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# STRUCTURE PRESERVING SPLITTING TECHNIQUES FOR EBOLA REACTION–DIFFUSION EPIDEMIC SYSTEM

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## Abstract

In this paper, we deal with the numerical solution of the reaction–diffusion Ebola epidemic model. The diffusion which is an important phenomenon for the epidemic model is included in the model. This inclusion has made the model more comprehensive for studying the disease dynamics in the human population. The quantities linked with the model indicate the population sizes which are taken as absolute, therefore, the numerical schemes utilized to solve the underlying Ebola epidemic system should sustain the positivity. The numerical approaches used to solve the underlying epidemic models are explicit nonstandard finite difference operator splitting (ENSFD-OS) and implicit nonstandard finite difference operator splitting (INSFD-OS) techniques. These schemes preserve all the physical features of the state variables, i.e. projected schemes hold the positive solution acquired by the Ebola diffusive epidemic model. The underlying epidemic model illustrates two stable steady states, a virus-free state, and a virus existence state. The suggested approaches retain the stability of each of the steady states possessed by the assumed epidemic model. A numerical example and simulations for validation of all the characteristics of suggested techniques are also investigated.

*Keywords*: Ebola Infection; Reaction–diffusion System; Splitting Techniques; Nonstandard Finite Differences; Simulations.

## 1. INTRODUCTION

Ebola is one of the scarce and fatal viruses which causes diarrhea, body aches, bleeding, and fever. Various organs and the immune system of a person are damaged when a virus diffuses in the body of that person. It is called Ebola hemorrhagic fever, commonly named as Ebola virus. The recent outbreak of this infection particularly in Africa led to over 28,000 cases and more than 11,000 people were deceased by the year 2015.[1] Also, the Ebola epidemic is growing and a recent outbreak was detected by the World Health Organization in Congo.

Another study of this fatal disease outbreak described that 34.7% was the rate of case fatality.[2] The current inventions of Ebola medication are declared by the World Medical Association which gives hope to the countries mainly infected by this disease. Several studies on the modeling of the Ebola epidemic are presented with the aid of various

approaches and attain the successful analysis of the reproduction value of the Ebola outbreak.[3–9]

Many researchers who are fascinated by the epidemiological modeling get attention to examine the complex dynamics of Ebola virus infection models.[10–27] Sharomi and Malik[28] presented various disease models to exhibit a comprehensive survey on optimal control. The infection model for controlling the spread of Ebola disease was investigated by Zakary *et al.*[29] A deterministic Ebola infection model is investigated mathematically and numerically.[30] A mathematical model for optimal control of Ebola disease is analyzed by Ahmad *et al.*[31] Many researchers worked on various mathematical models of epidemic diseases. They developed the classical models by using ordinary differential equations. These models do not describe the infection dynamics accurately.[32–34] Keeping in view such types of issues, the diffusion process is included in the model to describe the disease dynamics more accurately. Hence, the partial differential equations describe the disease dynamics more precisely. Fractional calculus is gaining the attention of many researchers and scientists. They are focusing now on the noninteger order mathematical modeling of many real-world situations, for instance in the field of engineering, fluid dynamics, epidemiology, and many more.[35–41] The researchers who have considered the infection disease models did not focus on the advection and diffusion processes simultaneously. So, this paper presents a novel and attractive model of cut-throat Ebola virus disease. Moreover, the consequences of the study paper are of great importance. This work will be favorable for the health departments to devise effective policies for controlling the disease. The advantages of integer order partial derivatives in epidemic models are that they describe the advection and diffusion processes. The first-order partial derivative $\frac{\partial}{\partial x}$ represents the advection and the second-order partial derivative represents the diffusion process. It is quite rational to include the diffusion factor in the epidemic models. These models can predict the disease dynamics in a better way.[42–49] A vaccinated Ebola epidemic system is proposed and studied by Area *et al.*[50] Grigorieva *et al.*[51] designed and investigated a controlled dynamical Ebola virus infection system. Many researchers who are fascinated by the epidemiological modeling get attention to examine the complex dynamics of Ebola virus infection models. Some other important research studies are given in Refs. 52–64.

## 2. PRELIMINARIES

In this section, some important definitions are described that can help us to understand this work.

**Definition 1.** Suppose that $f$ is a continuously differentiable function defined on an open subset $O$ of $R^n$ to $R^n$. Then $x^* \in O$ is called an equilibrium point of the dynamical system $x'(t) = f(x)$ if, $f(x) = 0$.

Here, $O$ is the state space of the physical system that may be biological, engineering or economics, etc.

It is important to note that all the notations and assumptions regarding the dynamical system are the same in the forthcoming definitions in the preliminaries section.

**Definition 2.** Suppose that $x^* \in O$ is an equilibrium point for $x'(t) = f(x)$. Then $x^*$ is a stable equilibrium if, for every neighborhood $N$ of $x^*$ in $O$, there is a neighborhood $N_1$ of $x^*$ in $N$ such that every solution $x(t)$ with $x(0)$ in $N_1$ is defined and in $N$ for all $t > 0$.

**Definition 3.** If $N_1$ can be taken so that

$$\lim_{x\to\infty} x(t) = x^*$$

along with the other assumptions defined in Definition 2, then $x^*$ is called the asymptotically stable equilibrium.

