}\iota\left(D\left(x\right)\right)\binom{0}{0}\iota\left(\frac{1}{x}\right)\iota\left(\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{1;0}}\frac{\partial \kappa_{1;0}}{\partial u_{1;0}}\right)\\
&= -\iota\left(\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa_{1;0}}\frac{\partial \kappa_{1;0}}{\partial u_{1;0}}\right)\\
&= -\frac{1}{\kappa_{0;1}}\,.
\end{aligned}
$$

Therefore, the invariant Euler–Lagrange equation of L is

$$
\begin{aligned}
\iota\left(H_0^0\right) + \iota\left(H_1^1\right) - \mathcal{P} &= -\frac{\kappa_{1;-1}}{\left(\kappa_{0;0}\right)^2} - \frac{1}{\kappa_{0;1}} + \frac{\kappa_{1;1}}{\left(\kappa_{0;1}\right)^2} - \left(-\frac{1}{\kappa_{0;1}}\right)\\
&= -\frac{\kappa_{1;-1}}{\left(\kappa_{0;0}\right)^2} + \frac{\kappa_{1;1}}{\left(\kappa_{0;1}\right)^2} = 0.
\end{aligned}
$$

Remark 4.4.4. The terms H_j^j and P are invariant here, however, the invariantization operator is needed here to write these (using the replacement rule Theorem 2.4.9) in terms of shifts and invariant derivatives of the generating invariant.

4.4.4 A group action on all continuous variables

Here for a more general case, we assume that there is a group action on both the independent continuous variable and the dependent variable. We also restrict attention to actions for which a projectable normalization (Definition 4.1.2) is possible and is used.

Proposition 4.4.5. The invariantization of the original Euler–Lagrange equation with a group action on both the independent and dependent variable is of the form

$$\iota\left(\mathrm{E}_u\left(\mathrm{L}\right)\right)=\sum_j\left(\mathcal{H}^{\beta}\right)^{\dagger}\iota\left(H^j_{j,\beta}\right)-\mathcal{P}$$

where $\left(\mathcal{H}^{\beta}\right)^{\dagger}$ is a linear differential-difference operator,

$$H^j_{j,\beta}=(-1)^j\underbrace{D[\ldots[D}_{j\text{ times}}[H^j_{0,\beta}\underbrace{\mathcal{J}]\ldots]\mathcal{J}]}_{j\text{ times}},\quad H^j_{0,\beta}=\mathrm{S}_{-k}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa^{\beta}_{j;k}},\tag{4.19}$$

and the correction term is

$$\mathcal{P}=\sum_{j\geq 1}\sum_{l=0}^{j-1}(-1)^j\,\iota\left(D_{j-l}\left(\mathcal{J}\right)\right)\sum_{s=0}^{l}\binom{l}{s}\iota\left(D_sJ\right)\iota\left(D_{l-s}\mathrm{S}_{-k}\left(\frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa^{\beta}_{i;r}}\frac{\partial\kappa^{\beta}_{i;r}}{\partial u_{j;k}}\right)\right).$$

Proof. Again, use (4.12) to split the calculations into two parts. First find $\iota\left(\mathrm{E}_u\left(\overline{\mathrm{L}}\right)\right)$. Here using that $L^{\boldsymbol{\kappa}}\left(n,[\boldsymbol{\kappa}]\right)=L^{\boldsymbol{\kappa}}=\overline{\mathrm{L}}$,

$$\sum\int\overline{\mathrm{L}}\iota\left(\mathrm{d}x\right)\wedge\Delta=\sum\int L^{\boldsymbol{\kappa}}\iota\left(\mathrm{d}x\right)\wedge\Delta.$$

Then

$$\begin{aligned}\mathrm{d}_v\sum\int L^{\boldsymbol{\kappa}}\iota\left(\mathrm{d}x\right)\wedge\Delta&=\sum\int\mathrm{d}_vL^{\boldsymbol{\kappa}}\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta\\&=\sum\int\frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa^{\beta}_{j;k}}\mathrm{d}_v\kappa^{\beta}_{j;k}\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta\\&=\sum\int\frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa^{\beta}_{j;k}}\mathrm{S}_k\mathcal{D}_j\mathrm{d}_v\kappa^{\beta}_{0;0}\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta.\end{aligned}$$

As we only consider projectable normalizations the vertical derivative of $\iota\left(\mathrm{d}x\right)$ is zero. Summation by parts gives

$$\mathrm{d}_v\sum\int L^{\boldsymbol{\kappa}}\iota\left(\mathrm{d}x\right)\wedge\Delta=\sum\int\left(\mathrm{S}_{-k}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa^{\beta}_{j;k}}\right)\mathcal{D}_j\mathrm{d}_v\kappa^{\beta}_{0;0}\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta,$$

and integrating by parts, the terms

$$H^j_{0,\beta}=\mathrm{S}_{-k}\frac{\partial L^{\boldsymbol{\kappa}}}{\partial\kappa^{\beta}_{j;k}},\qquad H^j_{1,\beta}=-D\left[H^j_{0,\beta}\mathcal{J}\right],\qquad\ldots,$$
$$H^j_{j,\beta}=(-1)^j\,D\left[\ldots\left[D\left[H^j_{0,\beta}\mathcal{J}\right]\ldots\right]\mathcal{J}\right],$$

reappear. After several iterations, we obtain

$$\mathrm{d}_v \sum \int L^{\boldsymbol{\kappa}} \iota\left(\mathrm{d}x\right) \wedge \Delta = \sum \int \sum_j H^j_{j,\beta} \mathrm{d}_v \kappa^{\beta}_{0;0} \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta.$$

Then using the replacement rule (Theorem 2.4.9) to invariantize this gives

$$\mathrm{d}_v \sum \int L^{\boldsymbol{\kappa}} \iota\left(\mathrm{d}x\right) \wedge \Delta = \sum \int \sum_j \iota\left(H^j_{j,\beta}\right) \iota\left(\mathrm{d}_v \kappa^{\beta}_{0;0}\right) \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta. \tag{4.20}$$

Recall that

$$\mathrm{d}_v \kappa^{\beta}_{0;0} = \frac{\partial \kappa^{\beta}_{0;0}}{\partial u_{i;l}} \mathrm{d}_v u_{i;l},$$

so

$$\iota\left(\mathrm{d}_v \kappa^{\beta}_{0;0}\right) = \iota\left(\frac{\partial \kappa^{\beta}_{0;0}}{\partial u_{i;l}}\right) \iota\left(\mathrm{d}_v u_{i;l}\right).$$

Again,

$$\begin{aligned} \iota\left(\mathrm{d}_v u_{0;0}\right) &= \left(\left.\frac{\partial\left(g \cdot u_{0;0}\right)}{\partial u_{0;0}}\right|_{g=\rho_{;0}(x,[u])}\right) \mathrm{d}_v u_{0;0} \\ &= \vartheta \mathrm{d}_v u_{0;0} \end{aligned}$$

and so

$$\mathrm{d}_v u_{0;0} = \theta_{0;0} \iota\left(\mathrm{d}_v u_{0;0}\right),$$

with $\theta_{0;0} = \vartheta^{-1}$. Then by shifting by l, taking the ith derivative and using the general Leibniz rule gives

$$\mathrm{d}_v u_{i;l} = \sum_{r=0}^{i} \binom{i}{r} \theta_{i-r;l} \left(D_r \mathrm{S}_l \iota\left(\mathrm{d}_v u_{0;0}\right)\right),$$

as before. Invariantizing (2.31) this gives

$$\iota\left(\mathrm{d}_v u_{i;l}\right) = \sum_{r=0}^{i} \binom{i}{r} \iota\left(\theta_{i-r;l}\right) \iota\left(D_r \mathrm{S}_l \iota\left(\mathrm{d}_v u_{0;0}\right)\right),$$

but as the derivatives are not invariant here, the term $D_r \mathrm{S}_l \iota\left(\mathrm{d}_v u_{0;0}\right)$ is not invariant. It is possible to find $\iota\left(D_r \mathrm{S}_l \iota\left(\mathrm{d}_v u_{0;0}\right)\right)$ in terms of the invariant derivatives, shifts and $\iota\left(\mathrm{d}_v u_{0;0}\right)$, but this can involve some arduous calculations. By letting

$$\mathcal{H}^\beta \iota\left(\mathrm{d}_v u_{0;0}\right) = \iota\left(\frac{\partial \kappa^\beta_{0;0}}{\partial u_{i;j}}\right) \iota\left(\mathrm{d}_v u_{i;l}\right),$$

the linear differential-difference operator $\mathcal{H}^\beta$ is

$$\mathcal{H}^\beta \iota\left(\mathrm{d}_v u_{0;0}\right) = \sum_r \sum_l \sum_{r=0}^{i} \binom{i}{r} \iota\left(\frac{\partial \kappa^\beta_{0;0}}{\partial u_{i;l}}\right) \iota\left(\theta_{i-r;l}\right) \iota\left(D_r \mathrm{S}_l \iota\left(\mathrm{d}_v u_{0;0}\right)\right),$$

and so

$$\mathrm{d}_v \sum \int L^{\boldsymbol{\kappa}} \iota\left(\mathrm{d}x\right) \wedge \Delta = \sum \int \sum_j \iota\left(H^j_{j,\beta}\right) \mathcal{H}^\beta \iota\left(\mathrm{d}_v u_{0;0}\right) \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta.$$

The last step is to take the adjoint of the linear differential-difference operator $\mathcal{H}^\beta$ in the formula above, to give the formula for the invariantization of the Euler operator of $\overline{\mathrm{L}}$ with respect to u:

$$\mathrm{d}_v \sum \int L^{\boldsymbol{\kappa}} \iota\left(\mathrm{d}x\right) \wedge \Delta = \sum \int \sum_j \left(\mathcal{H}^\beta\right)^\dagger \iota\left(H^j_{j,\beta}\right) \iota\left(\mathrm{d}_v u_{0;0}\right) \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta.$$

Hence,

$$\sum \int \iota\left(\mathrm{E}_u\left(\overline{\mathrm{L}}\right)\right) \iota\left(\mathrm{d}x\right) \wedge \Delta = \sum \int \sum_j \left(\mathcal{H}^\beta\right)^\dagger \iota\left(H^j_{j,\beta}\right) \iota\left(\mathrm{d}_v u_{0;0}\right) \wedge \iota\left(\mathrm{d}x\right) \wedge \Delta.$$

For the correction term, we use the same calculations as in the proof of the previous proposition and find that

$$\mathcal{P} = \sum_{j\geq 1} \sum_{l=0}^{j-1} (-1)^j \iota\left(D_{j-l}\left(\mathcal{J}\right)\right) \sum_{s=0}^{l} \binom{l}{s} \iota\left(D_s J\right) \iota\left(D_{l-s} \mathrm{S}_{-k}\left(\frac{\partial L^{\boldsymbol{\kappa}}}{\partial \kappa^\beta_{i;r}} \frac{\partial \kappa^\beta_{i;r}}{\partial u_{j;k}}\right)\right),$$

where there is more than one generating invariant for this case. With these results, the invariant formulation of the Euler–Lagrange equation of L is found from (4.12) in the same way as before. □

This formula for the invariant Euler–Lagrange equation is less explicit than the others that we have found. So here is an example of how one could use this formula.

Example 4.4.3. Consider the Lagrangian

$$\mathrm{L} = \frac{u_{1;0}}{u_{0;1}}$$

with the Lagrangian functional

$$\sum \int \mathrm{L}\,\mathrm{d}x \wedge \Delta.$$

Here use the 2-parameter Lie group action

$$g \cdot x = bx, \qquad g \cdot u_{0;0} = au_{0;0},$$

with the normalization equations (2.16),

$$g \cdot x = 1, \quad g \cdot u_{0;0} = 1.$$

The invariantization (2.31) of $u_{j;k}$ is

$$\iota\left(u_{j;k}\right) = \frac{x^j u_{j;k}}{u_{0;0}}$$

and the two generating invariants are

$$\kappa^1 = \iota\left(u_{1;0}\right) = \frac{xu_{1;0}}{u_{0;0}}, \qquad \kappa^2 = \iota\left(u_{0;1}\right) = \frac{u_{0;1}}{u_{0;0}}.$$

The Lagrangian functional can be written as

$$\sum \int \overline{\mathrm{L}}\,\iota\left(\mathrm{d}x\right) \wedge \Delta,$$

with

$$\overline{\mathrm{L}} = \frac{xu_{1;0}}{u_{0;1}}, \qquad \iota\left(\mathrm{d}x\right) = \frac{\mathrm{d}x}{x}.$$

However, for us the important invariant form of the Lagrangian and Lagrangian functional is

$$L^{\boldsymbol{\kappa}} = \frac{\kappa^1}{\kappa^2} \qquad \text{and} \qquad \sum \int L^{\boldsymbol{\kappa}}\,\iota\left(\mathrm{d}x\right) \wedge \Delta.$$

Using the formula in Proposition 4.4.5 the first step is to find the terms $H^j_{j,\beta}$ in this example. As there are no shifts or derivatives of the generating invariants,

$$\iota\left(H^0_{0,1}\right) = H^0_{0,1} = \frac{1}{\kappa^2}, \qquad \iota\left(H^0_{0,2}\right) = H^0_{0,2} = -\frac{\kappa^2}{\left(\kappa^2\right)^2}. \tag{4.21}$$

Next we need the adjoint of the linear differential-difference operators $\mathcal{H}^\beta$ for $\beta = 1, 2$. Using the method in the proof of Proposition 4.4.5,

$$\begin{aligned}
\mathrm{d}_v\kappa^1 &= \frac{\partial\kappa^1}{\partial u_{0;0}}\mathrm{d}_v u_{0;0} + \frac{\partial\kappa^1}{\partial u_{1;0}}\mathrm{d}_v u_{1;0} \\
&= \iota\left(\frac{\partial\kappa^1}{\partial u_{0;0}}\right)\iota\left(\mathrm{d}_v u_{0;0}\right) + \iota\left(\frac{\partial\kappa^1}{\partial u_{1;0}}\right)\iota\left(\mathrm{d}_v u_{1;0}\right)
\end{aligned}$$

with

$$\iota\left(\mathrm{d}_v u_{1;0}\right) = \iota\left(\theta_{1;0}\right)\iota\left(\mathrm{d}_v u_{0;0}\right) + \iota\left(\theta_{0;0}\right)\iota\left(D\iota\left(\mathrm{d}_v u_{0;0}\right)\right).$$

Then the identity $\mathcal{D} = \mathcal{J} D$ gives

$$\iota\left(D\iota\left(\mathrm{d}_v u_{0;0}\right)\right) = \iota\left(J\left(\mathcal{D}\iota\left(\mathrm{d}_v u_{0;0}\right)\right)\right) = \iota\left(\mathcal{D}\iota\left(\mathrm{d}_v u_{0;0}\right)\right) = \mathcal{D}\iota\left(\mathrm{d}_v u_{0;0}\right),$$

and therefore

$$\begin{aligned}
\mathrm{d}_v\kappa^1 &= \iota\left(\frac{\partial\kappa^1}{\partial u_{0;0}}\right)\iota\left(\theta_{0;0}\right)\iota\left(\mathrm{d}_v u_{0;0}\right) \\
&\quad + \iota\left(\frac{\partial\kappa^1}{\partial u_{1;0}}\right)\left[\iota\left(\theta_{1;0}\right)\iota\left(\mathrm{d}_v u_{0;0}\right) + \iota\left(\theta_{0;0}\right)\mathcal{D}\iota\left(\mathrm{d}_v u_{0;0}\right)\right] \\
&= \left[\left[\iota\left(\frac{\partial\kappa^1}{\partial u_{0;0}}\right)\iota\left(\theta_{0;0}\right) + \iota\left(\frac{\partial\kappa^1}{\partial u_{1;0}}\right)\iota\left(\theta_{1;0}\right)\right]\mathrm{id}\right. \\
&\quad \left. + \left[\iota\left(\frac{\partial\kappa^1}{\partial u_{1;0}}\right)\iota\left(\theta_{0;0}\right)\right]\mathcal{D}\right]\iota\left(\mathrm{d}_v u_{0;0}\right) \\
&= \mathcal{H}^1\iota\left(\mathrm{d}_v u_{0;0}\right).
\end{aligned}$$

The adjoint of the operator $\mathcal{H}^1$ on a smooth function f is

$$\begin{aligned}
\left(\mathcal{H}^1\right)^\dagger(f) &= \left[\iota\left(\frac{\partial\kappa^1}{\partial u_{0;0}}\right)\iota\left(\theta_{0;0}\right) + \iota\left(\frac{\partial\kappa^1}{\partial u_{1;0}}\right)\iota\left(\theta_{1;0}\right)\right] f \\
&\quad - \mathcal{D}\left[\iota\left(\frac{\partial\kappa^1}{\partial u_{1;0}}\right)\iota\left(\theta_{0;0}\right) f\right] - D\left(\mathcal{J}\right)\left[\iota\left(\frac{\partial\kappa^1}{\partial u_{1;0}}\right)\iota\left(\theta_{0;0}\right) f\right].
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{d}_v\kappa^2 &= \iota\left(\frac{\partial\kappa^2}{\partial u_{0;0}}\right)\iota\left(\theta_{0;0}\right)\iota\left(\mathrm{d}_v u_{0;0}\right)+\iota\left(\frac{\partial\kappa^2}{\partial u_{0;1}}\right)\iota\left(\theta_{0;1}\right)\mathrm{S}\iota\left(\mathrm{d}_v u_{0;0}\right)\\
&=\left[\iota\left(\frac{\partial\kappa^2}{\partial u_{0;0}}\right)\iota\left(\theta_{0;0}\right)\mathrm{id}+\iota\left(\frac{\partial\kappa^2}{\partial u_{0;1}}\right)\iota\left(\theta_{0;1}\right)\mathrm{S}\right]\iota\left(\mathrm{d}_v u_{0;0}\right)\\
&=\mathcal{H}^2\iota\left(\mathrm{d}_v u_{0;0}\right),
\end{aligned}$$

so

$$\left(\mathcal{H}^2\right)^{\dagger}(f)=\left[\iota\left(\frac{\partial\kappa^2}{\partial u_{0;0}}\right)\iota\left(\theta_{0;0}\right)f\right]+\mathrm{S}_{-1}\left[\iota\left(\frac{\partial\kappa^2}{\partial u_{0;1}}\right)\iota\left(\theta_{0;1}\right)f\right].$$

Putting together the necessary components,

$$\vartheta=\left.\frac{\partial\left(g\cdot u_{0;0}\right)}{\partial u_{0;0}}\right|_{g=\rho_{;0}(x,[u])}=\frac{1}{u_{0;0}},$$

and hence

$$\theta_{0;0}=u_{0;0},\qquad \theta_{1;0}=u_{1;0},\qquad \theta_{0;1}=u_{0;1},$$

with invariantizations (2.31)

$$\iota\left(\theta_{0;0}\right)=1,\qquad \iota\left(\theta_{1;0}\right)=\kappa^1,\qquad \iota\left(\theta_{0;1}\right)=\kappa^2.$$

Then $\mathcal{J}=x$, so $D(\mathcal{J})=1$. Finally,

$$\frac{\partial\kappa^1}{\partial u_{0;0}}=-\frac{x u_{1;0}}{\left(u_{0;0}\right)^2},\qquad \frac{\partial\kappa^1}{\partial u_{0;0}}=\frac{x}{u_{0;0}},\qquad \frac{\partial\kappa^2}{\partial u_{0;0}}=-\frac{u_{0;1}}{\left(u_{0;0}\right)^2},\qquad \frac{\partial\kappa^1}{\partial u_{0;1}}=\frac{1}{u_{0;0}},$$

and their invariantizations (2.31) are

$$\iota\left(\frac{\partial\kappa^1}{\partial u_{0;0}}\right)=-\kappa^1,\qquad \iota\left(\frac{\partial\kappa^1}{\partial u_{0;0}}\right)=1,\qquad \iota\left(\frac{\partial\kappa^2}{\partial u_{0;0}}\right)=\kappa^2,\qquad \iota\left(\frac{\partial\kappa^1}{\partial u_{0;1}}\right)=1.$$

Using

$$\mathrm{d}_v\sum\int L^{\boldsymbol{\kappa}}\iota\left(\mathrm{d}x\right)\wedge\Delta=\sum\int\sum_j\left(\mathcal{H}^{\beta}\right)^{\dagger}\iota\left(H^j_{j,\beta}\right)\iota\left(\mathrm{d}_v u_{0;0}\right)\wedge\iota\left(\mathrm{d}x\right)\wedge\Delta,$$

gives the invariantization (2.31) of the Euler operator of $\overline{\mathrm{L}}$ with respect to u.

