***Paradoxical partners: semantical brides and set-theoretical grooms***

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Some twenty years ago, I published in this journal (Goldstein 1994) a set-theoretical counterpart of Yablo’s paradox. Finding this new paradox was so easy that I felt a bit guilty about putting my name to a paper most of the credit for which was due to Stephen Yablo. That my task proved to be such an undemanding one should have given me pause, and caused me to ask the following question: Is there a simple key for ‘translating’ *each* semantical paradox into a partner set-theoretical paradox, and *vice-versa*? If there are at least some such partnerships, then the prospect of a unified solution to a range of logico-semantical paradoxes looks healthy. On the present occasion, I hope to make amends for my earlier mistake of not asking the right question.

Yablo’s paradox consists of an infinite sequence of sentences. For any positive n, the nth sentence, Yn, in the sequence is

For all k > n, ¬(Yk is true).

We can state the truth-conditions for Yn as follows:

1. Yn is true ↔ for all k > n, ¬(Yk is true)

Any assignment of truth value to any member of the Yablo sequence leads to contradiction. In the set-theoretical counterpart of the Yablo paradox, we have an infinite sequence of definitions of classes. For any n, the nth definition, Dn, in the sequence is

(x)((x ϵ Y\*n) ↔ for all k > n, ¬(x ϵ Y\*k))

Where ‘Yi\*’ is a class, the ‘partner’ of the proposition that Yi. So, for some arbitrary instantiation α of x, we have

1. (α ϵ Y\*n) ↔ for all k > n, ¬( α ϵ Y\*k)

The structural similarity between 1. and 2. is striking, and, by simple lexical replacement or ‘translation’, *α ϵ Ψ\** for *Ψ is true*, the two become interchangeable. Both seem to be stating conditions — 1., for a certain statement to be true; 2., for a certain class to have some object as a member.

Let us turn now to Curry’s paradox. As is well known, this paradox has a set-theretical and a semantical incarnation. An indexical version of the latter argues to absurdity from the sentence

1. If this sentence is true, then it is true that p

where ‘that p’ refers to any proposition you choose. For simplicity, we shall take the conditional in 3. to be material implication. Clearly 3. cannot be false for, suppose it were; its antecedent would be false, rendering the material conditional true, contrary to supposition. So 3. must be true, which means that its antecedent and hence its consequent must be true. But its consequent was any proposition you like — for any proposition, we seem, absurdly, to have a proof that it must be true. By similar reasoning about

1. This sentence is false and it is true that p

we seem able to prove that its second conjunct must be false. This is, in essence, the Epimenidean version of the Liar (Goldstein 1986).

In order to sharpen the paradoxicality of 3., we can choose for the proposition that p a necessary falsity. The mediaevals were fond of using ‘God does not exist’ for this purpose but we, more cautiously, could choose the proposition that 2 + 2 = 5, or the proposition that the successor of 0 equals 0, and represent it as ‘⊥’. The modified Curry statement C then becomes

1. C is true → ⊥

So the truth-conditions for C may be written as

1. C is true ↔ ((C is true→ ⊥)

or, alternatively,

1. C is true ↔ ¬(C is true) v ⊥

Since the disjunction ‘v ⊥’ does not affect the truth condition of the RHS of the biconditional, it can be deleted, and the relationship between Curry’s paradox and the (strengthened) Liar then becomes manifest. If C is our semantical bride then, by applying to 7. the translation key identified above, we have

1. (α ϵ C\*) ↔ ¬( α ϵ C\*) v ⊥

and, by Universal Generalization and definition of ‘→’

1. (x) (x ϵ C\*) ↔ (( x ϵ C\*) → ⊥)

Thus the (modified) set-theoretical Curry paradox is generated by the following instantiation of the naïve axiom of comprehension

1. (∃y)(x) (x ϵ y) ↔ (( x ϵ y) → ⊥)

We remarked above on the close similarity between the semantical Curry paradox and the Strengthened Liar S which says of itself that it is not true. The truth-conditions for S are given by

1. S is true ↔ S is not true

Using our translation key, the set-theoretical groom for the Strengthened Liar is

1. (∃y)(x) (x ϵ y ↔ ¬( x ϵ y))