To define the consistency stability and convergence of a numerical scheme, it is important to denote some notions. For this, suppose that $\Phi(x(t_n))$ represents the exact solution of the differential equation $x'(t) = f(x)$, $\Phi_n$ stands for the exact solution of the discrete system corresponding to $x'(t) = f(x)$ and $\Phi_n^*$ denotes the actually computed solution. By applying a numerical scheme to the continuous model, mainly, two types of errors occur. One of them is called the discretization error, i.e. the difference between the exact solutions of the continuous system and corresponding discrete system is denoted by

$$|\Phi(x(t_n)) - \Phi_n|,$$

and the other is called the solution error which is the difference between the exact solution of the discrete system and the actually computed solution, which is denoted by

$$|\Phi_n^* - \Phi_n|.$$

**Definition 4.** A numerical scheme is said to be consistent with the corresponding continuous model if the discrete model converges to the continuous model by applying the limit as $\Delta t \to 0$.

**Definition 5.** A discretized numerical scheme is said to be stable if the solution error $|\Phi_n^* - \Phi_n|$ remains bounded for all time steps in the domain.

In the next section, the numerical study of the Ebola virus infectious disease model with the inclusion of diffusion is studied numerically.

## 3. MATHEMATICAL MODEL

In this study, the Ebola virus infection $Z_1, Z_2, Z_3$, and $Z_4$ reaction–diffusion epidemic model is proposed as follows:

$$\frac{\partial Z_1}{\partial t} = \alpha - \eta Z_1 Z_2 - \alpha Z_1 + D_{Z_1}\frac{\partial^2 Z_1}{\partial \chi^2}, \quad (1)$$

$$\frac{\partial Z_2}{\partial t} = \eta Z_1 Z_2 - (\zeta + \alpha) Z_2 + D_{Z_2}\frac{\partial^2 Z_2}{\partial \chi^2}, \quad (2)$$

$$\frac{\partial Z_3}{\partial t} = \zeta Z_2 - (\pi + \alpha) Z_3 + D_{Z_3}\frac{\partial^2 Z_3}{\partial \chi^2}, \quad (3)$$

$$\frac{\partial Z_4}{\partial t} = \pi Z_3 - \alpha Z_4 + D_{Z_4}\frac{\partial^2 Z_4}{\partial \chi^2}. \quad (4)$$

In the above system, the state variables $Z_1$, $Z_2$, $Z_3$, and $Z_4$ are susceptible, exposed, infected, and recovered subpopulations, respectively, while the parameters $\alpha$, $\eta$, $\zeta$, and $\pi$ are birth as well as death rate, contact rate, transmission rate from $Z_2$ to $Z_3$, and treatment rate, respectively. The values $D_{Z_1} - D_{Z_2}$ are diffusion coefficients. As the state variable $Z_4$ is not part of Eqs. (1)–(3), we can write

$$\frac{\partial Z_1}{\partial t} = \alpha - \eta Z_1 Z_2 - \alpha Z_1 + D_{Z_1}\frac{\partial^2 Z_1}{\partial \chi^2}, \quad (5)$$

$$\frac{\partial Z_2}{\partial t} = \eta Z_1 Z_2 - (\zeta + \alpha) Z_2 + D_{Z_2}\frac{\partial^2 Z_2}{\partial \chi^2}, \quad (6)$$

$$\frac{\partial Z_3}{\partial t} = \zeta Z_2 - (\pi + \alpha) Z_3 + D_{Z_3}\frac{\partial^2 Z_3}{\partial \chi^2}, \quad (7)$$

which have the following initial conditions

$$Z_1(\chi, 0) = \vartheta_1(\chi), \quad Z_2(\chi, 0) = \vartheta_2(\chi),$$
$$Z_3(\chi, 0) = \vartheta_3(\chi), \quad 0 \le \chi \le \ell, \quad (8)$$

and no-flux boundary conditions are

$$Z_{1_\chi}(0, t) = Z_{1_\chi}(\ell, t) = 0, \quad (9)$$

$$Z_{2_\chi}(0, t) = Z_{2_\chi}(\ell, t) = 0, \quad (10)$$

$$Z_{3_\chi}(0, t) = Z_{3_\chi}(\ell, t) = 0. \quad (11)$$

The main idea of this paper is to formulate a numerical scheme that is easy to implement and maintain the positivity of the solution as the variables presented in the $Z_1 Z_2 Z_3$ system are absolute. Numerous investigators examine the numerical solution of various physical models containing differential equations with the assistance of structure-preserving approaches.[58–63] The operator splitting approaches extensively used numerical techniques for solving ordinary and partial differential equations.[52–56] In this paper, a hybrid splitting approach with a nonstandard finite difference (NSFD) technique is used to find the numerical solution to the Ebola reaction–diffusion epidemic model. The nonstandard approach proposed by Mickens[57] is a class of numerical schemes that helps to construct structure-preserving techniques. Several researchers employed different NSFD approaches for solving the differential equations.

The rest of this paper is organized as follows. Section 4 is given for the investigation of steady states and reproductive number for the proposed epidemic model. The proposed numerical techniques are presented in Sec. 5. The accuracy, stability, and positivity of both designed ENSFD-OS and INSFD-OS schemes are studied and analyzed in this section. Section 6 is dedicated to the numerical test and graphical simulations of suggested techniques for both steady states. Finally, Sec. 7 concludes.