As

$$\begin{aligned}
\left(\mathcal{H}^1\right)^\dagger \iota\left(H^0_{0,1}\right) &= \left[\iota\left(\frac{\partial \kappa^1}{\partial u_{0;0}}\right)\iota\left(\theta_{0;0}\right) + \iota\left(\frac{\partial \kappa^1}{\partial u_{1;0}}\right)\iota\left(\theta_{1;0}\right)\right]\iota\left(H^0_{0,1}\right) \\
&\quad -\mathcal{D}\left[\iota\left(\frac{\partial \kappa^1}{\partial u_{1;0}}\right)\iota\left(\theta_{0;0}\right)\iota\left(H^0_{0,1}\right)\right] - D\left(\mathcal{J}\right)\left[\iota\left(\frac{\partial \kappa^1}{\partial u_{1;0}}\right)\iota\left(\theta_{0;0}\right)\iota\left(H^0_{0,1}\right)\right] \\
&= \left[-\kappa^1 \cdot 1 + 1 \cdot \kappa^1\right]\left(\frac{1}{\kappa^2}\right) - \mathcal{D}\left[1 \cdot 1 \cdot \frac{1}{\kappa^2}\right] - 1 \cdot \left[1 \cdot 1 \cdot \frac{1}{\kappa^2}\right] \\
&= -\mathcal{D}\left(\frac{1}{\kappa^2}\right) - \frac{1}{\kappa^2} \\
&= -\frac{\kappa^1}{\kappa^2} + \frac{\kappa^1_{0;1}}{\kappa^2} - \frac{1}{\kappa^2}\,,
\end{aligned}$$

and

$$\begin{aligned}
\left(\mathcal{H}^2\right)^\dagger \iota\left(H^0_{0,2}\right) &= \left[\iota\left(\frac{\partial \kappa^2}{\partial u_{0;0}}\right)\iota\left(\theta_{0;0}\right)\iota\left(H^0_{0,2}\right)\right] + \mathrm{S}_{-1}\left[\iota\left(\frac{\partial \kappa^2}{\partial u_{0;1}}\right)\iota\left(\theta_{0;1}\right)\iota\left(H^0_{0,2}\right)\right] \\
&= -\kappa^2 \cdot 1 \cdot \left(-\frac{\kappa^1}{\left(\kappa^2\right)^2}\right) + \mathrm{S}_{-1}\left[1 \cdot \kappa^2 \cdot \left(-\frac{\kappa^1}{\left(\kappa^2\right)^2}\right)\right] \\
&= \frac{\kappa^1}{\kappa^2} - \frac{\kappa^1_{0;-1}}{\kappa^2_{0;-1}}\,,
\end{aligned}$$

the invariantization (2.31) of $\mathrm{E}_u\left(\overline{\mathrm{L}}\right)$ is the following

$$\left(\mathcal{H}^1\right)^\dagger \iota\left(H^0_{0,1}\right) + \left(\mathcal{H}^2\right)^\dagger \iota\left(H^0_{0,2}\right) = \frac{\kappa^1_{0;1}}{\kappa^2} - \frac{1}{\kappa^2} - \frac{\kappa^1_{0;-1}}{\kappa^2_{0;-1}}\,.$$

The correction term $\mathcal{P}$ for this example is

$$\begin{aligned}
\mathcal{P} &= -\iota\left(\frac{\partial L^{\kappa}}{\partial \kappa^1_{0;0}}\frac{\partial \kappa^1_{0;0}}{\partial u_{1;0}}\right) \\
&= -\frac{1}{\kappa^2}\,.
\end{aligned}$$

Therefore, the invariant formulation of the Euler–Lagrange equation for L is

$$\left(\mathcal{H}^1\right)^\dagger \iota\left(H^0_{0,1}\right) + \left(\mathcal{H}^2\right)^\dagger \iota\left(H^0_{0,2}\right) - \mathcal{P} = \frac{\kappa^1_{0;1}}{\kappa^2} - \frac{\kappa^1_{0;-1}}{\kappa^2_{0;-1}}\,.$$

Chapter 5

Reductions of ordinary difference equations

This chapter broadens the moving frame symmetry reduction method to different types of OΔEs that have a Lie group action involving the independent variable. Additionally, the applicability of the method is extended by showing how it works for partitioned OΔEs. A comprehensive comparison of this method and canonical coordinates is also given.

5.1 Introduction to moving frame reductions

This chapter aims to show how to reduce and solve OΔEs, systems of OΔEs and partitioned OΔEs using moving frames. This chapter follows closely the method used by Benson and Valiquette [4] on OΔE reductions. In [4], a given OΔE is reduced multiple times by a technique called inductive moving frames, which was developed in several different papers [4, 18, 30, 33, 34]. This technique and the general method of moving frame reductions are outlined in this chapter.

Benson and Valiquette's approach does not allow the OΔE to depend explicitly on the independent variable. This limits the applicability of the method as many OΔEs have this property. Also, the current literature does not address partitioned OΔEs, for which the group action may differ between partitions. Here we extend the method to OΔEs that depend on the independent variable and show how to apply the method for partitioned OΔEs in an example. Furthermore, we discuss some of the benefits and disadvantages of using moving frames in comparison to the canonical coordinates method, which is described in [15].

Many OΔEs require multiple reductions to solve them. The key component for multiple symmetry reductions using inductive moving frames or canonical coordinates is the existence of a solvable Lie symmetry group which leaves the OΔE invariant.

Definition 5.1.1. Let $\widetilde{G}$ be an r-dimensional Lie group with Lie algebra $\mathfrak{g}$. The Lie group $\widetilde{G}$ is said to be solvable if there exists a chain of Lie subgroups

$$\{e\} = G^{(0)} \subset G^{(1)} \subset G^{(2)} \subset \cdots \subset G^{(r-1)} \subset G^{(r)} = \widetilde{G} \tag{5.1}$$

such that for $l = 1, \ldots, r$, $G^{(l)}$ is a l-dimensional subgroup of $\widetilde{G}$ and $G^{(l-1)}$ is a normal subgroup of $G^{(l)}$. At the infinitesimal level, the Lie algebra $\mathfrak{g}$ is solvable if there exists a chain of Lie subalgebras

$$\{0\} = \mathfrak{g}^{(0)} \subset \mathfrak{g}^{(1)} \subset \mathfrak{g}^{(2)} \subset \cdots \subset \mathfrak{g}^{(r-1)} \subset \mathfrak{g}^{(r)} = \mathfrak{g},$$

such that for $l = 1, \ldots, r$, $\dim \mathfrak{g}^{(l)} = l$ and $\mathfrak{g}^{(l-1)}$ is a ideal of $\mathfrak{g}^{(l)}$, that is, $\left[\mathfrak{g}^{(l-1)}, \mathfrak{g}^{(l)}\right] \subset \mathfrak{g}^{(l-1)}$. Here $[\ ,\ ]$ represents the Lie bracket, see [28].

We use the difference prolongation space (with one independent variable) as described in Section 2.1. Consequently, when we construct a moving frame it is a difference moving frame over a particular fixed n (see Section 2.4).

Let $G = G^{(s+1)}$ in a solvable chain (5.1) and let $H = G^{(s)}$ in the same chain. The Lie group action of H on the prolongation space $P_n(\mathbb{R})$ is obtained by pulling back the action $h \cdot u$, $h \in H$, on the total space to $P_n(\mathbb{R})$.

A moving frame for H is given by the set of normalization equations

$$\mathcal{K}^H = \{u_{n^l} = c_l \,|\, l = 1, \ldots, s = \dim H\}, \quad \text{with}\, n^l \in \mathbb{Z}\, \text{for all}\, l, \tag{5.2}$$

defining its coordinate cross-section (2.29). Then, let $\rho^H([u])$ be the corresponding right moving frame, and let ι^H denote the induced invariantization map with

$$\iota^H(u_l) = \rho^H([u]) \cdot u_l, \tag{5.3}$$

denoting the H-normalized invariants.

The key observation to make for inductive moving frames is that

$$g \cdot u_l = \left(gh^{-1}\right) h \cdot u_l = \widetilde{g} \cdot (h \cdot u_l), \quad \widetilde{g} = gh^{-1} \in G. \tag{5.4}$$

This means the group G naturally acts on the H-lifted invariants $(h \cdot u_l)$, for all $h \in H$, and more importantly the Lie group G acts on the H-normalized invariants (5.3) as follows

$$\widetilde{g} \cdot \iota^H (u_l) = \widetilde{g} \cdot \left(\rho^H ([u]) \cdot u_l\right). \tag{5.5}$$

To implement the moving frame construction for G, let

$$\mathcal{K}^G = \{\iota^H (u_{n^1}) = c_1, \ldots, \iota^H (u_{n^s}) = c_s, \iota^H (u_{n^{s+1}}) = c_{s+1}\} \subset \mathcal{K}^H \tag{5.6}$$

be the cross-section (2.29) of the Lie group action (5.4). Here it is easier to work with the group action of G in (5.5) than in (5.4) because the first s equations in (5.6) are constant under invariantization (2.31) by H. Accordingly, the first s constants remain the same as in (5.2). Solving the normalization equations

$$\widetilde{g} \cdot \left(\iota^H (u_{n^1})\right) = c_1, \quad \ldots, \quad \widetilde{g} \cdot \left(\iota^H (u_{n^{s+1}})\right) = c_{s+1},$$

for $\widetilde{g} \in G$ gives the right moving frame $\widetilde{\rho^G} : \mathcal{K}^H \to G$. The right moving frame $\rho^G : P_n (\mathbb{Z}) \to G$ corresponding to the original Lie group action (5.4) is then

$$\rho^G ([u]) = \widetilde{\rho^G} ([u]) \, \rho^H ([u]).$$

Here is a simple example of how to use the inductive moving frame construction.

Example 5.1.1. Let G be the group generated by the infinitesimal generators ∂_u and $u\partial_u$, and let H be the subgroup generated by the infinitesimal generator ∂_u. The action of the Lie (sub)group H is

$$g \cdot u = u + b, \quad b \in \mathbb{R}.$$

Let the cross-section (2.29) for H be

$$\mathcal{K}^H = \{u_0 = 0\}. \tag{5.7}$$

This gives the value of the parameter on the frame as $b = -u_0$. Therefore, the H-normalized invariants are

$$\iota^H(u_l) = u_l - u_0,$$

for $l \in \mathbb{Z}$. Now the action of the Lie group G is

$$g \cdot u = au + b, \quad \text{where } a \in \mathbb{R}^+.$$

Using the cross-section (2.29) for G given by

$$\mathcal{K}^G = \{\iota^H(u_0) = 0, \iota^H(u_1) = 1\}$$

the equations we need to solve for the parameters are

$$\widetilde{g} \cdot \iota^H(u_0) = \widetilde{a}(u_0 - u_0) + \widetilde{b} = 0, \quad \widetilde{g} \cdot \iota^H(u_1) = \widetilde{a}(u_1 - u_0) + \widetilde{b} = 1.$$

This yields

$$\widetilde{a} = \frac{1}{u_1 - u_0}, \quad \widetilde{b} = 0.$$

So, the invariants of the group G are

$$\iota^G(u_l) = \rho^G([u]) \cdot u_l = \widetilde{\rho^G}([u])\,\rho^H([u]) \cdot u_l = \widetilde{\rho^G}([u]) \cdot (u_l - u_0) = \frac{u_l - u_0}{u_1 - u_0}.$$

It is easy to check this gives the same result as the moving frame with the coordinate cross-section $\mathcal{K}^G = \{u_0 = 0, u_1 = 1\}$.

A schematic of the inductive moving frame construction is shown in Figure 5.1. (As the cross-section $\mathcal{K}^G$ is a submanifold of the cross-section $\mathcal{K}^H$ it is difficult to give an accurate visual representation of this method in two dimensions.)

For the symmetry reduction algorithm, the subgroup H is a normal subgroup of G, so the quotient group G/H induces an action on the space of H-normalized

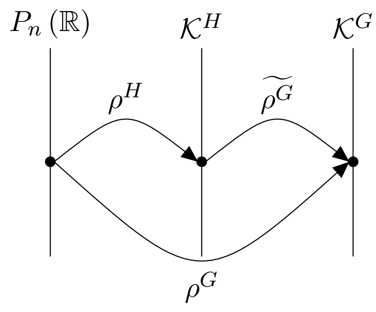


Figure 5.1: Inductive moving frame.

invariants (5.3). This action is obtained symbolically by looking at the partial cross-section

$$\widetilde{\mathcal{K}}^H = \{\iota^H(u_{n^1}) = c_1, \ldots, \iota^H(u_{n^s}) = c_s\}.$$

Solving the normalization equations,

$$\widetilde{g} \cdot \iota^H(u_{n^1}) = c_1, \quad \ldots, \quad \widetilde{g} \cdot \iota^H(u_{n^s}) = c_s,$$

for the group parameters of the normal subgroup H, one obtains a partial moving frame $\widetilde{\rho}^H : \widetilde{\mathcal{K}}^H \to H \subset G$. Then the action of the quotient Lie group G/H on the H-normalized invariants is obtained by substituting the values of the partial moving frame $\widetilde{\rho}^H([u])$ into (5.5).

The partial cross-section $\widetilde{\mathcal{K}}^H$ is a direct result of the cross-section in $\mathcal{K}^H$ (5.2). When considering group actions that depend explicitly on the independent variable, it is important to note that the partial moving frame on $P_n(\mathbb{R})$ depends on the independent variable n. Therefore, when finding the action of the quotient Lie group G/H on the H-normalized invariants one substitutes the values of the partial moving frame $\widetilde{\rho}^H([u]) = \widetilde{\rho_0}^H([u])$ (over n) into (5.5), as this is the moving frame associated to the partial cross-section. The Lie group does not change the discrete variable n; thus, n is a fixed value in the partial moving frame calculations.

Remark 5.1.2. The normalized invariants $\iota_0(u_1)$ and $\iota_1(u_1)$ are related by the recurrence relation

$$\iota_1(u_1) = K_0 \cdot \iota_0(u_1), \tag{5.8}$$

where $K_0 = \rho_1([u])\,\rho_0([u])^{-1}$ and the product operator $(\cdot)$ in (5.8) stands for the group product in G. Note that K_0 denotes $K_{(1)}$ (in the notation used in previous chapters), as there is only one independent variable. The normalized invariants $\iota_0(u_{-1})$ and $\iota_{-1}(u_{-1})$ are related by the recurrence relation

$$\iota_0(u_{-1}) = K_{-1} \cdot \iota_{-1}(u_{-1})\,, \tag{5.9}$$

where $K_{-1} = \mathrm{S}_{-1}K_0$. In general, $K_j = \mathrm{S}_j K_0$ is the jth shift of the Maurer–Cartan invariant K_0. Thus, for $l \geq 1$, the recurrence relations are the following:

$$\begin{aligned} \iota_0(u_l) &= K_0^{-1}K_1^{-1}\ldots K_{l-2}^{-1}K_{l-1}^{-1} \cdot \iota_l(u_l)\,,\\ \iota_0(u_{-l}) &= K_{-1}K_{-2}\ldots K_{-(l-1)}K_{-l} \cdot \iota_{-l}(u_{-l})\,. \end{aligned} \tag{5.10}$$

Consequently, the normalized invariants $\iota_0(u_l)$, with $l \in \mathbb{Z}$, can be expressed in terms of shifts of the normalized invariant $\iota_0(u_0)$ and shifts of the Maurer–Cartan invariant K_0 (or its inverse). For convenience, let $\mathfrak{m}_j = K_j^{-1}$ (in line with Benson and Valiquette's notation).