One cannot help but notice that this groom is no match for his bride in terms of depth. The Strengthened Liar has puzzled logicians for hundreds of years. By contrast, it is no surprise that one can derive a contradiction from 12., so 12. hardly qualifies for the epithet ‘paradox’. Let us suppose (or pretend) that 12. defines a class K. Then we have it that any object is a member of K if and only if it is not. Clearly, 12. does not provide a recipe for determining, for any object, whether or not it is a member of K, so, contrary to what we supposed (or pretended), 12. does not define a class; there is no class K. The reason for this failure to define a class becomes even clearer if we consider the groom of the Truth-teller sentence, ‘This statement is true’, viz.,

1. (∃y)(x) (x ϵ y ↔ x ϵ y)

As in the case of 12., the attempt to define a class by identifying its members is viciously circular. Some object is a member if and only if ….. it’s a member!

Unfortunately, our simple translation key does not provide us with a semantical bride for the Russell paradox

1. (∃y)(x)(x ϵ R\* & ~(x ϵ x))

but we have encountered, above, a ‘paradox’ so easily solvable as hardly to merit the title ‘paradox’, so perhaps examining the solution to that will lead to the solution of harder paradoxes like Russell’s.

The easy ‘paradox’ to which I am referring is 13. A class is properly defined when the conditions (or biconditions) of its membership supply a recipe for determining, for any object, whether that object is or is not a member of that class. Clearly, the tautological biconditional in 13. provides no such recipe; it is vacuous. This is another way of saying that it is circular: we have that p is true if and only if it is true that p. The same circularity afflicts 11., which consequently defines no class. This suggests, then, a simple restriction on the naïve comprehension axiom

1. (∃y)(x)(x ϵ y ↔ Fx)

The restriction forbids substituends for x, y and F in the scope*(x ϵ y ↔ Fx)* such that the result of any such substitution entails a tautology or a contradiction; for tautologies and contradictions fail to state conditions or to give a recipe for determining whether any given object belongs in the set. The rationale is simply to avoid the vacuity of circular definition. So the rationale is quite different from that of an axiomatic system such as ZFC which is driven by the imperative to avoid contradiction. Note that the definition of the empty set is not inhibited by this restriction; it may be defined as

1. (∃y)(x)(x ϵ y ↔ ⊥)

but Russell’s sentence 14. does not define a class because the result of substituting R\* / y, ¬(x ϵ x) / Fx in 15. entails a contradiction — instantiate x with R\*.

The naïve comprehension axiom captures a very clear conception of sets, and it makes sense to tamper with it as little as possible, identifying and justifying the restriction that must be made to its application, rather than building up set theory from entirely new foundations. One might compare the justification for this restriction with the kind of justification we give for restricting the rules *Universal Generalization* and *Existential Instantiation* in natural deduction systems. These restrictions are not offered merely as prophylactics that inexplicably prevent our going astray. The restriction on substitution in the axiom of comprehension is explained by explaining what is wrong with circular definition.

According to our translation key, the counterpart semantical restriction prevents circularity in truth (bi-)conditionals of the form

1. *Ψ* is true ↔ p

by prohibiting substitutions that entail tautology and contradiction. The biconditional 11. is thus forbidden, so there are no legitimate truth conditions for the Liar. What distinguishes a sentence from a proposition is that the latter is a semantical entity that makes a claim and has truth-conditions, so it follows that sentences such as the Liar sentence, the truth-teller sentence, the Curry sentence and all the sentences in the Yablo sequence fail to express propositions. As sentences, they are syntactically in order, but they cannot be used to say or to state anything. In particular, no sentence that fails the aforementioned test for propositionhood says of itself that it cannot be used to express a proposition.[[1]](#endnote-1) In set theory, there are well-formed formulae that fail to define any sets, and, on the bridal side, there are sentences that fail to express propositions. So if someone tries to tell you that the Strengthened Liar sentence says that it is either false or fails to express a proposition, you should stop them right there. The sentence in question does not say anything at all, so the conclusion that it is true — because it fails to express a proposition and *says just that* — does not follow. There is no revenge paradox.

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1. For discussion of the distinction between propositions that refer to themselves and propositions that refer to the sentences used to express them, see Kripke (1975: 693). [↑](#endnote-ref-1)