## 4. STEADY STATES OF THE PROPOSED EBOLA MODEL

The underlying system (5)–(7) has two steady states, virus absent steady state (VASS) and virus existence steady state (VESS). VASS is

$$({Z_1}^0, {Z_2}^0, {Z_3}^0) = (1, 0, 0), \quad (12)$$

and VESS is

$$({Z_1}^*, {Z_2}^*, {Z_3}^*) = \left(\frac{\alpha + \zeta}{\eta}, \frac{\zeta Z_2^*}{\alpha + \pi}, \frac{\alpha - \alpha Z_1^*}{\eta Z_1^*}\right), \quad (13)$$

and

$$R_0 = \frac{\eta}{(\alpha + \zeta)}, \quad \text{when } D_{Z_1} = D_{Z_2} = D_{Z_3} = 0,$$

where $R_0$ is the reproduction number and if $R_0 < 1$ then the underlying Ebola epidemic model illustrates VFSS and if, $R_0 > 1$ then it depicts VESS.

## 5. NUMERICAL METHODS

In this section, finite difference approximation approaches are proposed by using an operator-splitting environment. These numerical schemes govern the nonlinearity and intricacy of the reaction–diffusion system because they split the solution of reaction and diffusion terms. The techniques which we applied to the system (5)–(7) are ENSFD-OS and INSFD-OS schemes. Several numerical mathematicians used operator-splitting approaches on various ordinary and partial differential equations and systems.[52–56,64] The proposed Ebola epidemic system is split into two systems of equations. First, we consider the reaction step which is nonlinear and given as follows:

$$\frac{1}{2}\frac{\partial Z_1}{\partial t} = \alpha - \eta Z_1 Z_2 - \alpha Z_1, \tag{14}$$

$$\frac{1}{2}\frac{\partial Z_2}{\partial t} = \eta Z_1 Z_2 - (\zeta + \alpha) Z_2, \tag{15}$$

$$\frac{1}{2}\frac{\partial Z_3}{\partial t} = \zeta Z_2 - (\pi + \alpha) Z_3. \tag{16}$$

At the second stage, the linear diffusion equation is illustrated as follows:

$$\frac{1}{2}\frac{\partial Z_1}{\partial t} = D_{Z_1}\frac{\partial^2 Z_1}{\partial \chi^2}, \tag{17}$$

$$\frac{1}{2}\frac{\partial Z_2}{\partial t} = D_{Z_2}\frac{\partial^2 Z_2}{\partial \chi^2}, \tag{18}$$

$$\frac{1}{2}\frac{\partial Z_3}{\partial t} = D_{Z_3}\frac{\partial^2 Z_3}{\partial \chi^2}. \tag{19}$$

At the first stage, the discrete model of the reaction step is designed by incorporating the guidelines given by Mickens[57] for the construction of structure-preserving NSFD schemes.

$$\bar{Z_1}_p^{q+\frac{1}{2}} = \frac{Z_1{}_p^q + \Delta t\alpha}{1 + \Delta t\eta Z_2{}_p^q + \Delta t\alpha}, \tag{20}$$

$$\bar{Z_2}_p^{q+\frac{1}{2}} = \frac{Z_2{}_p^q + \Delta t\eta Z_1{}_p^q Z_2{}_p^q}{1 + \Delta t(\zeta + \alpha)}, \tag{21}$$

$$\bar{Z_3}_p^{q+\frac{1}{2}} = \frac{Z_3{}_p^q + \Delta t\zeta Z_2{}_p^q}{1 + \Delta t(\pi + \alpha)}, \tag{22}$$

where $Z_1{}_p^q, Z_2{}_p^q$, and $Z_3{}_p^q$ interpret the numerical values of $Z_1, Z_2$, and $Z_3$, respectively, at $0 + p\Delta\chi$, $p \in \{0, 1, \cdots\}$ and time $q\Delta t, q \in \{0, 1, \cdots\}$ and $\bar{Z_1}_p^{q+\frac{1}{2}}, \bar{Z_2}_p^{q+\frac{1}{2}}$ and $\bar{Z_3}_p^{q+\frac{1}{2}}$ demonstrate the representative values at the half time step.

In the first stage, ENSFD-OS and INSFD-OS methods have the same design which is given above. The discretization of ENSFD-OS for diffusion equation is described as follows:

$$Z_1{}_p^{q+1} = (1 - 2\Gamma_1)\bar{Z_1}_p^{q+\frac{1}{2}} + \Gamma_1(\bar{Z_1}_{p-1}^{q+\frac{1}{2}} + \bar{Z_1}_{p+1}^{q+\frac{1}{2}}), \tag{23}$$

$$Z_2{}_p^{q+1} = (1 - 2\Gamma_2)\bar{Z_2}_p^{q+\frac{1}{2}} + \Gamma_2(\bar{Z_2}_{p-1}^{q+\frac{1}{2}} + \bar{Z_2}_{p+1}^{q+\frac{1}{2}}), \tag{24}$$

$$Z_3{}_p^{q+1} = (1 - 2\Gamma_3)\bar{Z_3}_p^{q+\frac{1}{2}} + \Gamma_3(\bar{Z_3}_{p-1}^{q+\frac{1}{2}} + \bar{Z_3}_{p+1}^{q+\frac{1}{2}}). \tag{25}$$