Therefore, when $\iota_0(u_0)$ is a phantom invariant, that is when $\iota_0(u_0) = c$ (a constant), the recurrence relations (5.10) provide expressions for all normalized invariants $\iota_0(u_l)$ in terms of the Maurer–Cartan invariant K_0 (or $\mathfrak{m}_0$) and its shifts as follows. For $l \geq 1$,

$$\begin{aligned} \iota_0(u_l) &= \mathfrak{m}_0\mathfrak{m}_1\ldots\mathfrak{m}_{l-2}\mathfrak{m}_{l-1} \cdot c,\\ \iota_0(u_{-l}) &= K_{-1}K_{-2}\ldots K_{-(l-1)}K_{-l} \cdot c. \end{aligned} \tag{5.11}$$

5.2 Symmetry reduction using moving frames

By incorporating the moving frame machinery into the symmetry reduction algorithm, it is possible to implement the algorithm without relying on the coordinate expressions for the canonical variables and the difference invariants. First, we explain how to do this by describing the constructions for OΔEs with a one-dimensional Lie symmetry group. We then extend this to OΔEs with a multi-parameter solvable Lie symmetry group.

5.2.1 One-parameter symmetry groups

We use what is known about the canonical coordinates method (see [15]) to help describe the moving frame method. Doing this will also show some of the similarities and differences between the two approaches. Consider a kth order O∆E,

$$F(n, u_0, \ldots, u_k) = 0, \tag{5.12}$$

that is invariant under the action of a one-parameter Lie point symmetry group. The local action of this one-parameter Lie group is induced by the flow of the infinitesimal generator $\mathbf{v} = Q\,(n, u)\,\partial_u$, that is,

$$g \cdot u_0 = \exp\left[\epsilon \mathbf{v}\right] \cdot u_0, \quad g \in G.$$

Here ϵ is the parameter that we use below for the moving frame. In the canonical coordinates method, one seeks an equivariant component, s_0, and an invariant component r_0, which satisfy

$$g \cdot s_0 = s_0 + \epsilon, \quad g \cdot r_0 = r_0. \tag{5.13}$$

To find these one can use the formula

$$s_0 = \int \frac{1}{Q\,(n, u_0)} \mathrm{d}u_0$$

for the equivariant component and then for the invariant component

$$r_0 = s_1 - s_0, \quad (\text{as} \quad g \cdot s_1 - g \cdot s_0 = s_1 - s_0)\,.$$

Note that other forms of the equivariant and invariant components are

$$\widetilde{s} = s + a\,(r)\,, \quad \widetilde{r} = b\,(r)\,,$$

where a and b are smooth functions and b is invertible. For the moving frame method, the equivariant component is the moving frame $\rho_0\,([u])$. To show this we go through in detail the calculations for the moving frame. Let the cross-section

(2.29) be

$$\mathcal{K} = \{u_0 = \widehat{c}_1\},$$

with normalization equation (2.16) $g \cdot u_0 = \widehat{c}_1$, which must be solved for the group parameter $\epsilon = \rho_0([u])$. Then, for the general group element g parameterized by ϵ,

$$\rho_0(g \cdot [u]) = \rho_0([u])\, g^{-1} = \rho_0([u]) - \epsilon.$$

For the cross-section with $\widehat{c}_1 = 0$ the canonical coordinate is $s_0 = -\rho_0([u])$. The invariant component for the moving frame is, up to a sign, the inverse Maurer–Cartan invariant

$$\mathfrak{m}_0 = \rho_0([u])\, \rho_1([u])^{-1} = \rho_0([u]) - \rho_1([u]),$$

with $\rho_1([u]) = \mathrm{S}\rho_0([u])$. This is invariant for the same reason as r_0 is in the canonical coordinates method, that is,

$$\rho_0([u]) - \epsilon - (\rho_1([u]) - \epsilon) = \rho_0([u]) - \rho_1([u]) = \mathfrak{m}_0.$$

To reduce the OΔE the (normalized) invariants $\iota_0(u_l)$, for $l = 1, \dots, k$, are written in terms of $k-1$ shifts of the inverse Maurer–Cartan invariants. Using the recurrence relations (5.11) and the fact that the group product is addition, the recurrence relations (5.11) become

$$\iota(u_l) = [\mathfrak{m}_0 + \mathfrak{m}_1 + \cdots + \mathfrak{m}_{l-2} + \mathfrak{m}_{l-1}] \cdot \widehat{c}_1, \tag{5.14}$$

Finally by invariantizing (2.31) the OΔE, (5.12), and using the recurrence relations (5.14) we obtain the $(k-1)$th order reduced OΔE

$$F^1(n, \mathfrak{m}_0, \dots, \mathfrak{m}_{k-1}) = 0. \tag{5.15}$$

All the computations are performed symbolically meaning without relying on the coordinate expressions (in terms of u and n) of canonical variable (moving frame) or the difference invariants (Maurer–Cartan invariants). Therefore, the

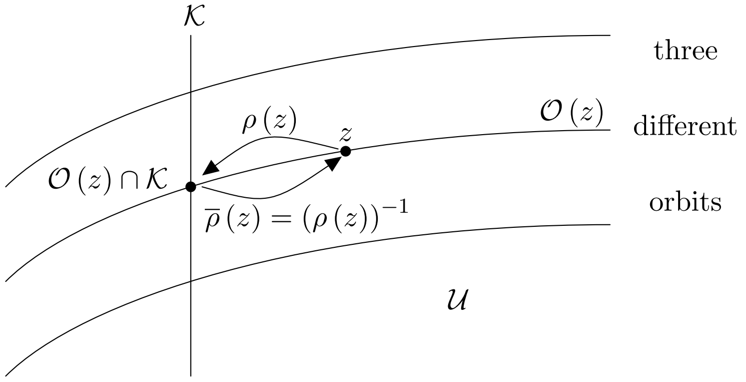


Figure 5.2: Right and left moving frame defined by a cross-section.

coordinate expressions for the moving frame $\rho_0([u])$, the invariants

$$\iota_0(u_l), \quad l = 0, \ldots, k,$$

and different shifts of the Maurer–Cartan invariant are unnecessary to find the reduced OΔE (5.15).

From here to find the solution $u_0 = f(n)$ of the original OΔE (5.12) one needs the solution to the reduced OΔE (5.15). For first-order OΔEs (i.e. equations of the form $F(n, u_0, u_1) = 0$) the moving frame method reduces to the solution for $\mathfrak{m}_0$. For other OΔEs there may be other methods to find the solution for $\mathfrak{m}_0$, including but not restricted to multiple symmetry reductions (as explained in the next subsection). By construction of the right moving frame, $\widehat{c}_1 = \rho_0([u]) \cdot u_0$. Therefore, the solution is $u_0 = \overline{\rho_0}([u]) \cdot \widehat{c}_1$, where $\overline{\rho_0}([u]) = \rho_0([u])^{-1}$ is the left moving frame (see Figure 5.2). As the group product is the usual sum, its inverse is the additive inverse, so

$$\mathfrak{m}_0 = \rho_0([u]) - \rho_1([u]) = \overline{\rho_1}([u]) - \overline{\rho_0}([u]). \tag{5.16}$$

Equation (5.16) is called the reconstruction equation for the left moving frame $\overline{\rho_0}([u])$. Once $\mathfrak{m}_0$ is known, (5.16) yields a first-order linear difference equation for $\overline{\rho_0}([u])$. The general solution to the reconstruction equation (5.16) is

$$\overline{\rho_0}([u]) = c_1 + \sigma_l\{\mathfrak{m}_l; n_0, n\},$$

where c_1 is an arbitrary constant of integration and

$$\sigma_l\{\mathfrak{m}_l; n_0, n\} = \begin{cases} \sum_{l=n_0}^{n-1} \mathfrak{m}_l, & n > n_0, \\ 0, & n = n_0, \\ -\sum_{l=n}^{n_0-1} \mathfrak{m}_l, & n < n_0. \end{cases}$$

Therefore, the general solution to the OΔE (5.12) is

$$u_0 = \overline{\rho_0}\left([u]\right) \cdot \widehat{c}_1 = [c_1 + \sigma_l\{\mathfrak{m}_l; n_0, n\}] \cdot \widehat{c}_1.$$

Example 5.2.1. We show how to solve a first-order OΔE which depends on n. This example comes from [15], where canonical coordinates are used to solve it. The OΔE is

$$u_1 = \frac{u_0 - n}{n u_0 - 1}, \quad n \geq 2, \tag{5.17}$$

and is invariant under the group action

$$g \cdot u_0 = -\frac{e^{2(-1)^n \epsilon} u_0 - e^{2(-1)^n \epsilon} + u_0 + 1}{e^{2(-1)^n \epsilon} u_0 - e^{2(-1)^n \epsilon} - u_0 - 1}, \tag{5.18}$$

coming from the characteristic $Q(n, u) = (-1)^n \left(u_0^2 - 1\right)$. Here we are interested in the solution for $n \geq 2$ with $u_0 = u\,(2)$. Let the cross-section (2.29) be

$$\mathcal{K} = \{u_0 = 0\}.$$

This means that $\iota_0\left(u_0\right) = 0$ and $\mathrm{S}\iota_0\left(u_0\right) = \iota_1\left(u_1\right) = 0$. Invariantizing (2.31) u_1 and using the recurrence relations for the Maurer–Cartan equations (5.14) gives

$$\begin{aligned} \iota_0\left(u_1\right) &= \mathfrak{m}_0 \cdot \iota_1\left(u_1\right) \\ &= \mathfrak{m}_0 \cdot 0 \\ &= \frac{1 - e^{2(-1)^{n+1} \mathfrak{m}_0}}{e^{2(-1)^{n+1} \mathfrak{m}_0} + 1}. \end{aligned}$$

Therefore, the reduced OΔE is

$$\frac{1 - e^{2(-1)^{n+1} \mathfrak{m}_0}}{e^{2(-1)^{n+1} \mathfrak{m}_0} + 1} = n,$$

which is the invariantization of the original OΔE (5.17). Solving for, $\mathfrak{m}_0$, the inverse Maurer–Cartan invariant is

$$\mathfrak{m}_0 = \frac{(-1)^{n+1}}{2} \mathrm{Log}\left(\frac{1-n}{n+1}\right),$$

where Log is the principal value of the complex logarithm:

$$\mathrm{Log}\,(z) = \ln\left(|z|\right) + i\mathrm{Arg}\,(z)\,, \quad \mathrm{Arg}\,(z) \in (-\pi, \pi]\,.$$

The fact that $n \geq 2$ means that $(1-n)\,/\,(n+1) < 0$, which implies

$$\mathfrak{m}_0 = \frac{(-1)^{n+1}}{2} \left(\ln\left(\frac{n-1}{n+1}\right) + i\pi\right),$$

by the formula for a complex logarithm (see Example 2.10 in [15]). To find the solution u_0 to the original OΔE we first need to solve $\mathfrak{m}_0 = \overline{\rho_1}\,([u]) - \overline{\rho_0}\,([u])$ for

$$\begin{aligned} \overline{\rho_0}\,([u]) &= c_1 + \sigma_k\left\{\frac{(-1)^{k+1}}{2}\left(\ln\left(\frac{k-1}{k+1}\right) + i\pi\right); 2, n\right\} \\ &= c_1 + \frac{(-1)^n}{2}\left(\ln\left(\frac{n-1}{n}\right) + \frac{i\pi}{2}\right). \end{aligned} \tag{5.19}$$

Then using $u_0 = \overline{\rho_0}\,([u]) \cdot 0$ gives

$$u_0 = \frac{1 - e^{2(-1)^n \overline{\rho_0}([u])}}{e^{2(-1)^n \overline{\rho_0}([u])} + 1}, \tag{5.20}$$

the general solution of the OΔE (5.17) when substituting in the left moving frame (5.19), $\overline{\rho_0}\,([u])$. However, we are looking for the solution for $n \geq 2$ with $u_0 = u\,(2)$. This needs to be split into two cases. This is because of the form of the left moving frame $\overline{\rho_0}\,([u])$ which is

$$\overline{\rho_0}\,([u]) = \frac{(-1)^n}{2} \mathrm{Log}\left(\frac{1-u_0}{u_0-1}\right), \tag{5.21}$$

from the rearranged formula of the general solution (5.20). The first case we look at is when $(1-u_0)\,/\,(u_0+1) > 0$. So, first to find the value of the constant c_1 in terms of $u\,(2)$, compare the left moving frame (5.19) with the formula (5.21) for the left moving frame $\overline{\rho_0}\,([u])$, with $u_0 = u\,(2)$. This gives the value of the constant

$$c_1 = \frac{1}{2}\ln\left(\frac{2\left(1-u\left(2\right)\right)}{u\left(2\right)+1}\right) - \frac{i\pi}{4}.$$

Substituting this into the equation for the left moving frame $\overline{\rho_0}\left([u]\right)$ (5.19) gives

$$\overline{\rho_0}\left([u]\right) = \frac{1}{2}\ln\left(\left(\frac{n-1}{n}\right)^{(-1)^n}\left(\frac{2\left(1-u\left(2\right)\right)}{u\left(2\right)+1}\right)\right) + \frac{i\pi}{2}\left(\frac{(-1)^n+1}{2}\right).$$

Then by substituting this into the equation for u_0 (5.20) the solution is

$$u_0 = \frac{1-\frac{n-1}{n}\left(\frac{2(1-u(2))}{u(2)+1}\right)^{(-1)^n}\exp\left(i\pi\left(\frac{1-(-1)^n}{2}\right)\right)}{\frac{n-1}{n}\left(\frac{2(1-u(2))}{u(2)+1}\right)^{(-1)^n}\exp\left(i\pi\left(\frac{1-(-1)^n}{2}\right)\right)+1}. \tag{5.22}$$

The solution can be written as

$$u_0 = \begin{cases} \frac{(u(2)+1)n+2(u(2)-1)(n-1)}{(u(2)+1)n-2(u(2)-1)(n-1)}, & n \quad \text{even}, \\ \frac{2(1-u(2))n+(u(2)+1)(n-1)}{2(1-u(2))n-(u(2)+1)(n-1)}, & n \quad \text{odd}. \end{cases} \tag{5.23}$$

For the case when $\left(1-u_0\right)/\left(u_0+1\right) < 0$,

$$c_1 = \frac{1}{2}\ln\left(\frac{2\left(u\left(2\right)-1\right)}{1+u\left(2\right)}\right) + \frac{i\pi}{4}.$$

so from (5.19),

$$\overline{\rho_0}\left([u]\right) = \frac{1}{2}\ln\left(\left(\frac{n-1}{n}\right)^{(-1)^n}\left(\frac{2\left(u\left(2\right)-1\right)}{1+u\left(2\right)}\right)\right) + \frac{i\pi}{2}\left(\frac{(-1)^n-1}{2}\right).$$

Finally, (5.20) gives the solution

$$u_0 = \frac{1-\frac{n-1}{n}\left(\frac{2(u(2)-1)}{1+u(2)}\right)^{(-1)^n}\exp\left(i\pi\left(\frac{1+(-1)^n}{2}\right)\right)}{\frac{n-1}{n}\left(\frac{2(u(2)-1)}{1+u(2)}\right)^{(-1)^n}\exp\left(i\pi\left(\frac{1+(-1)^n}{2}\right)\right)+1}. \tag{5.24}$$

This is equivalent to u_0 found in the formula (5.22). As expected, the solution by moving frames agrees with what is found using the canonical coordinates method in [15].

Remark 5.2.1. Any canonical coordinate s_0 that meets the requirement that one must be able to invert the map from u_0 to s_0 (at least, for all points (n, u) that occur in any solution of the original OΔE and satisfy $Q(n,u) \neq 0$) is called compatible with the OΔE (see [15]). In the moving frame method, we can take the inverse of the right moving frame to give us the solution $u_0 = \overline{\rho_0}([u]) \cdot \widehat{c}_1$. Therefore, if we can find a cross-section that works in some domain $\mathcal{U}$ effectively, we have a compatible canonical coordinate with the OΔE on $\mathcal{U}$.

5.2.2 Solvable symmetry groups

Now we show how to reduce and solve an OΔE with a multi-parameter solvable Lie group using the moving frame method. This time the kth order OΔE

$$A(n, u_0, \ldots, u_k) = 0, \tag{5.25}$$

is invariant under an r-dimensional solvable symmetry group G. Let (5.1) be a corresponding chain of normal subgroups. At the infinitesimal level, let $\mathfrak{g}$ be the corresponding r-dimensional solvable Lie algebra spanned by the infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$ such that

$$\mathfrak{g}^{(j)} = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}, \quad j = 1, \ldots, r.$$

Here let G_j denote the one-parameter Lie group for the infinitesimal generator $\mathbf{v}_j$. The local group action of this one-parameter Lie group is induced by the flow of $\mathbf{v}_j$:

$$g_j \cdot u_l = \exp(\epsilon_j \mathbf{v}_j) \cdot u_l, \quad g_j \in G_j, \quad l \in \mathbb{Z}.$$

Using the chain of solvable groups (5.1) the first reduction of the OΔE (5.25) is done using the one-parameter Lie group $G_1 = G^{(1)}$. Therefore, let the cross-section of the one-parameter group action G_1 be $\mathcal{K}^1 = \{u_0 = \widehat{c}_1\}$, with $\rho_0^1([u])$ the corresponding right moving frame and ι_0^1 denoting the induced invariantization map. Also the corresponding Maurer–Cartan invariant is $\mathfrak{m}_0^1 = \rho_0^1([u]) - \rho_1^1([u])$. To find the order $k-1$ reduced OΔE we invariantize (2.31) the kth order OΔE (5.25) by ι_0^1 and use the recurrence relations

$$\iota_0^1(u_l) = \left[\mathfrak{m}_0^1 + \mathfrak{m}_1^1 + \cdots + \mathfrak{m}_{l-2}^1 + \mathfrak{m}_{l-1}^1\right] \cdot \widehat{c}_1, \quad l \geq 1,$$

to obtain

$$F^1\left(n, \mathfrak{m}_0^1, \ldots, \mathfrak{m}_{k-1}^1\right) = 0. \tag{5.26}$$

Nothing up to this point has changed from the one-parameter group case.

For a second iteration of the symmetry reduction algorithm use the one-parameter group $G_2 \simeq G^{(2)}/G^{(1)}$ to reduce the order of the reduced OΔE (5.26). To find the induced action of G_2 on the G_1-normalized invariants apply the inductive moving frame construction. Once the G_2-action on $\iota_0^1(u_l)$ is known for $l = 1, \ldots, k$, the action on the inverse Maurer–Cartan invariant $\mathfrak{m}_0^1$ is deduced from $\iota_0^1(u_1) = \mathfrak{m}_0^1 \cdot \widehat{c}_1$. Then for the G_2-action on the first $k-1$ forward shifts of $\mathfrak{m}_0^1$, i.e., $g_2 \cdot \mathfrak{m}_l^1$ for $l = \{0, 1, \ldots, k-1\}$, take shifts of the action on $\mathfrak{m}_0^1$, i.e., $\mathrm{S}_l\left(g_2 \cdot \mathfrak{m}_0^1\right)$. This allows us to implement the moving frame construction a second time. Let $\mathcal{K}^2 = \{\mathfrak{m}_0^1 = \widehat{c}_2\}$ be a cross–section, $\rho_0^2\left(\left[\mathfrak{m}^1\right]\right)$ the corresponding right moving frame, ι_0^2 the induced invariantization map, and $\mathfrak{m}_0^2 = \rho_0^2\left(\left[\mathfrak{m}^1\right]\right) - \rho_1^2\left(\left[\mathfrak{m}^1\right]\right)$ the corresponding Maurer–Cartan invariant. Invariantizing (5.26) with respect to ι_0^2, we obtain the order $k-2$ reduced OΔE

$$F^2\left(n, \mathfrak{m}_0^2, \ldots, \mathfrak{m}_{k-2}^2\right) = 0.$$

Reducing this OΔE using $G_3 \simeq G^{(3)}/G^{(2)}$, followed by G_4 and so on up to G_r will leave in the end an order $k-r$ reduced equation

$$F^r\left(n, \mathfrak{m}_0^r, \ldots, \mathfrak{m}_{k-r}^r\right) = 0. \tag{5.27}$$

At each iteration of the symmetry reduction algorithm by moving frames a cross-section is introduced. Therefore, in the end, there are r cross-sections (2.29)

$$\mathcal{K}^1 = \{u_0 = \widehat{c}_1\}, \quad \mathcal{K}^{l+1} = \left\{\mathfrak{m}_0^l = \widehat{c}_{l+1}\right\},$$

with normalization equations

$$g_1 \cdot u_0 = \widehat{c}_1, \quad g_2 \cdot \mathfrak{m}_0^1 = \widehat{c}_2, \quad \ldots, \quad g_r \cdot \mathfrak{m}_0^{r-1} = \widehat{c}_r,$$

for $l = 1, \ldots, r-1$,. There are also r associated right moving frames

$$\rho_0^r\left(\left[\mathfrak{m}^{r-1}\right]\right), \quad \ldots, \quad \rho_0^2\left(\left[\mathfrak{m}^1\right]\right), \quad \rho_0^1\left([u]\right)$$

and consequently r inverse Maurer–Cartan invariants

$$\mathfrak{m}_0^l = \rho_0^l\left(\left[\mathfrak{m}^{l-1}\right]\right) - \rho_1^l\left(\left[\mathfrak{m}^{l-1}\right]\right), \quad \mathfrak{m}_0^1 = \rho_0^1\left([u]\right) - \rho_1^1\left([u]\right), \tag{5.28}$$

for $l = 2, \ldots, r$.

If $k = r$ then the original OΔE (5.25) reduces to the solution for $\mathfrak{m}_0^k$. If $k < r$ then one can use the first k symmetries to reduce to $\mathfrak{m}_0^k$. If $k > r$ then one needs to solve the resulting OΔE for $\mathfrak{m}_0^r$ to then find the solution to the original OΔE (5.25). In all these cases the method can be stopped if one can find a solution for an inverse Maurer–Cartan invariant along the way.

Suppose that $\mathfrak{m}_0^r$ is the general solution for the reduced OΔE (5.27). The solution u_0 to the original OΔE (5.25) is obtained using an iterative form of the reconstruction procedure in the one-parameter case. First, solve the reconstruction equation

$$\overline{\rho_1^r}\left(\left[\mathfrak{m}^{r-1}\right]\right) - \overline{\rho_0^r}\left(\left[\mathfrak{m}^{r-1}\right]\right) = \mathfrak{m}_0^r$$

for the left moving frame $\overline{\rho_0^r}\left(\left[\mathfrak{m}^{r-1}\right]\right)$. Then, using the cross-section of the left moving frame $\overline{\rho_0^r}\left(\left[\mathfrak{m}^{r-1}\right]\right)$ and the definition of the left moving frame gives

$$\mathfrak{m}_0^{r-1} = \overline{\rho_0^r}\left(\left[\mathfrak{m}^{r-1}\right]\right) \cdot \widehat{c}_r,$$

where the group product is the action of G_r on $\mathfrak{m}_0^{r-1}$. Knowing $\mathfrak{m}_0^{r-1}$ allows us to solve the reconstruction equation for $\overline{\rho_0^{r-1}}\left(\left[\mathfrak{m}^{r-2}\right]\right)$. Iterating the reconstruction procedure enables us to obtain the left moving frames

$$\overline{\rho_0^{r-1}}\left(\left[\mathfrak{m}^{r-2}\right]\right), \quad \overline{\rho_0^{r-2}}\left(\left[\mathfrak{m}^{r-3}\right]\right), \quad \ldots, \quad \overline{\rho_0^1}\left([u]\right).$$

Once $\overline{\rho_0^1}\left([u]\right)$ is known, the solution to the original OΔE (5.25) is $u_0 = \overline{\rho_0^1}\left([u]\right) \cdot \widehat{c}_1$.

Example 5.2.2. The second-order OΔE,

$$u_2 = \frac{(n+1)\, u_0 u_1}{(n+1)\, u_1 + 2u_0}\,, \tag{5.29}$$

has two symmetries with infinitesimal generators

$$\mathbf{v}_1 = n u_0^2 \partial_{u_0}, \qquad \mathbf{v}_2 = u_0 \partial_{u_0}.$$

These symmetries are commutative and their group actions are

$$g_1 \cdot u_0 = \frac{u_0}{1 - n\epsilon_1 u_0}\,, \qquad g_2 \cdot u_0 = e^{\epsilon_2} u_0,$$

respectively. We start the reduction with the first symmetry group $G^1 = G^{(1)}$ (containing g_1) and use the cross-section $\mathcal{K}^1 = \{u_0 = 1\}$ which gives the following recurrence relations

$$\begin{aligned}
\iota_0^1\,(u_0) &= 1,\\
\iota_0^1\,(u_1) &= \mathfrak{m}_0^1 \cdot \iota_1^1\,(u_1)\\
&= \mathfrak{m}_0^1 \cdot 1\\
&= \frac{1}{1 - (n+1)\,\mathfrak{m}_0^1}\,,\\
\iota_0^1\,(u_2) &= \mathfrak{m}_0^1 \mathfrak{m}_1^1 \cdot \iota_2^1\,(u_2)\\
&= \mathfrak{m}_0^1 \mathfrak{m}_1^1 \cdot 1\\
&= \frac{1}{1 - (n+2)\left(\mathfrak{m}_0^1 + \mathfrak{m}_1^1\right)}\,.
\end{aligned}$$

Invariantizing (2.31) the OΔE (5.29) with respect to ι_0^1 gives the reduced OΔE

$$\mathfrak{m}_1^1 = -\frac{\mathfrak{m}_0^1 n^2 + \mathfrak{m}_0^1 n + 2}{n^2 + 3n + 2}\,. \tag{5.30}$$

For the second iteration we use the one-parameter group $G_2 \simeq G^{(2)}/G^{(1)}$. The action of this one-parameter group G_2 on the inverse Maurer–Cartan invariants is found by looking at the partial cross-section $\mathcal{K} = \{\iota_0^1\,(u_0) = 1\}$, which has the normalization equation (2.16)

$$g \cdot \iota_0^1(u_0) = \frac{1}{\frac{e^{\epsilon_2}}{\iota_0^1(u_0)} + \widetilde{\epsilon_1} n} = 1.$$

Using $\iota_0^1(u_0) = 1$, solving for the group parameter of $G^{(1)}$ yields the partial moving frame parameter

$$\widetilde{\epsilon_1} = \frac{1 - e^{\epsilon_2}}{n}.$$

From

$$g \cdot \iota_0^1(u_1) = \frac{1}{\frac{e^{\epsilon_2}}{\iota_0^1(u_1)} + \widetilde{\epsilon_1}(n+1)},$$

we can use the partial moving frame parameter and recurrence relations to get

$$g \cdot \iota_0^1(u_1) = -\frac{n}{e^{\epsilon_2}\mathfrak{m}_0^1 n^2 + e^{\epsilon_2}\mathfrak{m}_0^1 n + e^{\epsilon_2} - n - 1}. \tag{5.31}$$

Comparing (5.31) with

$$g \cdot \iota_0^1(u_1) = \frac{1}{1 - (n+1)\, g \cdot \mathfrak{m}_0^1},$$

yields

$$g \cdot \mathfrak{m}_0^1 = \frac{e^{\epsilon_2}\mathfrak{m}_0^1 n^2 + e^{\epsilon_2}\mathfrak{m}_0^1 n + e^{\epsilon_2} - 1}{n(n+1)}.$$

As a consequence,

$$g \cdot \mathfrak{m}_1^1 = \frac{-1 + \left(\mathfrak{m}_1^1 n^2 + 3n\mathfrak{m}_1^1 + 2\mathfrak{m}_1^1 + 1\right) e^{\epsilon_2}}{n^2 + 3n + 2}.$$

Using the cross-section $\mathcal{K}^2 = \{\mathfrak{m}_0^1 = 0\}$ gives the recurrence relations

$$\iota_0^2\left(\mathfrak{m}_0^1\right) = 0,$$
$$\iota_0^2\left(\mathfrak{m}_1^1\right) = \frac{-1 + e^{\mathfrak{m}_0^2}}{n^2 + 3n + 2}.$$

Therefore, invariantizing (2.31) the reduced OΔE (5.30) by ι_0^2 gives after simplification

$$\mathfrak{m}_0^2 = i\pi.$$

We now reconstruct the solution to the original OΔE by using the relation $\mathfrak{m}_0^2 = \overline{\rho_1^2}\left(\left[\mathfrak{m}^1\right]\right) - \overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right)$ to solve for the left moving frame:

$$\begin{aligned}\overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right) &= c_1 + \sigma_k\{i\pi; 1, n\} \\ &= c_1 + (n-1)\, i\pi.\end{aligned}$$

Then using $\mathfrak{m}_0^1 = \overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right) \cdot 0$ implies that

$$\begin{aligned}\mathfrak{m}_0^1 &= \frac{e^{\overline{\rho_0^2}([\mathfrak{m}^1])} - 1}{n\,(n+1)} \\ &= \frac{-e^{c_1}\,(-1)^n - 1}{n\,(n+1)}.\end{aligned}$$

Finally, using the relation $\mathfrak{m}_0^1 = \overline{\rho_1^1}([u]) - \overline{\rho_0^1}([u])$, it follows that

$$\begin{aligned}\overline{\rho_0^1}([u]) &= c_2 + \sigma_k\left\{\frac{-e^{c_1}\,(-1)^k - 1}{k\,(k+1)}; 1, n\right\} \\ &= \frac{1}{n}\Bigg(- e^{c_1}\,(-1)^n\,\Psi\left(\frac{n}{2} + \frac{1}{2}\right) n + e^{c_1}\,(-1)^n\, n\Psi\left(\frac{n}{2}\right) \\ &\qquad + e^{c_1}\,(-1)^n + \left(-2\ln(2)\, n + n\right) e^{c_1} - n + 1\Bigg) + c_2,\end{aligned}$$

where $\Psi(n)$ is the digamma function; consequently, the solution is

$$\begin{aligned}u_0 &= \overline{\rho_0^1}([u]) \cdot 1 \\ &= \frac{1}{1 - n\overline{\rho_0^1}([u])} \\ &= -\Bigg(- e^{c_1}\,(-1)^n\,\Psi\left(\frac{n}{2} + \frac{1}{2}\right) n + e^{c_1}\,(-1)^n\, n\Psi\left(\frac{n}{2}\right) \\ &\qquad + e^{c_1}\,(-1)^n + n\left(\left(-2\ln(2) + 1\right) e^{c_1} + c_2 - 1\right)\Bigg)^{-1}.\end{aligned}$$

5.3 Higher-order examples

In this section, the moving frame method of symmetry reduction is used to reduce and solve two third-order OΔEs. For the first example, we reduce the OΔE by its two symmetries and find that the resulting reduced OΔE is of a particular solvable form. For the second example, a three-parameter symmetry group completely reduces the OΔE. For both, iterating the reconstruction procedure gives the solution to the original OΔE.

Example 5.3.1. The OΔE

$$u_3 = \frac{u_1 (u_2 - u_0)}{u_2 - 2u_0} \qquad (5.32)$$

has two Lie symmetries given by the infinitesimal generators

$$\mathbf{v}_1 = u_0 \partial_{u_0}, \qquad \mathbf{v}_2 = (-1)^n u_0 \partial_{u_0}. \qquad (5.33)$$

This Lie group is commutative, giving us a choice of which symmetry to start with (see Hydon [16] for further details). We start by reducing with the first symmetry which has the group action,

$$g_1 \cdot u_k = e^{\epsilon_1} u_k, \quad \epsilon_1 \in G_1,$$

and take the cross-section (2.29) to be

$$\mathcal{K}^1 = \{u_0 = 1\}.$$

This implies that

$$\begin{aligned}
\iota_0^1 (u_0) &= 1, \\
\iota_0^1 (u_1) &= e^{\mathtt{m}_0^1}, \\
\iota_0^1 (u_2) &= e^{\mathtt{m}_0^1 + \mathtt{m}_1^1}, \\
\iota_0^1 (u_3) &= e^{\mathtt{m}_0^1 + \mathtt{m}_1^1 + \mathtt{m}_2^1}.
\end{aligned}$$

These recurrence relations allow us to invariantize (2.31) the OΔE (5.32) with respect to ι_0^1, which gives the reduced OΔE

$$e^{\mathtt{m}_1^1 + \mathtt{m}_2^1} = \frac{e^{\mathtt{m}_0^1 + \mathtt{m}_1^1} - 1}{e^{\mathtt{m}_0^1 + \mathtt{m}_1^1} - 2}. \qquad (5.34)$$

The action of $G_2 \simeq G^{(2)}/G^{(1)}$ on the inverse Maurer–Cartan invariants is found using the partial cross-section $\mathcal{K} = \{\iota_0^1 (u_0) = 1\}$ with normalization equation (2.16)

$$g \cdot \iota_0^1 (u_0) = e^{\widetilde{\epsilon_1} + \epsilon_2 (-1)^n} \iota_0^1 (u_0) = 1.$$

Using $\iota_0^1(u_0) = 1$ and solving for the group parameter of $G_1\left(G^{(1)}\right)$ yields

$$\widetilde{\epsilon_1} = \epsilon_2 (-1)^{n+1}.$$

From,

$$g \cdot \iota_0^1(u_1) = e^{\widetilde{\epsilon_1} + \epsilon_2(-1)^{n+1}} \iota_0^1(u_1),$$

substituting the partial moving frame parameter and recurrence relations gives

$$\begin{aligned} g \cdot \iota_0^1(u_1) &= e^{-2\epsilon_2(-1)^n + \mathfrak{m}_0^1} \\ &= g \cdot e^{\mathfrak{m}_0^1}. \end{aligned}$$

As a result, $g \cdot \mathfrak{m}_0^1 = \mathfrak{m}_0^1 - 2\epsilon_2(-1)^n$. For the calculations the group action on $e^{\mathfrak{m}_0^1}$ is not only sufficient but also more convenient than the group action on $\mathfrak{m}_0^1$. Accordingly, the action of the second group on $e^{\mathfrak{m}_1^1}$ and $e^{\mathfrak{m}_2^1}$ is

$$g \cdot e^{\mathfrak{m}_1^1} = e^{2\epsilon_2(-1)^n + \mathfrak{m}_1^1}, \qquad g \cdot e^{\mathfrak{m}_2^1} = e^{-2\epsilon_2(-1)^n + \mathfrak{m}_2^1}.$$

Next by choosing the cross-section $\mathcal{K}^2 = \{\mathfrak{m}_0^1 = 0\}$, the recurrence equations for the exponentials of the inverse Maurer–Cartan invariants are

$$\begin{aligned} \iota_0^2\left(e^{\mathfrak{m}_0^1}\right) &= 1, \\ \iota_0^2\left(e^{\mathfrak{m}_1^1}\right) &= e^{2\mathfrak{m}_0^2(-1)^n}, \\ \iota_0^2\left(e^{\mathfrak{m}_2^1}\right) &= e^{2\left(\mathfrak{m}_0^2 + \mathfrak{m}_1^2\right)(-1)^{n+1}}. \end{aligned}$$

Therefore, to find the final reduced OΔE invariantize (2.31) the reduced OΔE (5.34) by ι_0^2, and substitute in the recurrence relations above and simplify:

$$e^{2\mathfrak{m}_1^2(-1)^{n+1}} = \frac{e^{2\mathfrak{m}_0^2(-1)^n} - 1}{e^{2\mathfrak{m}_0^2(-1)^n} - 2}. \tag{5.35}$$

Using the substitution $v_0 = e^{2\mathfrak{m}_0^2(-1)^n}$, we see that this is a Riccati equation, which is an equation of the form

$$v_1 = \frac{av_0 + b}{v_0 + c}, \quad a, b, c \in \mathbb{R}.$$

The solution to a Ricatti equation is obtained by using the substitution

$$v_0 = \frac{w_0}{w_1} + a,$$

and solving the resulting linear OΔE, (see [27]). Here using the substitution $v_0 = e^{2\mathfrak{m}_0^2(-1)^n}$ the Ricatti equation is

$$v_1 = \frac{v_0 - 1}{v_0 - 2},$$

and if we linearize using the substitution

$$v_0 = \frac{w_0}{w_1} + 1, \tag{5.36}$$

this gives the linear OΔE

$$w_2 + w_1 - w_0 = 0.$$

The solution to this linear OΔE is

$$w_0 = c_1 \left(\frac{-1 - \sqrt{5}}{2} \right)^n + c_2 \left(\frac{-1 + \sqrt{5}}{2} \right)^n,$$

which is substituted into (5.36) to find

$$v_0 = \frac{c_1 \left(\frac{-1-\sqrt{5}}{2} \right)^{n-1} + c_2 \left(\frac{-1+\sqrt{5}}{2} \right)^{n-1}}{c_1 \left(\frac{-1-\sqrt{5}}{2} \right)^{n+1} + c_2 \left(\frac{-1+\sqrt{5}}{2} \right)^{n+1}}. \tag{5.37}$$