The discretization for the INSFD-OS technique at second stage is given as follows:

$$-\Gamma_1 Z_1{}_{p-1}^{q+1} + (1 + 2\Gamma_1)Z_1{}_p^{q+1} - \Gamma_1 Z_1{}_{p+1}^{q+1} = \bar{Z_1}_p^{q+\frac{1}{2}}, \tag{26}$$

$$-\Gamma_2 Z_2{}_{p-1}^{q+1} + (1 + 2\Gamma_2)Z_2{}_p^{q+1} - \Gamma_2 Z_2{}_{p+1}^{q+1} = \bar{Z_2}_p^{q+\frac{1}{2}}, \tag{27}$$

$$-\Gamma_3 Z_3{}_{p-1}^{q+1} + (1 + 2\Gamma_3)Z_3{}_p^{q+1} - \Gamma_3 Z_3{}_{p+1}^{q+1} = \bar{Z_3}_p^{q+\frac{1}{2}}, \tag{28}$$

where

$$\Gamma_1 = d_{Z_1}\frac{\Delta t}{\Delta\chi^2}, \quad \Gamma_2 = d_{Z_2}\frac{\Delta t}{\Delta\chi^2}, \quad \Gamma_3 = d_{Z_3}\frac{\Delta t}{\Delta\chi^2}.$$

### 5.1. Accuracy and Stability of the Numerical Methods

The consistency and stability of ENSFD-OS and INSFD-OS techniques depend on their split solution.[52,53,64] Hence, reaction stage is solved exactly with $O(\Delta t)$ accuracy of time step[52,53,64] and $O(\Delta\chi^2)$ accuracy is contained by diffusion step. As for as stability is concerned, the stage in which the reaction equation is involved has unconditional stability. For the stage where diffusive step is involved ENSFD-OS (23)–(25) has the following

stability conditions

$$\Gamma_j \leq \frac{1}{2}, \quad (j \in \{1,2,3\}). \tag{29}$$

On the other hand, the INSFD-OS technique has unconditional stability at the diffusion step.

## 5.2. Positivity of ENSFD-OS and INSFD-OS Methods

**Theorem 1.** *ENSFD-OS and INSFD-OS discretization techniques at reaction steps* (20)–(22) *hold the positive solution provided that*

$$\begin{aligned} &Z_{1_p}^{\,q} \geq 0, \quad Z_{2_p}^{\,q} \geq 0, \quad Z_{3_p}^{\,q} \geq 0 \\ \Rightarrow\ &\bar{Z}_{1_p}^{\,q+\frac{1}{2}} \geq 0, \quad \bar{Z}_{2_p}^{\,q+\frac{1}{2}} \geq 0, \quad \bar{Z}_{3_p}^{\,q+\frac{1}{2}} \geq 0. \end{aligned} \tag{30}$$

**Proof.** The proof is obvious as each term involved on the right-hand side of the discretization formulas (20)–(22) has positive sign. □

**Remark 6.** The proposed ENSFD-OS technique (23)–(25) exhibits the positive solution if

$$1 - 2\Gamma_j \geq 0, \quad j \in \{1,2,3\},$$

which implies that

$$\Gamma_j \leq \frac{1}{2}, \quad (j \in \{1,2,3\}).$$

Above is the condition (29) of stability for ENSFD-OS technique (23)–(25), which verifies that this technique sustains the positivity of the solution in its region of stability.

To validate the positivity of INSFD-OS technique (26)–(28), the results of M-matrix theory are adopted.

**Theorem 2.** *For every* $\Delta\chi > 0$ *and* $\Delta t > 0$, *the system* (26)–(28) *is positive, i.e.* ${Z_1}^q > 0, {Z_2}^q > 0$ *and* ${Z_3}^q > 0, \forall\, q \in \{0,1,2,\ldots\}$.

**Proof.** The discretization formulas (26)–(28) can be written as follows:

$$\mathcal{A}{Z_1}^{q+1} = {Z_1}^q, \tag{31}$$

$$\mathcal{B}{Z_2}^{q+1} = {Z_2}^q, \tag{32}$$

$$\mathcal{C}{Z_3}^{q+1} = {Z_3}^q, \tag{33}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are the square matrices as follows:

$$\mathcal{A} = \begin{pmatrix} a_3 & a_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ a_2 & a_3 & a_2 & \ddots & & & & \vdots \\ 0 & a_2 & a_3 & a_2 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_2 & a_3 & a_2 & 0 \\ \vdots & & & & \ddots & a_2 & a_3 & a_2 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & a_1 & a_3 \end{pmatrix}, \tag{34}$$

$$\mathcal{B} = \begin{pmatrix} b_3 & b_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ b_2 & b_3 & b_2 & \ddots & & & & \vdots \\ 0 & b_2 & b_3 & b_2 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & b_2 & b_3 & b_2 & 0 \\ \vdots & & & & \ddots & b_2 & b_3 & b_2 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & b_1 & b_3 \end{pmatrix}, \tag{35}$$

and

$$\mathcal{C} = \begin{pmatrix} c_3 & c_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ c_2 & c_3 & c_2 & \ddots & & & & \vdots \\ 0 & c_2 & c_3 & c_2 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & c_2 & c_3 & c_2 & 0 \\ \vdots & & & & \ddots & c_2 & c_3 & c_2 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & c_1 & c_3 \end{pmatrix}. \tag{36}$$

The off-diagonal values of $\mathcal{A}$ are $a_1 = -2\Gamma_1$, $a_2 = -\Gamma_1$ whereas the diagonal values are $a_3 = 1 + 2\Gamma_1$. The off-diagonal values of $\mathcal{B}$ are $b_1 = -2\Gamma_1$, $b_2 = -\Gamma_1$ and the diagonal values are $b_3 = 1 + 2\Gamma_2$. The off-diagonal values of $\mathcal{C}$ are $c_1 = -2\Gamma_3$, $c_2 = -\Gamma_3$

while the diagonal values are $\mathfrak{c}_3 = 1 + 2\Gamma_3$. Therefore, the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are M-matrices. So Eqs. (31)–(33) become

$$Z_1{}^{q+1} = \mathcal{A}^{-1} Z_1{}^{q}, \tag{37}$$

$$Z_2{}^{q+1} = \mathcal{B}^{-1} Z_2{}^{q}, \tag{38}$$

$$Z_3{}^{q+1} = \mathcal{C}^{-1} Z_3{}^{q}. \tag{39}$$

Now, if we suppose that $Z_1{}^{q} > 0, Z_2{}^{q} > 0$ and $Z_3{}^{q} > 0$, then by using the M-matrix theory and the expressions (37)–(39), we have $Z_1{}^{q+1} > 0, Z_2{}^{q+1} > 0$ and $Z_3{}^{q+1} > 0$. Hence, by the induction, the theorem is proved. □

The above theorem validates that the INSFD-OS technique retains the positive solution unconditionally.

## 6. NUMERICAL EXAMPLE AND SIMULATIONS

In this section, we illustrate a numerical example for both steady states by using the ENSFD-OS technique and the INSFD-OS technique. For this, the following parametric values are chosen for VFSS

$$\eta = 0.5, \quad \zeta = 0.18187, \quad \pi = 0.1, \quad \alpha = 0.5.$$

For VESS, values of the parameters are given as follows:

$$\eta = 0.5, \quad \zeta = 0.18187, \quad \pi = 0.1, \quad \alpha = 0.5.$$

In this experiment, we take $D_{Z_1} = 0.05, D_{Z_2} = 0.05$ and $D_{Z_3} = 0.05$. The initial conditions for the Ebola virus model (5)–(7) are given as follows:

$$Z_1(\chi, 0) = \begin{cases} 0.4\chi & \text{if } 0 \le \chi < 0.5, \\ 0.4(1-\chi) & \text{if } 0.5 \le \chi \le 1, \end{cases} \tag{40}$$

$$Z_2(\chi, 0) = \begin{cases} 0.3\chi & \text{if } 0 \le \chi < 0.5, \\ 0.3(1-\chi) & \text{if } 0.5 \le \chi \le 1, \end{cases} \tag{41}$$

$$Z_3(\chi, 0) = \begin{cases} 0.2\chi & \text{if } 0 \le \chi < 0.5, \\ 0.2(1-\chi) & \text{if } 0.5 \le \chi \le 1. \end{cases} \tag{42}$$

### 6.1. Virus-Free Steady State

In this section, we consider the parametric values such that $R_0 < 1$ and the Ebola epidemic system under consideration converges to VASS. For all the simulation behaviors in this section, we assume the step size values and diffusion constant as $\Gamma_1 = \Gamma_2 =$