This splits the example into two cases, that is, $c_1 \neq 0$ and $c_1 = 0$. When $c_1 \neq 0$, the solution for $\mathfrak{m}_0^2$ is

$$\mathfrak{m}_0^2 = \frac{(-1)^n}{2} \ln \left(\frac{\left(\frac{-1-\sqrt{5}}{2} \right)^{n-1} + k_1 \left(\frac{-1+\sqrt{5}}{2} \right)^{n-1}}{\left(\frac{-1-\sqrt{5}}{2} \right)^{n+1} + k_1 \left(\frac{-1+\sqrt{5}}{2} \right)^{n+1}} \right),$$

with $k_1 = c_2/c_1$. Using the relation $\mathfrak{m}_0^2 = \overline{\rho_1^2}\left([\mathfrak{m}^1]\right) - \overline{\rho_0^2}\left([\mathfrak{m}^1]\right)$ and observing that

$$
\begin{aligned}
\mathfrak{m}_0^2 = &\frac{(-1)^n}{2}\ln\left(\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^n + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^n}{\left(\frac{-1-\sqrt{5}}{2}\right)^{n+1} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}\right)\\
&-\frac{(-1)^{n-1}}{2}\ln\left(\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n-1} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n-1}}{\left(\frac{-1-\sqrt{5}}{2}\right)^{n} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n}}\right),
\end{aligned}
$$

allows us to solve for the left moving frame:

$$
\overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right) = \frac{(-1)^{n-1}}{2}\ln\left(\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n-1} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n-1}}{\left(\frac{-1-\sqrt{5}}{2}\right)^{n} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n}}\right) + k_2.
$$

As $\iota_0^2\left(\mathfrak{m}_0^1\right) = 0$,

$$
\begin{aligned}
\mathfrak{m}_0^1 &= \overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right)\cdot 0\\
&= 2\overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right)(-1)^{n+1}\\
&= \ln\left(\frac{\left(\frac{-1-\sqrt{5}}{2}\right)^{n-1} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n-1}}{\left(\frac{-1-\sqrt{5}}{2}\right)^{n} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n}}\right) + 2k_2(-1)^{n+1}.
\end{aligned}
$$

From here solve the Maurer–Cartan invariant

$$
\begin{aligned}
\mathfrak{m}_0^1 &= \overline{\rho_1^1}([u]) - \overline{\rho_0^1}([u])\\
&= \ln\left(\frac{1}{\left(\frac{-1-\sqrt{5}}{2}\right)^{n} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n}}\right)\\
&\quad - \ln\left(\frac{1}{\left(\frac{-1-\sqrt{5}}{2}\right)^{n-1} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n-1}}\right) + 2k_2(-1)^{n+1},
\end{aligned}
$$

for $\overline{\rho_0^1}([u])$ to obtain the left moving frame

$$
\overline{\rho_0^1}([u]) = \ln\left(\frac{1}{\left(\frac{-1-\sqrt{5}}{2}\right)^{n-1} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n-1}}\right) + k_2(-1)^n + k_3.
$$

Finally, using $\iota_0^1(u_0) = 1$ implies that the solution, for $c_1 \neq 0$, to the original

OΔE (5.32) is

$$\begin{aligned} u_0 &= \overline{\rho_0^1}\left([u]\right) \cdot 1 \\ &= e^{\overline{\rho_0^1}([u])} \\ &= \frac{\exp\left(k_2\left(-1\right)^n + k_3\right)}{\left(\frac{-1-\sqrt{5}}{2}\right)^{n-1} + k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n-1}}. \end{aligned}$$

For the case when $c_1 = 0$, (5.37) and $v_0 = e^{2\mathfrak{m}_0^2(-1)^n}$ gives

$$\mathfrak{m}_0^2 = (-1)^{n+1} \ln\left(\frac{-1+\sqrt{5}}{2}\right).$$

Using the relation $\mathfrak{m}_0^2 = \overline{\rho_1^2}\left(\left[\mathfrak{m}^1\right]\right) - \overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right)$, the left moving frame is

$$\overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right) = \frac{(-1)^n}{2} \ln\left(\frac{-1+\sqrt{5}}{2}\right) + k_4.$$

The inverse Maurer–Cartan invariant is

$$\begin{aligned} \mathfrak{m}_0^1 &= \overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right) \cdot 0 \\ &= 2\overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right)(-1)^{n+1} \\ &= -\ln\left(\frac{-1+\sqrt{5}}{2}\right) + 2k_4\left(-1\right)^{n+1}. \end{aligned}$$

Therefore, the left moving frame is

$$\overline{\rho_0^1}\left([u]\right) = -\ln\left(\frac{-1+\sqrt{5}}{2}\right) n + k_4\left(-1\right)^n + k_5.$$

So the solution for $c_1 = 0$ of the original OΔE (5.32) is

$$\begin{aligned} u_0 &= \overline{\rho_0^1}\left([u]\right) \cdot 1 \\ &= \exp\left(\overline{\rho_0^1}\left([u]\right)\right) \\ &= \left(\frac{-1+\sqrt{5}}{2}\right)^{-n} \exp\left(k_4\left(-1\right)^n + k_5\right). \end{aligned}$$

Hence, the full solution to (5.32) is

$$u_0 = \begin{cases} \dfrac{\exp\left(k_2(-1)^n+k_3\right)}{\left(\frac{-1-\sqrt{5}}{2}\right)^{n-1}+k_1\left(\frac{-1+\sqrt{5}}{2}\right)^{n-1}}\,, & c_1 \neq 0, \\ \left(\frac{-1+\sqrt{5}}{2}\right)^{-n} \exp\left(k_4\left(-1\right)^n + k_5\right), & c_1 = 0, \end{cases} \tag{5.38}$$

with $k_1 = c_2/c_1$.

Example 5.3.2. Here we show how to reduce the third-order OΔE

$$u_3 = u_1 + \frac{1}{u_2 - u_0}\,, \tag{5.39}$$

and solve for the original variable u_0. This OΔE has three symmetries whose infinitesimal generators are

$$\mathbf{v}_1 = \partial_{u_0}, \qquad \mathbf{v}_2 = (-1)^n\,\partial_{u_0}, \qquad \mathbf{v}_3 = (-1)^n\,u_0\partial_{u_0}.$$

To begin the reduction we need to start with the first symmetry which has the group action

$$g_1 \cdot u_0 = u_0 + \epsilon_1, \quad \epsilon_1 \in G_1.$$

Using the cross-section $\mathcal{K}^1 = \{u_0 = 0\}$ gives

$$\begin{aligned} \iota_0^1\left(u_0\right) &= 0, \\ \iota_0^1\left(u_1\right) &= \mathfrak{m}_0^1, \\ \iota_0^1\left(u_2\right) &= \mathfrak{m}_0^1 + \mathfrak{m}_1^1, \\ \iota_0^1\left(u_3\right) &= \mathfrak{m}_0^1 + \mathfrak{m}_1^1 + \mathfrak{m}_2^1. \end{aligned}$$

These recurrence relations allow us to invariantize (2.31) the OΔE (5.39) with respect to ι_0^1, to find the reduced OΔE

$$\mathfrak{m}_1^1 + \mathfrak{m}_2^1 = \frac{1}{\mathfrak{m}_0^1 + \mathfrak{m}_1^1}\,. \tag{5.40}$$

To iterate the algorithm a second time we use the action $G_2 \simeq G^{(2)}/G^{(1)}$. Using the partial moving frame $\mathcal{K} = \{\iota_0^1(u_0) = 0\}$ the normalization equation (2.16) is

$$g \cdot \iota_0^1(u_0) = \iota_0^1(u_0) + \widetilde{\epsilon_1} + (-1)^n \epsilon_2 = 0,$$

solving for the parameter in $G_1\left(G^{(1)}\right)$ gives

$$\widetilde{\epsilon_1} = \epsilon_2 (-1)^{n+1}.$$

Then looking at

$$g \cdot \iota_0^1(u_1) = \iota_0^1(u_1) + \widetilde{\epsilon_1} + (-1)^{n+1} \epsilon_2,$$

substituting the recurrence relations and $\widetilde{\epsilon_1}$ yields

$$g \cdot \mathfrak{m}_0^1 = \mathfrak{m}_0^1 - 2(-1)^n \epsilon_2.$$

Therefore,

$$g \cdot \mathfrak{m}_1^1 = \mathfrak{m}_1^1 + 2(-1)^n \epsilon_2, \qquad g \cdot \mathfrak{m}_2^1 = \mathfrak{m}_2^1 - 2(-1)^n \epsilon_2.$$

Now using the cross-section $\mathcal{K}^2 = \{\mathfrak{m}_0^1 = 0\}$,

$$\begin{aligned}
\iota_0^2\left(\mathfrak{m}_0^1\right) &= 0,\\
\iota_0^2\left(\mathfrak{m}_1^1\right) &= 2(-1)^n \mathfrak{m}_0^2,\\
\iota_0^2\left(\mathfrak{m}_2^1\right) &= 2(-1)^{n+1}\left(\mathfrak{m}_0^2 + \mathfrak{m}_1^2\right).
\end{aligned}$$

Then if we invariantize (2.31) the reduced OΔE (5.40), with respect to ι_0^2, this gives the second reduced OΔE

$$\mathfrak{m}_1^2 = -\frac{1}{4\mathfrak{m}_0^2}. \tag{5.41}$$

This reduced OΔE is solvable, but to find a further reduced OΔE we use the algorithm a third time by considering the group action $G_3 \simeq G^{(3)}/G^{(2)}$. Using the partial moving frame $\mathcal{K} = \{\iota_0^2\left(\iota_0^1(u_0)\right) = 0, \iota_0^2\left(\iota_0^1(u_1)\right) = 0\}$ the first normalization equation (2.16) is

$$g \cdot \iota_0^2\left(\iota_0^1\left(u_0\right)\right) = e^{(-1)^n \epsilon_3} \iota_0^2\left(\iota_0^1\left(u_0\right)\right) + \widehat{\epsilon_1} + \widehat{\epsilon_2}\left(-1\right)^n = 0, \tag{5.42}$$

and using $\iota_0^2\left(\iota_0^1\left(u_0\right)\right) = 0$ (as $\iota_0^1\left(u_0\right) = 0$) implies that

$$\widehat{\epsilon_1} = \widehat{\epsilon_2}\left(-1\right)^{n+1}. \tag{5.43}$$

The second normalization equation is

$$g \cdot \iota_0^2\left(\iota_0^1\left(u_1\right)\right) = e^{(-1)^{n+1} \epsilon_3} \iota_0^2\left(\iota_0^1\left(u_1\right)\right) + \widehat{\epsilon_1} + \widehat{\epsilon_2}\left(-1\right)^{n+1} = 0,$$

and using $\iota_0^2\left(\iota_0^1\left(u_1\right)\right) = 0$ (as $\iota_0^1\left(u_1\right) = \mathfrak{m}_0^1$ and $\iota_0^2\left(\mathfrak{m}_0^1\right) = 0$) implies that

$$\widehat{\epsilon_1} = \widehat{\epsilon_2}\left(-1\right)^n. \tag{5.44}$$

Comparing the equations (5.43) and (5.44) it is clear to see that $\widehat{\epsilon_1} = \widehat{\epsilon_2} = 0$, and so considering $\iota_0^2\left(\iota_0^1\left(u_2\right)\right)$ yields the group action

$$g \cdot \mathfrak{m}_0^2 = e^{\epsilon_3 (-1)^n} \mathfrak{m}_0^2.$$

Note this action can also be found by considering the quotient group action $G^{(3)}/G^{(1)}$ which yields the normalization equation (2.16)

$$g \cdot \iota_0^1\left(u_0\right) = e^{\epsilon_3 (-1)^n} \iota_0^1\left(u_0\right) + \widetilde{\epsilon_1} + \epsilon_2\left(-1\right)^n = 0$$

with solution

$$\widetilde{\epsilon_1} = \epsilon_2\left(-1\right)^{n+1}.$$

Therefore,

$$\begin{aligned} g \cdot \iota_0^1\left(u_1\right) &= e^{\epsilon_3 (-1)^{n+1}} \iota_0^1\left(u_1\right) + \widetilde{\epsilon_1} + \epsilon_2\left(-1\right)^{n+1} \\ &= e^{\epsilon_3 (-1)^{n+1}} \mathfrak{m}_0^1 - 2\epsilon_2\left(-1\right)^n \\ &= g \cdot \mathfrak{m}_0^1. \end{aligned}$$

From here the cross-section $\mathcal{K}^2 = \{\mathfrak{m}_0^1 = 0\}$ is used which has the associated partial cross-section $\mathcal{K} = \{\iota_0^2\left(\mathfrak{m}_0^1\right) = 0\}$ with normalization equation (2.16)

$$g \cdot \iota_0^2\left(\mathfrak{m}_0^1\right) = e^{\epsilon_3(-1)^{n+1}}\iota_0^2\left(\mathfrak{m}_0^1\right) - 2\widetilde{\epsilon_2}\left(-1\right)^n = 0.$$

This normalization equation yields $\widetilde{\epsilon_2} = 0$. Hence considering $\iota_0^1\left(\mathfrak{m}_1^1\right)$ and using the recurrence relations achieves the same result.

By taking the final cross-section to be $\mathcal{K}^3 = \{\mathfrak{m}_0^2 = 1\}$, we obtain

$$\begin{aligned}\iota_0^3\left(\mathfrak{m}_0^2\right) &= 1,\\ \iota_0^3\left(\mathfrak{m}_1^2\right) &= e^{\mathfrak{m}_0^3(-1)^{n+1}}.\end{aligned}$$

Invariantizing (2.31) the reduced OΔE (5.41) with respect to ι_0^3, the final reduced OΔE is

$$e^{\mathfrak{m}_0^3(-1)^{n+1}} = -\frac{1}{4}. \tag{5.45}$$

We start the reconstruction procedure by solving the above equation for $\mathfrak{m}_0^3$:

$$\mathfrak{m}_0^3 = (-1)^n \ln(4) + (-1)^{n+1} i\pi.$$

Using the relation $\mathfrak{m}_0^3 = \overline{\rho_1^3}\left(\left[\mathfrak{m}^2\right]\right) - \overline{\rho_0^3}\left(\left[\mathfrak{m}^2\right]\right)$ gives

$$\begin{aligned}\overline{\rho_0^3}\left(\left[\mathfrak{m}^2\right]\right) &= c_1 + \sigma_k\{(-1)^k \ln(4) + (-1)^{k+1} i\pi; 0, n\}\\ &= c_1 + \left(\frac{(-1)^n - 1}{2}\right) i\pi + \ln(2)\left((-1)^{n+1} + 1\right).\end{aligned}$$

As $\mathfrak{m}_0^2 = \overline{\rho_0^3}\left(\left[\mathfrak{m}^2\right]\right) \cdot 1$, it follows that

$$\begin{aligned}\mathfrak{m}_0^2 &= \exp\left(c_1(-1)^n + \left(\frac{1-(-1)^n}{2}\right) i\pi + \ln(2)\left((-1)^n - 1\right)\right)\\ &= 2^{(-1)^n - 1}\left(\sinh(c_1) + (-1)^n \cosh(c_1)\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right) &= c_2 + \sigma_k\{2^{(-1)^k-1}\left(\sinh\left(c_1\right) + (-1)^k\cosh\left(c_1\right)\right); 0, n\} \\ &= c_2 + \sinh\left(c_1\right)\left(-\frac{3\,(-1)^n}{16} + \frac{5n}{8} + \frac{3}{16}\right) \\ &\quad + \cosh\left(c_1\right)\left(-\frac{5\,(-1)^n}{16} + \frac{3n}{8} + \frac{5}{16}\right).\end{aligned}$$

Then $\mathfrak{m}_0^1 = \overline{\rho_0^2}\left(\left[\mathfrak{m}^1\right]\right)\cdot 0$ gives

$$\begin{aligned}\mathfrak{m}_0^1 = 2\,(-1)^{n+1}&\left(c_2 + \sinh\left(c_1\right)\left(-\frac{3\,(-1)^n}{16} + \frac{5n}{8} + \frac{3}{16}\right)\right. \\ &\left.+ \cosh\left(c_1\right)\left(-\frac{5\,(-1)^n}{16} + \frac{3n}{8} + \frac{5}{16}\right)\right).\end{aligned}$$

Finally, using $\mathfrak{m}_0^1 = \overline{\rho_1^1}\left([u]\right) - \overline{\rho_0^1}\left([u]\right)$ the solution for the left moving frame is

$$\begin{aligned}\overline{\rho_0^1}\left([u]\right) &= c_3 + \sigma_k\left\{2\,(-1)^{k+1}\left(c_2 + \sinh\left(c_1\right)\left(-\frac{3\,(-1)^k}{16} + \frac{5k}{8} + \frac{3}{16}\right)\right.\right. \\ &\quad\left.\left.+ \cosh\left(c_1\right)\left(-\frac{5\,(-1)^k}{16} + \frac{3k}{8} + \frac{5}{16}\right)\right); 0, n\right\} \\ &= \frac{\left((1+3n)\cosh\left(c_1\right) + (5n-1)\sinh\left(c_1\right) + 8c_2\right)(-1)^n}{8} \\ &\quad + \frac{(5n-1)\cosh\left(c_1\right)}{8} + \frac{(1+3n)\sinh\left(c_1\right)}{8} - c_2 + c_3.\end{aligned}$$

Therefore, the solution to the original OΔE (5.39) is

$$\begin{aligned}u_0 &= \overline{\rho_0^1}\left([u]\right)\cdot 0 \\ &= \overline{\rho_0^1}\left([u]\right) \\ &= \frac{\left((1+3n)\cosh\left(c_1\right) + (5n-1)\sinh\left(c_1\right) + 8c_2\right)(-1)^n}{8} \\ &\quad + \frac{(5n-1)\cosh\left(c_1\right)}{8} + \frac{(1+3n)\sinh\left(c_1\right)}{8} - c_2 + c_3.\end{aligned}$$

5.4 Reductions of systems

Now we discuss the moving frame reduction theory for systems of OΔEs. This theory for group actions that do not depend on n was again developed by Benson and Valiquette [4]. We extend the method to systems of OΔEs with group actions that depend on the independent variable. The systems we consider depend on shifts of the dependent variables $\mathbf{u}_0 = \left(u_0^1, \ldots, u_0^q\right)$ and potentially the independent variable n. Here we only consider systems of OΔEs invariant under a one-parameter Lie group. It is possible to use a solvable group to achieve multiple reductions; however, the notation becomes cumbersome. Let

$$F_\alpha\left(n, \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_k\right) = 0, \quad \alpha = 1, \ldots, q, \tag{5.46}$$

be a system of OΔEs invariant under a one-parameter Lie group action. The Lie group must act on at least one dependent variable. Without loss of generality assume the group acts on the dependent variable u^1 (among maybe others). As before, we start the moving frame construction by choosing a cross-section (2.29)

$$\mathcal{K} = \{u_0^1 = \widehat{c}_1\}.$$

Then, as the group action may act on other dependent variables, we introduce the order zero invariants

$$\mathbf{t}_0 = \left(t_0^1, t_0^2, \ldots, t_0^{q-1}\right) = \left(\iota_0^1\left(u_0^2\right), \ldots, \iota_0^1\left(u_0^q\right)\right).$$

The recurrence relations from before are replaced with the equations

$$\iota_0^1\left(u_l^1\right) = \left[\mathfrak{m}_0^1 + \mathfrak{m}_1^1 + \cdots + \mathfrak{m}_{l-2}^1 + \mathfrak{m}_{l-1}^1\right] \cdot \left(c, \mathbf{t}_l\right)$$

With these, all invariants $\iota_0^1\left(u_l\right)$ can be written in terms of shifts of the inverse Maurer–Cartan invariants $\mathfrak{m}_0^1$ and shifts of the order zero invariants $\mathbf{t}_0$. Then as before, we invariantize (2.31) the system of OΔEs (5.46) to find the new reduced system of OΔEs

$$F_\alpha^1\left(\mathfrak{m}_0^1, \ldots, \mathfrak{m}_{k-1}^1, \mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_k\right) = 0, \quad \alpha = 1, \ldots, q. \tag{5.47}$$

This new system of OΔEs is reduced by one for the inverse Maurer–Cartan invariants. From here if there are more Lie group symmetries one can reduce this system of OΔEs further using the method for solvable groups (see Section 5.2). If a general solution to the reduce OΔE (5.47) is found, that is, $\left(\mathfrak{m}_0^1, \mathbf{t}_0\right)$ then the reconstruction procedure can be applied. Therefore, solving the reconstruction equation $\mathfrak{m}_0^1 = \overline{\rho_1^1}([\mathbf{u}]) - \overline{\rho_0^1}([\mathbf{u}])$ for the left moving frame $\overline{\rho_0^1}([\mathbf{u}])$, the general solution to the original system (5.46) is

$$u_0^1 = \overline{\rho_0^1}([\mathbf{u}]) \cdot c, \quad u_0^2 = \overline{\rho_0^1}([\mathbf{u}]) \cdot t_0^1, \quad \dots, \quad u_0^q = \overline{\rho_0^1}([\mathbf{u}]) \cdot t_0^{q-1}.$$

Now consider an example of a system that is dependent on the independent variable n.

Example 5.4.1. Let $f_\pm = 2 \pm (-1)^n$ and consider the system of OΔEs

$$x_1 = \frac{f_+ x_0 - 3y_0}{f_+ x_0^2 + f_- y_0^2}, \qquad y_1 = \frac{3x_0 + f_- y_0}{f_+ x_0^2 + f_- y_0^2}, \tag{5.48}$$

which has a two-dimensional abelian Lie group. The characteristics of this Lie group are linear homogeneous in x and y:

$$Q = \left(Q^{(x)}, Q^{(y)}\right), \; Q^{(x)} = \widetilde{c_1}(-1)^n x - \widetilde{c_2} f_- y, \; Q^{(y)} = \widetilde{c_1}(-1)^n y + \widetilde{c_2} f_+ x,$$

where $\widetilde{c_1}$ and $\widetilde{c_2}$ are arbitrary constants. The group action of the two-parameter Lie group is $(x_0, y_0) \mapsto (\tilde{x_0}, \tilde{y_0})$ where

$$\begin{aligned} \tilde{x_0} &= \exp\{\epsilon_2(-1)^n\} \left\{ \cos(\epsilon_1) x_0 - \sqrt{\frac{f_-}{f_+}} \sin(\epsilon_1) y_0 \right\}, \\ \tilde{y_0} &= \exp\{\epsilon_2(-1)^n\} \left\{ \sqrt{\frac{f_+}{f_-}} \sin(\epsilon_1) x_0 + \cos(\epsilon_1) y_0 \right\}. \end{aligned} \tag{5.49}$$

Here ϵ_1 and ϵ_2 are the Lie group parameters. Note that $f_+ f_- = 3$ and $\mathrm{S} f_\pm = f_\mp$, which will be used in the calculations. To start the reduction of this system we take the cross-section $\mathcal{K}^1 = \{y_0 = 0\}$. The recurrence relations for this cross-section are then

$$\begin{aligned}
\iota_0^1(y_0) &= 0, \\
\iota_0^1(y_1) &= \sqrt{\frac{f_-}{f_+}} \sin\left(\mathfrak{m}_0^1\right) t_1^1, \\
\iota_0^1(x_0) &= t_0^1, \\
\iota_0^1(x_1) &= \cos\left(\mathfrak{m}_0^1\right) t_1^1.
\end{aligned}$$

Now we use these to invariantize (2.31) the system of OΔEs (5.49), with respect to ι_0^1, which gives the following reduced system of OΔEs

$$\cos\left(\mathfrak{m}_0^1\right) t_1^1 = \frac{1}{t_0^1}, \qquad \sqrt{\frac{f_-}{f_+}} \sin\left(\mathfrak{m}_0^1\right) t_1^1 = \frac{3}{f_+ t_0^1}. \tag{5.50}$$

Next, instead of using the second symmetry, we seek two equations, one of which only depends on the inverse Maurer–Cartan invariant, $\mathfrak{m}_0^1$, and another which only depends on the order zero invariant and its first shift, i.e., t_0^1 and t_1^1. This can be done at any stage during the process so long as we can find the order zero invariants and Maurer–Cartan invariants explicitly. The first of these equations is

$$\cot\left(\mathfrak{m}_0^1\right) = \frac{\iota_0^1(x_1)}{\sqrt{\frac{f_+}{f_-}}\,\iota_0^1(y_1)} = \frac{1}{\sqrt{3}}.$$

This gives

$$\mathfrak{m}_0^1 = \frac{\pi}{3} + k_1\pi, \quad \text{where} \quad k_1 \in \mathbb{Z}.$$

Therefore, using the relation $\mathfrak{m}_0^1 = \overline{\rho_1^1}([x],[y]) - \overline{\rho_0^1}([x],[y])$,

$$\overline{\rho_0^1}([x],[y]) = \left(\frac{1}{3} + k_1\right)\pi n + c_1.$$

Also

$$\left(t_1^1\right)^2 = \iota_0^1(x_1)^2 + \frac{f_+}{f_-}\iota_0^1(y_1)^2 = \frac{4}{\left(t_0^1\right)^2}.$$

We seek real-valued solutions, which correspond to the positive square root:

$$t_1^1 = \frac{2}{t_0^1}. \tag{5.51}$$

Next to find the solution to the original system of O∆Es (5.48) one must find the solution for the order zero invariant t_0^1. By taking the natural logarithm we obtain

$$v_1 + v_0 = \ln(2), \tag{5.52}$$

where $v_0 = \ln\left(t_0^1\right)$. Then using that

$$\ln(2) = \frac{1}{2}\ln(12) + \frac{1}{2}\ln\left(\frac{1}{f_+ f_-}\right)$$

the O∆E

$$v_1 + v_0 = \frac{1}{2}\ln(12) + \frac{1}{2}\ln\left(\frac{1}{f_+ f_-}\right)$$

has the general solution

$$v_0 = \frac{1}{2}\ln\left(\frac{1}{f_+}\right) + \frac{1}{4}\ln(12) + b_1(-1)^n.$$

Taking the exponential of this gives the solution for the order zero invariant:

$$t_0^1 = \frac{\sqrt[4]{12}}{\sqrt{f_+}} e^{b_1(-1)^n}. \tag{5.53}$$

Finally, using the identities $x_0 = \overline{\rho_0^1}([x],[y]) \cdot t_0^1$ and $y_0 = \overline{\rho_0^1}([x],[y]) \cdot 0$ allows us to find the solutions

$$\begin{aligned} x_0 &= \cos\left(\overline{\rho_0^1}([x],[y])\right) t_0^1 = \frac{\sqrt[4]{12}}{\sqrt{f_+}} \cos\left(\left(\frac{1}{3} + k_1\right)\pi n + c_1\right) e^{b_1(-1)^n}, \\ y_0 &= \sin\left(\overline{\rho_0^1}([x],[y])\right) t_0^1 = \frac{\sqrt[4]{12}}{\sqrt{f_-}} \sin\left(\left(\frac{1}{3} + k_1\right)\pi n + c_1\right) e^{b_1(-1)^n}, \end{aligned}$$

for $k_1 = \{0, 1\}$. The case $k_1 = 1$ does not solve the system, so the only solution is

$$\begin{aligned} x_0 &= \frac{\sqrt[4]{12}}{\sqrt{f_+}} \cos\left(\frac{\pi n}{3} + c_1\right) e^{b_1(-1)^n}, \\ y_0 &= \frac{\sqrt[4]{12}}{\sqrt{f_-}} \sin\left(\frac{\pi n}{3} + c_1\right) e^{b_1(-1)^n}. \end{aligned}$$

5.5 Partitioned equations

As defined in [15] a partitioned OΔE is an OΔE of the form,

$$u_{qL} = w\left(n, u_0, u_L, u_{2L}, \ldots, u_{(q-1)L}\right), \tag{5.54}$$

where n, L, $q \in \mathbb{Z}$. An OΔE of this form can be split into L OΔEs of the form

$$u_q^{(l)} = w\left(mL + l, u_0^{(l)}, u_1^{(l)}, u_2^{(l)}, \ldots, u_{(q-1)}^{(l)}\right), \tag{5.55}$$

where m, L, l, $q \in \mathbb{Z}$ and $u^{(l)}(m) = u(mL+l)$. One can apply the moving frames method for a solvable Lie group to find the solutions $u_0^{(l)}$ for each of the L OΔEs separately. Then the solution to the original OΔE is

$$u_0 = \begin{cases} u_0^{(0)}, & \text{where } n^{(0)} = Lm, \\ \vdots & \\ u_0^{(L-1)}, & \text{where } n^{(L-1)} = Lm + (L-1). \end{cases}$$

To illustrate the method of moving frame reductions for partitioned OΔEs consider the following example.

Example 5.5.1. Let the OΔE be

$$u_4 = \frac{2(u_2)^2}{u_0 + (1 + (-1)^n)u_2}. \tag{5.56}$$

This OΔE can be split into the two OΔEs

$$u_2^{(0)} = \frac{2\left(u_1^{(0)}\right)^2}{u_0^{(0)} + 2u_1^{(0)}}, \tag{5.57}$$

$$u_2^{(1)} = \frac{2\left(u_1^{(1)}\right)^2}{u_0^{(1)}}. \tag{5.58}$$

Both of these OΔEs are symmetric under the scaling symmetry with the infinitesimal generator

$$\mathbf{v} = u\partial_u$$

and the group action

$$g \cdot u = e^{\epsilon} u.$$

These two OΔEs are completely separate; therefore, we could use two different cross-sections (2.29) for the moving frame reduction. However, here we use the same cross-section for both OΔEs, namely $\mathcal{K} = \{u_0^{(0)} = u_0^{(1)} = 1\}$. This leads to the recurrence relations

$$\begin{aligned}
\iota_0^1\left(u_0^{(l)}\right) &= 1,\\
\iota_0^1\left(u_1^{(l)}\right) &= e^{\mathfrak{m}_0^{(l)}},\\
\iota_0^1\left(u_2^{(l)}\right) &= e^{\mathfrak{m}_0^{(l)}+\mathfrak{m}_1^{(l)}},
\end{aligned}$$

for $l = \{0, 1\}$. Invariantizing (2.31) the odd partitioned OΔE (5.58) with respect to ι_0^1 gives the reduced OΔE

$$e^{\mathfrak{m}_1^{(1)}} = 2e^{\mathfrak{m}_0^{(1)}}.$$

This is a simple linear OΔE for $e^{\mathfrak{m}_1^{(1)}}$, which has the solution

$$\mathfrak{m}_0^{(1)} = n^{(1)} \ln(2) + k_1.$$

Using the relation $\mathfrak{m}_0^{(1)} = \overline{\rho_1^{(1)}}\left(\left[u^{(1)}\right]\right) - \overline{\rho_0^{(1)}}\left(\left[u^{(1)}\right]\right)$ gives

$$\overline{\rho_0^{(1)}}\left(\left[u^{(1)}\right]\right) = \ln\left(\sqrt{2}\right) n^{(1)}\left(n^{(1)} - 1\right) + k_1 n^{(1)} + c_1.$$

Then

$$\begin{aligned}
u_0^{(1)} &= \overline{\rho_0^{(1)}}\left(\left[u^{(1)}\right]\right) \cdot 1\\
&= e^{\overline{\rho_0^{(1)}}\left(\left[u^{(1)}\right]\right)}\\
&= k_2\left(\sqrt{2}\right)^{n^{(1)}\left(n^{(1)}-1\right)}\left(e^{k_1 n^{(1)}}\right),
\end{aligned}$$

where $k_2 = e^{c_1}$. This is the solution for the odd part of the lattice.

The even partitioned OΔE (5.57), invariantized (2.31) with respect to ι_0^1, simplifies to a Ricatti equation :

$$e^{\mathfrak{m}_1^{(0)}} = \frac{e^{\mathfrak{m}_0^{(0)}}}{e^{\mathfrak{m}_0^{(0)}} + \frac{1}{2}}.$$

Using the same method as in Example 5.3.1 gives the solution

$$\mathfrak{m}_0^{(0)} = \ln\left(\frac{b_2 2^{n^{(0)}}}{b_1 + b_2 2^{n^{(0)}+1}}\right). \tag{5.59}$$

We can simplify this by letting $k_3 = b_2/b_1$. However, as before this splits the problem into two cases, the first being when $b_1 \neq 0$. Using the reconstruction equation $\mathfrak{m}_0^{(0)} = \overline{\rho_1^{(0)}}\left(\left[u^{(0)}\right]\right) - \overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right)$ we then try to solve for the left moving frame $\overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right)$. This cannot be solved easily in closed form so we leave it as

$$\overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right) = \sigma_k\left\{\ln\left(\frac{k_3 2^k}{1 + k_3 2^{k+1}}\right); 0, n^{(0)}\right\} + k_4. \tag{5.60}$$

Then to find the solution for the even part when $b_1 \neq 0$ we use

$$\begin{aligned} u_0^{(0)} &= \overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right) \cdot 1 \\ &= e^{\overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right)}. \end{aligned}$$

For the second case, when $b_1 = 0$, the inverse Maurer–Cartan invariant is

$$\mathfrak{m}_0^{(0)} = \ln\left(\frac{1}{2}\right),$$

which gives

$$\overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right) = \ln\left(\frac{1}{2}\right) n^{(0)} + k_5.$$

Thus,

$$\begin{aligned} u_0^{(0)} &= \overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right) \cdot 1 \\ &= \left(\frac{1}{2}\right)^{n^{(0)}} e^{k_5}. \end{aligned}$$

Therefore, for the even part of the lattice, the solution is

$$u_0^{(0)} = \begin{cases} e^{\overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right)}, & \text{for } b_1 \neq 0, \\ \left(\frac{1}{2}\right)^{n^{(0)}} e^{k_5}, & \text{for } b_1 = 0, \end{cases}$$

where $\overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right) = \sigma_k \left\{ \ln\left(\frac{k_3 2^k}{1+k_3 2^{k+1}}\right); 0, n^{(0)} \right\} + k_4$.

Finally, we can give the solution for u_0 as a piecewise solution as follows

$$u_0 = \begin{cases} e^{\overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right)}, & \text{for } b_1 \neq 0, \text{ where } n^{(0)} = 2m, \\ \left(\frac{1}{2}\right)^{n^{(0)}} e^{k_5}, & \text{for } b_1 = 0, \\ k_2 \left(\sqrt{2}\right)^{n^{(1)}\left(n^{(1)}-1\right)} \left(e^{k_1 n^{(1)}}\right), & \text{where } n^{(1)} = 2m+1, \end{cases}$$

with $\overline{\rho_0^{(0)}}\left(\left[u^{(0)}\right]\right) = \sigma_k \left\{ \ln\left(\frac{k_3 2^k}{1+k_3 2^{k+1}}\right); 0, n^{(0)} \right\} + k_4$. Another way in which this can be represented is as follows

$$u_0 = u_0^{(0)} \pi_{\mathrm{E}}(n) + u_0^{(1)} \pi_{\mathrm{O}}(n),$$

where $\pi_{\mathrm{E}}(n) = (1 + (-1)^n)/2$ and $\pi_{\mathrm{O}}(n) = (1 - (-1)^n)/2$.

In this example, the even and odd partitions admit the same symmetry group. This is not necessarily true for general partitioned equations.