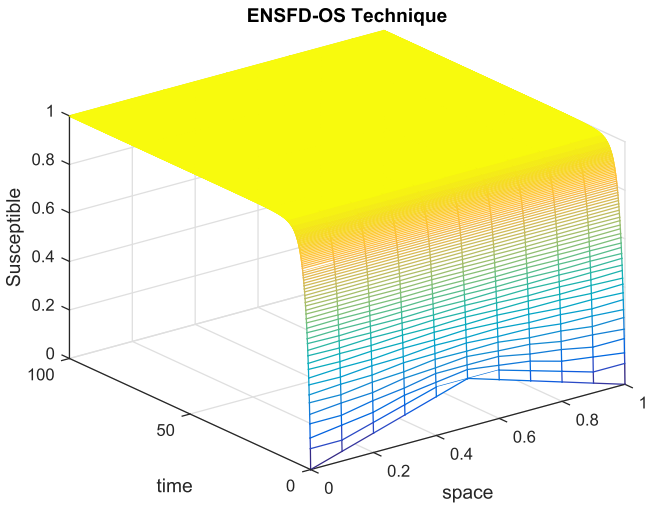


**Fig. 1** Numerical solution of $Z_1(\chi, t)$ (susceptible populace) by employing ENSFD-OS numerical scheme.

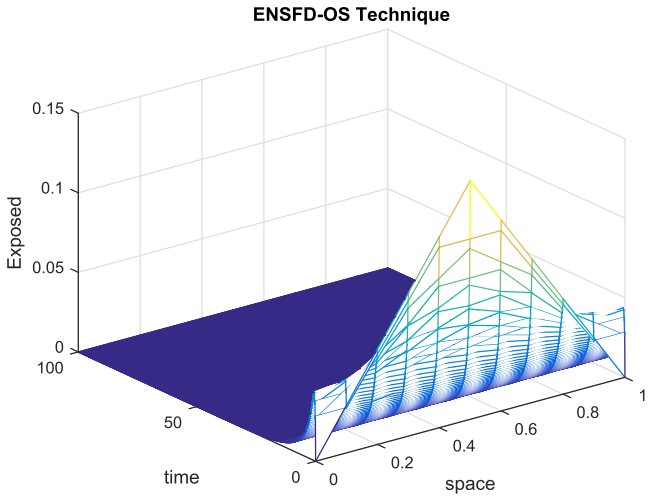


**Fig. 2** Numerical solution of $Z_2(\chi, t)$ (exposed populace) by employing ENSFD-OS numerical scheme.

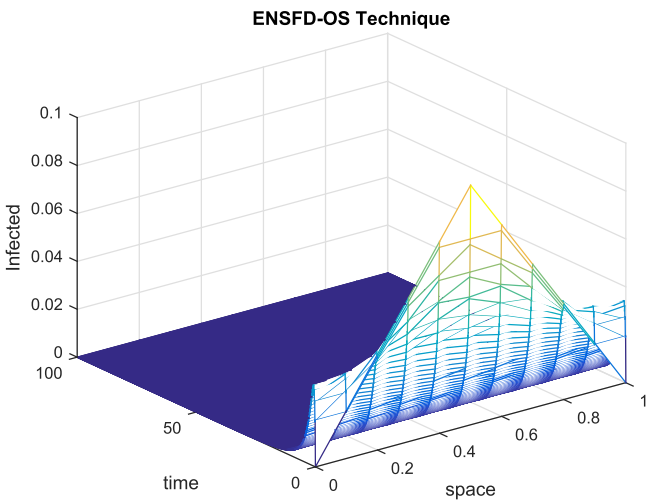


**Fig. 3** Numerical solution of $Z_3(\chi, t)$ (infected populace) by employing ENSFD-OS numerical scheme.

$\Gamma_3 = 0.5$ and $D_{Z_1} = D_{Z_2} = D_{Z_3} = 0.05$, respectively. First, the graphical solution of the suggested ENSFD-OS numerical technique is discussed.

Figures 1–3 manifest the solution behavior of susceptible, exposed, and infected populace, respectively, for VFSS. It is figured out that the graphs demonstrate the convergence exactly towards VFSS $({Z_1}^0, {Z_2}^0, {Z_3}^0) = (1, 0, 0)$. These graphs also express the positive behavior which is essential because the state variables are taken as absolute. This behavior can be observed at all step sizes of time and space.

Next, we depict the graphical results of $Z_1 Z_2 Z_3$ with the aid of the proposed INSFD-OS numerical technique.