5.6 Comparison with canonical coordinates

The moving frame method gives an alternative to canonical coordinates for reducing and solving OΔEs. Despite some of their similarities, there are also some important differences between the two methods. The first of these is in how each calculates the equivariant component. For the canonical coordinates method, to find the equivariant component there is an integration step using the formula

$$s_0 = \int \frac{1}{Q(n,u)} \mathrm{d}u.$$

By contrast, the moving frame method makes use of the fact that the frame itself is an equivariant component. This makes the choice of cross-section, defined by the normalization equations (2.16), comparable to the integration step for the canonical coordinates method. Therefore, one of the potential benefits of the moving frame method is that there is no integration step to be done. A possible disadvantage of the moving frame method is the need to pick a normalization equation. Is there a best choice of the normalization equation? To answer this question it is wise to do the calculations for an arbitrary constant, $\widehat{c}_1$, and then see what the resulting O∆E will look like for different values of the constant. If the resulting O∆E is solvable for a particular value then it is sensible to use that. For most simple examples, both the integration step and choice of the cross-section (2.29) are particularly easy. But some examples may be easier to solve by one of the two methods.

Next, we consider compatibility. For this, we need to summarise some of the calculations of Example 5.2.1 using the canonical coordinates method. In this example the canonical coordinate s_0 is valid but the coordinate s_1 is not as it becomes complex, therefore, there is no compatible s_0 that exists on $\mathbb{R}$. This is due to the fact that if $|u_0|$ is greater (less) that 1 then $|u_1|$ is less (greater) than 1 (with $u \neq 1/n$). To overcome this problem in [15] the canonical coordinate

$$s_0\left(n,u\right) = \frac{(-1)^n}{2}\mathrm{Log}\left(\frac{u_0-1}{u_0+1}\right) = \begin{cases} \frac{(-1)^n}{2}\ln\left(\frac{u_0-1}{u_0+1}\right), & \frac{u_0-1}{u_0+1} > 0; \\ \frac{(-1)^n}{2}\left(\ln\left(\frac{1-u_0}{u_0+1}\right) + i\pi\right), & \frac{u_0-1}{u_0+1} < 0; \end{cases}$$

is used. Therefore, as no real-valued compatible s_0 exists to get passed this problem one needs to extend the space from the reals, $\mathbb{R}$, to the complex numbers, $\mathbb{C}$. This splits the problem in to two cases where the canonical coordinate s_0 is different depending on if $(u_0-1)/(u_0+1)$ is positive or negative. Looking back at the moving frame approach the extension to complex numbers is more natural as the inverse Maurer–Cartan invariant is always complex. It is also unnecessary at this stage to consider the two different cases. In fact, it is only necessary when finding the solution to the initial value problem with $u_0 = u(2)$. Whether $(u_0-1)/(u_0+1)$ is positive or negative the O∆E amounts to

$$s_1 - s_0 = \frac{(-1)^{n+1}}{2}\left(\ln\left(\frac{n-1}{n+1}\right) + i\pi\right),$$

which is the same as the inverse Maurer–Cartan invariant $\mathfrak{m}_0^1$. From here to find the general solution in both methods one solves the OΔE above for s_0 or the left moving frame $\overline{\rho_0}\left([u]\right)$. Then in the moving frame method, we look at the left moving frame and the original cross-section to find the solution for u_0. In the canonical coordinates method one writes the equivariant canonical coordinates, s_0, in terms of u_0 and solves for u_0. In this case, there are two equivariant canonical coordinates, s_0; therefore, both need to be checked.

Finally, for solvable Lie groups, the action of the inherited symmetries on the invariants is found differently for each method. In the moving frame method, the inductive moving frame is used to find the Lie group action on the inverse Maurer-Cartan invariants. For the canonical coordinates method, one has to find the new canonical coordinates for each reduced OΔE. To do this a new characteristic is found by applying the unused symmetry generators to the invariant r_0. If the symmetry is inherited then this is of the form

$$\mathbf{v}\left(r_0\right) = f\left(n, r_0\right),$$

and the new characteristic is $Q = f\left(n, r_0\right)$. The equivariant and invariant components are then found in the usual manner.

Chapter 6

Conclusions and further research

In this thesis, we have explored several different applications of difference moving frames. To begin with, we looked at extending the theory on variational problems for OΔEs to PΔEs on a rectangular mesh. This includes the problem of finding the Euler–Lagrange equations directly in terms of invariants. This led to several new discoveries in the form of Proposition 2.6.5 and Lemma 2.6.7 giving direct formulas for the invariant Euler–Lagrange equations for PΔEs. Then by creating a difference prolongation space for the non-rectangular case (with two independent steps) we were able to look at variational problems for this case and found that the resulting formula remains essentially the same (Proposition 3.4.1). (The difference prolongation space for two independent steps is new, however.) Then we extended this approach to DΔEs and found the invariant formulation of the Euler–Lagrange equations for several different types of Lie group actions (Proposition 4.4.1, 4.4.2, 4.4.3 and 4.4.5).

Finally, we examined the problem of finding the solutions to OΔEs using moving frames. Benson and Valiquette had already looked at the problem and produced a method which works in lots of examples. However, two considerations that were not looked at were when the Lie group depends on the independent variable and partitioned OΔEs. So we extended the applicability of the method to include these considerations, with several examples to illustrate this.

Some potential research ideas which could be looked into in the future include the following.

- Develop a Maple package for the difference (differential-difference) invariant Euler–Lagrange equations, in which one can choose the normalization and generating invariants. This would lead to a quicker computation of the best reduction.

- The difference prolongation space for non-rectangular mesh could be extended to a completely free mesh. This would have several applications, particularly for finite difference approximations of PDEs.

- Obtain the general formula for all differential-difference invariant Euler–Lagrange equations. In this thesis the formula for a group action on only one independent continuous variable has been found. However, a formula for a group action on more than one independent continuous variable has not been found. Neither has a formula been found for group actions on both variables where there is more than one group action on the independent variables. Additionally, one can drop the need for a projectable normalization (Definition 4.1.2) allowing for more Lie group actions. The general formula could potentially be found using a similar method to that in the paper by Kogan and Olver [19] with the differential-difference structure in the paper by Peng and Hydon [32].

- Develop the invariant differential-difference variational bicomplex. This can potentially be used to examine the form of the equivariant conservation laws.

- Develop a Maple package for symmetry reductions using moving frames. This would need to include finding the best set of normalization equations at each stage.

- Find an equivariant formulation for conservation laws of DΔEs.

Appendix A

Additional details

A.1 Symmetry condition applied to the Lagrangian

The Lagrangian (2.17) in the running example is

$$\mathrm{L} = \frac{1}{2} \ln \left| \frac{(u_{2,0} - u_{1,1})(u_{1,-1} - u_{0,0})}{(u_{2,0} - u_{1,-1})(u_{1,1} - u_{0,0})} \right|.$$

Using the symmetry condition (2.20) and (2.19) for $\mathbf{v}_1$ we have

$$\mathbf{v}_1(\mathrm{L}) = u_{0,0} \frac{\partial \mathrm{L}}{\partial u_{0,0}} + u_{2,0} \frac{\partial \mathrm{L}}{\partial u_{2,0}} + u_{1,1} \frac{\partial \mathrm{L}}{\partial u_{1,1}} + u_{1,-1} \frac{\partial \mathrm{L}}{\partial u_{1,-1}}.$$

The partial derivatives are given in (2.22). Consequently,

$$\begin{aligned}
\mathbf{v}_1(\mathrm{L}) = &- \frac{u_{0,0}(u_{1,1} - u_{1,-1})}{2(u_{0,0} - u_{1,-1})(u_{0,0} - u_{1,1})} \\
&+ \frac{u_{2,0}(u_{1,1} - u_{1,-1})}{2(u_{1,1} - u_{2,0})(u_{1,-1} - u_{2,0})} + \frac{u_{1,1}(u_{0,0} - u_{2,0})}{2(u_{1,1} - u_{2,0})(u_{0,0} - u_{1,1})} \\
&- \frac{u_{1,-1}(u_{0,0} - u_{2,0})}{2(u_{0,0} - u_{1,-1})(u_{1,-1} - u_{2,0})} \\
= &\ 0
\end{aligned}$$

Similar calculations can be done for the other infinitesimal generators $\mathbf{v}_i$ for $i = 2, \ldots, 6$ (2.18).

A.2 Lagrangians equivalent up to a divergence

The Lagrangians L (2.17) and L_0 (2.21) are equivalent up to a divergence. To prove this we find the divergence term $\mathrm{Div}\,(A)$, such that,

$$\mathrm{L} = \mathrm{L}_0 + \mathrm{Div}\,(A)\,.$$

Here we see that

$$\begin{aligned}\mathrm{Div}\,(A) &= \mathrm{L} - \mathrm{L}_0 \\ &= \frac{1}{2}\ln\left|\frac{(u_{2,0}-u_{1,1})\,(u_{1,-1}-u_{0,0})}{(u_{2,0}-u_{1,-1})\,(u_{1,1}-u_{0,0})}\right| - \ln\left|\frac{u_{1,0}-u_{0,1}}{u_{1,1}-u_{0,0}}\right| \\ &= \frac{1}{2}\ln\left|\frac{(u_{2,0}-u_{1,1})\,(u_{1,-1}-u_{0,0})\,(u_{1,1}-u_{0,0})}{(u_{2,0}-u_{1,-1})\,(u_{1,0}-u_{0,1})^2}\right|.\end{aligned}$$

After some rearranging, we see the divergence term can be written in the form

$$\begin{aligned}\mathrm{Div}\,(A) = (\mathrm{S}_1 - \mathrm{id})\left(\frac{1}{2}\ln|u_{1,0}-u_{0,1}|\right) + (\mathrm{S}_2 - \mathrm{id})\left(-\frac{1}{2}\ln|u_{1,-1}-u_{0,0}|\right) \\ + \left(\mathrm{S}_1\mathrm{S}_2^{-1} - \mathrm{id}\right)\left(-\frac{1}{2}\ln|u_{1,1}-u_{0,0}|\right).\end{aligned}$$

A.3 Checking the invariant Euler–Lagrange equations

The invariant Euler–Lagrange equation is

$$\begin{aligned}\mathcal{H}_\lambda^\dagger \mathrm{E}_\lambda\left(L^{\boldsymbol{\kappa}}\right) + \mathcal{H}_\kappa^\dagger \mathrm{E}_\kappa\left(L^{\boldsymbol{\kappa}}\right) &= \frac{\lambda_{-1,1}}{2\,(\kappa_{-1,1})} + \frac{\kappa - \lambda_{-1,-1} - 1}{2\kappa} \\ &\quad - \frac{(\kappa_{-2,0}-1)\,(\kappa_{-1,1}-\lambda_{-1,1})}{2\,(\kappa_{-2,0}-\lambda_{-2,0})\,\kappa_{-1,1}} \\ &= 0.\end{aligned} \tag{A.1}$$

In the original variables,

$$\kappa = \iota\,(u_{1,-1}) = \frac{u_{1,-1}-u_{0,0}}{u_{1,1}-u_{0,0}}\,, \quad \lambda = \iota\,(u_{2,0}) = \frac{u_{2,0}-u_{0,0}}{u_{1,1}-u_{0,0}}\,.$$

Therefore,

$$\begin{aligned}\lambda_{-1,1} &= \frac{u_{1,1}-u_{-1,1}}{u_{0,2}-u_{-1,1}}, \quad \kappa_{-1,1} = \frac{u_{0,0}-u_{-1,1}}{u_{0,2}-u_{-1,1}},\\ \lambda_{-2,0} &= \frac{u_{0,0}-u_{-2,0}}{u_{-1,1}-u_{-2,0}}, \quad \kappa_{-2,0} = \frac{u_{-1,-1}-u_{-2,0}}{u_{-1,1}-u_{-2,0}},\\ \lambda_{-1,-1} &= \frac{u_{1,-1}-u_{-1,-1}}{u_{0,0}-u_{-1,-1}}.\end{aligned}$$

Substituting these into (A.1) yields

$$\begin{aligned}&\mathcal{H}_\lambda^\dagger \mathrm{E}_\lambda\left(L^{\boldsymbol{\kappa}}\right) + \mathcal{H}_\kappa^\dagger \mathrm{E}_\kappa\left(L^{\boldsymbol{\kappa}}\right) =\\ &\frac{\left(u_{1,1}-u_{1,-1}-u_{-1,1}-u_{-1,-1}\right)u_{0,0}^2 + \left(-2u_{-1,-1}u_{1,1}+2u_{1,-1}u_{-1,1}\right)u_{0,0}}{\left(u_{0,0}-u_{-1,1}\right)\left(u_{0,0}-u_{1,-1}\right)\left(u_{0,0}-u_{-1,-1}\right)}\\ &+\frac{\left(\left(u_{1,1}-u_{-1,1}\right)u_{1,-1}+u_{-1,1}u_{1,1}\right)u_{-1,-1}-u_{1,1}u_{1,-1}u_{-1,1}}{\left(u_{0,0}-u_{-1,1}\right)\left(u_{0,0}-u_{1,-1}\right)\left(u_{0,0}-u_{-1,-1}\right)}.\end{aligned} \tag{A.2}$$

The original Euler–Lagrange equation is

$$\mathrm{E}_u\left(\mathrm{L}\right) = \frac{1}{u_{1,1}-u_{0,0}} - \frac{1}{u_{-1,1}-u_{0,0}} - \frac{1}{u_{1,-1}-u_{0,0}} + \frac{1}{u_{-1,-1}-u_{0,0}} = 0. \tag{A.3}$$

The invariants

$$\begin{aligned}&\iota\left(u_{0,0}\right)=0, \quad \iota\left(u_{1,1}\right)=1, \quad \iota\left(u_{-1,1}\right)=\frac{u_{-1,1}-u_{0,0}}{u_{1,1}-u_{0,0}},\\ &\iota\left(u_{1,-1}\right)=\frac{u_{1,-1}-u_{0,0}}{u_{1,1}-u_{0,0}}, \quad \iota\left(u_{-1,-1}\right)=\frac{u_{-1,-1}-u_{0,0}}{u_{1,1}-u_{0,0}},\end{aligned}$$

are used to find the invariantization (2.31) of (A.3),

$$\begin{aligned}\iota\left(\mathrm{E}_u\left(\mathrm{L}\right)\right) &= 1 - \frac{1}{\iota\left(u_{-1,1}\right)} - \frac{1}{\iota\left(u_{1,-1}\right)} + \frac{1}{\iota\left(u_{-1,-1}\right)}\\ &= \frac{\left(u_{1,1}-u_{1,-1}-u_{-1,1}-u_{-1,-1}\right)u_{0,0}^2 + \left(-2u_{-1,-1}u_{1,1}+2u_{1,-1}u_{-1,1}\right)u_{0,0}}{\left(u_{0,0}-u_{-1,1}\right)\left(u_{0,0}-u_{1,-1}\right)\left(u_{0,0}-u_{-1,-1}\right)}\\ &\quad+\frac{\left(\left(u_{1,1}-u_{-1,1}\right)u_{1,-1}+u_{-1,1}u_{1,1}\right)u_{-1,-1}-u_{1,1}u_{1,-1}u_{-1,1}}{\left(u_{0,0}-u_{-1,1}\right)\left(u_{0,0}-u_{1,-1}\right)\left(u_{0,0}-u_{-1,-1}\right)}.\end{aligned} \tag{A.4}$$

Hence, comparing (A.2) and (A.4) gives

$$\mathcal{H}_\lambda^\dagger \mathrm{E}_\lambda\left(L^{\boldsymbol{\kappa}}\right) + \mathcal{H}_\kappa^\dagger \mathrm{E}_\kappa\left(L^{\boldsymbol{\kappa}}\right) = \iota\left(\mathrm{E}_u\left(\mathrm{L}\right)\right).$$

Appendix B

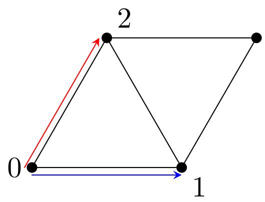
Other regular and semi-regular tilings

B.1 Path generators for various tilings

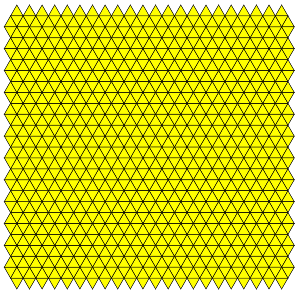
In this section, we explore how one can find a basis for describing paths in various regular (Figure B.1 and B.2) and semi-regular (Figure B.3, B.4, B.5, B.6 and B.7) tilings of the plane. To do this, we give one choice of the two independent steps needed for each of the different tilings. For the different tilings there are several choices for the two independent steps. The regular square tiling and stub square tiling have already been explained in detail and so are left out of this section.

Included in the explanation of each tiling will be a figure of the basic shape which tessellates the plane, and the tiling itself. On the figures of the basic shape, arrows in blue and red describe the translation component of the two independent steps. The base point, which is the point we apply the steps from, is labelled 0. The other important points are ones that the two steps map 0 to; these are labelled by the number of the step. On the figures, neighbours are joined by edges. The regular square tiling is an example of a locally rectangular tiling, which means it has 4 neighbours for each point.

Triangle tiling



(a) Standard template.



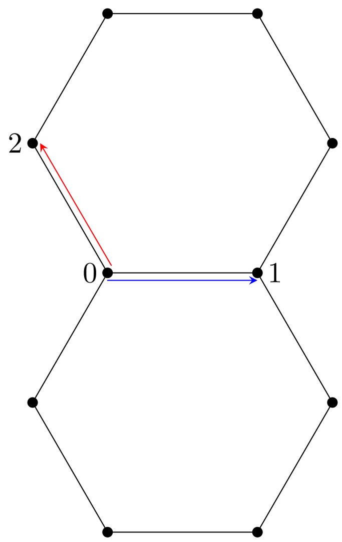
(b) Tiling[1].

Figure B.1: Triangle standard template and tiling.

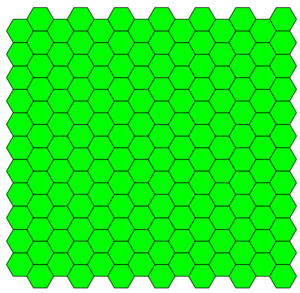
The two independent steps (see Figure B.1a) for the triangle tiling are the translation from 0 to 1, along the horizontal edge of the upright triangle, and the translation from 0 to 2, along the edge angled $\pi/3$ anti-clockwise from the horizontal edge. To get to all points in the tiling (see Figure B.1b) we can take steps and inverse steps of these two translations. These steps are similar to how we move around the square mesh as they do not require a rotation component.