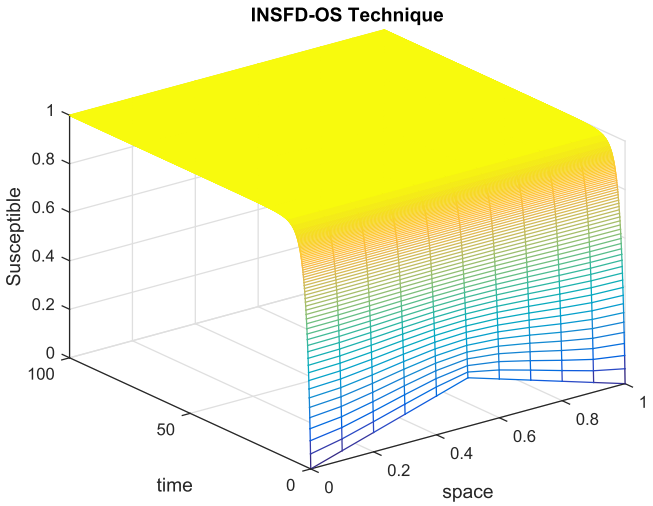


**Fig. 4** Numerical solution of $Z_1(\chi, t)$ (susceptible populace) by employing INSFD-OS numerical scheme.

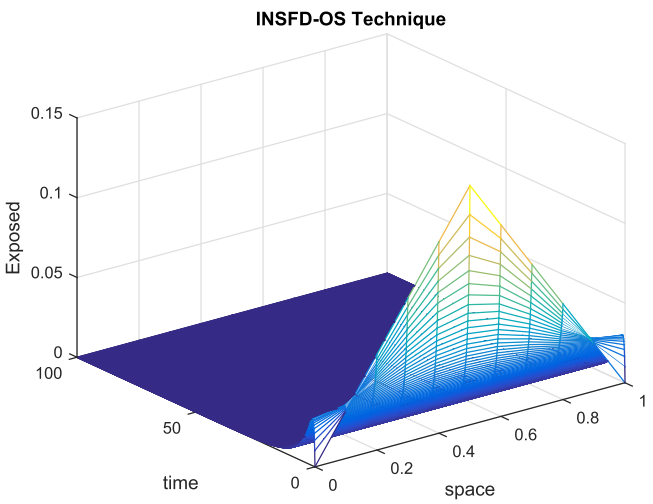


**Fig. 5** Numerical solution of $Z_2(\chi, t)$ (exposed populace) by employing INSFD-OS numerical scheme.

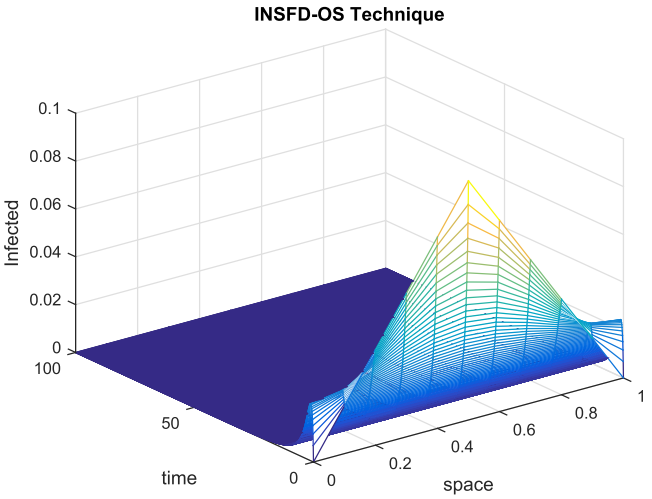


**Fig. 6** Numerical solution of $Z_3(\chi, t)$ (infected populace) by employing INSFD-OS numerical scheme.

It is observed that the Ebola reaction–diffusion system (5)–(7) exhibits some important phenomena like the positivity of the state variable and stability of steady states. The proposed INSFD-OS technique also illustrates the positive solution which is clearly shown in Figs. 4–6. One can see this positive behavior at various step sizes of time and space. This technique also retains the convergence of the system towards VFSS as preserved by ENSFD-OS.

## 6.2. Virus Existence Steady State

This section is devoted to exhibiting the graphical solution of all the sub-population by taking the values of parameters in such a manner that $R_0 > 1$