[1]1-uniform_n11 by Tomruen, retrieved from "`https://commons.wikimedia.org/wiki/File:1-uniform_n11.svg`" is licensed under CC BY-SA 4.0.

Hexagonal tiling



(a) Standard template.



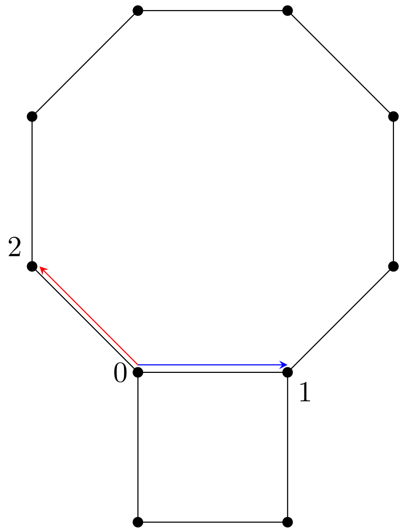
(b) Tiling[2].

Figure B.2: Hexagonal standard template and tiling.

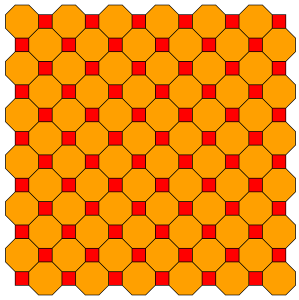
For the only other regular tiling of the plane, the first step is from 0 to 1 (along the horizontal edge of the hexagon) followed by a rotation $\pi/3$ clockwise. The second step is from 0 to 2 (along the edge angled $2\pi/3$ anti-clockwise from the horizontal edge) followed by a rotation by π. We always look at the image (see Figure B.2a) from the point of view with 0 in the bottom left corner of a hexagon (Figure B.2b).

[2] 1-uniform_n1 by Tomruen, retrieved from "`https://commons.wikimedia.org/wiki/File:1-uniform_n1.svg`" is licensed under CC BY-SA 4.0.

Truncated square tiling



(a) Standard template.



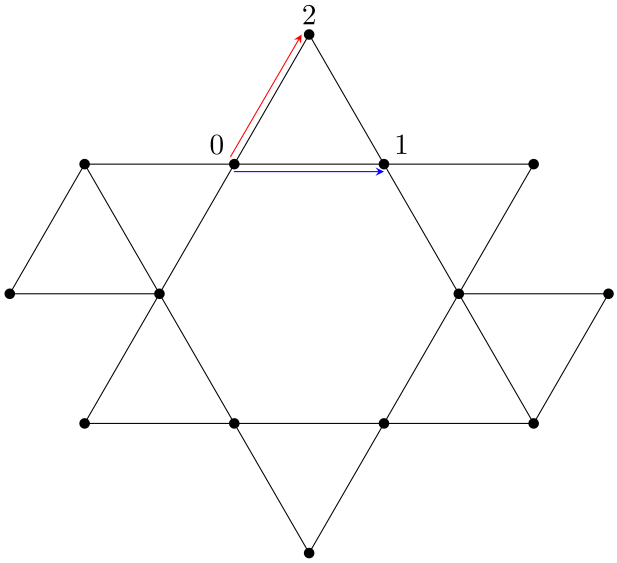
(b) Tiling[3].

Figure B.3: Truncated square standard template and tiling.

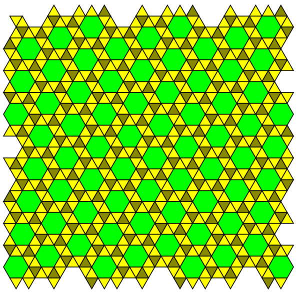
For the truncated square tiling (Figure B.3b), the first step is from the point 0 to 1 (along the horizontal edge between the square and octagon) followed by a rotation of $\pi/2$ clockwise. The second step is from 0 to 2 (between two octagons angled $3\pi/4$ anti-clockwise from horizontal) followed by a rotation of π. We always look at the image (Figure B.3a) from the point of view with 0 in the bottom left corner of the octagon. This tiling is very similar to the snub square tiling, as both have one step which used twice takes one back to the same point and another step that when it is used four times takes one back to the same point.

[3] 1-uniform_n2 by Tomruen, retrieved from "`https://commons.wikimedia.org/wiki/File:1-uniform_n2.svg`" is licensed under CC BY-SA 4.0.

Snub trihexagonal tiling



(a) Standard template.



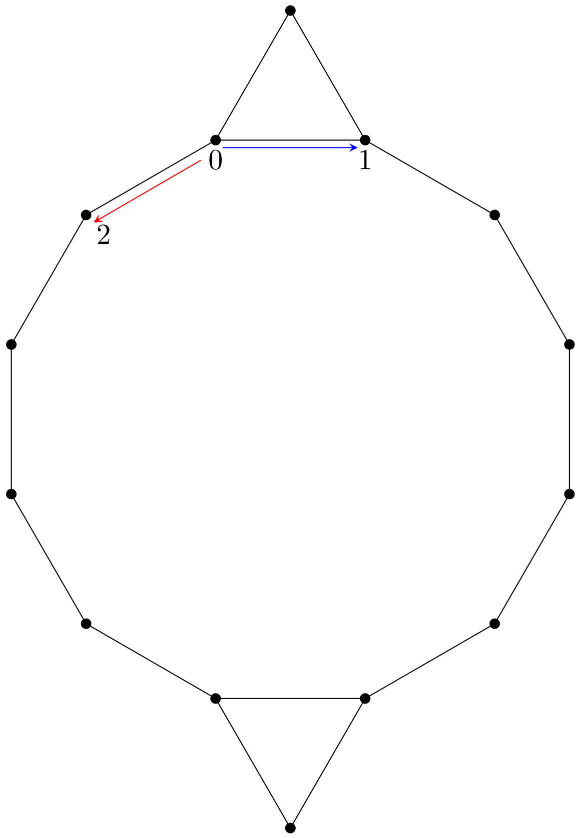
(b) Tiling[4].

Figure B.4: Snub trihexagonal standard template and tiling.

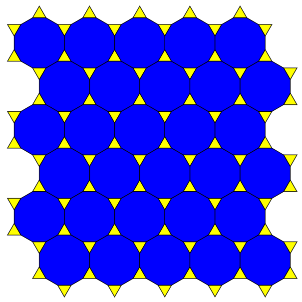
For the snub trihexagonal tiling (Figure B.4b), the first step is from 0 to 1 (along the horizontal edge between the hexagon and the top triangle) followed by a rotation of $\pi/3$ clockwise. The second step is from 0 to 2 (along the edge angled $\pi/3$ anti-clockwise from the horizontal edge) followed by a rotation of $2\pi/3$ anti-clockwise. We always look at the image (Figure B.4a) from the point of view with 0 in the top left corner of the hexagon.

[4] 1-uniform_n10 by Tomruen, was retrieved from the site "`https://commons.wikimedia.org/wiki/File:1-uniform_n10.svg`" and is licensed under CC BY-SA 4.0.

Truncated hexagonal tiling



(a) Standard template.



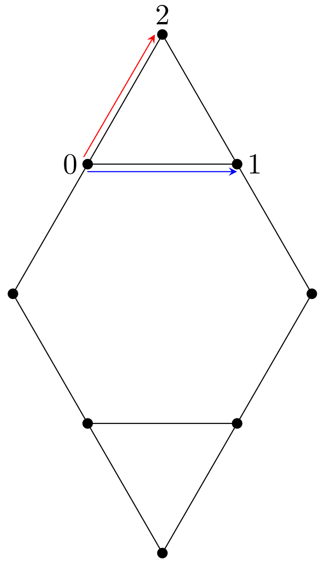
(b) Tiling[5].

Figure B.5: Truncated hexagonal standard template and tiling.

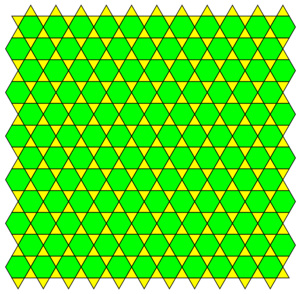
For the truncated hexagonal tiling (Figure B.5b), the first step is 0 to 1 (along the horizontal edge between the triangle and dodecagon) followed by a rotation of $2\pi/3$ anti-clockwise. The second step is from 0 to 2 (along the edge between two dodecagons at an angle of $5\pi/6$ clockwise from the horizontal edge) followed by a rotation of π. We always look at the image (Figure B.5a) from the point of view with 0 in the bottom left corner of the top triangle.

[5]1-uniform_n4 by Tomruen, retrieved from "`https://commons.wikimedia.org/wiki/File:1-uniform_n4.svg`" is licensed under CC BY-SA 4.0.

Trihexagonal tiling



(a) Standard template.



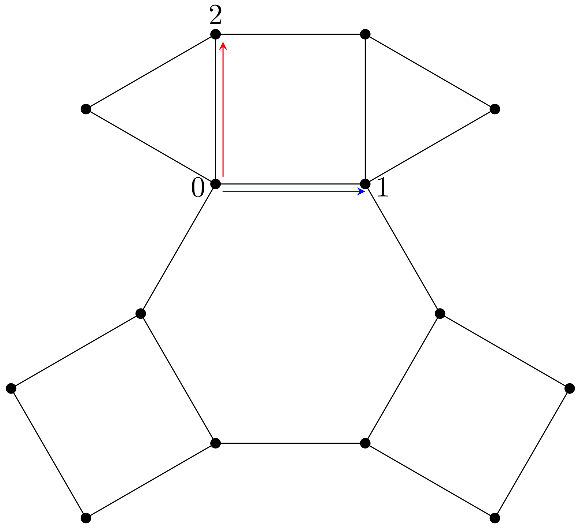
(b) Tiling[6].

Figure B.6: Trihexagonal standard template and tiling.

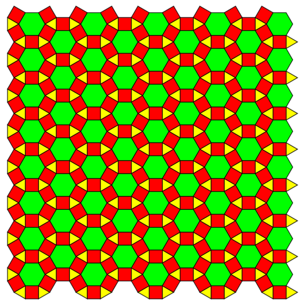
For the trihexagonal tiling (Figure B.6b), the first step is from 0 to 1 (along the horizontal edge of the triangle) then a rotation of $\pi/3$ clockwise. The second step is 0 to 2 (along the edge angled $\pi/3$ anti-clockwise from the horizontal edge) then a rotation $\pi/3$ anti-clockwise. We always look at the image (Figure B.6a) from the point of view with 0 in the bottom left corner of a triangle.

[6]1-uniform_n7 by Tomruen, retrieved from "`https://commons.wikimedia.org/wiki/File:1-uniform_n7.svg`" is licensed under CC BY-SA 4.0.

Rhombitrihexagonal tiling



(a) Standard template.



(b) Tiling[7].

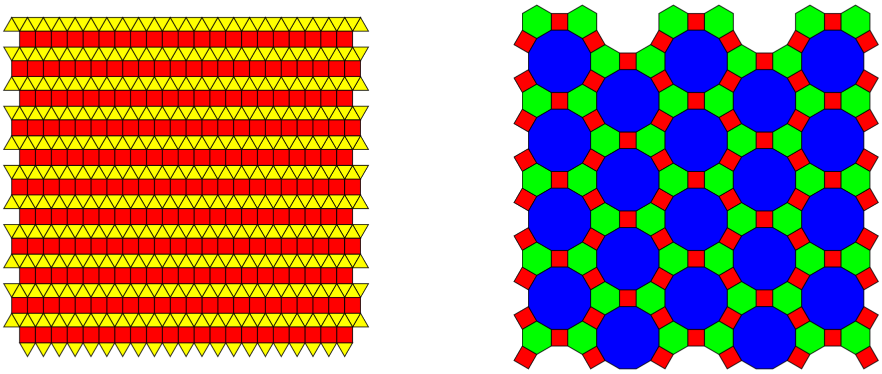
Figure B.7: Rhombitrihexagonal standard template and tiling.

For the rhombitrihexagonal tiling (Figure B.7b), the first step is 0 to 1 (along the horizontal edge) then a rotation by $\pi/3$ clockwise. The second step is from 0 to 2 (vertically up the square edge) then a rotation by $2\pi/3$ anti-clockwise. We always look at the image (Figure B.7a) from the point of view with 0 in the bottom left corner of the square.

As stated at the start, for most of these tilings there are other choices for the two steps to move around the plane. However, the ones chosen here represent possibly the easiest choices for the different tilings.

[7] 1-uniform_n6 by Tomruen, retrieved from "`https://commons.wikimedia.org/wiki/File:1-uniform_n6.svg`" is licensed under CC BY-SA 4.0.

Semi-regular tilings not considered



(a) Elongated triangular tiling[8].

(b) Truncated trihexagonal tiling[9].

Figure B.8: The other semi-regular tilings.

The elongated triangular tiling (Figure B.8a) is the only tiling which we have looked at so far which requires 3 different steps to create all possible paths. The main reason for this is that there are three types of edges. There are edges between squares and squares; triangles and triangles; and squares and triangles. There is no way to describe one of these edges in terms of the other two, because of the way they tessellate the plane. Therefore, as we only consider tilings which need two independent steps, this example is omitted from the discussion.

The other semi-regular tiling (Figure B.8) of the plane the method does not work for is the truncated trihexagonal tiling (Figure B.8b). For this, one needs to use reflections which we do not consider due to the need to preserve the orientation of the mesh. Reflections could potentially be used but would make the calculations more complex.

For the prolongation space, we restrict attention to tilings with only two independent steps.

B.2 Prolongation space for two independent steps

The two independent steps T_1 and T_2 are of two types. The first type is a translation without an ensuing rotation. For this type of step the only $k \in \mathbb{Z}$ such

[8] 1-uniform_n8 by Tomruen, retrieved from "`https://commons.wikimedia.org/wiki/File:1-uniform_n8.svg`" is licensed under CC BY-SA 4.0.

[9] 1-uniform_n3 by Tomruen, retrieved from "`https://commons.wikimedia.org/wiki/File:1-uniform_n3.svg`" is licensed under CC BY-SA 4.0.

that $\mathrm{T}_r^k = \mathrm{id}$ is $k = 0$. This type of step is used in triangle tilings and regular square tilings. The other type of step is a translation followed by a rotation. This type of step takes us along a closed path (either an edge or polygon) meaning there exists $k \in \mathbb{Z}^+/\{1\}$ such that $\mathrm{T}_r^k = \mathrm{id}$. This variety of step is used for all the other types of regular and semi-regular tilings we consider.

The steps map the tiling to itself, so their inverses exist at every point in the tiling. The inverse of the step which is a translation only is just a translation in the opposite direction, i.e., $(\mathrm{T}_r)^{-1} = \mathrm{T}_r^{-1}$. For a step which is a translation followed by a rotation, the inverse step is the opposite rotation followed by a translation back in the opposite direction, i.e., $(\mathrm{T}_r)^{-1} = \mathrm{T}_r^{k-1}$, where $\mathrm{T}_r^k = \mathrm{id}$.

A path between 0 and any other point $\mathbf{J}$, which may be anywhere in the tiling, is obtained by applying the steps T_r successively to move along edges between vertices in the tiling. Two paths P_1 and P_2 differ by a trivial path (either the identity or a cycle, i.e., a closed loop) if and only if they map 0 to the same point $\mathbf{J}$. Note there may be multiple ways one can get from 0 to $\mathbf{J}$ in the tiling, but, it is important for the calculations that only one path is used to describe each path from 0 to any other vertex $\mathbf{J}$ in the tiling.

Using the equivalence of paths enables one to write complicated paths in a shortest-possible form.

For a function u (or, more generally, a difference form) on the tiling, the prolongation space over the base point $\mathbf{n}$ is the product space giving the values of u at all points in the tiling. Here we are not thinking about any particular function, but rather the space of all functions, so each $u_{\mathbf{J}}$ can take any value in $\mathbb{R}$ (or $\mathbb{C}$ if needed). The variable $u_{\mathbf{J}}$ is the value of u at the point $\mathbf{nJ}$. The pullback of u on $\mathbf{nJ}$ by a path $P_{\mathbf{J}}$ is $u_{\mathbf{J}}$ on $\mathbf{n}$. The general notation of the pullback by $P_{\mathbf{K}}$ is $P_{\mathbf{K}}^*$. For u on $\mathbf{nJ}$ where $\mathbf{J} = \underbrace{r\ldots r}_{k\text{ times}}$ the pullback by T_r^k is $\left(\mathrm{T}_r^k\right)^* u = u_{\mathbf{J}}$ on $\mathbf{n}$. We use the snub square tiling case as an example of this new notation.

Example B.2.1. Let u be a function on $\mathbf{nJ}$, where the path from 0 to $\mathbf{J}$ is $P_{\mathbf{J}} = \mathrm{T}_2\mathrm{T}_1^3\mathrm{T}_2\mathrm{T}_1\mathrm{T}_2\mathrm{T}_1$. Then the pullback of u by $P_{\mathbf{J}}$ is the function

$$u_{\mathbf{J}} = P_{\mathbf{J}}^* u = \mathrm{T}_1^*\mathrm{T}_2^*\mathrm{T}_1^*\mathrm{T}_2^*\left(\mathrm{T}_1^3\right)^*\mathrm{T}_2^* u = u_{1:1:3:}\,,$$

on **n**. Using the general notation this is

$$u_{\mathbf{J}} = u_{12121112}.$$

The indices used in going from 0 to **J** are read left to right in the pullback. Therefore, the pullback of $P_{\mathbf{K}}P_{\mathbf{J}}$ is $P_{\mathbf{J}}^{*}P_{\mathbf{K}}^{*}u = u_{\mathbf{JK}}$, where **JK** represents the concatenation of indices, simplified by whatever relations apply. If $f_{\mathbf{nJ}}$ is a function on **nJ**, the pullback is related to the shift operator as follows: $\mathrm{S}_{\mathbf{K}}f_{\mathbf{n}} := \mathrm{T}_{\mathbf{K}}^{*}f_{\mathbf{nk}}$.

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