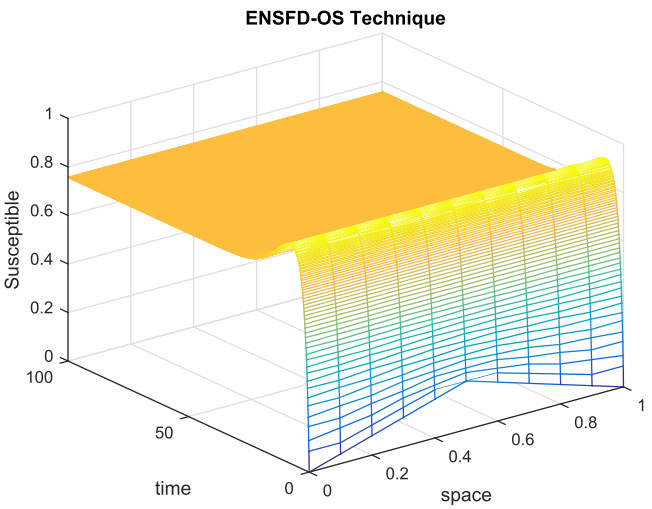


**Fig. 7** Numerical solution of $Z_1(\chi, t)$ (susceptible populace) by employing ENSFD-OS numerical scheme.

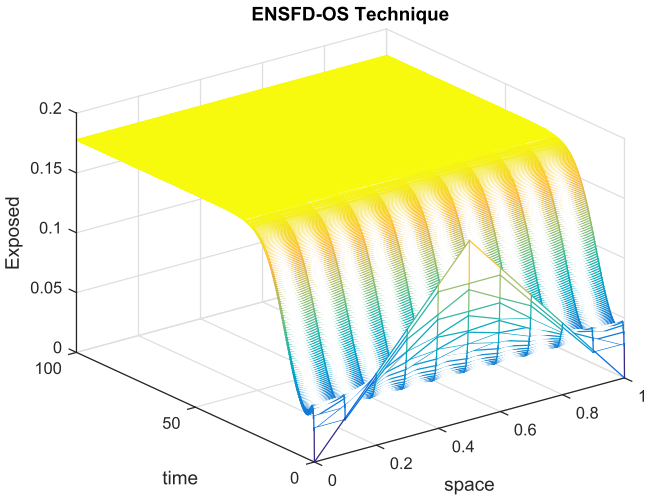


**Fig. 8** Numerical solution of $Z_2(\chi, t)$ (exposed populace) by employing ENSFD-OS numerical scheme.

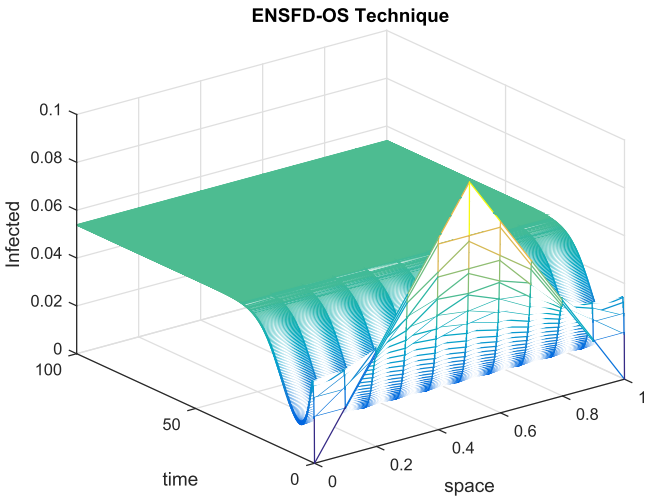


**Fig. 9** Numerical solution of $Z_3(\chi, t)$ (infected populace) by employing ENSFD-OS numerical scheme.

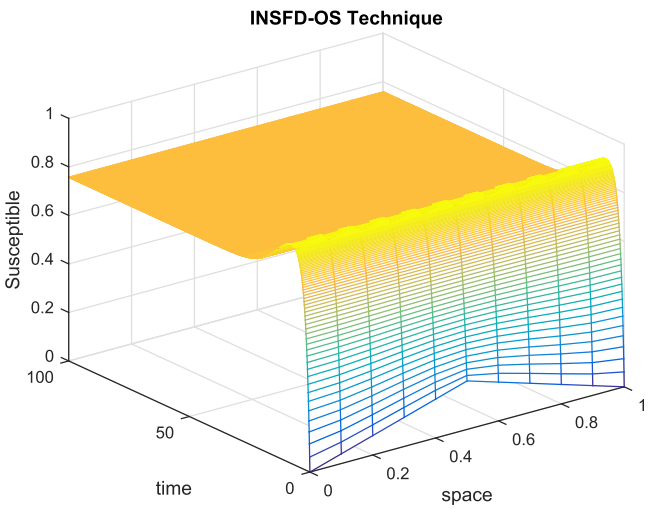


**Fig. 10** Numerical solution of $Z_1(\chi, t)$ (susceptible populace) by employing INSFD-OS numerical scheme.

and the underlying Ebola disease reaction–diffusion model shows that the system is stable at VESS.

Again, the step size and diffusion constant as $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0.5$ and $D_{Z_1} = D_{Z_2} = D_{Z_3} = 0.05$, respectively, are considered same as given in previous section.

The solution graphs depicted in Figs. 7–9 reveal the simulation behavior of susceptible, exposed, and infected populace, respectively, for VESS. It is evident from the solution behavior that the proposed ENSFD-OS approach sustains the stability of VESS $(Z_1{}^*, Z_2{}^*, Z_3{}^*)$. It can easily be concluded that the proposed ENSFD-OS approach is consistent with the continuous Ebola model and retains all the

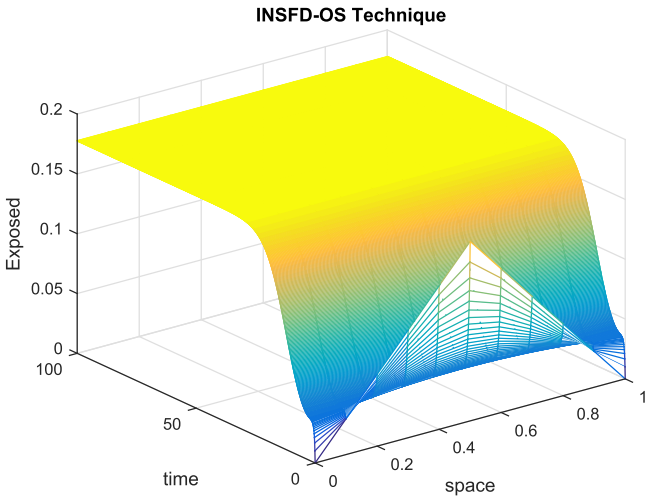


**Fig. 11** Numerical solution of $Z_2(\chi, t)$ (exposed populace) by employing INSFD-OS numerical scheme.

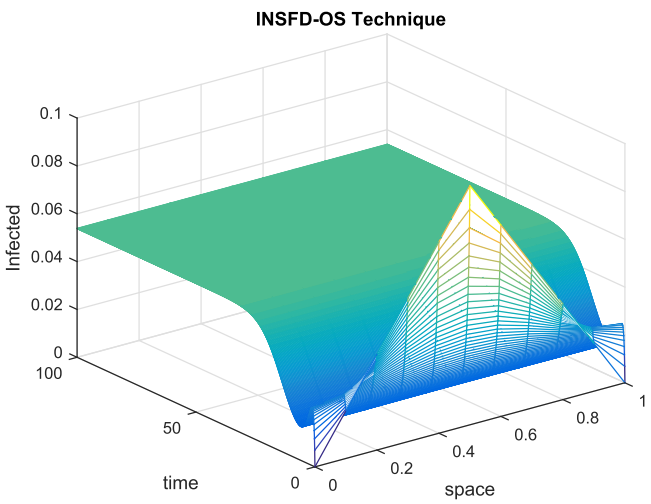


**Fig. 12** Numerical solution of $Z_3(\chi, t)$ (infected populace) by employing INSFD-OS numerical scheme.

essential phenomena exhibited by the underlying system.

Figures 10–12 clearly show that the designed INSFD-OS approach is also consistent with the continuous system under consideration as shown by the ENSFD-OS approach. Both the suggested schemes hold a positive solution as well as the stability of both steady states.

## 7. CONCLUSION

In this work, we proposed a reaction–diffusion epidemic model of Ebola infection dynamics and solved this infectious disease model with the support of two structure-preserving approaches, the ENSFD-OS numerical scheme and the INSFD-OS numerical scheme. The model under study is the population model therefore, we propose such numerical techniques for the solution of this model which sustain all-important structural properties like positivity and, the stability of steady states. All the important properties of both techniques are discussed mathematically and validated with graphical simulations. The M-matrix theory is applied to demonstrate the positivity of the numerical solutions. The philosophy of the operating splitting schemes is discussed, moreover, the order of accuracy and stability conditions for the numerical methods are described. The steady state of the model is presented i.e. virus-free state and virus existing state. Numerical simulations for both steady states are presented. All the graphs reflect the fact that the schemes converge towards the exact steady states, with positive values against each variable which is an important feature for a numerical scheme in the theory of dynamical systems. In the future, these techniques can be employed on delay reaction–diffusion models, advection–reaction–diffusion models, advection-reaction models, etc.



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